1-D Wave Equations

Transmission lines

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The electrical case

Series elements:

$e$ is the electromotive force

From Ohm’s Law

$$\lim_{\delta x \rightarrow 0} \frac{e_1 - e_2}{x_1 - x_2} = -\frac{\partial e}{\partial x} = L \frac{\partial i}{\partial t}$$
The electrical case, cont.

Shunt elements:

$i$ is the flow of charge

\[ \frac{i_1 - i_2}{\delta x} = C \frac{\partial e}{\partial t} \]

\[ i_1(x, t) \quad e_1 = e_2 \quad e_2(x + \delta x, t) \]

\[ i_2(x + \delta x, t) \quad C \quad (F/m) \]

thus from Ohm’s Law

(1)

\[ - \frac{\partial i}{\partial x} = C \frac{\partial e}{\partial t}. \]
The electrical case, cont.

- Remove $i$ to obtain an expression for $e$

- From Eq. 1, operating with $\partial/\partial x$

\[- \frac{\partial^2 i}{\partial x \partial t} = \frac{1}{L} \frac{\partial^2 e}{\partial x^2}.\]

- From Eq. 2, operating with $\partial/\partial t$

\[- \frac{\partial^2 i}{\partial t \partial x} = C \frac{\partial^2 e}{\partial t^2}.\]

- Equate the RHSs to obtain the wave equation in $e$:

$$\frac{1}{c^2} \frac{\partial^2 e}{\partial t^2} = \frac{\partial^2 e}{\partial x^2}; \quad c = 1/\sqrt{LC}, \quad z_0 = \sqrt{L/C}$$
The acoustic case

Pressure $p(x, t)$ is similar to $e(x, t)$ (scaler force, Ohm’s Law)

\[- \frac{\partial p}{\partial x} = \rho_0 \frac{\partial u}{\partial t}\]

Volume velocity $u$ similar to $i$ (vector flow, Ohm’s Law)

\[-\gamma P_0 \frac{\partial u}{\partial x} = \frac{\partial p}{\partial t}.\]

thus

\[
\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 p}{\partial x^2}
\]

Defs: $P_0 = 10^5$ [Pa], $\gamma = 1.4$, $\rho_0 = 1.18$ [kgm/m$^3$];

\[c = \sqrt{\frac{\gamma P_0}{\rho_0}} \text{[m/s]}, \ z_0 = \sqrt{\rho_0 \gamma P_0} \text{[acoustic ohms]}\]
The mechanical case

- Force $f$ is similar to $e$ and $p$: From Newton’s Law
  
  $$-\frac{\partial f}{\partial x} = m \frac{\partial u}{\partial t}$$

- Particle velocity $u$ similar to $i$: From Hooke’s Law
  
  $$-k \frac{\partial u}{\partial x} = \frac{\partial f}{\partial t}.$$ 

- thus, as before:
  
  $$\frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}$$

- where $c = \sqrt{\frac{k}{m}}$ m/s; $z_0 = \sqrt{km}$, with $k$ [Nt/m$^2$]; $m$ [kgm/m].
Solution to the wave equation

Assume two arbitrary functions $f(\zeta)$ and $g(\zeta)$, traveling with velocity $c$ and $-c$, respectively:

$$\Phi(x, t) = f(ct - x) + g(ct + x)$$

The proof of this is nice. Let $f' \equiv \partial f(\zeta)/\partial \zeta$.

$$\frac{\partial \Phi(x, t)}{\partial x} = -f' + g'; \quad \frac{\partial^2 \Phi(x, t)}{\partial x^2} = (-1)^2 f'' + g''$$

$$\frac{\partial \Phi(x, t)}{\partial t} = c(f' + g'); \quad \frac{\partial^2 \Phi(x, t)}{\partial t^2} = c^2(f'' + g'')$$

Thus:

$$\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = \frac{\partial^2 \Phi}{\partial x^2}$$
Summary

- The wave equation describes any *linear system* that allows waves (arbitrary functions) to travel at the wave speed, in different directions, at the same time.
- It obeys linearity (superposition).
- Note that this proof applies to the 1-dimension case.
- All transmission line theory can be understood with these simple ideas (i.e., two waves traveling in opposite directions).

**Diagram:**

- Pressure \( p(t) \) [voltage \( v(t) \), force \( f(t) \)]
- Volume velocity \( u(t) \) [current \( i(t) \), particle velocity \( v(t) \)]
- Load \( Z_{load} \)