An Invitation to Mathematical Physics
and Its History

Jon B. Allen
<table>
<thead>
<tr>
<th>Week</th>
<th>Topics</th>
</tr>
</thead>
<tbody>
<tr>
<td>H.1</td>
<td>Week 1</td>
</tr>
<tr>
<td>H.2</td>
<td>Week 2</td>
</tr>
<tr>
<td>H.3</td>
<td>Week 3</td>
</tr>
<tr>
<td>H.4</td>
<td>Week 4</td>
</tr>
<tr>
<td>G.1</td>
<td>Week 2</td>
</tr>
<tr>
<td>G.2</td>
<td>Week 3</td>
</tr>
<tr>
<td>G.3</td>
<td>Week 4</td>
</tr>
<tr>
<td>H.1.1</td>
<td>Week 1</td>
</tr>
<tr>
<td>H.1.2</td>
<td>Week 1</td>
</tr>
<tr>
<td>H.1.3</td>
<td>Week 1</td>
</tr>
<tr>
<td>H.1.4</td>
<td>Week 1</td>
</tr>
<tr>
<td>H.2.1</td>
<td>Week 2</td>
</tr>
<tr>
<td>H.2.2</td>
<td>Week 2</td>
</tr>
<tr>
<td>H.2.3</td>
<td>Week 2</td>
</tr>
<tr>
<td>H.2.4</td>
<td>Week 2</td>
</tr>
<tr>
<td>H.3.1</td>
<td>Week 3</td>
</tr>
<tr>
<td>H.3.2</td>
<td>Week 3</td>
</tr>
<tr>
<td>H.3.3</td>
<td>Week 3</td>
</tr>
<tr>
<td>H.3.4</td>
<td>Week 3</td>
</tr>
<tr>
<td>H.4.1</td>
<td>Week 4</td>
</tr>
<tr>
<td>H.4.2</td>
<td>Week 4</td>
</tr>
<tr>
<td>H.4.3</td>
<td>Week 4</td>
</tr>
<tr>
<td>H.4.4</td>
<td>Week 4</td>
</tr>
<tr>
<td>G.1.1</td>
<td>Week 2</td>
</tr>
<tr>
<td>G.1.2</td>
<td>Week 2</td>
</tr>
<tr>
<td>G.1.3</td>
<td>Week 2</td>
</tr>
<tr>
<td>G.1.4</td>
<td>Week 2</td>
</tr>
<tr>
<td>G.2.1</td>
<td>Week 3</td>
</tr>
<tr>
<td>G.2.2</td>
<td>Week 3</td>
</tr>
<tr>
<td>G.2.3</td>
<td>Week 3</td>
</tr>
<tr>
<td>G.2.4</td>
<td>Week 3</td>
</tr>
<tr>
<td>G.3.1</td>
<td>Week 4</td>
</tr>
<tr>
<td>G.3.2</td>
<td>Week 4</td>
</tr>
<tr>
<td>G.3.3</td>
<td>Week 4</td>
</tr>
<tr>
<td>G.3.4</td>
<td>Week 4</td>
</tr>
<tr>
<td>H.1.1.1</td>
<td>Week 1.1</td>
</tr>
<tr>
<td>H.1.1.2</td>
<td>Week 1.1</td>
</tr>
<tr>
<td>H.1.1.3</td>
<td>Week 1.1</td>
</tr>
<tr>
<td>H.1.1.4</td>
<td>Week 1.1</td>
</tr>
<tr>
<td>H.1.2.1</td>
<td>Week 1.2</td>
</tr>
<tr>
<td>H.1.2.2</td>
<td>Week 1.2</td>
</tr>
<tr>
<td>H.1.2.3</td>
<td>Week 1.2</td>
</tr>
<tr>
<td>H.1.2.4</td>
<td>Week 1.2</td>
</tr>
<tr>
<td>H.1.3.1</td>
<td>Week 1.3</td>
</tr>
<tr>
<td>H.1.3.2</td>
<td>Week 1.3</td>
</tr>
<tr>
<td>H.1.3.3</td>
<td>Week 1.3</td>
</tr>
<tr>
<td>H.1.3.4</td>
<td>Week 1.3</td>
</tr>
<tr>
<td>H.1.4.1</td>
<td>Week 1.4</td>
</tr>
<tr>
<td>H.1.4.2</td>
<td>Week 1.4</td>
</tr>
<tr>
<td>H.1.4.3</td>
<td>Week 1.4</td>
</tr>
<tr>
<td>H.1.4.4</td>
<td>Week 1.4</td>
</tr>
<tr>
<td>H.2.1.1</td>
<td>Week 2.1</td>
</tr>
<tr>
<td>H.2.1.2</td>
<td>Week 2.1</td>
</tr>
<tr>
<td>H.2.1.3</td>
<td>Week 2.1</td>
</tr>
<tr>
<td>H.2.1.4</td>
<td>Week 2.1</td>
</tr>
<tr>
<td>H.2.2.1</td>
<td>Week 2.2</td>
</tr>
<tr>
<td>H.2.2.2</td>
<td>Week 2.2</td>
</tr>
<tr>
<td>H.2.2.3</td>
<td>Week 2.2</td>
</tr>
<tr>
<td>H.2.2.4</td>
<td>Week 2.2</td>
</tr>
<tr>
<td>H.2.3.1</td>
<td>Week 2.3</td>
</tr>
<tr>
<td>H.2.3.2</td>
<td>Week 2.3</td>
</tr>
<tr>
<td>H.2.3.3</td>
<td>Week 2.3</td>
</tr>
<tr>
<td>H.2.3.4</td>
<td>Week 2.3</td>
</tr>
<tr>
<td>H.2.4.1</td>
<td>Week 2.4</td>
</tr>
<tr>
<td>H.2.4.2</td>
<td>Week 2.4</td>
</tr>
<tr>
<td>H.2.4.3</td>
<td>Week 2.4</td>
</tr>
<tr>
<td>H.2.4.4</td>
<td>Week 2.4</td>
</tr>
<tr>
<td>H.3.1.1</td>
<td>Week 3.1</td>
</tr>
<tr>
<td>H.3.1.2</td>
<td>Week 3.1</td>
</tr>
<tr>
<td>H.3.1.3</td>
<td>Week 3.1</td>
</tr>
<tr>
<td>H.3.1.4</td>
<td>Week 3.1</td>
</tr>
<tr>
<td>H.3.2.1</td>
<td>Week 3.2</td>
</tr>
<tr>
<td>H.3.2.2</td>
<td>Week 3.2</td>
</tr>
<tr>
<td>H.3.2.3</td>
<td>Week 3.2</td>
</tr>
<tr>
<td>H.3.2.4</td>
<td>Week 3.2</td>
</tr>
<tr>
<td>H.3.3.1</td>
<td>Week 3.3</td>
</tr>
<tr>
<td>H.3.3.2</td>
<td>Week 3.3</td>
</tr>
<tr>
<td>H.3.3.3</td>
<td>Week 3.3</td>
</tr>
<tr>
<td>H.3.3.4</td>
<td>Week 3.3</td>
</tr>
<tr>
<td>H.3.4.1</td>
<td>Week 3.4</td>
</tr>
<tr>
<td>H.3.4.2</td>
<td>Week 3.4</td>
</tr>
<tr>
<td>H.3.4.3</td>
<td>Week 3.4</td>
</tr>
<tr>
<td>H.3.4.4</td>
<td>Week 3.4</td>
</tr>
<tr>
<td>H.4.1.1</td>
<td>Week 4.1</td>
</tr>
<tr>
<td>H.4.1.2</td>
<td>Week 4.1</td>
</tr>
<tr>
<td>H.4.1.3</td>
<td>Week 4.1</td>
</tr>
<tr>
<td>H.4.1.4</td>
<td>Week 4.1</td>
</tr>
<tr>
<td>H.4.2.1</td>
<td>Week 4.2</td>
</tr>
<tr>
<td>H.4.2.2</td>
<td>Week 4.2</td>
</tr>
<tr>
<td>H.4.2.3</td>
<td>Week 4.2</td>
</tr>
<tr>
<td>H.4.2.4</td>
<td>Week 4.2</td>
</tr>
<tr>
<td>H.4.3.1</td>
<td>Week 4.3</td>
</tr>
<tr>
<td>H.4.3.2</td>
<td>Week 4.3</td>
</tr>
<tr>
<td>H.4.3.3</td>
<td>Week 4.3</td>
</tr>
<tr>
<td>H.4.3.4</td>
<td>Week 4.3</td>
</tr>
<tr>
<td>H.4.4.1</td>
<td>Week 4.4</td>
</tr>
<tr>
<td>H.4.4.2</td>
<td>Week 4.4</td>
</tr>
<tr>
<td>H.4.4.3</td>
<td>Week 4.4</td>
</tr>
<tr>
<td>H.4.4.4</td>
<td>Week 4.4</td>
</tr>
</tbody>
</table>

**CONTENTS**

1. Introduction to the Branch and Remainder of the Text
2. Algebraic Equations: Stream 2
3. Number Systems: Stream 1
4. Continued Fraction Expansion (CFA)
5. Transmission Line Example
6. Gaussian Elimination of Linear Equations
7. Eigen-Analysis of Fibonacci Matrix
8. Generalized Impedance and Transmission Lines
9. Physics of Linear vs. Nonlinear Complex Analytic Expressions
10. Introduction to the Branch Cut and Riemann Sheets
11. Greatest Common Divisor (GCD)
12. Introduction to Fourier and Laplace Transforms
13. Derivation of Euclid's Formula for Pythagorean Triplets (PTs)
14. Matrix Composition: Bilinear and ABCD
15. Differential Equations
16. Laplace Transform Table
17. Lagrange, 50
18. Kirchhoff, 58, 282
19. Newton's Laws of Motion, 140
22. Partial Fraction Expansion, 71, 118, 120, 279, 281
23. Passive, 262, 264
24. Impedance, 78, 92, 93, 100, 103, 140
26. Impedance, Brune, 78, 278
27. Algebra, 102
28. Non-Reciprocal, See Reciprocity, 102
29. Partial Fraction Expansions, 71, 118, 120, 279, 281
30. Critical Points, 118, 120, 279
31. Impedance, Poles and Zeros, 147
Abstract
An understanding of physics requires knowledge of mathematics. The converse is not true. By
making the assignments and the exams entirely based on the assignments. It is my phi-
osophy that, in principle, the course is about learning mathematics, not physics, that
students are learning through the process of doing the assignments and exams. The
exams are entirely based on the assignments. It is my philosophy that, in principle,
the course is about learning mathematics, not physics, that students are learning through the process of doing the assignments and exams.
the students can see the exam in advance of taking it. In a real sense they do, since each exam is based on the assignments.

Author’s Personal Statement

An expert is someone who has made all possible mistakes in a small field. I don’t know if I would be called an expert, but I certainly have made my share of mistakes. I openly state that “I love making mistakes, because I learn so much from them.” One might call that the “expert’s corollary.”

This book has been written out of both my love for the topic of mathematical physics, and a desire to share many key concepts, and many new ideas on these basic concepts. Over the years I have developed a certain physical sense of math, along with a related mathematical sense of physics. While doing my research, I have come across what I feel are certain conceptual holes that need filling, and sense many deep relationships between math and physics that remain unidentified. While what we presently teach is not wrong, it is missing these relationships. What is lacking is an intuition for how math “works.” Good scientists “listen” to their data. In the same way we need to start listening to the language of mathematics. We need to let mathematics guide us toward our engineering goals.

1https://auditorymodels.org/index.php/Main/Publications

Index

F: See fractional numbers, 30
Γ(s): reflectance, 147
∇: See gradient, 140
N: See counting numbers, 29, 185
P: See primes, 29
Q: See rational numbers, 30
F(s): see transmission matrix, 92
Y: See characteristic admittance, 150
Z: See integers, 30
Z: See characteristic impedance, 150
δ(t) function, 254
δ: Dirac function, 257
κ: propagation function, 57
⊥: See perp p. 186, 30
tan(s), inverse, 105
tan⁻¹(s), 105
ζ(s): See zeta function, 256
ζ(n), Riemann, 40
ζ(x). Euler, 40
ka < 1, 171
u(t) function, 254
u(t) step function, 210
u₄ Heaviside function, 257
zviz.m, see: plots, colorized, 255
ceil(c), 46
compan, 68
conv(), 77
conv(A, B), 67
decconv(A, B), 68
fix(c), 46
floor(c), 46
floor(x), 45
mod(x, y), 47
poly(R), 67
polyder(), 67
polyval(), 77
polyval(A, x), 67
rem(x, y), 47
residue(), 77
residue(N, D), 67
root(), 77
root, 74
round(c), 46
round(x), 45
roundb(c), 46
1-port, 103
2-port, 103
abacus, 29, 172
ABCD: see transmission matrix, 92
abscessa, see domain, 119
active/passive, 264
admittance, poles and zeros, 147
admittance: see impedance, 93
algebraic network, see Transmission matrix, 100
analytic functions, 71, 105
analytic geometry, 79, 83
analytic, complex, 38, 58, 73, 101
analytic, evolution operator, 57
anti-reciprocal, see reciprocity, 102
Bardeen, 74
Beranek, 174
Bernaulli, Daniel, 58
Bernaulli, Jakob, 25
Bernaulli, Johann, 25
Bessel function, 118, 123, 124
Bethe, 74
Bhaskara II, 50
bi-harmonic, 110
bilinear transformation, 77, 95, 188, 246, 247, 264
Bou, 174
Bode, 174
Bombelli, 19, 23, 33, 79, 106
Brahmagupta, 20, 29, 50, 52, 173
branch cut, 115, 116, 252
branch cuts, 116

221
This is foremost a math book, but not the typical math book. I could absorb the advanced material at a reasonable pace. This book soon followed. I had much more to master. It became clear that by teaching this material to first year engineers, I could succeed at something, take it as far as you can. But on the other hand "you should not do something when you are not ready." Pursuits were not only possible, they were openly encouraged. The idea was that if you are successful at something, take it as far as you can. The answer to such a cosmic question depends strongly on who you ask. Who is qualified to answer such a question? It is best answered by those who study mathematics applied to the physical world. The utility and accuracy of that answer depends critically on the depth of understanding of the physics. Mathematics is a powerful tool that we make progress in understanding the physical world. We must turn to mathematics and physics when trying to understand the universe. My views follow from a lifelong attempt to understand the cosmic clock. Bhaskara II (1114–1185 CE) had mastered the art well before Newton. Planets. To do this Newton needed mathematics, a tool he had mastered. It is widely accepted that Newton's grand treatise, When Halley asked Newton to explain how he knew, Newton responded "I calculated it." But when challenged to show the calculation, Newton was unable to reproduce it. This open challenge immediately answered "an ellipse." It is said that Halley was stunned by the response (Stillwell, 2002).

1919
1946
1949
1952
1964
1970
1973
1989
2002
2012
1643
1656–1742
1687
1727
2002
Third, the main goal of this book is to teach motivated engineers mathematics, in a way that it can be understood, mastered and remembered. How can this impossible goal be achieved? The answer is to fill in the gaps with Who did what, and when? Compared with the math, the historical record is easily mastered.

To be an expert in a field, one must know its history. This includes who the people were, what they did, and the credibility of their story. Do you believe the Pope or Galileo on the roles of the sun and the earth? The observables provided by science are clearly on Galileo’s side. Who were those first engineers? They are names we all know: Archimedes, Pythagoras, Leonardo da Vinci, Galileo, Newton, etc. All of these individuals had mastered mathematics. This book presents the tools taught to every engineer. Rather than memorizing complex formulas, make the relations “obvious” by mastering each simple underlying concept.

Fourth, when most educators look at this book, their immediate reactions are: Each lecture is a topic we spend a week on (in our math/physics/engineering class). And: You have too much material crammed into one semester. The first sentence is correct, the second is not. Tracking the students who have taken the course, looking at their grades, and interviewing them personally, demonstrate that the material presented here is appropriate for one semester.5

To write this book I had to master the language of mathematics (John D’Angelo language). I had already mastered the language of engineering, and a good part of physics. One of my secondary goals is to build this scientific Tower of Babel, by unifying the terminology and removing the jargon.

Acknowledgments

Besides thanking my parents, I would like to credit John Stillwell for his constructive, historical summary of mathematics. My close friend and colleague Steve Levinson somehow drew me into this project, without my even knowing it. My brilliant graduate student Sarah Robinson was constantly at my side, grading homeworks and exams, and tutoring the students. Without her, I would never have survived the first semester the material was taught. Her proofreading skills are amazing. Thank you Sarah for your infinite help. Finally I would like to thank John D’Angelo for putting up with my many silly questions. When it comes to the heavy hitting, John was always there to provide a brilliant explanation that I could easily translate into Engineersese (Matheering?) (i.e., engineer language).

My delightful friend Robert Fossum, emeritus professor of mathematics from the University of Illinois, who kindly pointed out my flawed use of mathematics. James (Jamie) Hutchinson’s precise use of the English language, dramatically raised the bar on my more than occasionally-casual writing style. To each of you, thank you!

Finally I would like to thank my wife Sheau Feng Jeng, aka Patricia Allen, for her unbelievable support and love. She delivered constant peace of mind, without which this project could never have been started, much less finished. Many others played important roles, but they must remain anonymous.

—Jont Allen, Mahomet II., Dec. 24, 2015 (Jan 1, 2018)

http://www.istem.illinois.edu/news/jont.allen.html

Menzies, G. (2008), 1434: The year a Magnificent Chinese fleet sailed to Italy and ignited the renaissance (HarperCollins, NYC).


CONTENTS

Mathematics and its History (Stillwell, 2002)

Figure 2: Table of contents of Stillwell (2002)

BIBLIOGRAPHY


It is widely acknowledged that interdisciplinary science is the backbone of modern scientific inquiry as embodied in the STEM (Science, Technology, Engineering, and Mathematics) programs. However, while communication networks are a core component of these programs, interdisciplinary science is not being taught, due to its inherent complexity and breadth. To create an interdisciplinary STEM program, the MEP core is critical. In my view, this core should be taught to all first-year students, not just those trained in mathematics, as it provides a common foundation for understanding the physical principles underlying modern technology.

To exercise interdisciplinary science, students need to be trained to deal with uncertainty. Richard Hamming expressed this thought succinctly: "You are only shown finished theorems and proofs. That is just plain wrong. We must give up the results, and teach the methods. You are only shown finished theorems and proofs. That is just plain wrong. We must give up the results, and teach the methods. You are only shown finished theorems and proofs. That is just plain wrong. We must give up the results, and teach the methods.

In my view, this core should be taught to every student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student. The core of this fundamental theorems of mathematics should be taught to every MEP student.
The present six-semester regime serves many students poorly, defined in terms of the needs of the student. The goal should not be to be drilling on surface integrals, at least not in the first year. As suggested by Hamming, these details must be self-taught, at the time they are needed. Furthermore, start with the students who place out of early math courses: they love math and are highly motivated to learn as much as possible. I contend that if first or second year students are given a comprehensive early conceptual understanding, they will end up at the top of their field.

As identified by Hamming, a key problem is the traditional university approach, a five to eight semester sequence of: Calc I, II, III, Linear Algebra IV, DiffEq V, Real analysis VI, Complex analysis VII and given near-infinite stamina, Number theory VIII, over a time frame of three or more years (six semesters). This was the way I learned math. The process simply took too long, and the concepts spread too thin. After following this regime, I felt I had not fully mastered the material, so I started over. I consider myself to be largely self-taught.

We need a more effective teaching method. I am not suggesting we replace the standard six semester math curriculum of Calc I, II, III, etc. Rather, I am suggesting a broad unified introduction to all these topics, based on an historical approach. The present approach is driving the talent away from science and mathematics, by focusing too much on the details (as clearly articulated by Hamming). One needs more than a high school education to succeed in college engineering courses. The key missing element in our present education system is teaching critical thought. Drilling facts does not do that.

By learning mathematics in the context of history, the student will fully and easily appreciate the underlying concepts. The history provides a uniform terminology for understanding the fundamentals of MEP. The present teaching method, using abstract proofs, with no (or few) figures or physical principles and units, by design, removes intuition and the motivation that was available to the creators of these fundamentals. The present six-semester regime serves many students poorly, leaving them with little insight (i.e., intuition) and an aversion to mathematics.

**Postscript Dec 5, 2017** How to cram five semesters of math into one semester, and leave the students with something they can remember? Here are some examples:

Maxwell’s equations (MEs) are a fundamental challenging topic, presented in one lecture (Section 1.5.15, p. 166). Here is how it works:

1. The development starts with Sections 1.4.1-1.4.9 (pp. 106-124), which develop complex integration (p. 108) and the Laplace transform (pp. 122-123).
2. Kennelly’s (1893) 1893 complex impedance, as defined by Ohm’s law, is the ratio of the force over the flow, the key elements being 1) capacitance (e.g., compliance) per unit area ($c_F$ [F/m²]), and 2) inductance (e.g., mass) per unit area ($l_F$ [Hz/m²]).
3. Section 1.5.1 (p. 127) develops analytic field theory, while Sect. 1.5.2 (p. 129) introduces Grad $\nabla \cdot$, Div $\nabla \cdot$, and Curl $\nabla \times$, starting from the scalar $A \cdot B$ and vector product $A \times B$ of two vectors.
4. On p. 161, second-order operators are introduced and given physical meanings (based on the physics of fluids), but most important they are given memorable names (DoC, CoG, Dog, and CoC: p. 129). Thanks to this somewhat quaint and gamy innovation, the students can both understand and easily remember the relationships between these confusing second-order vector calculus operations.

**Bibliography**


Campbell, G. (1903a), “On Loaded Lines in Telephonic Transmission,” Phil. Mag. 5, 313–331, see Campbell22a (footnote 2): In that discussion, ... it is tacitly assumed that the line is either an actual line with resistance, or of the limit such with $R=0$. 215
His communication skill was a result of his depth of understanding, and he was not afraid to question the present understanding of physics. He was always on a quest. He died in 1988, at the age of 70. Let us all be on his quest. Any belief that we have figured out the ways of the universe is absurd. We have a lot to learn. Major errors in our understanding must to be corrected. We cannot understand the world around us until we understand its creation. That is, we cannot understand where we are going until we understand where we came from.

– Jont Allen

Postscript Dec 15, 2017 As this book comes to completion, I’m reading and appreciating the Feynman lectures. We all know (I hope you know) that Feynman had a special lecture style, that was both entertaining, and informative. His communication skill and the present understanding of physics. He was always on a quest. He died in 1988, at the age of 70. Let us all be on his quest. Any belief that we have figured out the ways of the universe is absurd. We have a lot to learn. Major errors in our understanding must to be corrected. We cannot understand the world around us until we understand its creation. That is, we cannot understand where we are going until we understand where we came from.

– Jont Allen

Contents 15
5. Exercises and examples are interspersed throughout the lectures.
6. The foregoing carefully sets the stage for ME (p. 166), introduced using proper names and units of the electric and magnetic field intensity (strengths, and forces seen by charge) \( E, H \), and electric and magnetic flux (flow) \( D, B \), as summarized on page 167. The meanings of ME equations are next explored in integral form. After Sect. 1.5.2–1.5.15, the students are fully conversant with MEs. The conformation of this is in the final exam grade distributions.

Appendix F. Tables of Fourier and Laplace Transforms

Thus the companion matrix for the numerator polynomial \( N(s) \) is

\[
\begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 1 \\
-1 & 0 & \ldots & 0 & 0
\end{bmatrix}
\]

(E.2)

Thus the companion matrix for the denominator polynomial \( D(s) \) is

\[
\begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 1 \\
-1 & 0 & \ldots & 0 & 0
\end{bmatrix}
\]

(E.3)

while the denominator polynomial of \( Z(s) \) is \( D(s) = Y_N - \lambda N \).

2. Note that these equations need to include the determination of unknown \( L_n \) and \( i_n \), which in some cases will be zero.

3. Note that these equations need to include the determination of unknown \( L_n \) and \( i_n \), which in some cases will be zero.

4. General method to substitute \( s = s_n \) in \( Z(s) \), to define a non-degenerate linear system of equations in \( X_n \), having no zero determinant (\( \det(X_n) \neq 0 \)).

5. This method has close ties to the classic CPA, where it has been called the Cauer decomposition, named after its inventor Wilhelm Cauer (Cauer and Mathis, 1995; Cauer et al., 2000; Cauer, 1958; Cauer et al., 1958), who acted as the primary thesis advisor for Brune (Brune, 1931b; Van Valkenburg, 1964b).
F.1 Methods for automating the calculation of residues

In this appendix we shall set up the general problem of finding \( K_k \) given Eq. 1.52 (Gustavsen and Semlyen, 1999).

\[
Z(s) = \frac{N(s)}{D(s)} = sL_0 + R_0 + \sum_{k=0}^{K} \frac{K_k}{s - s_k},
\]

(F.1)

given the roots \( s_k \) of polynomial \( D(s) = \prod_{k=1}^{K} (s - s_k) = 0 \).

1. First discuss the general properties of \( Z(s) = K_{-1}s + K_0 + \sum_{k=1}^{K} \frac{K_k}{s - s_k} \).

Exercise: The impedance may be written as

\[
Z(s) = \frac{N(s)}{D(s)} = \frac{\sum_{m} n_m s^m}{\sum_k d_k s^k}
\]
Introduction

Chapter 1

1.1 Early Science and Mathematics

While early Asian mathematics is not fully documented, it clearly defined the course for mathematics for a thousand years. The first recorded mathematics were those of the Chinese (5000 BCE), followed by the Babylonians (Mesopotamia, 1800 BCE), and the Egyptians (2000 BCE). The Greeks, including Pythagoras, are well known for their early contributions to early music theory. We are largely ignorant of much of early mathematics dating back to 1600 BCE.

Much of early mathematics during 1600 BCE centered around the love of art and music. The Pythagorean view is relevant today: 

\[ \sum_{n=0}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \]

As acknowledged by Stillwell (2010, p. 16), the Pythagorean view is relevant today:

Pythagoras strongly believed that “all is number,” meaning that every number, and every mathematical relationship, is proportional to the sensations of light and sound. Our psychological sense of color and pitch are determined by the auditory organ (the cochlea).

\[ \frac{\text{frequency ratio}}{18} \approx 1.17 \] 

This innate sense of frequency ratios comes from the physiology of our psychological sense of color and pitch are determined by the auditory organ (the cochlea).
p. 235) provides a method for computing Pythagorean triplets, a formula believed to be due to the Chinese (Stillwell, 2010, pp. 4-9).2

Chinese bells and stringed musical instruments were exquisitely developed with tonal quality, as documented by ancient physical artifacts (Fletcher and Rossing, 2008). In fact this development was so rich that one must ask why the Chinese failed to initiate the industrial revolution. Specifically, why did Europe eventually dominate with its innovation when it was the Chinese who did the extensive early invention?

It could have been for the wrong reasons, but perhaps our best insight into the scientific history from China may have come from an American chemist and scholar from Yale, Joseph Needham, who learned to speak Chinese after falling in love with a Chinese woman, and ended up researching early Chinese science and technology for the US government. According to Lin (1995) this is known as the Needham question:

“Why did modern science, the mathematization of hypotheses about Nature, with all its implications for advanced technology, take its meteoric rise only in the West at the time of Galileo[, but] had not developed in Chinese civilization or Indian civilization?”

Needham cites the many developments in China:3

“Gunpowder, the magnetic compass, and paper and printing, which Francis Bacon considered as the three most important inventions facilitating the West’s transformation from the Dark Ages to the modern world, were invented in China.” (Lin, 1995)

“Needham’s works attribute significant weight to the impact of Confucianism and Taoism on the pace of Chinese scientific discovery, and emphasizes what it describes as the ‘diffusionist’ approach of Chinese science as opposed to a perceived independent inventiveness in the western world. Needham held that the notion that the Chinese script had inhibited scientific thought was ‘grossly overrated ’” (Grosswiler, 2004).

Lin was focused on military applications, missing the importance of non-military contributions. A large fraction of mathematics was developed to better understand the solar system, acoustics, musical instruments and the theory of sound and light. Eventually the universe became a popular topic, as it still is today.

Regarding the “Needham question,” I suspect the resolution is now clear. In the end, China withdrew from its several earlier expansions (Menzies, 2004, 2008).

1.1.1 Lec 1 The Pythagorean theorem

Thanks to Euclid’s Elements (c323 BCE) we have an historical record, tracing the progress in geometry, as established by the Pythagorean theorem, which states that for any right triangle

\[ c^2 = a^2 + b^2, \]  

having sides of lengths \((a, b, c) \in \mathbb{R}\) that are either positive real numbers, or more interesting, integers, such that \(c > a, b\) and \(a + b > c\). Early integer solutions were likely found by trial and error rather than by Euclid’s formula.

If \(a, b, c\) are lengths, then \(a^2, b^2, c^2\) are each the area of a square. Equation 1.1 says that the

In many physical applications, the Laplace transform takes the form of a ratio of two polynomials. In such case the roots of the numerator polynomial are called the zeros while the roots of the denominator polynomial are called the poles. For example the LT of \(u(t) \leftrightarrow 1/s\) has a pole at \(s = 0\), which represents integration, since

\[ u(t) \ast f(t) = \int_{-\infty}^{t} f(\tau)d\tau \leftrightarrow F(s) \]

12. The LT is quite different from the FT in terms of its analytic properties. For example, the step function \(u(t) \leftrightarrow 1/s\) is complex analytic everywhere, except at \(s = 0\). The FT of \(1 \leftrightarrow 2\pi \delta(\omega)\) is not analytic anywhere.

13. Dilated step function \((a \in \mathbb{R})\)

\[ u(at) \leftrightarrow \int_{-\infty}^{\infty} u(at)e^{-\tau}dt = \frac{1}{a} \int_{-\infty}^{\infty} u(\tau)e^{-(\tau/a)}d\tau = \frac{a}{|a|} \frac{1}{s} = \frac{\pm 1}{s}, \]

where we have made the change of variables \(\tau = at\). The only effect that \(a\) has on \(u(at)\) is the sign of \(t\), since \(u(t) = u(2t)\). However \(u(-t) \neq u(t)\), since \(u(t) - u(-t) = 0\), and \(u(t) + u(-t) = 1\), except at \(t = 0\), where it is not defined.

Once complex integration in the complex plane has been defined (Section 1.4.2, p. 108), we can justify the definition of the inverse LT (Eq. 1.79).2

2https://en.wikipedia.org/wiki/Laplace_transform#Table_of_selected_Laplace_transforms

1.1. EARLY SCIENCE AND MATHEMATICS
Chronological history pre 16th century

20th BCE Chinese (Primes; quadratic equation; Euclidean algorithm (GCD))
16th BCE Babylonian (Mesopotamian) (quadratic equation)
6th BCE Pythagoras (Thales) and the Pythagorean "tribe"
391 CE Copernicus

Early number theory: inspiration for Galileo, Descartes, Fermat and Newton.

Figure 1.1: Mathe matical time-line between 1500 BCE and 1650 CE. The western renaissance is considered to have occurred between the 15-17 centuries. However the Asian "renaissance" was likely well before the 1st century CE. One of the first true astronomers and mathematicians was Thales (c.1,500 BCE). The first clear evidence of complex arithmetic was not an option for the Greek mathematicians, since complex numbers were not yet developed.

(a) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

(b) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

Although complex arithmetic was not an option for the Greek mathematicians, since complex numbers were not yet developed.

However, complex arithmetic was also not an option for the Greek mathematicians, since complex numbers were not yet developed.

(a) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

(b) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

Although complex arithmetic was not an option for the Greek mathematicians, since complex numbers were not yet developed.

However, complex arithmetic was not an option for the Greek mathematicians, since complex numbers were not yet developed.

(a) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

(b) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

Although complex arithmetic was not an option for the Greek mathematicians, since complex numbers were not yet developed.

However, complex arithmetic was also not an option for the Greek mathematicians, since complex numbers were not yet developed.

(a) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

(b) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

Although complex arithmetic was not an option for the Greek mathematicians, since complex numbers were not yet developed.

However, complex arithmetic was also not an option for the Greek mathematicians, since complex numbers were not yet developed.

(a) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

(b) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

Although complex arithmetic was not an option for the Greek mathematicians, since complex numbers were not yet developed.

However, complex arithmetic was also not an option for the Greek mathematicians, since complex numbers were not yet developed.

(a) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

(b) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

Although complex arithmetic was not an option for the Greek mathematicians, since complex numbers were not yet developed.

However, complex arithmetic was also not an option for the Greek mathematicians, since complex numbers were not yet developed.

(a) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

(b) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

Although complex arithmetic was not an option for the Greek mathematicians, since complex numbers were not yet developed.

However, complex arithmetic was also not an option for the Greek mathematicians, since complex numbers were not yet developed.

(a) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

(b) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

Although complex arithmetic was not an option for the Greek mathematicians, since complex numbers were not yet developed.

However, complex arithmetic was also not an option for the Greek mathematicians, since complex numbers were not yet developed.

(a) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

(b) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

Although complex arithmetic was not an option for the Greek mathematicians, since complex numbers were not yet developed.

However, complex arithmetic was also not an option for the Greek mathematicians, since complex numbers were not yet developed.

(a) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

(b) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

Although complex arithmetic was not an option for the Greek mathematicians, since complex numbers were not yet developed.

However, complex arithmetic was also not an option for the Greek mathematicians, since complex numbers were not yet developed.

(a) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

(b) Complex numbers were not an option for the Greek mathematicians, since complex numbers were not yet developed.

Although complex arithmetic was not an option for the Greek mathematicians, since complex numbers were not yet developed.
Table F.2: A brief table of simple Fourier Transforms. Note $a > 0 \in \mathbb{R}$ has units [rad/s]. To flag this necessary condition, we use $|a|$ to assure this condition will be met. The other constant $T_0 \in \mathbb{R}$ [s] has no restrictions, other than being real. Complex constants may not appear as the argument to a delta function, since complex numbers do not have the order property.

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(\omega)$</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta(t)$</td>
<td>$1(\omega)$</td>
<td>Dirac</td>
</tr>
<tr>
<td>$1(t)$</td>
<td>$1 \forall t \leftrightarrow 2\pi \delta(\omega)$</td>
<td>Dirac</td>
</tr>
<tr>
<td>$\text{sgn}(t) = \frac{t}{</td>
<td>t</td>
<td>}$</td>
</tr>
<tr>
<td>$\tilde{u}(t) = \frac{1(t) + \text{sgn}(t)}{2}$</td>
<td>$\pi \delta(\omega) + \frac{1}{j \omega} \equiv \tilde{U}(\omega)$</td>
<td>step</td>
</tr>
<tr>
<td>$\delta(t - T_0)$</td>
<td>$e^{-j \omega T_0}$</td>
<td>delay</td>
</tr>
<tr>
<td>$\delta(t - T_0) \ast f(t) \rightarrow F(\omega)e^{-\omega T_0}$</td>
<td>delay</td>
<td></td>
</tr>
<tr>
<td>$\tilde{u}(t)e^{-j\omega t} \rightarrow \frac{1}{</td>
<td>\omega</td>
<td>}$</td>
</tr>
<tr>
<td>$\text{rec}(t) = \frac{1}{T_0} [\tilde{u}(t) - \tilde{u}(t - T_0)] \rightarrow \frac{1}{T_0} \left(1 - e^{-\omega T_0}\right)$</td>
<td>pulse</td>
<td></td>
</tr>
<tr>
<td>$\tilde{u}(t) \ast \tilde{u}(t) \rightarrow \tilde{\delta}^2(\omega)$</td>
<td>Not defined</td>
<td></td>
</tr>
</tbody>
</table>

It would be interesting to search the archives of the monasteries, where the records were kept, to determine exactly what happened during this religious blackout.

...
Math (the syntax) is a language: Numbers and the equal sign say they are equivalent, playing the role of a verb, or action. The rules of math are defined by algebra. For example, the sentence \( a = b \) has the same value as the number \( -1 \). The sentence is spoken as "a equals b." The symbols for minus and equal indicate two types of actions. The equal sign says they are equivalent, following the rules of algebra, this sentence may be rewritten as numbers are nouns and the equal sign says they are equivalent, playing the role of a verb, or action.

Table F.1: Summary of key properties of FTs.

<table>
<thead>
<tr>
<th>Property</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>( f(t) + g(t) \longleftrightarrow F(\omega) + G(\omega) )</td>
</tr>
<tr>
<td>Time Delay</td>
<td>( f(t - a) \longleftrightarrow F(\omega)e^{-j\omega a} )</td>
</tr>
<tr>
<td>Scaling</td>
<td>( f(at) \longleftrightarrow \frac{1}{</td>
</tr>
<tr>
<td>Differentiation</td>
<td>( \frac{df(t)}{dt} \longleftrightarrow j\omega F(\omega) )</td>
</tr>
<tr>
<td>Convolution</td>
<td>( f(t) * g(t) \longleftrightarrow F(\omega)G(\omega) )</td>
</tr>
<tr>
<td>Fourier Transform</td>
<td>( F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} , dt )</td>
</tr>
<tr>
<td>Inverse Fourier Transform</td>
<td>( f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} , d\omega )</td>
</tr>
</tbody>
</table>
rules of algebra and calculus. This language of mathematics is powerful, with deep consequences, known as theorems.

The writer of an equation should always translate (explicitly summarize the meaning of the expression), so the reader will not miss the main point, as a simply a matter of clear writing. Just as math is a language, so may language be thought of as mathematics. To properly write correct English it is necessary to understand the construction of the sentence. It is important to identify the subject, verb, object, and various types of modifying phrases. Look up the interesting distinction between *that* and *which.* Thus, like math, language has rules. Most individuals use what "sounds right," but if you’re learning English as a second language, it is necessary to understand the rules, which are arguably easier to master than the foreign speech sounds.

### 1.1.3 Early physics as mathematics: Back to Pythagoras

The role of mathematics is to summarize algorithms (i.e., sets of rules), and formalize the idea as a theorem. Pythagoras and his followers, the Pythagoreans, believed that there was a fundamental relationship between mathematics and the physical world. The Pythagoreans may have been the first to capitalize on the relationship between science and mathematics, to use mathematics to design things for profit. This may have been the beginning of capitalizing technology, based on the relationship between physics and math. This impacted commerce in many ways, such as map making, tools, implements of war (the wheel, gunpowder), art (music), water transport, sanitation, secure communication, food, etc. Of course the Chinese were the first to master many of these technologies.

Why is Eq. 1.1 called a theorem, and what exactly needs to be proved? We do not need to prove that \((a, b, c)\) obey this relationship, since this is a condition that is observed. We do not need to prove that \(a^2\) is the area of a square, as this is the definition of the area of a square. What needs to be proved is that the relation \(c^2 = a^2 + b^2\) holds *if, and only if,* the angle between the two shorter sides is 90°. The Pythagorean theorem (Eq. 1.1) did not begin with Euclid or Pythagoras, rather they appreciated its importance, and documented it.

In the end the Pythagoreans, who instilled fear in the neighborhood, were burned out, and murdered. This may be the fate of mixing technology with politics:

> "Whether the complete rule of number (integers) is wise remains to be seen. It is said that when the Pythagoreans tried to extend their influence into politics they met with popular resistance. Pythagoras fled, but he was murdered in nearby Mesopotamia in 497 BCE."

—Stillwell (2010, p. 16)

### 1.1.4 Modern mathematics is born

Modern mathematics (what we practice today) was born in the 15-16th centuries, in the minds of Leonardo da Vinci, Bombelli, Galileo, Descarte, Fermat, and many others (Stillwell, 2010). Many of these early masters were, like the Pythagoreans, secretive to the extreme about how they solved problems. This soon changed due to Galileo, Mersenne, Descarte and Newton, causing mathematics to blossom. During this time the developments were hectic, seemingly disconnected.

---

2. [https://en.oxforddictionaries.com/usage/that-or-which](https://en.oxforddictionaries.com/usage/that-or-which)
3. It seems likely that the Chinese and Egyptians also did this, but it is more difficult to document.
It seems likely that Bombelli's discovery of Diophantus's book "Arithmetic" in the Vatican library triggered many of...

2.1.5 Science meets mathematics

Table E.1: Pell equation for $N = 2, 3, M$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$M$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Note: $\gamma = \frac{\beta}{\delta}$ depending on the signs of $\beta$ and $\delta$.
CHAPTER 1. INTRODUCTION

It seems likely that Galileo was attracted to this model of two masses connected by a spring because he was also interested in planetary motion, which consist of masses (sun, earth, moon), and also mutually attracted by gravity.

Galileo also performed related experiments on pendulums, where he varied the length \( l \), mass \( m \), and and angle \( \theta \) of the swing. By measuring the period (periods/unit time) he was able to formulate precise rules between the variables. This experiment also measured the force exerted by gravity, so the experiments were related, but in very different ways. The pendulum served as the ideal clock, as it needed very little energy to keep it going, due to its very low friction (energy loss).

In a related experiment, Galileo measured the duration of a day by counting the number of swings of the pendulum in 24 hours, measured precisely by daily period of a star as it crossed the tip of a church steeple. The number of seconds in a day is precisely an integer (rounded to the nearest integer). It is the product of 24 [hr/day] \( \times \) 60 [s/hr] = 86400 = \( 2^4 \cdot 3^2 \cdot 5^2 \) [s/day]. This number may be reduced many ways, and remain precise, starting with seven factors of 2, 2 factors of 3 or 1 factor of 5. Factoring the number of days in a year (5*73) is impractical, thus cannot be easily treated as an integer.

Galileo also extended work on the relationship of wavelength and frequency of a sound wave in musical instruments. On top of these impressive accomplishments, Galileo greatly improved the telescope, which he needed for his observations of the planets.

Many of Galileo’s contributions resulted in new mathematics, leading to Newton’s discovery of the wave equation (c1687), followed 60 years later by its one-dimensional general solution by d’Alembert (c1747).

Mersenne: Mersenne (1588–1648) also contributed to our understanding of the relationship between the wavelength and the dimensions of musical instruments. At first Mersenne strongly disagreed with Galileo, partially due to errors in Galileo’s reports of his results. But once Mersenne saw the significance of Galileo’s conclusion, he being Galileo’s strongest advocate, helping to spread the word (Palmerino, 1999).

Newton: With the closure of Cambridge University due to the plague of 1665, Newton returned home to Woolsthorpe-by-Colsterworth (95 [mi] north of London), to worked by himself, for over a year. It was during this time he did his most creative work.

While Newton (1642–1726) may be best known for his studies on light, he was the first to predict the speed of sound. However his theory was in error\(^9\) by \( \sqrt{\gamma_0/\gamma_0} = \sqrt{\frac{T}{T'}} = 1.183 \). This famous error would not be resolved for over two hundred years, awaiting the formulation of thermodynamics and the equi-partition theorem, by Maxwell and Boltzmann, and others.

Just 11 years prior to Newton’s 1687 Principia, there was a basic understanding that sound and light traveled at very different speeds, due to the experiments of Ole Rømer.\(^9\)\(^10\)

\(^9\)The square root of the ratio of the specific heat capacity at constant pressure \( \gamma_0 \) to that at constant volume \( \gamma_v \).

\(^10\)https://www.youtube.com/watch?v=rZ0wx3uD2wo

 Appendix E

Analysis of Pell equation (N=2, 3, M)

Section G.2.3 (p. 237) showed that the solution \([x_n, y_n]T\) to Pell’s equation, for \( N = 2 \), is given by powers of Eq. 1.10. To find an explicit formula for \([x_n, y_n]T\), one must compute powers of

\[
A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.
\]

(E.1)

We wish to find the solution to Pell’s equation (Eq. 1.10), based on the recursive solution, Eq. 1.11 (p. 50). Thus we need is powers of \( A \), that is \( A^n \), which gives the a closed form expression for \([x_n, y_n]T\). By the diagonalization of \( A \), its powers are simply the powers of its eigenvalues. This diagonalization is called an eigenvalue analysis, a very general method rooted in linear algebra. This type of analysis allows us to find the solution to most of the linear the equations we encounter.

From Matlab with \( N = 2 \) the eigenvalues of Eq. E.1 are \( \lambda_1 \approx (2.4142, -0.4142) \) (i.e., \( \lambda_3 = 1(1 \pm \sqrt{2}) \)). The final solution to Eq. E.1 is given in Eq. G.11 (p. 237). The solution for \( N = 3 \) is provided in Appendix E.1.1 (p. 206).

Once the matrix has been diagonalized, one may compute powers of that matrix as powers of the eigenvalues. This results in the general solution given by

\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \gamma^n A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \gamma^n E \Lambda^n E^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

The eigenvalue matrix \( D \) is diagonal with the eigenvalues sorted, largest first. The Matlab command \([E,D]=eig(A)\) is helpful to find \( D \) and \( E \) given any \( A \). As we saw above,

\[
\Lambda = \begin{bmatrix} 1 + \sqrt{2} & 0 \\ 0 & 1 - \sqrt{2} \end{bmatrix} \approx \begin{bmatrix} 2.4142 & 0 \\ 0 & -0.4142 \end{bmatrix}.
\]

E.1 Pell equation eigenvalue-eigenvector analysis

Here we show how to compute the eigenvalues and eigenvectors for the 2x2 Pell matrix for \( N = 2 \)

\[
A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.
\]

The Matlab command \([E,D]=eig(A)\) returns the eigenvector matrix \( E \)

\[
E = \begin{bmatrix} e_+ & e_- \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} \approx \begin{bmatrix} 0.8165 & -0.8165 \\ 0.5774 & 0.5774 \end{bmatrix}.
\]
1.1. EARLY SCIENCE AND MATHEMATICS

Figure 1.4: Above: Jakob (1655–1705) and Johann (1667–1748) Bernoulli. Below: Leonhard Euler (1707-1783) and Jean le Rond d’Alembert (1717-1783). Euler was blind in his right eye, hence the left portrait view. The figure numbers are from Stillwell (2010).

APPENDIX D. SYMBOLIC ANALYSIS OF $T E = E \Lambda$
The idea behind Rømer’s discovery was that due to the large distance between earth and Io, there was a difference between the period of the moon when Jupiter was closest to earth vs. when it was furthest from earth. This difference in distance caused a delay or advance in the observed eclipse of Io as it went behind Jupiter, delayed by the difference in time due to the difference in distance. It is like watching a video of a clock, delayed or speed up. When the video is slowed down, the time will be inaccurate (it will indicate an earlier time).

**Studies of vision and hearing:** Since light and sound (music) played such a key role in the development of the early science, it was important to fully understand the mechanism of our perception of light and sound. There are many outstanding examples where physiology impacted mathematics. Leonardo da Vinci (1452–1519) is well known for his early studies of the human body. Exploring our physiological senses requires a scientific understanding of the physical processes of vision and hearing, first considered by Newton (1687) (1643–1727), but first properly researched much later by Helmholtz (1863a)\(^2\) (Stillwell, 2010, p. 261). Helmholtz’s (1821–1894) studies and theories of music and the perception of sound are fundamental scientific contributions (Helmholtz, 1863a). His best known mathematical contribution is today known as the fundamental theorem of vector calculus, or simply Helmholtz theorem.

**The amazing Bernoulli family:** The first individual who seems to have openly recognized the importance of mathematics, enough to actually teach it, was Jacob Bernoulli (1654–1705) (Fig. 1.4). Jacob worked on what is now viewed as the standard package of analytic “circular” (i.e., periodic) functions: \(\sin(x), \cos(x), \exp(x), \log(x)\).\(^3\) Eventually the full details were developed (for real variables) by Euler (Sections 1.3.1 p. 73 and H.1.1, p. 242).

From Fig. 1.2 we see that Jacob was contemporary with Descartes, Fermat, and Newton. Thus it seems likely that he was strongly influenced by Newton, who in turn was influenced by Descartes.\(^4\) Vile and Wallis (Stillwell, 2010, p. 175).

Jacob Bernoulli, like all successful mathematicians of the day, was largely self-taught. Yet Jacob was in a new category of mathematicians, because he was an effective teacher. Jacob taught his sibling Johann (1667–1748), who then taught his sibling Daniel (1700–1782). But most importantly, Johann taught Leonhard Euler (1707–1783) (Figs. 1.4 and 1.24, p. 104), the most prolific mathematician of all time. Euler’s papers continued to appear long after his death (Calinger, 2015). The power \(P\) is the real part of the voltage times the current

\[
2P = V^I + V^T = (ZI)^I + ZI^I = I^T Z^I + ZI^I
\]

??? Subtracting the two equations give

**D.2.6 Double roots**

For the 2x2 case of double roots the matrix has Jordan form

\[
T = \begin{bmatrix}
\lambda & 1 \\
0 & \lambda
\end{bmatrix}
\]

Then

\[
T^2 = \begin{bmatrix}
\lambda^2 & n\lambda \\
0 & \lambda^2
\end{bmatrix}
\]

This generalizes to \(n \times n\) matrices having arbitrary combinations of degeneracies (multiple roots), as in symmetric (square) drums, for example.
Three Streams from the Pythagorean Theorem

11.1. EARLY SCIENCE AND MATHEMATICS

Unfortunately, and perhaps somewhat unfairly, his rigor was criticized by Euler, mathematics grew exponentially. Figure 1.24 (p. 104) shows the time-line of the most famous mathematicians. Gauss was born at the end of Euler's long and productive life. I suspect that Gauss owed a great debt to Euler: surely he must have been a scholar of Euler. Gauss's most important achievement may have been his contribution to solving the precise general case is waiting for enlightenment. The impedances of physical systems are reciprocal. The determinant of the transmission matrix of a reciprocal network is both reversible A = D and reciprocal, the impedance matrix simplifies to

\[ Z(s) = \frac{1}{C} \begin{bmatrix} A & 1 \\ 1 & A \end{bmatrix} \]

\[ \text{Impedance matrix is symmetric} \]

\[ \Delta = \text{impedance matrix is symmetric} \]

\[ \Delta \]
1. **Introduction** Chapter 1 is intended to be a self-contained survey of basic pre-college mathematics, as a detailed overview of the fundamentals, presented as three streams:

1.2 Number systems (Stream 1)
1.3 Algebraic equations (Stream 2)
1.4 Scalar calculus (Stream 3a)
1.5 Vector calculus (Stream 3b)

If you’re a student, stick to Chapter 1.

Chapters 2-5 are disorganized rambling research ideas that have not yet found a home. Students, please stay out of these chapters.

2. **Number Systems** (Chapter G: Stream 1) Some uncertain ideas of number systems, starting with prime numbers, through complex numbers, vectors and matrices.

3. **Algebraic Equations** (Chapter H: Stream 2) Algebra and its development, as we know it today. The theory of real and complex equations and functions of real and complex variables. Complex impedance $Z(s)$ of complex frequency $s = \sigma + \omega j$ is covered with some care, developing the topic which is needed for engineering mathematics.

4. **Scalar Calculus** (Chapter I: Stream 3a) Ordinary differential equations. Integral theorems. Acoustics.


### 1.2 Stream 1: Number Systems

Number theory (discrete, i.e., integer mathematics) was a starting point for many key ideas. For example, in Euclid’s geometrical constructions the Pythagorean theorem for real $[a, b, c]$ was accepted as true, but the emphasis in the early analysis was on integer constructions, such as Euclid’s formula for Pythagorean triplets (Eq. 1.9, Fig. 1.9, p. 48).

As we shall see, the Pythagorean theorem is a rich source of mathematical constructions, such as composition of polynomials, and solutions of Pell’s equation by eigen-vector and recursive analysis methods. Recursive difference equation solutions predate calculus, going back at least to the Chinese (~2000 BCE). These are early (pre-limit) forms of differential equations, best analyzed using an eigen-vector or eigen-function expansion (a geometrical concept from linear algebra), as an orthogonal set of normalized (unit-length) vectors (Appendix D, p. 201).

The first use of zero and $\infty$: It is hard to imagine that one would not appreciate the concept of zero and negative numbers when using an abacus. It does not take much imagination to go from counting numbers $\mathbb{N}$ to the set of all integers $\mathbb{Z}$, including zero. On an abacus, subtraction is obviously the inverse of addition. Subtraction, to obtain zero abacus beads, is no different than subtraction from zero, giving negative beads. To assume the Romans who first developed counting from counting numbers $\mathbb{N}$ to the set of all integers $\mathbb{Z}$, including zero. On an abacus, subtraction is obviously the inverse of addition. Subtraction, to obtain zero abacus beads, is no different than subtraction from zero, giving negative beads. To assume the Romans who first developed counting

### Appendix D

#### Symbolic analysis of $TE = E\Lambda$

Here we derive the eigen-matrix $E$, and eigen-value matrix $\Lambda$ given a 2x2 Transmission matrix

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

such that $TE = E\Lambda$, using symbolic algebra methods, given by the Matlab/Octave’s script

```matlab
syms A B C D T E L %Use symbolic Matlab/Octave
T=[A B;C D] %Given matrix T
[E,L]=eig(T) %Find eigen-vector matrix E and
%eigen-value matrix L
```

#### D.1 General case

The eigenvectors $e_{\pm}$ are

$$e_{\pm} = \left( \frac{1}{\sqrt{2}} \left[ (A-D) \pm \sqrt{(A-D)^2 + 4BC} \right] \right)$$  \hspace{1cm} (D.1)

and eigenvalues are

$$\lambda_{\pm} = \frac{1}{2} \left( (A+D) - \sqrt{(A-D)^2 + 4BC} \right)$$ \hspace{1cm} (D.2)

The term under the radical (i.e., the discriminant) may be written in terms of the determinant of $T$

$$(A-D)^2 + 4BC = A^2 + D^2 - 4(AD - BC) = A^2 + D^2 - 4\Delta_T.$$  

This becomes especially important for the case of reciprocal systems where $\Delta_T = 1$, or for anti-reciprocal systems where $\Delta_T = -1$.

#### D.2 Special cases having symmetry

Each 2x2 matrix has four entries, each of which can be complex. This leads to 4x2=8 possible special symmetries (an eightfold way), discussed next, in quasi-order of their importance. Each
1.2. STREAM 1: NUMBER SYSTEMS (10 LECTURES)

sticks, or the Chinese who then deployed the concept using beads, did not understand negative numbers, is impossible. However, understanding the concept of zero (and negative numbers) is not the same as having a symbolic notation. The Roman number system has no such symbols. The first recorded use of a symbol for zero is said to be by Brahmagupta in 628 CE. Defining zero (c. 628 CE) depends on the concept of subtraction, which formally requires the creation of algebra (c. 830 CE, Fig. 1.1, p. 19). But apparently it takes more than 600 years, i.e., from the time Roman numerals were put into use, without any symbol for zero, to the time when the symbol for zero is first documented. Likely this delay is more about the political situation, such as government rulings, than mathematics.

The concept that caused much more difficulty was infinity (∞). First solved by Riemann in 1851 with the development of the extended plane, which mapped the plane to a sphere (Fig. 1.19, p. 94). His construction made it clear that the point at infinity (∞) is simply another point on the open plane, since rotating the sphere (extended plane) moves the point at infinity (∞) to a finite point on the plane, thereby closing the plane.

1.2.1 Lec 2: The Taxonomy of Numbers: $\mathbb{N}$, $\mathbb{P}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{F}$, $\mathbb{I}$, $\mathbb{R}$, $\mathbb{C}$

Once symbols for zero and negative numbers were accepted, progress could be made. To fully understand numbers, a transparent notation was required. First one must differentiate between the different classes (genus) of numbers, providing a notation that defines each of these classes, along with their relationships. It is logical to start with the most basic counting numbers, which we indicate with the double-bold symbol $\mathbb{N}$. For easy access, double-bold symbols and set-theory symbols, i.e., $\{\cdot\}$, $\cup$, $\cap$, $\in$ etc., are summarized in Appendix A.

Counting numbers $\mathbb{N}$: These are known as the “natural numbers” $\mathbb{N} = 1, 2, 3, \ldots$, denoted by the double-bold symbol $\mathbb{N}$. For clarity we shall refer to the natural numbers as counting numbers, since natural here means integer. The mathematical sentence “$2 \in \mathbb{N}$” is read as “2 is a member of the set of counting numbers.” The word set is defined as the collection of any objects that share a specific property. Typically the set may be defined either as a sentence, or by example.

Primes $\mathbb{P}$: A number is prime ($\pi_n \in \mathbb{P}$) if its only factors are 1 and itself. The set of Primes $\mathbb{P}$ is a subset of the counting numbers ($\mathbb{P} \subset \mathbb{N}$). A somewhat amazing fact, well known to the earliest mathematicians, is that every integer may be written as a unique product of primes. A second key mathematical function is the density of primes $\rho_{\pi}(\mathbb{N}) \sim 1/\log(\mathbb{N})$ (Eq. G.1, p. 228), an observation first made by Gauss (Goldstein, 1973). A third is that there is a prime between every integer $\mathbb{N}$ and $2 \mathbb{N}$, excluding $2 \mathbb{N}$.

Exercise: Write out the first 10 to 20 integers in factored form.

Solution:

1, 2, 3, $2^2$, 5, $2 \cdot 3$, 7, $2^3$, 11, $3 \cdot 2^2$, 13, $2 \cdot 7$, $3 \cdot 5$, $2^4$, 17, $2 \cdot 3^2$, 19, $2^2 \cdot 5$.
CHAPTER 1. INTRODUCTION

Exercise: Write integers 2 to 20 in terms of \( \pi_n \). Here is a table to assist you:

\[
\begin{array}{cccccccccccccccc}
\hline
n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & \cdots \\
\hline
\pi_n & 2 & \pi_2 & \pi_3 & \pi_4 & \pi_5 & \pi_6 & \pi_7 & \pi_8 & \pi_9 & \pi_{10} & \pi_{11} & \pi_{12} & \pi_{13} & \pi_{14} & \cdots \\
\hline
\end{array}
\]

We shall use the convenient notation \( \pi_n \) for the prime numbers, indexed by \( \mathbb{N} \). The first 12 primes \((n = 1, \ldots, 12)\) are \( \pi_n = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37 \). Since \( 4 = 2^2 \) and \( 6 = 2 \cdot 3 \) may be factored, \( 4, 6 \notin \mathbb{P} \) (read as: \( 4 \) and \( 6 \) are not in the set of primes). Given this definition, multiples of a prime, i.e., \( [2, 3, 4, 5, \ldots] \cdot \pi_k \) of any prime \( \pi_k \), cannot be prime. It follows that all primes except 2 must be odd and every integer \( N \) is unique in its factorization.

Coprimes are two numbers with no common prime factors. For example, \( 4 = 2 \cdot 2 \) and \( 6 = 2 \cdot 3 \) are not coprime, as they have \( 2 \) as a common factor, whereas \( 21 = 3 \cdot 7 \) and \( 10 = 2 \cdot 5 \) are. By definition all pairs of primes are coprime. We shall use the notation \( m \perp n \) to indicate that \( m, n \) are coprime. The ratio of two coprimes is reduced, as it has no factors to cancel. The ratio of two numbers that are not coprime may always be reduced by canceling the common factors. This is called the reduced form, or a irreducible fraction. When doing numerical work, for computational accuracy it is always beneficial to work with the reduced form.

The fundamental theorem of arithmetic states that each integer may be uniquely expressed as a product of primes. The Prime Number Theorem estimates the mean density of primes over \( \mathbb{N} \).

Integers \( \mathbb{Z} \): These include positive and negative counting numbers and zero. Notionally we might indicate this using set notation as \( \mathbb{Z} = \{-N, \{0\}, N \} \). Read this as The integers are in the set composed of the negative of the natural numbers \( -N \), zero, and \( N \). Note that \( N \subset \mathbb{Z} \).

Rational numbers \( \mathbb{Q} \): These are defined as numbers formed from the ratio of two integers. Given two numbers \( n, d \in \mathbb{N} \), then \( n/d \in \mathbb{Q} \). Since \( d \) may be 1, it follows that the rationals include the counting numbers as a subset. For example, the rational number \( 3/1 \in \mathbb{N} \). The main utility of rational numbers is that they can efficiently approximate any number on the real line, to any precision. For example, the rational approximation \( \pi \approx 22/7 \), has a relative error of \( \approx 0.04\% \).

Fractional number \( \mathbb{F} \): A fractional number \( \mathbb{F} \) is defined as the ratio of coprimes. If \( n, d \in \mathbb{P} \), then \( n/d \in \mathbb{F} \). Given this definition, \( \mathbb{F} \subset \mathbb{Q} = \mathbb{Z}/\mathbb{P} \). Because of the powerful approximating power of rational numbers, the fractional set \( \mathbb{F} \) has special utility. For example, \( \pi \approx 22/7, 1/\pi \approx 7/22 \) (to 0.04%), \( e \approx 19/7 \) to 0.15%, and \( \sqrt{2} \approx 7/5 \) to 1%.

Irrational numbers \( \mathbb{I} \): Every real number that is not rational is irrational \((\mathbb{Q} \perp \mathbb{I})\). Irrational numbers include \( \pi, e \) and the square roots of primes. These are decimal numbers that never repeat, thus requiring infinite precision in their representation. Such numbers cannot be represented on a computer, as they would require an infinite number of bits (precision).

The rationals \( \mathbb{Q} \) and irrationals \( \mathbb{I} \) split the reals \((\mathbb{R} = \mathbb{Q} \cup \mathbb{I}, \mathbb{Q} \perp \mathbb{I})\), thus each is a subset of the reals \((\mathbb{Q} \subset \mathbb{R}, \mathbb{I} \subset \mathbb{R})\). This relation is analogous to that of the integers \( \mathbb{Z} \) and fractions \( \mathbb{F} \), which split the rationals \((\mathbb{Q} = \mathbb{Z} \cup \mathbb{F}, \mathbb{Z} \perp \mathbb{F})\) (thus each is a subset of the rationals \((\mathbb{Z} \subset \mathbb{Q}, \mathbb{F} \subset \mathbb{Q})\)).

Verify that \( \lambda = E^{-1}AE \): To find the inverse of \( E \), 1) swap the diagonal values, 2) change the sign of the off diagonals, and 3) divide by the determinant \( \Delta = 2/\sqrt{2}/\sqrt{3} \) (see Appendix ??)

\[
E^{-1} = \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & \sqrt{3} \\ -1 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} 0.6124 & 0.866 \\ -0.6124 & 0.866 \end{bmatrix}
\]

By definition for any matrix \( E^{-1}E = EE^{-1} = I_2 \). Taking the product gives

\[
E^{-1}E = \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & \sqrt{3} \\ -1 & \sqrt{3} \end{bmatrix} \cdot \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{3} \\ 1 & -\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2
\]

We wish to show that \( \lambda = E^{-1}AE \)

\[
\begin{bmatrix} 1 + \sqrt{2} & 0 \\ 0 & 1 - \sqrt{2} \end{bmatrix} = \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & \sqrt{3} \\ -1 & \sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdot \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix},
\]

which is best verified with Matlab.

Verify that \( A = E\Lambda E^{-1} \): We wish to show that

\[
\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1 + \sqrt{2} & 0 \\ 0 & 1 - \sqrt{2} \end{bmatrix} \cdot \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix},
\]

which is best verified with Matlab (or Octave). All the above equations have been verified both with Matlab and Octave.
Real numbers are the union of rational and irrational numbers, namely $\mathbb{Q}$ and $\mathbb{R}$ respectively. Many people assume that lengths in Euclidean geometry are reals. Many people assume that $\mathbb{R}$ is a natural order. For example, $1 < 2$. All real numbers have a natural order on the real line. Complex numbers do not have a natural order. For example, $1 + i > 1$. The concept of an integer length in Euclid’s geometry was not defined.

Cartesian multiplication of complex numbers follows the basic rules of real algebra, for example $(1 + i) \cdot (2 + 3i) = 5 + 5i$. It is a convention to order the eigenvalues from largest to smallest. The eigenvectors are orthogonal and normalized. This gives two identical equations $\pm \vec{e} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where $\vec{e}$ is a unit vector. The two eigenvectors are $\pm \vec{e} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

However, the concept of an integer length in Euclid’s geometry was not defined. Many people assume that $\mathbb{R}$ is a natural order. For example, $1 < 2$. All real numbers have a natural order on the real line. Complex numbers do not have a natural order. For example, $1 + i > 1$.
Appendix C

Eigen Analysis

In the following discussion we show how to determine $E$ and $D$ (i.e., $\Lambda$), given $A$.

Calculating the eigenvalue matrix ($\Lambda$): The matrix equation for $E$ is

$$AE = E\Lambda.$$  \hfill (C.1)

Pre-multiplying by $E^{-1}$ diagonalizes $A$, given the eigenvalue matrix (in Matlab)

$$\Lambda = E^{-1}AE.$$  \hfill (C.2)

Post-multiplying by $E^{-1}$ recovers $A$

$$A = E\Lambda E^{-1}.$$  \hfill (C.3)

Matrix power formula: This last relation is the entire point of the eigenvector analysis, since it shows that any power of $A$ may be computed from powers of the eigenvalues. Specifically

$$A^n = E\Lambda^n E^{-1}.$$  \hfill (C.4)

For example, $A^2 = AA = E\Lambda (E^{-1}E) \Lambda E^{-1} = E\Lambda^2 E^{-1}$.

Equations C.1, C.2 and C.3 are the key to eigenvector analysis, and you need to memorize them. You will use them repeatedly throughout this course, and possibly for a long time after it is over.

Showing that $A - \lambda_2 I_2$ is singular: If we restrict Eq. C.1 to a single eigenvector (one of $e_\pm$), along with the corresponding eigenvalue $\lambda_\pm$, we obtain a matrix equations

$$Ae_\pm = E_\pm \lambda_\pm = \lambda_\pm E_\pm$$

Note the important swap in the order of $E_\pm$ and $\lambda_\pm$. Since $\lambda_\pm$ is a scalar, this is legal (and critically important), since this allows us to remove (factored out) $E_\pm$

$$(A - \lambda_2 I_2)E_\pm = 0.$$  \hfill (C.5)

This means that the matrix $A - \lambda_2 I_2$ must be singular, since when it operates on $E_\pm$, which is not zero, it gives zero. It immediately follows that its determinant is zero (i.e., $|A - \lambda_2 I_2| = 0$). This equation is used to uniquely determine the eigenvalues $\lambda_\pm$. Note the important difference between $\lambda_2 I_2$ and $\Lambda$ (i.e., $|A - \Lambda| \neq 0$).
B.3 Inverse of the 2x2 matrix

We shall now apply Gaussian elimination to find the solution \( x_1, x_2 \) for the 2x2 matrix equation \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \). We wish to show that the intersection (solution) is given by the equation on the right.

Here we wish to prove that the left equation (i) has an inverse given by the right equation (ii):

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.
\]

Solution: The best way may be using numbers. Below is symbolic code, independent of numerics:

\[
syms a b c d A B
A = \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.
\]

Numerical taxonomy:

A simplified taxonomy of numbers is given by the mathematical sentence

\[
\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.
\]

This sentence says:

1. Every prime number \( p \) is in the set of primes \( \mathbb{P} \).
2. Which is a subset of the set of counting numbers \( \mathbb{N} \).
3. Which is a subset of the set of integers \( \mathbb{Z} \).
4. Which is a subset of the set of rationals \( \mathbb{Q} \) (ratio of counting numbers \( \mathbb{N} \)).
5. Which is a subset of the set of complex numbers \( \mathbb{C} \).
6. Which is a subset of the set of complex numbers \( \mathbb{C} \).

The numerals \( \mathbb{Q} \) may be further decomposed into the fractions \( \mathbb{F} \) and the integers \( \mathbb{Z} \) into \( \mathbb{N} \). The classification nicely defines all the numbers used in engineering and physics.

The taxonomy structure may be summarized with the single compact sentence, starting with the prime numbers \( \mathbb{P} \) and ending with the complex numbers \( \mathbb{C} \).

\[
\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.
\]

Summary: This is a lot of messy algebra, that is why it is essential you memorize it.
As discussed in Appendix A (p. 185), all numbers may be viewed as complex. Namely, every real number is complex if we take the imaginary part to be zero (Boas, 1987). For example, 2 \in \mathbb{R} \subseteq \mathbb{C}. Likewise every purely imaginary number (e.g., 0 + 1j) is complex with zero real part.

Finally, note that complex numbers \mathbb{C}, much like vectors, do not have "rank-order," meaning one complex number cannot be larger or smaller than another. It makes no sense to say that \( j > 1 \) or \( j = 1 \) (Boas, 1987). The real and imaginary parts and the magnitude and phase, have order. If time \( t \) were complex, there could be no yesterday and tomorrow.\(^{25}\)

### Applications of integers

The most relevant question at this point is "Why are integers important?" First, we count with them, so we can keep track of "how much." But there is much more to numbers than counting: We use integers for any application where absolute accuracy is essential, such as banking transactions (making change), the precise computing of dates (Stillwell, 2010, p. 70) and location ("I'll meet you at 34 and Vine at noon on Jan. 1, 2034.").), building roads or buildings out of bricks (objects built from a unit size).

To navigate we need to know how to predict the tides, the location of the moon and sun, etc. Integers are important because they are precise: Once a month there is a full moon, easily recogniz-

False. Today all computers compute floating point numbers as fractionals. However, in theory they were wrong. Today all computers compute floating point numbers as fractionals. However, in theory they were wrong. The difference is a matter of precision.

### Numerical Representations of \( \mathbb{I}, \mathbb{R}, \mathbb{C} \):

When doing numerical work, one must consider how we may compute within the set of reals (i.e., which contain irrationals). There can be no irrational number representation on a computer. The international standard of computation, IEEE floating point numbers,\(^{26}\) are actually rational approximations. The mantissa and the exponent are both integers, having sign and magnitude. The size of each integer depends on the precision of the number being represented. An IEEE floating-point number is rational because it has a binary (integer) mantissa, multiplied by 2 raised to the power of a binary (integer) exponent. For example, \( \pi \approx 2^{a} \) with \( a, b \in \mathbb{Z} \). In summary, IEEE floating-point numbers are not, and cannot, be irrational, because numerical representations would imply an infinite number of bits.

True floating point numbers contain irrational numbers, which must be approximated by rational numbers. This leads to the concept of *fractional representation*, which requires the definition of the *mantissa*, *base* and *exponent*, where both the mantissa and the exponent are signed. Numerical results must not depend on the base. One could dramatically improve the resolution of the numerical representation by the use of the *fundamental theorem of arithmetic* (Section 1.2.2, page 37). For example, one could factor the exponent into its primes and then represent the number as \( a^{2} 3^{5} 5^{7} \) \( (a, b, c, d, e \in \mathbb{Z}) \), etc. Such a representation would improve the resolution of the representation. But even so, the irrational numbers would be approximated. For example, base ten\(^{27}\) is

---

\(^{25}\)One can define \( \xi = x + j \) to be complex \((x, t \in \mathbb{R})\), with \( x \) in meters [m], \( t \) in seconds [s], and the speed of light \( c \) [m/s].


\(^{27}\)Base 10 is the natural world-wide standard simply because we have 10 fingers.

---

2. Find a second GE matrix, \( G_{2} \), to put \( G_{1}A \) in upper triangular form. Identify the elementary row operations that this matrix performs. **Solution:**

\[
G_{2} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1 & 1
\end{bmatrix}
\]

or \([2) \leftarrow -(2) + (3)] \). Thus we have

\[
G_{2}G_{1}[A] = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & -1 \\
1 & -2 & 4 \\
1 & -1 & 4
\end{bmatrix} = \begin{bmatrix}
1 & 1 & -1 \\
0 & -2 & 4 \\
0 & 0 & 1
\end{bmatrix}
\]

3. Find a third GE matrix, \( G_{3} \), which scales each row so that its leading term is 1. Identify the elementary row operations that this matrix performs. **Solution:**

\[
G_{3} = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1/2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

which scales the second row by \(-1/2\). Thus we have

\[
G_{3}G_{2}G_{1}[A] = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1/2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & -1 \\
1 & -2 & 4 \\
1 & -1 & 4
\end{bmatrix} = \begin{bmatrix}
1 & 1 & -1 \\
0 & 1 & -2 \\
0 & 0 & 1
\end{bmatrix}
\]

4. Finally, find the last GE matrix, \( G_{4} \), that subtracts a scaled version of row 3 from row 2, and scaled versions of rows 2 and 3 from row 1, such that you are left with the identity matrix \((G_{4}G_{3}G_{2}G_{1}A = I)\). **Solution:**

\[
G_{4} = \begin{bmatrix}
1 & -1 & -1 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{bmatrix}
\]

Thus we have

\[
G_{4}G_{3}G_{2}G_{1}[A] = \begin{bmatrix}
1 & -1 & -1 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & -1 \\
1 & -2 & 4 \\
1 & -1 & 4
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

5. Solve for \([x_{1}, x_{2}, x_{3}]^{T}\) using the augmented matrix format \(G_{4}G_{3}G_{2}G_{1}[A] \) (where \([A] \) is the augmented matrix). Note that if you’ve performed the preceding steps correctly, \( x = G_{4}G_{3}G_{2}G_{1}b \). **Solution:** From the preceding problems, we see that \([x_{1}, x_{2}, x_{3}]^{T} = [3, -1, 1]^{T}\).
I.2. STREAM 1: NUMBER SYSTEMS (10 LECTURES)

The IEEE representation is

\[\pi \approx 314 \ldots\] constructing buildings or roads made from bricks of uniform size.

\[x = \frac{\pi \cdot 10^5}{\approx 314000}\]

http://www.h-schmidt.net/FloatConverter/IEEE754.html

4 The second row of \(C\) selects the first row from \(\mathbf{I}\) and adds it to the second row.

\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & -1 \\
1 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 \\
1 & 1 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

The shorthand for this operation is Gaussian elimination. As per our convention, the second row of \(C\) selects the first row from \(\mathbf{I}\) and adds it to the second row.

The concept of the decimal point is replaced by an integer, having the desired number of digits.

The shorthand for this operation is Gaussian elimination. As per our convention, the second row of \(C\) selects the first row from \(\mathbf{I}\) and adds it to the second row.

Consider the GE matrix:

\[
G = \begin{bmatrix}
1 & 0 \\
1 & -1 \\
\end{bmatrix}
\]

(a) This pre-multiplication adds and scales columns. (b) Post-multiplication adds and scales columns. (c) Then we have column \(C\). (d) Which won’t change our ability to multiply.

\[
G = \begin{bmatrix}
1 & 0 \\
1 & -1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

The second row of \(C\) selects the first row from \(\mathbf{I}\) and adds it to the second row.

\[
\left[\begin{array}{ccc}
1 & 0 & -1 \\
0 & 1 & -1 \\
1 & 1 & 1 \\
\end{array}\right]
= \left[\begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{array}\right]
\]

Compute the determinant by elimination. The matrix form this equation.

\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & -1 \\
1 & 1 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & -1 \\
1 & 1 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

The concept of this operation is Gaussian elimination. As per our convention, the second row of \(C\) selects the first row from \(\mathbf{I}\) and adds it to the second row.

The shorthand for this operation is Gaussian elimination. As per our convention, the second row of \(C\) selects the first row from \(\mathbf{I}\) and adds it to the second row.

The second row of \(C\) selects the first row from \(\mathbf{I}\) and adds it to the second row.

\[
\left[\begin{array}{ccc}
1 & 0 & -1 \\
0 & 1 & -1 \\
1 & 1 & 1 \\
\end{array}\right]
= \left[\begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{array}\right]
\]

Consider the GE matrix.
Public-key Security: An important application of prime numbers is public-key encryption, essential for internet security applications (e.g., online banking). Most people assume encryption is done by a personal login and passwords. Passwords are fundamentally insecure, for many reasons. Decryption depends on factoring large integers, formed from products of primes having thousands of bits.\(^1\) The security is based on the relative ease of multiplying large primes, along with the virtual impossibility of factoring their products.

When a computation is easy in one direction, but its inverse is impossible, it is called a trapdoor function. We shall explore trapdoor functions in Chapter G. If everyone were to switch from passwords to public-key encryption, the internet would be much more secure.

Puzzles: Another application of integers is imaginative problems that use integers. An example is the classic Chinese four stone problem: “Find the weight of four stones that can be used with a scale to weigh anything (e.g., salt, gold) between 0, 1, 2, ..., 40 [gm].” As with the other problems, the answer is not as interesting as the method, since the problem may be easily cast into a related one. This type of problem can be found in airline magazines as amusement on a long flight. This puzzle is best cast as a linear algebra problem, with integer solutions. Again, once you know the trick, it is “easy.”\(^2\)

1.2.2 Lec 3: The role of physics in mathematics

Bells, chimes and eigen-modes: Integers naturally arose in art, music and science. Examples include the relations between musical notes, the natural eigen-modes (tones) of strings and other musical instruments. These relations were so common and well studied, it appeared that to understand the physical world (aka, the Universe), one needed to understand integers. This was a seductive view, but not actually correct. As will be discussed in Sections 1.3.1 (p. 56) and H.1.1 (p. 241), it is best to view the relationship between acoustics, music and mathematics as historical, since these topics played such an important role in the development of mathematics. Also interesting is the role that integers play in quantum mechanics, also based on eigen-modes, but in this case, those of atoms. Eigen-modes follow from solutions of the wave equation, which has natural indexing property, which is essential for making lists that are ordered, so that one can quickly look things up. The alphabet also has this property (e.g., a book’s index).

\(^1\) It would seem that public-key encryption could work by having two numbers with a common prime, and then using the Euclidean algorithm, the greatest common divisor (GCD) could be worked out. One of the integers could be the public-key and the second could be the private key.

\(^2\) Whenever someone tells you something is “easy,” you should immediately appreciate that it is very hard, but once you learn a concept, the difficulty evaporates.

\(^3\) Check out the history of 1729 = 1\(^3\) + 12\(^3\) = 9\(^3\) + 10\(^3\).
The fundamental theorem of arithmetic states that every integer \( n \geq 1 \) can be expressed uniquely as a product of primes, up to the order of the factors, in the form
\[
\prod_{i=1}^{k} p_i^{e_i} = n,
\]
where the primes \( p_i \) are distinct and the exponents \( e_i \) are non-negative integers. This theorem is crucial in number theory and has far-reaching implications in various fields of mathematics.

We now know that the code-breaking effort during the war effort, the credit was only acknowledged in the 1970s when the project was finally declassified. (The Colossus) used to break the WWII German "Enigma" code. Due to the high secrecy of this operation, it remained unknown to the public until recently.

For Stream 1 there is the fundamental theorem of arithmetic, which states that every integer greater than 1 can be uniquely represented as a product of primes. For Stream 2 there is the fundamental theorem of algebra, which says that every non-constant single-variable polynomial with complex coefficients has at least one complex root.

### Definitions:

1. **Scalar**: A quantity having magnitude only.
2. **Matrix**: A rectangular array of numbers, symbols, or expressions, arranged in rows and columns. Matrices are used to represent linear transformations and systems of linear equations.
3. **Linear system of equations**: A set of linear equations involving the same set of variables. A linear system of equations may be written in matrix form as
\[
A\mathbf{x} = \mathbf{b},
\]
where \( A \) is the coefficient matrix, \( \mathbf{x} \) is the variable vector, and \( \mathbf{b} \) is the constant vector.

### Equivalence:

- \( a \equiv b \) means that \( a \) and \( b \) are equivalent in some sense. For example, in modular arithmetic, \( a \equiv b \pmod{n} \) means that \( a \) and \( b \) have the same remainder when divided by \( n \).

### Inverse:

- \( A^{-1} \) denotes the inverse of a matrix \( A \). If \( A \) is invertible, then
\[
A^{-1}A = AA^{-1} = I,
\]
where \( I \) is the identity matrix.

### Determinant:

The determinant of a square matrix \( A \) is denoted either as \( \det(A) \) or \( |A| \). It is a scalar value that can be computed from the elements of a square matrix and encodes certain properties of the matrix.

### Matrix Algebra of Systems

#### Linear system of equations

When writing a complex number we shall adopt the convention of using a letter with an arrow, \( \mathbf{v} \), for a vector. A vector is a quantity having both magnitude and direction. A tacit notation in Matlab is that \( \mathbf{v} \) vectors are treated as column vectors. Matlab defines the notation \( 1:4 \) as the "row-vector" \( \mathbf{1} \times 4 \). The inverse of a square matrix is \( A^{-1} \). For example, the inverse of the identity matrix is the identity matrix itself:
\[
A^{-1} = I.
\]

The determinant of a matrix is denoted either as \( \det(A) \) or \( |A| \). It is a scalar value that can be computed from the elements of a square matrix and encodes certain properties of the matrix. For example, the determinant of a 2x2 matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is \( ad - bc \).

\[
\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.
\]

### Equivalence:

Two matrices are equivalent if they have the same number of rows and columns, and if there exists a permutation matrix \( P \) such that \( PAP^{-1} \) is a diagonal matrix. Equivalence is an equivalence relation on the set of matrices.

### Inverse:

- \( A^{-1} \) denotes the inverse of a matrix \( A \). If \( A \) is invertible, then
\[
A^{-1}A = AA^{-1} = I,
\]
where \( I \) is the identity matrix.

### Determinant:

The determinant of a square matrix \( A \) is denoted either as \( \det(A) \) or \( |A| \). It is a scalar value that can be computed from the elements of a square matrix and encodes certain properties of the matrix. For example, the determinant of a 2x2 matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is \( ad - bc \).

\[
\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.
\]

### Matrix Algebra of Systems

#### Linear system of equations

When writing a complex number we shall adopt the convention of using a letter with an arrow, \( \mathbf{v} \), for a vector. A vector is a quantity having both magnitude and direction. A tacit notation in Matlab is that \( \mathbf{v} \) vectors are treated as column vectors. Matlab defines the notation \( 1:4 \) as the "row-vector" \( \mathbf{1} \times 4 \). The inverse of a square matrix is \( A^{-1} \). For example, the inverse of the identity matrix is the identity matrix itself:
\[
A^{-1} = I.
\]

The determinant of a matrix is denoted either as \( \det(A) \) or \( |A| \). It is a scalar value that can be computed from the elements of a square matrix and encodes certain properties of the matrix. For example, the determinant of a 2x2 matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is \( ad - bc \).

\[
\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.
\]

### Equivalence:

Two matrices are equivalent if they have the same number of rows and columns, and if there exists a permutation matrix \( P \) such that \( PAP^{-1} \) is a diagonal matrix. Equivalence is an equivalence relation on the set of matrices.

### Inverse:

- \( A^{-1} \) denotes the inverse of a matrix \( A \). If \( A \) is invertible, then
\[
A^{-1}A = AA^{-1} = I,
\]
where \( I \) is the identity matrix.

### Determinant:

The determinant of a square matrix \( A \) is denoted either as \( \det(A) \) or \( |A| \). It is a scalar value that can be computed from the elements of a square matrix and encodes certain properties of the matrix. For example, the determinant of a 2x2 matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is \( ad - bc \).

\[
\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.
\]

### Matrix Algebra of Systems

#### Linear system of equations

When writing a complex number we shall adopt the convention of using a letter with an arrow, \( \mathbf{v} \), for a vector. A vector is a quantity having both magnitude and direction. A tacit notation in Matlab is that \( \mathbf{v} \) vectors are treated as column vectors. Matlab defines the notation \( 1:4 \) as the "row-vector" \( \mathbf{1} \times 4 \). The inverse of a square matrix is \( A^{-1} \). For example, the inverse of the identity matrix is the identity matrix itself:
\[
A^{-1} = I.
\]

The determinant of a matrix is denoted either as \( \det(A) \) or \( |A| \). It is a scalar value that can be computed from the elements of a square matrix and encodes certain properties of the matrix. For example, the determinant of a 2x2 matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is \( ad - bc \).

\[
\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.
\]

### Equivalence:

Two matrices are equivalent if they have the same number of rows and columns, and if there exists a permutation matrix \( P \) such that \( PAP^{-1} \) is a diagonal matrix. Equivalence is an equivalence relation on the set of matrices.

### Inverse:

- \( A^{-1} \) denotes the inverse of a matrix \( A \). If \( A \) is invertible, then
\[
A^{-1}A = AA^{-1} = I,
\]
where \( I \) is the identity matrix.

### Determinant:

The determinant of a square matrix \( A \) is denoted either as \( \det(A) \) or \( |A| \). It is a scalar value that can be computed from the elements of a square matrix and encodes certain properties of the matrix. For example, the determinant of a 2x2 matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is \( ad - bc \).

\[
\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.
\]
CHAPTER 1. INTRODUCTION

Table 1.1: The fundamental theorems of mathematics

1. Fundamental theorems of:
   (a) Number systems: Stream 1
      • arithmetic
      • prime number
   (b) Geometry: Stream 2
      • algebra
   (c) Calculus: Stream 3*
      • Leibniz \( \mathbb{R}^1 \)
      • complex \( \mathbb{C} \subset \mathbb{R}^2 \)
      • vectors \( \mathbb{R}^3, \mathbb{R}^n, \mathbb{R}^\infty \)
         - Gauss’s law (divergence theorem)
         - Stokes’s law (curl theorem, or Green’s theorem)
         - Vector calculus (Helmholtz’s theorem)

2. Other key concepts:
   • Complex analytic functions (complex roots are finally accepted!)
     - Complex Taylor series (complex analytic functions)
     - Region of convergence (RoC) of complex analytic series
       (p. 71)
     - Laplace transform, and its inverse
     - Causal time \( \Rightarrow \) complex frequency \( s \)
     - Cauchy integral theorem
     - Residue integration (i.e., Green’s thm in \( \mathbb{R}^2 \))
   • Riemann mapping theorem (Gray, 1994; Walsh, 1973)
   • Complex impedance (Ohm’s law) (Kennelly, 1893)

---


B.2. NXM MATRICES

Complex power: In this special case, the complex power \( P(s) \in \mathbb{R}(s) \) is defined, in the complex frequency domain \( s \) as

\[
P(s) = I(s)V(s) = I(s)Z(s)I(s) \leftrightarrow p(t). \quad [W]
\]

The real part of the complex power must be positive. The imaginary part corresponds to available stored energy.

GIVE MORE EXAMPLES

The case of three-dimensions is special, allowing definitions that are not defined in more dimensions. A vector in \( \mathbb{R}^3 \) labels the point having the coordinates of that vector.

B.1.1 Vectors in \( \mathbb{R}^3 \)

Dot product in \( \mathbb{R}^3 \): The dot product of the difference between two vectors \( A - B \) is the Euclidean distance between the points they define:

\[
||A - B|| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}.
\]

Cross product: and cross product \( A \times B = ||A|| ||B|| \sin(\theta) \) are defined between the two vectors \( A \) and \( B \).

The triple product: This is defined between three vectors as

\[
A \cdot (B \times C) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}
\]

also defined in Fig. 1.16. This may be indicated without the use of parentheses, since there can be no other meaningful interpretation. However for rigor, parentheses should be used. The triple product is the volume of the parallelepiped (3D-crystal shape) outlined by the three vectors, shown in Fig. 1.16, p. 81.

B.2 NxM Matrices

When working with matrices, the role of the weights and vectors can change, depending on the context. A useful way to view a matrix is as a set of column vectors, weighted by the elements of the column-vector of weights multiplied from the right. For example

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & \cdots & a_{1M} \\
  a_{21} & a_{22} & a_{23} & \cdots & a_{2M} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{N1} & a_{N2} & a_{N3} & \cdots & a_{NM} \\
\end{bmatrix}
\begin{bmatrix}
  w_1 \\
  w_2 \\
  \vdots \\
  w_N \\
\end{bmatrix}
= 
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & \cdots & a_{1M} \\
  a_{21} & a_{22} & a_{23} & \cdots & a_{2M} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{N1} & a_{N2} & a_{N3} & \cdots & a_{NM} \\
\end{bmatrix}
\begin{bmatrix}
  w_1 \\
  w_2 \\
  \vdots \\
  w_N \\
\end{bmatrix}
\]
When the ratio of two frequencies (pitch) is 2, the relationship is called an octave. There are 12 intervals called semitones, which nicely summarize theorem 3. There are exactly 12 primes per octave, thus one might wonder how many primes there are per semitone? In the end, it is a question of the density of primes on a log (i.e., ratio) scale.

3. There are 12 primes per octave.

When the ratio of two frequencies (pitch) is 2, the relationship is called an octave. Thus, one might say, with a slight stretch of terminology, there is at least one prime per octave. In modern music, the octave is further divided into 12 intervals called semitones (frets). As the product of 12 semitones is an octave, one might ask how many primes there are per semitone? In the end, it is a question of the density of primes on a log (i.e., ratio) scale.

This theorem states that every polynomial in a is of degree N. Hence mathematics plays a key role in physics, as does physics in mathematics.

When Laplace transformed, the fields become functions of space and frequency, typically written as $\mathbf{E}(x, y, z, t)$. When given a system of equations (a mechanical or electrical circuit), one may define an impedance matrix.

Stream 2: Fundamental theorem of algebra

This theorem states that every polynomial in $a$ is of degree $N$. Thus, one might ask how many primes there are per semitone? In the end, it is a question of the density of primes on a log (i.e., ratio) scale.

In Sections 15.13 and 15.14, we will deal with the complex elements, which are related by an impedance matrix. For this case when the elements are complex, the dot product is a real number which is always non-negative. Such a concept is useful when $a$ and $b$ are related by an impedance matrix, which is always non-negative.

Theorem 3: Fundamental theorem of calculus

This theorem states that every polynomial in $a$ is of degree $N$. Thus, one might ask how many primes there are per semitone? In the end, it is a question of the density of primes on a log (i.e., ratio) scale.

In Sections 15.13 and 15.14, we will deal with the complex elements, which are related by an impedance matrix. For this case when the elements are complex, the dot product is a real number which is always non-negative. Such a concept is useful when $a$ and $b$ are related by an impedance matrix, which is always non-negative.
The widely recognized Cauchy integral theorem is an excellent example, since it is a stepping stone to Green’s theorem and the fundamental theorem of complex calculus. In Section 1.5.6 (p. 143) we clarify the contributions of each of these special theorems.

Once these fundamental theorems of integration (Stream 3) have been mastered, the student is ready for the complex frequency domain, which takes us back to Stream 2 and the complex frequency plane \((s = \sigma + \omega) \in \mathbb{C}\). While the Fourier and Laplace transforms are taught in mathematics courses, the concept of complex frequency is rarely mentioned. The complex frequency domain (p. 1) and causality are fundamentally related (Sects. 1.4.6–1.4.8, p. 121–123), and are critical for the analysis of signals and systems, and especially for the concept of impedance (Sect. 1.4.3, p. 110).

Without the concept of time and frequency, one cannot develop an intuition for the Fourier and Laplace transforms, especially within the context of engineering and mathematical physics. The Fourier transform covers signals, while the Laplace transform describes systems. Separating these two concepts, based on their representations as Fourier and Laplace transforms, is an important starting place for understanding physics and the role of mathematics. However, these methods, by themselves, do not provide the insight into physical systems necessary to be productive, or better, creative with these tools. One needs to master the tools of differential equations, and then partial differential equations to fully appreciate the world that they describe. Electrical and mechanical networks, composed of inductors, capacitors and resistors, are isomorphic to mechanical systems composed of masses, springs and dashpots. Newton’s laws are analogous to those of Kirchhoff, which are the rules needed to analyze simple physical systems composed of linear (and nonlinear) sub-components. When lumped-element systems are taken to the limit, in several dimensions, we obtain Maxwell’s partial differential equations, or the laws of continuum mechanics, and beyond.

The ultimate goal of this book is to make you aware of and productive in using these tools. This material can be best absorbed by treating it chronologically through history, so you can see how this body of knowledge came into existence, through the minds and hands of Galileo, Newton, Maxwell and Einstein. Perhaps one day you too can stand on the shoulders of the giants who went before you.

### Appendix B

#### Matrix algebra: Systems

##### B.1 Vectors

Vectors as columns of ordered sets of scalars \(\in \mathbb{C}\). When we write then out in text, we typically use row notation, with the transpose symbol:

\[
[a, b, c]^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.
\]

This is strictly to save space on the page. The notation for conjugate transpose is \(\dagger\), for example

\[
[a] = [a^* b^* c^*].
\]

The above example is said to be a 3 dimensional vector, because it has three components.

**Row vs. column vectors:** With rare exceptions, vectors are columns, denoted *column-major.*

To avoid confusion, it is a good rule to make your mental default column-major, in keeping with most signal processing (vectorized) software. Column vectors are the unstated default of Matlab/Octave, only revealed when matrix operations are performed. The need for the column (or row) major is revealed as a consequence of efficiency when accessing long sequences of numbers from computer memory. For example, when forming the sum of many numbers using the Matlab/Octave command `sum(A)`, where \(A\) is a matrix, by default Matlab/Octave operates on the columns, returning a row vector, of column sums. Specifically

\[
\text{sum} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = [4, 6].
\]

If the data were stored in “row-major” order, the answer would have been the column vector \(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\).

---

33The Riemann zeta function is known as the million dollar equation as there is a cash reward for a proof of the Riemann Hypothesis.
The importance of prime numbers:

Likely the first insight into the counting numbers started with the question, "What are numbers?" The answer comes from looking for irregular patterns in the counting numbers, by examining the multiples of each number, and observing that the multiples of 2 are products of the target prime (2 in our example) and every another integer (e.g., $2, 4, 6, 8, 10, 12, \ldots$). This idea is related to the Sieve of Eratosthenes, an ancient Greek method for finding prime numbers. Are all multiples of 2 removed? Yes, so multiples of 3 are the next to be removed, and so on. This process goes on forever, and the numbers that are left after every other number is removed are the prime numbers: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, and so on. These are the “prime numbers,” the building blocks of all natural numbers.

The history of prime numbers has a rich and complex history, with contributions from ancient mathematicians like Euclid, who proved that there are infinitely many prime numbers, and recent discoveries, such as the discovery of the largest known prime number by a computer in 2018. The study of prime numbers continues to be an active area of research in mathematics, with applications in cryptography, computer science, and other fields.

A prime number is a natural number greater than 1 that has no positive divisors other than 1 and itself. For example, the first six prime numbers are 2, 3, 5, 7, 11, and 13. Every natural number greater than 1 is either a prime number itself or can be uniquely factored into prime numbers. This fundamental theorem of arithmetic is a cornerstone of number theory, and the study of prime numbers continues to be an active area of research in mathematics, with applications in cryptography, computer science, and other fields.

A prime number is a natural number greater than 1 that has no positive divisors other than 1 and itself. For example, the first six prime numbers are 2, 3, 5, 7, 11, and 13. Every natural number greater than 1 is either a prime number itself or can be uniquely factored into prime numbers. This fundamental theorem of arithmetic is a cornerstone of number theory, and the study of prime numbers continues to be an active area of research in mathematics, with applications in cryptography, computer science, and other fields.
CHAPTER 1. INTRODUCTION

The final set of primes is displayed in step 4 of Fig. 1.6.

Once a prime greater than \( \sqrt{N} \) has been identified, the recursion stops, since twice that prime is greater than \( N \), the maximum number under consideration. Once \( \sqrt{N} \) has been reached all the primes have been identified (this follows from the fact that the next prime \( \pi_n \) is multiplied by an integer \( n = 1, \ldots, N \)).

When using a computer, memory efficiency and speed are the main considerations. There are various schemes for making the sieve more efficient. For example, the recursion \( n\pi_k = (n-1)\pi_k + \pi_k \) will speed up the process by replacing the multiply with an addition of \( \pi_k \).

Two fundamental theorems of primes: Early theories of numbers revealed two fundamental theorems (there are many more than two), as discussed in Section 1.2.2 and G.1.1 (p. 227). The first of these is the fundamental theorem of arithmetic, which says that every integer greater than 1 may be uniquely factored into a product of primes

\[
n = \prod_{k=1}^{K} \pi_k^{\beta_k},
\]

where \( k = 1, \ldots, K \) indexes the integer’s \( K \) prime factors \( \pi_k \in P \). Typically prime factors appear more than once, for example \( 25 = 5^2 \). To make the notation compact we define the multiplicity \( \beta_k \) of each prime factor \( \pi_k \). For example \( 2312 = 2^3 \cdot 17^2 = \pi_1^3 \pi_2^2 \) (i.e., \( \pi_1 = 2, \beta_1 = 3; \pi_2 = 17, \beta_2 = 2 \)) and \( 2313 = 3^3 \cdot 257 = \pi_3^3 \pi_6^1 \) (i.e., \( \pi_3 = 3, \beta_3 = 3; \pi_6 = 257, \beta_6 = 1 \)). Our demonstration of this is empirical, using the Matlab/Octave \texttt{factor} (N) routine, which factors \( N \).

What seems amazing is the unique nature of this theorem. Each counting number is uniquely represented as a product of primes. No two integers can share the same factorization. Once multiply the factors out, the result is unique (N). Note that it’s easy to multiply integers (e.g., primes), but expensive to factor them. And factoring the product of three primes is significantly more difficult than factoring two.

Factoring is much more expensive than division. This is not due to the higher cost of division over multiplication, which is less than a factor of 2.\textsuperscript{36} Dividing the product of two primes, given one, is trivial, slightly more expensive that multiplying. Factoring the product of two primes is nearly impossible, as one needs to know what to divide by. Factoring means dividing by some integer and obtaining another integer with remainder zero. Thus one could factor a product of primes \( N = \pi_0 \pi_1 \) by doing \( M \) divisions, where \( M \) is the number of primes less than \( N \). This assumes the list of primes less than \( N \) is known.

But the utility has to do with the density of the primes (the prime number theorem, i.e., Gauss’s hypothesis). If we were simply looking up a few numbers from a short list of primes, it would be easy, but the density of primes among the integers, is logarithmic (>1 per octave, Section G.1.1, p. 229).

This brings us to the prime number theorem (PNT). The security problem is the reason why these two theorems are so important: 1) Every integer has a unique representation as a product of primes, and 2) the density of primes is high (Section G.1.1, p. 227 and discussion on p. 37). Security reduces to the “needle in the haystack” problem, the cost of a search. A more formal way to measure the density is known as Shannon entropy, couched in terms of the expected value of the log-probability of events: “What is the probability of finding a prime between \( N \) and \( 2N \)?\textsuperscript{37}

\textsuperscript{36}If you wish to be a mathematician, you need to learn how to prove theorems. If you’re a physicist, you are happy that someone else has already proved them, so that you can use the result.

\textsuperscript{37}https://streamcomputing.eu/blog/2012-07-16-how-expensive-is-an-operation-on-a-cpu/

\textsuperscript{37}When I understand this better, I’ll do a better job of explaining it.

A.1. NUMBER SYSTEMS

A.1.5 Double-Bold notation

Table A.1 indicates the symbol followed by a page number indication where it is discussed, and the Genus (class) of the number type. For example, \( N > 0 \) indicates the infinite set of counting numbers \( \{1, 2, 3, \ldots\} \), not including zero. Starting from any counting number, you get the next one by adding 1. Counting numbers are also known as the Cardinal numbers.

We say that a number is in the set with the notation \( \beta \in N \subset \mathbb{R} \), which is read as “\( \beta \) is in the set of counting numbers, which in turn in the set of real numbers,” or in vernacular language “\( \beta \) is a real counting number.”

Prime numbers \( \{P \subset N \} \) are taken from the counting numbers, but do not include 1.

The signed integers \( \mathbb{Z} \) include 0 and negative integers. Rational numbers \( \mathbb{Q} \) are historically defined to include \( \mathbb{Z} \), a somewhat inconvenient definition, since the more interesting class are the fractionals \( \mathbb{F} \), a subset of rationals \( \mathbb{F} \subset \mathbb{Q} \) that exclude the integers (i.e., \( \mathbb{F} \cap \mathbb{Z} \)). This is a useful definition because the rationals \( \mathbb{Q} = \mathbb{Z} \cup \mathbb{F} \) are formed from the union of integers and fractionals.

The rationals may be defined, using set notation (a very sloppy language, with incomprehensible syntax) as

\[
\mathbb{Q} = \{p/q : q \neq 0 \& p, q \in \mathbb{Z}\}
\]

which may be read as “the set ‘\( \{\ldots\} \)’ of all \( p/q \) such that ‘\( \cdot\)’ \( q \neq 0 \)’, and ‘\( p, q \subset \mathbb{Z} \)’. The translation of the symbols is in single (‘\( \cdot \)’) quotes.

Irrational numbers \( \hat{x} \) are very special: They are formed by taking a limit of fractionals, as the numerator and denominator \( \rightarrow \infty \), and approach a limit point. It follows that irrational numbers must be approximated by fractionals.

The reals \( \mathbb{R} \) include complex numbers \( \mathbb{C} \) having a zero imaginary part (i.e., \( \mathbb{R} \subset \mathbb{C} \)).

The size of a set is denoted by taking the absolute value (e.g., \( |\mathbb{N}| \)). Normally in mathematics this symbol indicates the cardinality, so we are defining it differently from the standard notation.

Classification of numbers: From the above definitions there exists a natural heretical structure of numbers:

\[
\mathbb{P} \subset \mathbb{N}, \quad \mathbb{Z} : \{\mathbb{N}, 0, -\mathbb{N}\}, \quad \mathbb{F} \cap \mathbb{Z}, \quad \mathbb{Q} : \mathbb{Z} \cup \mathbb{F}, \quad \mathbb{R} : \mathbb{Q} \cup \hat{x} \subset \mathbb{C}
\]

1. The primes are a subset of the counting numbers: \( \mathbb{P} \subset \mathbb{N} \).
2. The signed integers \( \mathbb{Z} \) are composed of \( \pm \mathbb{N} \) and 0, thus \( \mathbb{N} \subset \mathbb{Z} \).
3. The fractionals \( \mathbb{F} \) do not include of the signed integers \( \mathbb{Z} \).
4. The rationals \( \mathbb{Q} = \mathbb{Z} \cup \mathbb{F} \) are the union of the signed integers and fractionals
5. Irrational numbers \( \hat{x} \) have the special properties \( \hat{x} \subseteq \mathbb{Q} \).
6. The reals \( \mathbb{R} : \mathbb{Q} \cup \hat{x} \) are the union of rationals and irrational \( \hat{x} \).
7. Reals \( \mathbb{R} \) may be defined as a subset of those complex numbers \( \mathbb{C} \) having zero imaginary part.
Given the factors, we see that the largest common factor is $\pi$ they are said to be prime. One can define sets of sets and subsets of sets, and this is prone (in my experience) to error.

Of course if we divide 582 into 873 we will numerically obtain the answer $582/873 = 1 + \frac{291}{873}$. Since the two numbers are equal, at which point the GCD equals that final number. If we were to take one more step, the final numbers would be the GCD and zero. For our example step 1 gives $99/10 = 9 + \frac{9}{10}$ has a GCD of 9 and a remainder of $\frac{9}{10}$. Thus we all learned how to compute the GCD in grade school, when we learned long division.

**Exercise:** Show that in Matlab/Octave the

$$\text{gcd}(873, 582) = \text{factor}(873)/\text{factor}(582) = 3/2 = 1 + \frac{1}{2},$$

and the null set $\{\}$ is low cost, compared to factoring, which is extremely expensive. This utility surfaces when the numbers are equal, at which point the GCD equals that final number. If we were to take one more step, the final numbers would be the GCD and zero. For our example step 1 gives $99/10 = 9 + \frac{9}{10}$ has a GCD of 9 and a remainder of $\frac{9}{10}$. Thus we all learned how to compute the GCD in grade school, when we learned long division.

**Exercise:** Show that in Matlab/Octave

$$\text{gcd}(873, 582) = \text{factor}(873)/\text{factor}(582) = 3/2 = 1 + \frac{1}{2},$$

and the null set $\{\}$ is low cost, compared to factoring, which is extremely expensive. This utility surfaces when the numbers are equal, at which point the GCD equals that final number. If we were to take one more step, the final numbers would be the GCD and zero. For our example step 1 gives $99/10 = 9 + \frac{9}{10}$ has a GCD of 9 and a remainder of $\frac{9}{10}$. Thus we all learned how to compute the GCD in grade school, when we learned long division.

**Exercise:** Show that in Matlab/Octave

$$\text{gcd}(873, 582) = \text{factor}(873)/\text{factor}(582) = 3/2 = 1 + \frac{1}{2},$$

and the null set $\{\}$ is low cost, compared to factoring, which is extremely expensive. This utility surfaces when the numbers are equal, at which point the GCD equals that final number. If we were to take one more step, the final numbers would be the GCD and zero. For our example step 1 gives $99/10 = 9 + \frac{9}{10}$ has a GCD of 9 and a remainder of $\frac{9}{10}$. Thus we all learned how to compute the GCD in grade school, when we learned long division.

**Exercise:** Show that in Matlab/Octave

$$\text{gcd}(873, 582) = \text{factor}(873)/\text{factor}(582) = 3/2 = 1 + \frac{1}{2},$$

and the null set $\{\}$ is low cost, compared to factoring, which is extremely expensive. This utility surfaces when the numbers are equal, at which point the GCD equals that final number. If we were to take one more step, the final numbers would be the GCD and zero. For our example step 1 gives $99/10 = 9 + \frac{9}{10}$ has a GCD of 9 and a remainder of $\frac{9}{10}$. Thus we all learned how to compute the GCD in grade school, when we learned long division.
**CHAPTER 1. INTRODUCTION**

**Greatest common divisor: \( k = \gcd(m, n) \)**

- **Examples** \((m, n, k \in \mathbb{Z})\):
  - \( \gcd(13 \cdot 5, 11 \cdot 5) = 5 \) (The common 5 is the gcd)
  - \( \gcd(13 \cdot 10, 11 \cdot 10) = 10 \) (The gcd\((13, 110) = 10 \cdot 2 \cdot 5, \) is not prime)
  - \( \gcd(1234, 1024) = 2 \) (1234 = 2 \cdot 617, 1024 = 2^10)
  - \( \gcd(x \pi_n, x \pi_n) = \pi_k \)
  - \( k = \gcd(n, n) \) is the part that cancels in the fraction \( m/n \in \mathbb{F} \)
  - \( m/\gcd(m, n) \in \mathbb{Z} \)

- **Coprimes** \((m \perp n)\) are numbers with no distinct common factors: i.e., \( \gcd(m, n) = 1 \)
  - The gcd of two primes is always 1: \( \gcd(13, 11) = 1 \), \( \gcd(\pi_m, \pi_n) = 1 \) (\( m \neq n \))
  - If \( m \perp n \) then \( \gcd(m, n) = 1 \)
  - If \( \gcd(m, n) = 1 \) then \( m \perp n \)

- **The GCD may be extended to polynomials:** e.g., \( \gcd(x^2 + bx + c, ax^2 + \beta x + \gamma) \)
  - \( \gcd(x^3 - 3x - 4, (x - 3)(x - 5)) \) \( = (x - 3) \)
  - \( \gcd(x^3 - 3x - 4, 3(x^2 - 8x + 15)) \) \( = 3(x - 3) \)
  - \( \gcd(x^3 - 3x - 4, (3x^2 - 24x + 45)) \) \( = 3(x - 3) \)
  - \( \gcd(x - 2\pi)(x - 4), (x - 2\pi)(x - 5) \) \( = (x - 2\pi) \) (Needs long division)

**Figure 1.7:** The Euclidean algorithm for finding the GCD of two numbers is one of the oldest algorithms in mathematics, and is highly relevant today. It is both powerful and simple. It was used by the Chinese during the Han dynasty (Stillwell, 2010, p. 70) for reducing fractions. It may be used to find pairs of integers that are coprime (their GCD must be 1), and it may be used to identify factors of polynomials by long division. It has an important sister algorithm called the continued fraction algorithm (CFA), that generalizes the Euclidean algorithm.

---

**Appendix A**

**Notation**

**A.1 Number systems**

The notation used in this book is defined in this appendix so that it may be quickly accessed. The definition is sketchy, page numbers are provided where these concepts are fully explained, along with many other important and useful definitions. For example a discussion of \( N \) may be found on page 29. Math symbols such as \( N \) may be found at the top of the index, since they are difficult to alphabetize.

**A.1.1 Units**

Strangely, or not, classical mathematics (as taught today in schools) does not contain the concept of units. It seems units have been abstracted away. This makes mathematics distinct from physics, where almost everything has units. Presumably this makes mathematics more general (i.e., abstract). But for the engineering mind, this is not ideal, as it necessarily means that important physical meaning has been surgically removed, by design. We shall stick to SI units whenever possible. Spatial coordinates are quoted in meters \((m)\), and time in seconds \((s)\). Angles in degrees have no units, whereas radians have units of inverse-seconds \((s^{-1})\).

**A.1.2 Symbols and functions**

We use \( \ln \) as the log function base \( e \), \( \log \) as base \( 2 \), and \( \pi_k \) to indicate the \( k \)th prime (e.g., \( \pi_1 = 2, \pi_2 = 3 \)).

When working with Fourier \( \mathcal{F} \) and Laplace \( \mathcal{L} \) transforms, lower case symbols are in the time domain while upper case indicates the frequency domain, as \( f(t) \leftrightarrow F(\omega) \). An important exception are Maxwell’s equations, because they are so widely used as upper case bold letters (e.g., \( \mathbf{E}(x, \omega) \)). It seems logical to change this to conform to lower case, with \( e(x, t) \leftrightarrow \mathbf{E}(x, \omega) \) as the preferred notation.

**A.1.3 Special symbols common to mathematical:**

There are many pre-defined symbols in mathematics, too many to summarize here. We shall only use a small subset, defined here.

---

\[ 1\text{https://en.wikipedia.org/wiki/List_of_mathematical_symbols_by_subject#Definition_symbols} \]
This program loops until \( m = 0 \). A much more efficient method is described in Section G.1.3, using the `floor()` function, which is called division with rounding.

### 1.2.5 Lec 6: Continued fraction algorithm

In its simplest form, the continued fraction algorithm (CFA) starts from a single real decimal number \( x \in \mathbb{R} \), and recursively expands it as a fraction \( x \in \mathbb{F} \). Thus the CFA is used in finding rational approximations to any real number. For example, \( \pi \approx \frac{22}{7} \), which was well known by the Chinese mathematicians. The GCD (the Euclidean algorithm) on the other hand operates on a pair of integers \( m, n \in \mathbb{N} \) and finds their greatest common divisor \( k \in \mathbb{N} \). Thus \( m/k, n/k \in \mathbb{N} \), reducing the ratio to its irreducible form, since \( m/k \perp n/k \).

Despite this seemingly large difference between the two algorithms, apparently the CFA is closely related to the Euclidean algorithm (the GCD), so closely in fact, that (see Fig. 1.7) Gauss referred to the Euclidean algorithm as the Continued fraction algorithm (Stillwell, 2010, P. 48). At first glance it is not clear why Gauss would call the Euclidean algorithm the CFA. One must assume that Gauss had some deeper insight into this relationship. If so, that insight would be valuable to understand.

In the following we refine the description of the CFA and give examples that go beyond the simple cases of expanding numbers. The CFA of any positive number, say \( x_0 \in \mathbb{R}^+ \), is defined as follows:

1. Start with \( n = 0 \) and a positive input target \( x_0 \in \mathbb{R} \).
2. Define \( a_n = \text{round}(x_n) \), which rounds to the nearest integer.
3. Define \( r_n = x_n - a_n \), thus \( -0.5 \leq r_n \leq 0.5 \).
4. If \( r_n = 0 \), the recursion terminates.
5. Define \( x_{n+1} = \frac{1}{r_n} \) and return to step 2, with \( n = n + 1 \).

The recursion may continue to any desired accuracy, since convergence is guaranteed.

An example: Let \( x_0 \equiv \pi \approx 3.14159 \ldots \). Thus \( a_0 = 3 \), \( r_0 = 0 \). This approximation of \( \pi \approx \frac{22}{7} \) has a relative error of 0.04%.

For the next approximation we continue by reciprocating the remainder \( 1/0.0625 \approx 16 \). which rounds to 16 giving a negative remainder of \( \approx -1/300 \), resulting in the second approximation

\[
\hat{\pi}_3 \approx 3 + \frac{1}{7 + \frac{1}{16}} = 3 + \frac{16}{(7 \cdot 16 + 1)} = 3 + \frac{16}{113} = \frac{355}{113}.
\]

The resolution of this interrelationship is still unresolved.
Note that if we had truncated 15.9966 to 15, the remainder would have been much larger, but always positive, resulting in a much less accurate rational approximation for the same number of terms. It follows that there can be a dramatic difference depending on the rounding scheme, which, for clarity, is best specified rather than inferred.

### Rational approximation examples

\[
\begin{align*}
\hat{x}_2 &= \frac{22}{7} = [3, 7] \quad \Rightarrow \hat{x}_2 + \mathcal{O}(1.3 \times 10^{-3}) \\
\hat{x}_3 &= \frac{355}{113} = [3, 7, 16] \quad \Rightarrow \hat{x}_3 + \mathcal{O}(2.7 \times 10^{-7}) \\
\hat{x}_4 &= \frac{103348}{33215} = [3, 7, 16, -249] \quad \Rightarrow \hat{x}_4 + \mathcal{O}(3.3 \times 10^{-10})
\end{align*}
\]

Figure 1.8: The expansion of \( \pi \) to various orders, using the CFA, along with the order of the error of each rational approximation, with rounding. For example, \( \hat{x}_2 = \frac{22}{7} \) has an absolute error \(|22/7 - \pi|\) of about 0.13%.

**Notation:** Writing out all the fractions can become tedious. For example, expanding \( e = 2.7183 \cdots \) using the Matlab/Octave command `rat(exp(1))` gives the approximation

\[
\exp(1) = 3 + 1/(−4 + 1/(2 + 1/(5 + 1/(−2 + 1/(−7)))))) = \mathcal{O}(1.75 \times 10^{-6}) ,
\]
\[
= [3, −4, 2, 5, −2, −7] − \mathcal{O}(1.75 \times 10^{-6}).
\]

Since many entries are negative, we may deduce that rounding arithmetic is being used by Matlab (but this is not documented). Note that the leading integer part may be noted by an optional decimal point or semicolon.\(^{39}\) If the process is carried further, the values of \( a_n \in \mathbb{N} \) give increasingly more accurate rational approximations.

**Rounding schemes:** In Matlab/Octave there are five different rounding schemes (i.e., mappings): `round(c)`, `fix(c)`, `floor(c)`, `ceil(c)`, `roundb(c)` with \( c \in \mathbb{C} \). If the rounding-down (`floor()`) is used \( \hat{x}_2 = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1] \), whereas true rounding to the nearest integer (`round()`) gives \( \hat{x}_8 = [3, 7, 16, −294, 3, −4, 5, −15] \). Thus `round()` introduces negative coefficients when a number rounds up to the nearest integer.

**Exercise:** Based on several examples, which rounding scheme is the most accurate? Explain why. **Solution:** Rounding will give a smaller remainder at each iteration, resulting in a smaller net error and thus faster convergence.

When the CFA is applied and the expansion terminates \( (r_n = 0) \), the target is rational. When the expansion does not terminate (which is not always easy to determine), as the remainder may be ill-conditioned due to small numerical rounding errors, the number is irrational. Thus the CFA

---

\(^{39}\)Unfortunately Matlab/Octave does not support the bracket notation.

---

### 1.6. Exercises

2. Write a Matlab program to compute \( x_n \) using the matrix equation above (you don’t need to turn in your code). Test your code using the first few values of the sequence. Using your program, what is \( x_{104348} \)?

**Note:** to make your program run faster, consider using the eigen decomposition of \( A \), described by Eq. 1.193 from the Pell’s equation problem.

3. Using the eigen decomposition of the matrix \( A \) (and a lot of algebra), it is possible to obtain the general formula for the Fibonacci sequence,

\[
x_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right].
\]

(1.193)

What are the eigenvalues \( \lambda_k \) of the matrix \( A \)? How is the formula for \( x_n \) related to these eigenvalues?

4. Consider Eq. 1.193 in the limit as \( n \to \infty \).

(a) What happens to each of the two terms \( (1 \pm \sqrt{5})/2 \)?

(b) What happens to the ratio \( x_{n+1}/x_n \)?

5. Prove that \(^{115}\)

\[
\sum_{i=1}^{N} f_n^2 = f_N f_{N+1}.
\]

6. Replace the Fibonacci sequence with

\[
x_n = \frac{x_{n-1} + x_{n-2}}{2},
\]

such that the value \( x_n \) is the average of the previous two values in the sequence.

(a) What matrix \( A \) is used to calculate this sequence?

(b) Modify your computer program to calculate the new sequence \( x_n \). What happens as \( n \to \infty \) ?

(c) What are the eigenvalues of your new \( A \)? How do they relate to the behavior of \( x_n \) as \( n \to \infty \) ? Hint: you can expect the closed-form expression for \( x_n \) to be similar to Eq. 1.193.

7. Now consider

\[
x_n = \frac{x_{n-1} + 1.01x_{n-2}}{2}.
\]

(a) What matrix \( A \) is used to calculate this sequence?

(b) Modify your computer program to calculate the new sequence \( x_n \). What happens as \( n \to \infty \) ?

(c) What are the eigenvalues of your new \( A \)? How do they relate to the behavior of \( x_n \) as \( n \to \infty \) ? Hint: you can expect the closed-form expression for \( x_n \) to be similar to Eq. 1.193.

\(^{115}\)I found this problem on a worksheet for Math 213 midterm (213practice.pdf).
10 numbers may be generated from the counting numbers using
\[ \mod(x,10) \]

1. Show how to generate a base-10 real number \( y \) from the counting numbers using
\[ \mod(x,10) \]

2. Find the first 3 values of \( \sqrt{2} \) to 10 decimal places.
\[ \sqrt{2} \approx 1.4142135624 \]

3. How would you generate hexadecimal numbers (base 16) using the
\[ \mod(x,b) \]

4. Write out the first 19 numbers in hex notation, starting from zero.
\[ \text{hex2dec('ff')} \]

5. What is \( \lambda \) in the CFA is
\[ \lambda = 0.070 \ldots \]

6. Now that you have diagonalized \( A \) and found \( \lambda \), verify that
\[ \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \]
In summary: Every rational number \( m/n \in \mathbb{F} \), with \( m > n > 1 \), may be uniquely expanded as a continued fraction, with coefficients \( a_k \) determined using the CFA. When the target number is irrational (\( x \in \mathbb{Q} \)), the CFA does not terminate; thus, each step produces a more accurate rational approximation, converging in the limit as \( n \to \infty \).

Thus the CFA expansion is an algorithm that can, in theory, determine when the target is rational, but with an important caveat: one must determine if the expansion terminates. This may not be obvious. The fraction \( 1/3 = 0.33333\ldots \) is an example of such a target, where the CFA terminates yet the fraction repeats. It must be that

\[
1/3 = 3 \times 10^{-1} + 3 \times 10^{-2} + 3 \times 10^{-3} + \ldots .
\]

Here \( 3^2 = 3 \). As a second example\(^{40}\)

\[
1/7 = 0.142857142857142857 \ldots = 142857 \times 10^{-6} + 142857 \times 10^{-12} + \ldots
\]

There are several notations for repeating decimals such as \( 1/7 = 0.1\overline{42857} \) and \( 1/7 = 0.1(142857) \). Note that \( 142857 = 999999/7 \). Related identities include \( 1/11 = 0.090909 \ldots \) and \( 10 \times 0.090909 = 999999 \). When the sequence of digits repeats, the sequence is predictable, and it must be rational. But it is impossible to be sure that it repeats, because the length of the repeat can be arbitrarily long.

1.2.6 Lec 7: Pythagorean triplets (Euclid’s formula)

Euclid’s formula is a method for finding three integer lengths \( [a, b, c] \in \mathbb{N} \), that satisfy Eq. 1.1. It is important to ask “Which set are the lengths \( [a, b, c] \) drawn from?” There is a huge difference, both practical and theoretical, if they are from the real numbers \( \mathbb{R} \), or the counting numbers \( \mathbb{N} \). Given \( p, q \in \mathbb{N} \) with \( p > q \), the three lengths \( [a, b, c] \in \mathbb{N} \) of Eq. 1.1 are given by

\[
a = p^2 - q^2, \quad b = 2pq, \quad c = p^2 + q^2.
\]  

(1.9)

This result may be directly verified, since

\[
\]

or

\[
p^4 + q^4 + 2p^2q^2 = p^4 - q^4 - 2p^2q^2 + 4p^2q^2.
\]

Thus, Eq. 1.9 is easily proven, once given. Deriving Euclid’s formula is obviously more difficult. A well-known example is the right triangle depicted in Fig. 1.9, defined by the integers \( [3, 4, 5] \in \mathbb{N} \), having angles \([0.54, 0.65, \pi/2]\) [rad], which satisfies Eq. 1.1 (p. 18). As quantified by Euclid’s formula (Eq. 1.9), there are an infinite number of Pythagorean triplets (PTs). Furthermore the seemingly simple triangle, having angles of \([30, 60, 90]\) \in \mathbb{N} [deg] (i.e., \([\pi/6, \pi/3, \pi/2]\) \in \mathbb{R} [rad]), has one irrational (\( \sqrt{2} \)) length (\( \sqrt{2} \)).

The technique for proving Euclid’s formula for PTs \( [a, b, c] \in \mathbb{Q} \), derived in Fig. G.3 (p. 235) of Section G.2.1, is much more interesting than the PTs themselves.

---

\(^{40}\)Taking the Fourier transform of the target number, represented as a sequence, could help to identify an underlying periodic component. The number \( 1/7 \leftrightarrow [1, 4, 2, 8, 5, 7] \), has a 50 [dB] notch at \( 0.8\pi \) [rad] due to its 6 digit periodicity, carried to 15 digits (Matlab/Octave maximum precision), Hamming windowed, and zero padded to 1024 samples.
Figure 1.9: Beads on a string form perfect right triangles when the number of unit lengths between beads for each side satisfies Eq. 1.1. For example, when $p = 2$, $q = 1$, the sides are $[3, 4, 5]$.

Figure 1.10: "Plimpton-322" is a stone tablet from 1800 BCE, displaying $a$ and $c$ values of the Pythagorean triplets $[a, b, c]$, with the property $b = \sqrt{c^2 - a^2} \in \mathbb{N}$. Several of the $c$ values are primes, but not the $a$ values. The stone is item 322 (item 3 from 1922) from the collection of George A. Plimpton. –Stillwell (2010)

The set from which the lengths $[a, b, c]$ are drawn was not missed by the Indians, Chinese, Egyptians, Mesopotamians, Greeks, etc. Any equation whose solution is based on integers is called a Diophantine equation, named after the Greek mathematician Diophantus of Alexandria (c. 250 CE).

A stone tablet having the numbers engraved on it, as shown in Fig. 1.10, was discovered in Mesopotamia, from the 19th century BCE, and cataloged in 1922 by George Plimpton. These numbers are $a$ and $c$ pairs from PTs $[a, b, c]$. Given this discovery, it is clear that the Pythagoreans were following those who came long before them. Recently a second similar stone, dating between 350 and 50 BCE has been reported, that indicates early calculus on the orbit of Jupiter's moons.


1.2. STREAM 1: NUMBER SYSTEMS (10 LECTURES)

Here we explore the continued fraction algorithm (CFA), as discussed in Lec. 6 (Chapters 1 and 2). In its simplest form the CFA starts with a real number, which we denote as $\alpha \in \mathbb{R}$. Let us work with an irrational real number, $\pi \in \mathbb{R}$, as an example, because its CFA representation will be infinitely long. We can represent the CFA coefficients $\alpha$ as a vector of integers $\mathbf{n} = [n_1, n_2, \ldots]$. Let $\mathbf{n} = [n_1, n_2, \ldots]$ be the vector of integers. Since we have a real number with a rational number, we can express the CFA representation with a real number $\alpha$.

As discussed in Section 1.2, the CFA is recursive with these steps:

$$\alpha = \frac{n_1}{1 + \frac{n_2}{1 + \frac{n_3}{1 + \ddots}}}$$

In Matlab (Octave) script

```matlab
alpha0 = pi;
K=10;
alpha=zeros(1,K);
n=zeros(1,K);
for k=2:K %k=1 to K
    n(k)=round(alpha(k-1));
    alpha(k)= 1/(alpha(k-1)-n(k));
end
disp([n; alpha]);
```

1. By hand (you may use Matlab as a calculator), find the first 3 values of $n_k$ for $\alpha = \pi$.

2. For part (1), what is the error (remainder) when you truncate the continued fraction after $n_1$, ..., $n_3$? Give the absolute value of the error, and the percentage error relative to the original $\alpha$.

3. Use the Matlab program provided to find the first 10 values of $n_k$ for $\alpha = \pi$, and verify your result using the Matlab command `rat()`.

4. Discuss the similarities and differences between the Euclidean algorithm (EA) and CFA.
1.7 Lec 8: Pell’s Equation

There is a venerable history for Pell’s equation

\[ x^2 - Ny^2 = 1, \quad (1.10) \]

with non-square \( N \in \mathbb{N} \) specified and \( x, y \in \mathbb{N} \) unknown. It is suspected that Pell’s equation is directly related to the Euclidean algorithm, as applied to polynomials having integer coefficients (Stillwell, 2010, 48). For example, with \( N = 2 \), one solution is \( x = 17, y = 12 \) (i.e., \( 17^2 - 2 \cdot 12^2 = 1 \)).

A 2x2 matrix recursion algorithm, likely due to the Chinese, was used by the Pythagoreans to investigate the \( \sqrt{N} \)

\[
\begin{bmatrix}
  x_n \\
  y_n
\end{bmatrix} = \begin{bmatrix} 1 & N \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_{n-1} \\
  y_{n-1}
\end{bmatrix},
\]

(1.11)

Starting with \([x_0, y_0]^T = [1, 0]^T\), results in solutions of Pell’s equations (Stillwell, 2010, p. 44).

Note that this is a composition method, of 2x2 matrices, since the output of one matrix multiply is the input to the next. They key question is what is the relationship between Pell’s equation and the linear recursion? Is it that Pell’s equation may be trivially factored? There must be some simple way to prove that Eqs. 1.11 and 1.10 are equivalent, as demonstrated on Section G.2.3 (p. 237).

**Example:** \(\gcd(2, 3) = 1\)

**Define**

\[
\begin{bmatrix}
  a_0 \\
  b_0
\end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \begin{bmatrix}
  a_{i+1} \\
  b_{i+1}
\end{bmatrix} = \begin{bmatrix} 0 & 1 \\
  1 & -[a_i/b_i] \end{bmatrix} \begin{bmatrix}
  a_i \\
  b_i
\end{bmatrix}
\]

**Example: \(\gcd(2, 3) = 1\):** For \((a, b) = (2, 3)\), the result is as follows:

\[
\begin{bmatrix}
  1 \\
  0
\end{bmatrix} = \begin{bmatrix} 0 & 1 \\
  1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\
  1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\
  3 & -2 \end{bmatrix}
\]

Thus from the above equation we find the solution \((m, n)\) to the integer equation

\[ 2m^2 + 3n^2 = \gcd(2, 3) = 1, \]

namely \((m, n) = (-1, 1)\) (i.e., \(-2 + 3 = 1\)). There is also a second solution \((3, -2)\), (i.e., \(3 \cdot 2 - 2 \cdot 3 = 0\)), which represents the terminating condition. Thus these two solutions are a pair and the solution only exists if \((a, b)\) are coprime \((a \perp b)\). Subtraction method: This method is more complicated than the division algorithm, because at each stage we must check if \(a < b\).

Define

\[
\begin{bmatrix}
  a_0 \\
  b_0
\end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -1 \\
  0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 \\
  1 & 0 \end{bmatrix}
\]

where \(Q\) sets \(a_{i+1} = a_i - b_i\) and \(b_{i+1} = b_i\) assuming \(a_i > b_i\), and \(S\) is a ‘swap-matrix’ which swaps \(a_i\) and \(b_i\) if \(a_i < b_i\). Using these matrices, the algorithm is implemented by assigning

\[
\begin{bmatrix}
  a_{i+1} \\
  b_{i+1}
\end{bmatrix} = Q \begin{bmatrix} a_i \\
  b_i\end{bmatrix} \text{ for } a_i > b_i, \quad \begin{bmatrix}
  a_{i+1} \\
  b_{i+1}
\end{bmatrix} = QS \begin{bmatrix} a_i \\
  b_i\end{bmatrix} \text{ for } a_i < b_i.
\]

The result of this method is a cascade of \(Q\) and \(S\) matrices. For \((a, b) = (2, 3)\), the result is as follows:

\[
\begin{bmatrix}
  1 \\
  0
\end{bmatrix} = \begin{bmatrix} 1 & -1 \\
  0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\
  1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\
  -1 & 2 \end{bmatrix}
\]

Thus we find two solutions \((m, n)\) to the integer equation \(2m^2 + 3n^2 = \gcd(2, 3) = 1\).

1. By inspection, find at least one integer pair \((m, n)\) that satisfies \(12m + 15n = 3\).
2. Using matrix methods for the Euclidean algorithm, find integer pairs \((m, n)\) that satisfy \(12m + 15n = 3\) and \(12m + 15n = 0\). Show your work!!!
3. Does the equation \(12m + 15n = 1\) have integer solutions for \(n\) and \(m\)? Why, or why not?
Consider the Euclidean algorithm to find the greatest common divisor (GCD; the largest integer that divides both of the given integers).

The GCD of two numbers is the largest positive integer that divides both of the numbers without leaving a remainder. It is often used to find the common factors of two or more numbers.

The Euclidean algorithm is a simple and efficient method for finding the GCD. It is based on the observation that the GCD of two numbers also divides their difference. In other words, the GCD of two numbers is the same as the GCD of the smaller number and the difference of the two numbers.

The algorithm works as follows:
1. If one of the numbers is 0, the other number is the GCD.
2. Otherwise, use the Euclidean algorithm recursively on the smaller number and the remainder of the division of the larger number by the smaller number.

For example, to find the GCD of 54 and 24:

1. $54 \div 24 = 2$ remainder $6$.
2. $24 \div 6 = 4$ remainder $0$.
3. Therefore, the GCD of 54 and 24 is 6.

The algorithm is based on the principle that the GCD of two numbers also divides their difference. This property is used to reduce the problem to a smaller one by subtracting the smaller number from the larger one and repeating the process until one of the numbers becomes 0.

The Euclidean algorithm is a widely used method for finding the GCD and is one of the most efficient algorithms for this purpose. It is also used in various other applications, such as cryptography, computer algebra, and number theory.
result in damped modes, which decay in time due to energy losses. Common examples include tuning forks, pendulums, bell, and strings of musical instruments, all of which have a characteristic frequency.

Cauchy’s residue theorem is used to find the time-domain response of each frequency-domain complex eigen-mode. Thus eigen-analysis and eigen-modes of physics are the same thing (see Sect. 1.4.3, p. 110), but are described using different (i.e., mutually unrecognizable) notional methods. The eigen-method method is summarized in Appendix D, p. 201.

Taking a simple example of a $2 \times 2$ matrix $T \in \mathbb{C}$, we start from the definition of the two eigen-equations

$$TE_k = \lambda_k e_k,$$

(1.13)

corresponding to two eigen-values $\lambda_k \in \mathbb{C}$ and two $2 \times 1$ eigen-vectors $e_k \in \mathbb{C}$. The eigen-values $\lambda_k$ may be merged into a $2 \times 2$ diagonal eigen-value matrix

$$\Lambda = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix},$$

while the two eigen-vectors $e_+$ and $e_-$ are merged into a $2 \times 2$ eigen-vector matrix

$$E = \begin{bmatrix} e_+ \\ e_- \end{bmatrix} = \begin{bmatrix} e_+^T \\ e_-^T \end{bmatrix},$$

(1.14)

corresponding to the two eigen-values. Using matrix notation, this may be compactly written as

$$TE = E\Lambda.$$

(1.15)

Note that while $\Lambda$ and $E$ commute, $E\Lambda \neq \Lambda E$. From Eq. 1.15 we may obtain two very important forms:

1. the diagonalization of $T$

$$\Lambda = E^{-1}TE,$$

(1.16)

and

2. the eigen-expansion of $T$

$$T = E^{-1}\Lambda E^{-1},$$

(1.17)

which is useful for computing power of $T$ (i.e., $T^{100} = E^{-1}\Lambda^{100}E$).

**Example:** If we take

$$T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

then the eigen-values are given by $(1 - \lambda_k)(1 + \lambda_k) = -1$, thus $\lambda_k = \pm \sqrt{2}$. This method of eigen-analysis is discussed in Section G.2.2 (p. 234) and Appendix E (p. 205).

The key idea of the $2 \times 2$ matrix solution, widely used in modern engineering, can be traced back to Brahmagupta’s solution of Pell’s equation, for arbitrary $N$. Brahmagupta’s recursion, identical to that of the Pythagoreans’ $N = 2$ case (Eq. 1.11), eventually led to the concept of linear algebra, defined by the simultaneous solutions of many linear equations. The recursion by the Pythagoreans

4During the discovery or creation of quantum mechanics, two alternatives were developed: Schrödinger’s differential equation method and Heisenberg’s matrix method. Eventually it was realized the two were equivalent.

### EXERCISES NS-2

**Topic of this homework:** Prime numbers, greatest common divisors, the continued fraction algorithm

**Deliverable:** Answers to questions.

1. According to the fundamental theorem of arithmetic, every integer may be written as a product of primes.

   (a) Put the numbers $1,000, 1,000, 1,000, 004$ and $999,999$ in the form $N = \prod_{k} \pi_k^{\beta_k}$ (you may use Matlab to find the prime factors).

   (b) Give a generalized formula for the natural logarithm of a number, $\ln(N)$, in terms of its primes $\pi_k$ and their multiplicities $\beta_k$. Express your answer as a sum of terms.

2. Explain why the following 2-line Matlab/Octave program returns the prime numbers $\pi_k$ between 1 and 100!

   ```matlab
   n=2:100; k = 1;isprime(n); n(k)
   ```

3. Prime numbers may be identified using ‘sieves’

   (a) By hand, perform the sieve of Eratosthenes for $n = 1 \ldots 49$. Circle each prime $p$ then cross out each number which is a multiple of $p$.

   (b) In part (a), what is the highest number you need to consider before only the primes remain?

   (c) Generalize: for $n = 1 \ldots N$, what is the highest number you need to consider before only the primes remain?

   (d) Write each of these numbers as a product of primes:

   22= \hspace{1cm} 30= \hspace{1cm} 34= \hspace{1cm} 43= \hspace{1cm} 44= \hspace{1cm} 48= \hspace{1cm} 49= \hspace{1cm} · · ·

4. Find the largest prime $\pi_k \leq 100$? Hint: Do not use matlab other than to check your answer.

   Hint: Write out the numbers starting with 100 and counting backwards: 100, 99, 98, 97, · · · . Cross off the even numbers, leaving 99, 97, 95, · · · . Pull out a factor (only 1 is necessary to show that it is not prime).

5. Find the largest prime $\pi_k \leq 1000$? Hint: Do not use matlab other than to check your answer.
Another classic problem, formulated by the Chinese, was the Fibonacci sequence, generated by the recursion

\[
F_n = F_{n-1} + F_{n-2}
\]

Suppose that the Fibonacci sequence recursion is replaced by

\[
F_n = F_{n-1} + 1
\]

This construction is called the Fibonacci spiral. Note how it is constructed out of squares having areas given by the square of the Fibonacci numbers. In this way, the spiral is smooth and the radius increases as the

\[
\frac{\sqrt{5}}{2}\approx 0.85
\]

Another classic problem, formulated by the Chinese, was the Fibonacci sequence, generated by the recursion

\[
F_n = F_{n-1} + F_{n-2}
\]

Suppose that the Fibonacci sequence recursion is replaced by

\[
F_n = F_{n-1} + 1
\]

This construction is called the Fibonacci spiral. Note how it is constructed out of squares having areas given by the square of the Fibonacci numbers. In this way, the spiral is smooth and the radius increases as the

\[
\frac{\sqrt{5}}{2}\approx 0.85
\]
Exercise: Find the 2x2 matrix corresponding to Eq. 1.20. Solution: The 2x2 matrix may be found using the companion matrix method (p. 66), giving
\[
\begin{bmatrix}
1 \\
[y]
\end{bmatrix}_{n+1} = \begin{bmatrix}
1 & 1 \\
2 & 0
\end{bmatrix}\begin{bmatrix}
x \\
y
\end{bmatrix}_n.
\]
(1.21)
The eigen-values of this matrix are \([1, -1/2]\) (i.e., the roots of the binomial equation \(\lambda^2 - \lambda/2 - 1/2 = 0\)). Thus \([x_n, y_n] = [1, 1]^T\) and \([x_n, y_n] = [1, 1]^T\) are both solutions.

Exercise: Starting from \([x_n, y_n]^T = [1, 0]^T\) compute the first 5 values of \([x_n, y_n]^T\). Solution: Here is a Matlab/Octave code for computing \(x_n\):
\[
x(1:1:2)=[1;0];
A=[1 1;2 0]/2;
for k=1:10; x(k+1)=A*x(:,k); end
\]
which gives the rational \((x_n \in \mathbb{Q})\) sequence: 1, 1/2, 3/4, 5/8, 11/16, 21/32, 43/64, 85/128, 171/256, 341/512, 683/1024, \ldots.

Exercise: Show that the solution to Eq. 1.20 is bounded, unlike that of the Fibonacci sequence, which diverges. Explain what is going on. Solution: Because the next value is the mean of the last two, the sequence is bounded. To see this one needs to compute the eigen-values of the matrix of Eq. 1.20.

Exercise: Use the formula for the generalized diagonalization of a matrix to find the general solution of the mean-Fibonacci sequence. Solution: The eigen-values are given by the roots of
\[
0 = -\lambda(1/2 - \lambda) - 1/2 = (\lambda - 1/4)^2 - 9/16
\]
which are \([1, -1/2]\).

By studying the eigen-values of Eq. 1.21 one finds that the steady state solution approaches 1. Namely \(f_n \to 1 = (f_{n-1} + f_{n-2})/2\) is the solution, as \(n \to \infty\). Namely the average of the last two values must approach 1 for large \(n\).

Exercise: Show that the geometric series formula holds for 2x2 matrices. Starting with the 2x2 identity matrix \(I_2\) and \(a \in \mathbb{C}\), with \(|a| < 1\), show that
\[
I_2(I_2 - aI_2)^{-1} = I_2 + aI_2 + a^2I_2^2 + a^3I_2^3 + \cdots.
\]
Solution: Since \(a^k I_2^k = a^k I_2\), we may multiply both sides by \(I_2 - aI_2^2\) to obtain
\[
I_2 = I_2 + aI_2 + a^2I_2^2 + a^3I_2^3 + \cdots - aI_2(aI_2 + a^2 I_2^2 + a^3 I_2^3 + \cdots)
= 1 + (a + a^2 + a^3 + \cdots) - (a + a^2 + a^3 + \cdots)I_2
= I_2
\]

EXERCISES NS-1

1.6 Exercises

Topic of this homework: Introduction to MATLAB/OCTAVE (see the Matlab or Octave tutorial for help).

Deliverable: Report with charts and answers to questions. Hint: Use LATEX\footnote{http://www.overleaf.com}. Plotting complex quantities in Matlab

Plot real, imaginary, magnitude and phase quantities.

1. Consider the functions \(f(s) = s^2 + 6s + 5\) and \(g(s) = s^2 + 6s + 5\).

(a) Find the zeros of functions \(f(s)\) and \(g(s)\) using the command roots.

(b) On a single plot, show the roots of \(f(s)\) as red circles, and the roots of \(g(s)\) as blue plus signs. The x-axis should display the real part of each root, and the y-axis should display the imaginary part. Use hold on and grid on when plotting the roots.

(c) Give your figure the title ‘Complex Roots of \(f(s)\) and \(g(s)\)’ using the command title. Label the x-axis ‘Real Part’ and the y-axis ‘Imaginary Part’ using xlabel and ylabel.

2. Consider the function \(h(t) = e^{2t}t^5\) for \(t = 5\) and \(t = [0:0.01:2]\).

(a) Use subplot to show the real and imaginary parts of \(h(t)\) as two graphs in one figure. Label the x-axis ‘Time (\(s\))’ and the y-axis ‘Real Part’ and ‘Imaginary Part’.

(b) Use subplot to plot the magnitude and phase parts of \(h(t)\). Use the command angle or unwrap(angle()) to plot the phase. Label the x-axis ‘Time (\(s\))’ and the y-axes ‘Magnitude’ and ‘Phase (radians)’.

1. Prime numbers in Matlab

(a) Use the Matlab function factor to find the prime factors of 123, 248, 1767, and 999,999.

(b) Use the Matlab function isprime to check if 2, 3 and 4 are prime numbers. What does the function isprime return when a number is prime, or not prime? Why?

(c) Use the Matlab function primes to generate prime numbers between 1 and \(10^6\) and save them in a vector \(x\). Plot this result using the command hist(x).

(d) Now try \([n, bin centers] = hist(x)\). Use length(n) to find the number of bins.

(e) Set the number of bins to 100 by using an extra input argument to the function hist. Show the resulting figure and give it a title and axes labels.

2. Inf, NaN and logarithms in Matlab
Summary: The GCD (Euclidean algorithm), Pell's equation and the Fibonacci sequence may all be written as compositions of 2x2 matrices. Thus Pell's equation and the Fibonacci sequence are special cases of 2x2 matrix composition

\[
\begin{pmatrix} x & y \\ n+1 \\ \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \\ \end{pmatrix} \begin{pmatrix} x & y \\ n \\ \end{pmatrix}
\]

This is an important and common thread of these early mathematical findings. It will turn out that this 2x2 matrix recursion plays a special role in physics, mathematics and engineering, because such equations are solved using the eigen-analysis method. The first several thousands of years of mathematical trial and error set the stage for this breakthrough, but this took a long time to fully appreciate.

Discuss importance of eigen-analysis

1.2.9 Lec 10: Exam I (In class)
1.3 Algebraic Equations: Stream 2

The era of 1600 to 1850 (Fig. 1.24, p. 104) produced a stream of fundamental theorems. A few of the individuals who played a notable role in this development, in chronological (birth) order, include Galileo, Mersenne, Newton, d’Alembert, Fermat, Huygens, Descartes and Helmholz. These individuals were some of the first to develop the basic ideas, in various forms, that were then later reworked into the proofs, that today we recognize as the fundamental theorems of mathematics.

1.3.1 Lec 11 The physics behind nonlinear Algebra (Euclidean geometry)

Following Stillwell’s history of mathematics, Stream 2 is geometry, which led to the merging of Euclid’s geometrical methods and the 9th century development of algebra by al-Khw¯ arizm¯ i (830 CE). This integration of ideas led Descartes and Fermat to develop analytic geometry. While not entirely a unique and novel idea, it was late in coming, given what was known at that time.

The mathematics up to the time of the Greeks, documented and formalized by Euclid, served students of mathematics for more than two thousand years. Algebra and geometry were, at first, independent lines of thought. When merged, the focus returned to the Pythagorean theorem, generalized as analytic conic sections rather than as geometry in Euclid’s Elements. With the introduction of algebra, numbers, rather than lines, could be used to represent a geometrical length. Thus the appreciation for geometry grew, given the addition of the rigorous analysis using numbers.

Physics inspires algebraic mathematics: The Chinese used music, art, and navigation to drive mathematics. Unfortunately much of their knowledge has been handed down either as artifacts, such as musical bells and tools, or mathematical relationships documented, but not created, by scholars such as Euclid, Archimedes, Diophantus, and perhaps Brahmagupta. With the invention of algebra by al-Khw¯ arizm¯ i (830CE), mathematics became more powerful, and blossomed. During the 16th and 17th century, it had become clear that differential equations (DEs), such as the wave equation, can characterize a law of nature, at a single point in space and time. This principle was not obvious. A desire to understand motions of objects and planets precipitated many new discoveries. This period is illustrated in Fig. 1.2. Galileo investigated gravity and invented the telescope. The law of gravity was first formulated by Galileo to explain the falling of two objects of different masses, and how they must obey conservation of energy. Kepler investigated the motion of the planets. While Kepler was the first to observe that orbit of planets is described by ellipses, it seems he under-appreciated the significance of his finding, and continued working on his incorrect epicycle planetary model. Following up on Galileo’s work, Newton (c1687) went on to show that there must be a gravitational potential between two masses \( m_1, m_2 \) of the form

\[
\phi_g(r(t)) \propto \frac{m_1m_2}{r(t)},
\]

where \( r = |x_1 - x_2| \) is the Euclidean distance between the two point masses at locations \( x_1 \) and \( x_2 \). Using algebra and his calculus, Newton formalized the equation of gravity, forces and motion (Newton’s three laws) and showed that Kepler’s discovery of planetary elliptical motion naturally follows from these laws. With the discovery of Uranus (1781) “Kepler’s theory was ruined” (i.e., proven wrong) (Stillwell, 2010, p. 23).

Once Newton proposed the basic laws of gravity, he proceed to calculate, for the first time, the fundamental theorem of algebra, and essential for solving the generalizations of the number systems.

Many of the concepts about numbers naturally evolved from music, where the length of a string (along with its tension) determined the pitch (Stillwell, 2010, pp. 11, 16, 153, 261). Cutting the string’s length by half increased the frequency by a factor of 2. One fourth of the length increases the frequency by a factor of 4. One octave is a factor of 2 and two octaves a factor of 4 while a half octave is \( \sqrt{2} \). The musical scale was soon factored into rational parts. This scale almost worked, but did not generalize (sometimes known as the Pythagorean comma\(^{13}\)), resulting in today’s well tempered scale, which is based on 12 equal geometric steps along one octave, or 1/12 octave (\( \sqrt[12]{2} \approx 1.05946 \approx 18/17 = 1 + 1/17 \)).

But the concept of a factor was clear. Every number may be written as either a sum or a product (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of \( x^2 = \phi \) (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same ...
The speed of sound is a key equation in mathematical physics. The speed of sound, \( v \), is defined as the ratio of the pressure change to the particle displacement.

\[
v = \frac{\Delta P}{\Delta x}
\]

where \( \Delta P \) is the pressure change and \( \Delta x \) is the particle displacement.

According to the known written record, the number zero (null) had no written symbol until the invention of the Roman numeral system. Zero was finally published in 1687, and the general solution to Newton's wave equation, \( \psi(x,t) = o_1 e^{i(x\omega t - kx)} + o_2 e^{i(x\omega t - kx)} \), was finally published 60 years later. The solution was only valid for waves in a linear medium.

The speed of sound is given by

\[
v = \sqrt{\frac{\partial^2 \psi}{\partial x^2}}
\]

This is the wave equation in its linear form. The solution to the wave equation is given by

\[
\psi(x,t) = o_1 e^{i(x\omega t - kx)} + o_2 e^{i(x\omega t - kx)}
\]

where \( o_1 \) and \( o_2 \) are constants.

The speed of sound is related to the pressure and density of the medium by

\[
v = \frac{\sqrt{k^2 \rho}}{\rho}
\]

where \( k \) is the wave number and \( \rho \) is the density of the medium.

The frequency of sound is given by

\[
f = \frac{k}{2\pi}
\]

This is the frequency domain concept of Fourier analysis, based on the idea that complex signals can be broken down into simpler harmonic components.

Today d'Alembert's analytic wave solution can be written as Eq. 1.24 with a real wave number, \( k = 1 \), and a real wave frequency, \( f = 1 \). This view is elegantly explained by Brillouin (1953, Chap. 1), in an historical context.
1881; Boyer and Merzbach, 2011), “trapping” the heat energy in the wave. There were several other physical enigmas, such as the observation that sound disappears in a vacuum and that a vacuum cannot draw water up a column by more than 34 feet. In air, assuming no visco-elastic losses, it is constant (i.e., $c_v = \sqrt{\frac{\rho c^2}{\rho + \rho_v}}$). When including losses the wave number becomes a complex function of frequency, leading to Eq. 1.25. In periodic structures, again the wave number becomes complex due to diffraction, as commonly observed in optics (e.g., diffraction gratings) and acoustics. Thus Eq. 1.26 only holds for the most simple cases, but in general it must be considered as a complex analytic function of $s$, as $n(s)$ in Eq. 1.25.

The corresponding discovery for the formula for the speed of light was made 174 years after Principia, by Maxwell (c1861). Maxwell’s formulation also required great ingenuity, as it was necessary to hypothesize an experimentally unmeasured term in his equations, to get the mathematics to correctly predict the speed of light.

**Chronological history from the 17th century**

17th
Newton 1642–1727, Bernoulli, Johann 1667–1748

18th
Bernoulli, Daniel, Cauchy 1789–1857, Euler 1707–1783, d’Alembert 1717–1783, Gauss 1777–1855

19th

20th
Sommerfeld 1868–1951, Einstein 1879–1955, Brillouin 1889–1969 ...

**Time-Line**

<table>
<thead>
<tr>
<th>1640</th>
<th>1700</th>
<th>1750</th>
<th>1800</th>
<th>1850</th>
<th>1900</th>
</tr>
</thead>
<tbody>
<tr>
<td>Newton</td>
<td>Bernoulli</td>
<td>Gauss</td>
<td>Stokes</td>
<td>Helmholtz</td>
<td>Kirchhoff</td>
</tr>
<tr>
<td>Johann</td>
<td>Daniel</td>
<td>Euler</td>
<td>d’Alembert</td>
<td>Kelvin</td>
<td>Riemann</td>
</tr>
<tr>
<td>Mozart</td>
<td>Beethoven</td>
<td>Rayleigh</td>
<td>Heaviside</td>
<td>Poincare</td>
<td>Sommerfeld</td>
</tr>
</tbody>
</table>

Figure 1.13: Time-line of the three centuries from the mid 17th to 20th CE, one of the most productive times of all, because mathematicians were sharing information. Figure 1.2 (p. 23) (Newton-Gauss) provides a closer look at the 15–18 CE, and Fig. 1.24 (p.104) (Bombelli-Einstein) provides the full view 16–20 CE. (fig:TimeLine19CE)

The first Algebra:

Prior to the invention of algebra, people worked out problems as sentences using an obtuse description of the problem (Stillwell, 2010, p. 93). Algebra changed this approach, resulting in a compact language of mathematics, where numbers are represented as abstract symbols (e.g., $x$ and $a$). The problem to be solved could be formulated in terms of sums of powers of smaller terms, the most

1.5. STREAM 3B: VECTOR CALCULUS (10 LECTURES)

of this equilibrium? The most satisfying answer is provided by looking at the internal forces on the air, due to the gradients in the pressure. The pressure $p(x,t)$ is a potential, thus its gradient is a force density $f(x,t) = -\nabla p(x,t)$. This equation tells us how the pressure wave evolves as it propagates down the horn. Any curvature in the pressure wave-front induces stresses, which lead to changes (strains) in the local wave velocity, in the directions of the force density. The main force is driving the wave-front forward (down the horn), but there are radial (transverse) forces as well, which tend to rapidly go to zero.

For example, if the tube has a change in area (or curvature), the local forces will create radial flow, which is immediately reflected by the walls, due to the small distance to the walls, causing the forces to average out. After traveling a few diameters, these forces will come to equilibrium. Because of the slower speed, the ear drum has low-frequency cross-modes, and the cross-modes are present, but we call upon the quasi-static approximation as a justification for ignoring them, to get closer to the first-order physics.

**Quasi-statics and Quantum Mechanics**

It is important to understand the meaning of Planck’s constant $h$, which appears in the relations of both photons (light “particles”) and electrons (mass particles). If we could obtain a handle on what exactly Planck’s constant means, we might have a better understanding of quantum mechanics, and physics in general. By cataloging the dispersion relations (the relation between the wavelength $\lambda(v)$ and the frequency $v$), between electrons and photons, this may be attainable.

Basic relations from quantum mechanics for photons and electrons include:

1. Photons (mass=0, velocity = $c$)
   - $c = \lambda \nu$: The speed of light $c$ is the product of its wavelengths $\lambda$ times its frequency $\nu$.
   - This relationship is only for mono-chromatic (single frequency) light.
   - The speed of light is
     \[ c = \frac{1}{\sqrt{\mu_0 \sigma_0}} = 0.3 \times 10^6 \text{[m/s]} \]
   - The characteristic resistance of light $r_o = \sqrt{\mu_0 / \sigma_0} = |E|/|H| = 377$ [ohms] is defined as the magnitude of the ratio of the electric $E$ and magnetic $H$ field, of a plane wave in-vacuo.
   - $E = h \nu$: the photon energy is given by Planck’s constant $h \approx 6.623 \times 10^{-34} \text{[m}^2 \text{kgm/s]}, times the frequency (or bandwidth) of the photon.
\[ \frac{\partial^2 x}{\partial t^2} - \nabla^2 x = 0 \]

Solving Eq. 1.28 in zero and applying the two forms, we get the quadratic formula:

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

The best way to proceed is to divide this effort as an obsession. Today the roots of any polynomial may be found, to high accuracy, by

\[ \hat{a} \]

The problem of factoring polynomials has a history more than a millennium in the making. While

\[ N \]

The acoustic wave equation describes how the scalar field pressure

\[ P \]

One of the very first insights into wave propagation was due to Huygens (c1640) (Fig. 1.24).

\[ \lambda \approx \frac{x}{y} \]

Quasi-statics and its implications:

\[ \lambda \]
The roots of \( \hat{P}_2(x) \), with \( a = 1 \), greatly simplify to
\[
x_\pm = -\frac{1}{2} b \pm \sqrt{\frac{b^2}{4} - c}.
\] (1.31)

This can be simplified even further. The term \( b^2 - c > 0 \) under the square root is called the discriminant, and in physics and engineering problems, 99.9% of the time it is negative. Finally \( b/2 \ll \sqrt{c} \), the most natural way (i.e., corresponding to the most common physical cases) of writing the solution is
\[
x_\pm = \frac{-b \pm \sqrt{c - (b/2)^2}}{b/2} \approx -b/2 \pm j\sqrt{c}.
\] (1.32)

This form separates the real and imaginary parts of the solution in a natural way. The term \( b/2 \) is called the damping, which accounts for losses in a resonant circuit, while the term \( \sqrt{c} \), for mechanical, acoustical and electrical networks, is called the resonant frequency, typically written as \( \omega_r \). The last approximation ignores the (typically) minor correction to the resonant frequency, which in engineering practice is typically always ignored. Knowing that there is a correction is highlighted by this formula, making one aware it exists.

**Summary:** The quadratic equation and its solution are ubiquitous in physics and engineering. It seems obvious that instead of memorizing the meaningless Eq. 1.30, one should learn the physically meaningful solution, Eq. 1.32, obtained via Eq. 1.29, with \( a = 1 \). Arguably, the factored and normalized form (Eq. 1.29) is easier to remember, as a method (completing the square), rather than as a formula to be memorized.

Additionally, the real \((b/2)\) and imaginary \(\sqrt{c}\) parts of the roots have physical significance as the damping and resonant frequency. Equation 1.30 has none.

No insight is gained by memorizing the quadratic formula. To the contrary, an important concept is gained by learning how to complete the square, which is typically easier than identifying \(a, b, c\) and blindly substituting them into Eq. 1.30. Thus it’s worth learning the alternate solution (Eq. 1.32) since it is more common in practice and requires less algebra to interpret the final answer.

**Exercise:** By direct substitution demonstrate that Eq. 1.30 is the solution of Eq. 1.28. Hint: Work with \( \hat{P}_2(x) \). **Solution:** Setting \( a = 1 \) the quadratic formula may be written
\[
x_\pm = \frac{-b \pm \sqrt{4c - b^2}}{2}
\] Substituting this into \( \hat{P}_2(x) \) gives
\[
\hat{P}_2(x_\pm) = x_\pm^2 + bx_\pm + c
\]
\[
= \left( -\frac{b \pm \sqrt{b^2 - 4c}}{2} \right)^2 + b \left( -\frac{b \pm \sqrt{b^2 - 4c}}{2} \right) + c
\]
\[
= \frac{1}{4} \left( b^2 \pm 2b\sqrt{b^2 - 4c} + (b^2 - 4c) \right) + \frac{1}{4} \left( -2b^2 \pm 2b\sqrt{b^2 - 4c} \right) + b
\]
\[
= 0.
\]

This is the case for mechanical and electrical circuits having small damping. Physically \( b > 0 \) is the damping coefficient and \( \sqrt{c} > 0 \) is the resonant frequency. One may then simplify the form as \( x^2 + 2bx + x^2 = (x + b + \sqrt{b^2 - 4c})(x + b - \sqrt{b^2 - 4c}) \).

15. STREAM 3B: VECTOR CALCULUS (10 LECTURES)

Recall the d’Alembert solutions of the scalar wave equation (Eq. 1.97, p. 110)
\[
E(x, t) = f(x - ct) + g(x + ct),
\]
where \( f, g \) are arbitrary vector fields. This result applies to the vector case since it represents three identical, yet independent, scalar wave equations, in the three dimensions.

In a like manner one may derive the wave equation in terms of \( H \)
\[
\nabla^2 H(x, t) = \frac{1}{c^2} \frac{\partial^2 H(x, t)}{\partial t^2} - \frac{s^2}{c^2} H(x, \omega).
\] (1.190)

This equation does not have the restriction that there is no free charge, because \( \nabla \cdot \vec{B} = 0 \). Thus both \( E, H \) obey the wave equation (thus they are locked together in space-time if we assume no free charge) (Sommerfeld, 1952).

**Poynting’s theorem:** The energy flux density \( \mathcal{P} \) [W/m²] is perpendicular to \( E \) and \( B \), denoted as
\[
\mathcal{P} = \frac{1}{\mu_0} E \times B.
\]

**Electrical impedance seen by an electron:** Up to now we have only considered the Brune impedance which is a special case with no branch points or branch cuts. We can define impedance for the case of diffusion, as in the case of the diffusion of heat. There is also the diffusion of electrical and magnetic fields, which is called the electrical skin effect, where the conduction currents are dominated by the conductivity of the metal rather than the displacement currents. In such cases the impedance is proportional to \( \sqrt{\tau} \), implying that it has a branch cut. Still in this case the real part of the impedance must be positive in the right half-plane, the required condition of all impedances, such that postulate P3 (p. 102) is satisfied.

**Example:** When we deal with Maxwell’s equations the force is defined by the Lorentz force
\[
\vec{f} = q \vec{E} + q \vec{v} \times \vec{B} = q (\vec{E} + \vec{C} \times \vec{B}),
\]
which is the force on a charge (e.g., electron) due to the electric \( E \) and magnetic \( B \) fields. The magnetic field plays a role when the charge has a velocity \( \vec{v} \). When a charge is moving with velocity \( \vec{v} \), it may be viewed as a current \( \vec{C} = q \vec{v} \).

In this case the impedance in a wire, where the current is constrained, the complex impedance density is
\[
Z(s) = \sigma + \text{se}_r, \quad [\Omega/m^2]
\]
which when integrated over an area is the impedance in ohms (Feynman, 1970c, p. 13-1). Here \( \sigma \) is the electrical conductivity and \( e_r \) is the electrical permittivity. Since \( \sigma \gg \omega e_r \), this reduces to the resistance of the wire, per unit length.

1.5.16 **Lec 42 The Quasi-static approximation**

There are a number of assumptions and approximations that result in special cases, many of which are classic. These manipulations are all done at the differential equation level, by making assumptions that change the basic equations that are to be solved. These approximations are distinct from assumptions made while solving a specific problem.\(^{90}\)

\(^{90}\)https://www.youtube.com/watch?v=p8IA0-r5A8
verify

With this simple rule I did not need to depend on my memory for the 9 times tables. By expanding

\[ (a+b)(c+d) = ac + ad + bc + bd \]

Thus we may truncate

\[ \frac{1}{2} \int s \, dP \]

This equation may be recursively iterated, defining a sequence of approximations to the root,

\[ P_{n+1} = P_n - \frac{F(P_n)}{G(P_n)} \]

is much more powerful than memorization of the 9 times tables. How one thinks about a problem

\[ \frac{\partial}{\partial t} \delta(x) = \phi(x) \]

Exercise: If our initial guess for the root

\[ P = \sqrt{\frac{1}{1} - (\frac{1}{x} \Delta) \phi(x) + \frac{\partial}{\partial \phi} \frac{1}{1} - (\frac{1}{x} \Delta) \phi(x)} \]

Putting

\[ \Delta = \frac{1}{1} - (\frac{1}{x} \Delta) \phi(x) + \frac{\partial}{\partial \phi} \frac{1}{1} - (\frac{1}{x} \Delta) \phi(x) \]

such that

\[ P = \frac{1}{1} \Delta \phi(x) - \frac{\partial}{\partial \phi} \frac{1}{1} \Delta \phi(x) \]

If we apply Gauss's law to the divergence equations, we find the total flux crossing the closed

\[ \Omega = \int_{\Sigma} \mathbf{H} \cdot d\mathbf{S} = \int_{\Sigma} \mathbf{H} \cdot d\mathbf{S} \]

where

\[ \int_{V} \rho \, dV = \int_{\Sigma} \mathbf{E} \cdot d\mathbf{S} \]

such that

\[ \mathbf{E} = \frac{\nabla \mathbf{D}}{\epsilon_0} \]

\[ \mathbf{B} = \nabla \times \mathbf{H} \]

\[ \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \]

\[ \mathbf{B} = \mu_0 \mathbf{H} \]

\[ \mathbf{F} = \nabla \times \mathbf{E} \]

\[ \mathbf{G} = \nabla \times \mathbf{B} \]

\[ \mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E} \]

\[ \mathbf{J} = \nabla \times \mathbf{D} \]

\[ \mathbf{K} = \nabla \times \mathbf{B} \]

\[ \mathbf{L} = \mathbf{E} \times \mathbf{B} \]

\[ \mathbf{M} = \mathbf{B} \times \mathbf{D} \]

\[ \mathbf{N} = \mathbf{B} \times \mathbf{E} \]

\[ \mathbf{O} = \mathbf{D} \times \mathbf{E} \]

\[ \mathbf{P} = \mathbf{E} \times \mathbf{B} \]

\[ \mathbf{Q} = \mathbf{B} \times \mathbf{D} \]

\[ \mathbf{R} = \mathbf{D} \times \mathbf{E} \]

\[ \mathbf{S} = \mathbf{E} \times \mathbf{B} \]

\[ \mathbf{T} = \mathbf{B} \times \mathbf{D} \]

\[ \mathbf{U} = \mathbf{D} \times \mathbf{E} \]

\[ \mathbf{V} = \mathbf{E} \times \mathbf{B} \]

\[ \mathbf{W} = \mathbf{B} \times \mathname{E} \]

\[ \mathbf{X} = \mathbf{D} \times \mathbf{E} \]

\[ \mathbf{Y} = \mathbf{E} \times \mathbf{B} \]

\[ \mathbf{Z} = \mathbf{B} \times \mathbf{D} \]

\[ \mathbf{A} = \mathbf{E} \times \mathbf{B} \]

\[ \mathbf{B} = \mathbf{B} \times \mathbf{D} \]

\[ \mathbf{C} = \mathbf{D} \times \mathbf{E} \]

\[ \mathbf{D} = \mathbf{E} \times \mathbf{B} \]

\[ \mathbf{E} = \mathbf{B} \times \mathbf{D} \]

\[ \mathbf{F} = \mathbf{D} \times \mathbf{E} \]

\[ \mathbf{G} = \mathbf{E} \times \mathbf{B} \]

\[ \mathbf{H} = \mathbf{B} \times \mathbf{D} \]

\[ \mathbf{I} = \mathbf{D} \times \mathbf{E} \]

\[ \mathbf{J} = \mathbf{E} \times \mathbf{B} \]

\[ \mathbf{K} = \mathbf{B} \times \mathbf{D} \]

\[ \mathbf{L} = \mathbf{D} \times \mathbf{E} \]

\[ \mathbf{M} = \mathbf{E} \times \mathbf{B} \]

\[ \mathbf{N} = \mathbf{B} \times \mathbf{D} \]

\[ \mathbf{O} = \mathbf{D} \times \mathbf{E} \]

\[ \mathbf{P} = \mathbf{E} \times \mathbf{B} \]

\[ \mathbf{Q} = \mathbf{B} \times \mathbf{D} \]

\[ \mathbf{R} = \mathbf{D} \times \mathname{E} \]

\[ \mathbf{S} = \mathbf{E} \times \mathbf{B} \]

\[ \mathbf{T} = \mathbf{B} \times \mathbf{D} \]

\[ \mathbf{U} = \mathbf{D} \times \mathbf{E} \]

\[ \mathbf{V} = \mathbf{E} \times \mathbf{B} \]

\[ \mathbf{W} = \mathbf{B} \times \mathbf{D} \]

\[ \mathbf{X} = \mathbf{D} \times \mathname{E} \]

\[ \mathbf{Y} = \mathbf{E} \times \mathbf{B} \]

\[ \mathbf{Z} = \mathbf{B} \times \mathbf{D} \]
With every step the expansion point moves closer to the root, converging to the root in the limit. As it comes closer, the linearity assumption becomes more accurate, ultimately resulting in the convergence to the root. Solving for \( x_n \) gives the key formula behind Newton’s famous root-finding method

\[
s_n = x_{n-1} - \frac{P'(x_{n-1})}{P'(x_{n-1})}
\]

Here \( s_{n-1} \) is the old expansion point and \( s_n \) is the next approximation to the root. This expression is related to the log-derivative \( d \log P(x)/dx = P'(x)/P(x) \). It follows that even for cases where fractional derivatives of roots are involved, Newton’s method should converge, since the log-derivative linearizes them.

\[49\text{https://en.wikipedia.org/wiki/Argument_principle}\]

Newton’s view: Newton believed that imaginary roots and numbers had no meaning (p. 106) and only sought real roots. In this case Newton’s relation may be explored as a graph, which shows the values of \( s_n \) as Newton’s method converges to the root. Different random starting points converge to different roots. The method always results in convergence to a root. Claims to the contrary are a result of forcing the roots to be real (Stewart, 2012, p. 347). For convergence, one must work with \( s_n \in \mathbb{C} \).

\[\text{Figure 1.14: Newton’s method applied to the polynomial having real roots (1, 2, 3, 4) (left) and 5 complex roots (right). A random starting point was chosen, and each curve shows the values of } s_n \text{ as Newton’s method converges to the root. Different random starting points converge to different roots. The method always results in convergence to a root. Claims to the contrary are a result of forcing the roots to be real (Stewart, 2012, p. 347). For convergence, one must work with } s_n \in \mathbb{C}.\]

### 1.5. STREAM 3B: VECTOR CALCULUS (10 LECTURES)

#### Maxwell’s equations

As shown in Fig. 1.18, Maxwell’s equations consist of two curl equations, operating on the field strengths \( EF \) and \( MF \), and two divergence equations, operating on the field fluxes \( ED \) and \( MI \). Stokes’s law may be applied to the curl equations and Gauss’s law may be used on the divergence equations. This should be logically obvious.

**Example:** When a static current is flowing in a wire in the \( \hat{z} \) direction, the magnetic flux is determined by Stokes’s theorem (Fig. 1.36). Thus just outside of the wire we have

\[
\mathcal{I}_{\text{enc}} = \oint_S \left( \nabla \times \mathbf{H} \right) \cdot \mathbf{n} \, dS = \oint_{B} \mathbf{H} \cdot d\mathbf{l} \quad [\text{A}]
\]

(1.186)

For this simple geometry, of the current in a wire is related to \( \mathbf{H}(x, t) \) by

\[
\mathcal{I}_{\text{enc}} = \oint_{B} \mathbf{H} \cdot d\mathbf{l} = H_0 2\pi r.
\]

Here \( H_0 \) is perpendicular to both the radius \( r \) and the direction of the current \( \hat{z} \). Thus

\[
\mathbf{H}_0 = \frac{\mathcal{I}_{\text{enc}}}{2\pi r}
\]

and we see that \( \mathbf{H} \), and thus \( \mathbf{B} = \mu_0 \mathbf{H} \), drop off as the reciprocal of the radius \( r \).

**Exercise:** Explain how Stokes’s law may be applied to \( \nabla \times \mathbf{E} = -\mathbf{B} \), and explain what it means. Hint: This is the identical argument given above for the current in a wire, but for the electric case. **Solution:** Integrating the left side of equation \( EF \) over an open surface results in a voltage (emf) induced in the loop closing the boundary \( B \) of the surface

\[
\phi_{\text{induced}} = \oint_S (\nabla \times \mathbf{E}) \cdot \mathbf{n} \, dS = \oint_{B} \mathbf{E} \cdot d\mathbf{l} \quad [\text{V}].
\]

The emf (electromagnetic force) is the same as the Thévenin source voltage induced by the rate of change of the flux. Integrating the Eq. 1.15 over the same open surface \( S \) results in the source of the induced voltage \( \phi_{\text{induced}} \), which is proportional to the rate of change of the flux \( \psi \)

\[
\phi_{\text{induced}} = \frac{\partial}{\partial t} \oint_S \mathbf{B} \cdot \mathbf{n} \, dA = L \psi \quad [\text{Wb/s}].
\]
Fractals, fractal sets, and fractal problems. The solution to the two-dimensional problem is:

\[ \text{Solution:} \quad \text{set} \quad \text{method} \quad \text{Maxwell's Equations: The unification of electricity and magnetism.} \]

1.3. STREAM 2: ALGEBRAIC EQUATIONS (12 LECTURES)

1. Let \( x \)

2. What if you let \( x \)

3. Write a Matlab script to check your answer for part (a).

4. Solve the equation \( x^2 + 1 = (x) \) which may be written in integral or vector form. An important difference is that with Maxwell's equations, we have the gradient, divergence and curl. Once you have mastered the three basic vector operations, you are ready to appreciate Maxwell's equations. Like the vector operations, these equations may be


7. \text{Fractal diagrams:} Fractal problems.

8. Fractal patterns:

9. Fractals, fractal sets, and fractal problems. The solution to the two-dimensional problem is:

\[ \text{Solution:} \quad \text{set} \quad \text{method} \quad \text{Maxwell's Equations: The unification of electricity and magnetism.} \]

1.3. STREAM 2: ALGEBRAIC EQUATIONS (12 LECTURES)

1. Let \( x \)

2. What if you let \( x \)

3. Write a Matlab script to check your answer for part (a).

4. Solve the equation \( x^2 + 1 = (x) \) which may be written in integral or vector form. An important difference is that with Maxwell's equations, we have the gradient, divergence and curl. Once you have mastered the three basic vector operations, you are ready to appreciate Maxwell's equations. Like the vector operations, these equations may be


8. Fractals, fractal sets, and fractal problems. The solution to the two-dimensional problem is:

\[ \text{Solution:} \quad \text{set} \quad \text{method} \quad \text{Maxwell's Equations: The unification of electricity and magnetism.} \]

1.3. STREAM 2: ALGEBRAIC EQUATIONS (12 LECTURES)

1. Let \( x \)

2. What if you let \( x \)

3. Write a Matlab script to check your answer for part (a).

4. Solve the equation \( x^2 + 1 = (x) \) which may be written in integral or vector form. An important difference is that with Maxwell's equations, we have the gradient, divergence and curl. Once you have mastered the three basic vector operations, you are ready to appreciate Maxwell's equations. Like the vector operations, these equations may be


8. Fractals, fractal sets, and fractal problems. The solution to the two-dimensional problem is:

\[ \text{Solution:} \quad \text{set} \quad \text{method} \quad \text{Maxwell's Equations: The unification of electricity and magnetism.} \]
Example: Assume that polynomial \( P_3(s) = (s - a)^2(s - b) \). Then
\[
\ln P_3(s) = 2 \ln s - a + \pi \ln s - b
\]
and
\[
\frac{d}{ds} \ln P_3(s) = \frac{2}{s - a} + \frac{\pi}{s - b}
\]
Reduction by logarithmic derivative to simple poles: As shown by the above trivial example, any polynomial, having zeros of arbitrary degree (i.e., \( \pi \) in the example), may be reduced to the ratio of two polynomials, by taking the logarithmic derivative, since
\[
Y_N(s) = \frac{N(s)}{D(s)} = \frac{d}{ds} \ln P_N(s) = \frac{P_N(s)}{P_N'(s)}.
\] (1.34)

Here the starting polynomial is the denominator \( D(s) = P_N(s) \) while the numerator \( N(s) = P_N'(s) \) is the derivative of \( D(s) \). Thus the logarithmic derivative can play a key role in analysis of complex-analytic functions, as it reduces higher order poles, even those of irrational degree, to simple poles.

The logarithmic derivative \( Y_N(s) \) has a number of special properties:

1. \( Y_N(s) \) has simple poles \( s_n \) and zeros \( s_z \).
2. The poles of \( Y_N(s) \) are the zeros of \( P_N(s) \).
3. The zeros of \( Y_N(s) \) (i.e., \( P_N'(s_z) = 0 \)) are the zeros of \( P_N'(s) \).
4. \( Y_N(s) \) is analytic everywhere other than its poles.
5. Since the zeros of \( P_N'(s) \) are simple (no second-order poles), it is obvious that the zeros of \( Y_N(s) \) always lie close to the line connecting the two poles. One may easily demonstrate the truth of the statement numerically, and has been quantified by the Gauss-Lucas theorem which specifies the relationship between the roots of a polynomial and those of its derivative. Specifically, the roots of \( P_N(s) \) lie inside the convex hull of the roots of \( P_N \).
6. Newton’s method may be expressed in terms of the logarithmic derivative, since
\[
s_{k+1} = s_k + \epsilon_k Y_N(s),
\]
where we call \( \epsilon_k \) the step size, which may be used to control the rate of convergence of the algorithm to the zeros of \( P_N(s) \). If the step size is too large, the root finding can jump to a different domain of convergence, thus a different root of \( P_N(s) \).

1.3.2 Matrix formulation of the polynomial

There is a one-to-one relationship between polynomials and matrix analysis. These are best described in terms of two methods from mathematical physics, the Vandermonde determinant and the companion matrix.

1.5. STREAM 3B: VECTOR CALCULUS (10 LECTURES)

momentum (rotational energy) is independent of the translational momentum. Once these forces are made clear, the vector operations all take on a very well defined meaning, and the mathematical constructions, centered around Helmholtz’s theorem, begin to provide some common-sense meaning. One could conclude that the physics is simply related to the geometry via the scalar and vector product.

Specifically, if we take the divergence of Eq. 1.183, and use the DoG, then
\[
\nabla \cdot E = \nabla \cdot \{-\nabla \phi + \nabla \times A\}^0 = -\nabla \cdot \nabla \phi = -\nabla^2 \phi,
\]
since the DoG zeros the vector potential \( A(x, y, z) \). If instead we use the CoG, then
\[
\nabla \times E = \nabla \times \{-\nabla \phi + \nabla \times A\} = \nabla \times \nabla \cdot A = \nabla (\nabla \cdot A) - \nabla^2 A,
\]
since the CoG zeros the scalar field \( \phi(x, y, z) \). The last expression requires GoD.

Table 1.7: The four possible classifications of scalar and vector potential fields: rotational/irrotational. compressible/incompressible. Rotational fields are generated by the vector potential (e.g., \( A(x, t) \)), while compressible fields are generated by the scalar potentials (e.g., voltage \( \phi(x, t) \), velocity \( \omega \), pressure \( p(x, t) \) or temperature \( T(x, t) \)).

<table>
<thead>
<tr>
<th>Field:</th>
<th>Compressible</th>
<th>Incompressible</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotational</td>
<td>( \nabla \cdot \psi \neq 0 )</td>
<td>( \nabla \cdot \psi = 0 )</td>
</tr>
<tr>
<td>Irrotational</td>
<td>Vector wave Eq. (EM)</td>
<td>Lubrication theory</td>
</tr>
<tr>
<td>Conservative</td>
<td>Acoustics</td>
<td>Statics</td>
</tr>
</tbody>
</table>

The four categories of linear fluid flow: The following is a summary of the four cases for fluid flow, as summarized in Fig. 1.7:

1.1 Compressible and rotational fluid (general case): \( \nabla \phi \neq 0 \), \( \nabla \times w \neq 0 \). This is the case of wave propagation in a medium where viscosity cannot be ignored, as in the case of acoustics close to the boundaries, where viscosity contributes to losses (Batchelor, 1967).

1.2 Incompressible, rotational, fluid (Lubrication theory): \( v = \nabla \times w \neq 0 \), \( \nabla \cdot w = 0 \), \( \nabla^2 \phi = 0 \). In this case the flow is dominated by the walls, while the viscosity and heat transfer introduce shear. This is typical of lubrication theory.

2.1 Fluid compressible irrotational flow (acoustics): \( v = \nabla \phi \), \( \nabla \times w = 0 \). Here losses (viscosity and thermal diffusion) are small (assumed to be zero). One may define a velocity potential \( \psi \), the gradient of which gives the air particle velocity, thus \( w = -\nabla \phi \). Thus for an irrotational fluid \( \nabla \times v = 0 \) (Greenberg, 1988, p. 826). This is the case of the conservative field, where \( \int \nabla \cdot w \, dl \) only depends on the end points, and \( \int_w \nabla \cdot w \, dl = 0 \). When a fluid may be treated as having no viscosity, it is typically assumed to be irrotational, since it is the viscosity that introduces the shear (Greenberg, 1988, p. 814). A fluid’s angular velocity is \( \Omega = \frac{1}{2} \nabla \times w = 0 \), thus irrotational fluids have zero angular velocity (\( \Omega = 0 \)).
1.3. STREAM 2: ALGEBRAIC EQUATIONS (12 LECTURES)

2. Case of every vector field may be split into its translational and rotational parts. If definitions, defined in Sect. 1.5.13, p. 155. The identities have a physical meaning, as stated above: of the hydrodynamic equations that correspond to Vortex motions (in German) (Helmholtz, 1978). Right: Gustav CoG

\begin{equation}
\frac{x}{t} + \frac{\partial}{\partial x}(x) = 0
\end{equation}

Figure 1.37:

CHAPTER 1. INTRODUCTION

\begin{equation}
V_n \pi n = 0
\end{equation}

\begin{equation}
V_n \pi n = 0
\end{equation}

\begin{equation}
V_n \pi n = 0
\end{equation}

\begin{equation}
V_n \pi n = 0
\end{equation}

\begin{equation}
V_n \pi n = 0
\end{equation}

\begin{equation}
V_n \pi n = 0
\end{equation}

Helmholtz's decomposition is expressed as the linear sum of a scalar potential \( \phi \) and a vector potential \( A \) (Helmholtz, 1863a).
To find the determinant we may expand in cofactors along the bottom row
\[
\Delta_3 = \begin{vmatrix} 1 & x_1 & \frac{1}{2}x_1^2 - x_1^2 - x_1^2 & 1 \ x_2 \frac{1}{2}x_2^2 - x_1^2 \ x_2 \end{vmatrix} = (x_2 - x_1)^2 - (x_1^2 - x_2^2) + x_1x_2(x_2 - x_1) = (x_2 - x_1)[x^2 - (x_1 + x_2) + x_1x_2]
\]

**Companion Matrix**

The companion matrix
\[
C_N = \begin{bmatrix} -c_{N-1} & c_{N-2} & \cdots & \cdots & c_1 \\
1 & 0 & \cdots & \cdots & 0 \\
0 & 1 & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 1 \end{bmatrix}
\]

is derived from the monic polynomial of degree \(N\)
\[
P_N(s) = s^N + c_{N-1}s^{N-1} + \cdots + c_1s + c_0 = s^N + \sum_{n=0}^{N-1} c_ns^n,
\]

having coefficient vector having coefficients
\[
e_n^T = [c_{N-1}, c_{N-2}, \cdots, c_0]^T.
\]

An alternate form is (Horn and Johnson, 1988, p. 146)
\[
C'_N = \begin{bmatrix} 0 & -c_0 \\
1 & 0 & 0 & -c_1 \\
0 & 1 & 0 & -c_2 \\
\vdots & \vdots & \ddots & \ddots \\
0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & -c_{N-2} \\
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & -c_{N-1} \end{bmatrix}
\]

The Companion matrix has the same eigen-values as the roots of the monic polynomial \(P_N(s)\).

**Exercise:** Show that the eigen-values of the 3x3 companion matrix are the same as the roots of \(P_3(s)\). **Solution:** Expanding the determinant of \(C_3 - sI_3\) along the right-most column:
\[
\begin{vmatrix} -s & 0 & -c_0 \\
1 & -s & -c_1 \\
0 & 1 & -(c_2 + s) \end{vmatrix} = c_0 + c_1s + (c_2 + s)^2 = s^3 + c_2s^2 + c_1s + c_0.
\]

This is the characteristic polynomial, which is equal to \(-P_3(s)\).

1.5. STREAM 3B: VECTOR CALCULUS (10 LECTURES)

while taking the divergence along with DoC=0
\[
\nabla \cdot \nabla \times \mathbf{E} = -\frac{\partial \nabla \cdot \mathbf{B}}{\partial t} = 0,
\]

requires that \(\nabla \cdot \mathbf{B}\) is independent of time, and therefore that
\[
\nabla \cdot \mathbf{B}(x, t) = 0.
\]

We would like to recover Maxwell’s magnetic equation \(\nabla \times \mathbf{H} = \mathbf{J}\) from the potential solution. Taking the curl of Eq. 1.114 gives
\[
\frac{1}{\mu_0} \nabla \times \mathbf{H} = \mathbf{J} = \nabla \times \mathbf{A} = \nabla \mathbf{A} - \nabla^2 \mathbf{A}.
\]

This says that the current \(C\) only depends on \(A\), which follows directly from Eq. 1.114 forward. Since \(A\) must satisfy the wave equation,
\[
\nabla^2 \mathbf{A} = \frac{1}{c^2} \mathbf{A} - C,
\]

which requires that
\[
\nabla \nabla \cdot \mathbf{A} = \frac{1}{c^2} \mathbf{A}.
\]

Taking the divergence of Eq. 1.113 gives an expression for \(\nabla \cdot \mathbf{A}\):
\[
\frac{1}{c_0} \nabla \cdot \mathbf{D} = \rho/c_0 = -\nabla^2 \phi - \partial_t \nabla \cdot \mathbf{A}.
\]

Here \(c_0\) [Col/m] and \(\mu_0\) [H/m].

**Exercise:** Starting from the values of the speed of light \(c_0 = 3 \times 10^8\) [m/s] and the characteristic resistance of light waves \(r_p = 377\) [ohms], use the formula for \(c_0 = 1/\sqrt{\mu_0\varepsilon_0}\) and \(r_p = \sqrt{\varepsilon_0/\mu_0}\) to find values for \(\epsilon_0\) and \(\mu_0\). **Solution:** Squaring \(c_0^2 = 1/\mu_0\varepsilon_0\) and \(r_p^2 = \mu_0/\varepsilon_0\) we may solve for the two unknowns: \(c_0^2 = 1/\mu_0\varepsilon_0\) and \(r_p^2 = \mu_0/\varepsilon_0\), thus \(c_0 = 1/\sqrt{\mu_0\varepsilon_0}\) and \(r_p = 377/\varepsilon_0 = 8.84 \times 10^{-12}\) [Fd/m]. Likewise \(\mu_0 = c_0/\varepsilon_0 = (377/3) \times 10^{-8} \approx 125.67 \times 10^{-8}\). The value of \(\mu_0\) is defined in the international SI standard as \(4 \pi \times 10^{-7} \approx 12.566077\) [H/m].

In conclusion, Eq. 1.113, along with DoC=0 and CoG=0, give Maxwell’s Eq. 1.179 and Eq. 1.181 result. It would appear that Eq. 1.113 is the key. This equation defines the magnetic component of the field, expressed in terms of its vector potential, in the same way as Eq. 1.112 describes \(\mathbf{E}(x, t)\) in terms of the scalar potential (voltage). Does the same argument apply for Eq. 1.114?

**Exercise:** Take the divergence of Maxwell’s equation for the magnetic intensity
\[
\nabla \times \mathbf{H}(x, t) = \mathbf{J}(x, t) + \frac{\partial}{\partial t} \mathbf{D}(x, t)
\]

and explain what results. **Solution:** The divergence of the curl is always zero (DoC=0), thus
\[
\nabla \cdot \nabla \times \mathbf{H}(x, t) = \nabla \cdot \mathbf{J}(x, t) + \frac{\partial}{\partial t} \rho(x, t) = 0.
\]
1.3. STREAM 2: ALGEBRAIC EQUATIONS (12 LECTURES)

### Example:
Matlab/Octave: A polynomial is represented in Matlab/Octave in terms of its coefficients. For instance, the Fibonacci companion matrix can be generated using the following commands:

```matlab
% Fibonacci companion matrix
N = [1 1]; % Fibonacci sequence
R = poly(N); % Companion matrix
```

### Exercise:
Find the companion matrix for the Fibonacci sequence, defined by the difference equation:

\[ a_{n+1} = a_n + a_{n-1} \]

### Solution:

- The coefficient vector is \( \mathbf{f} \) with \( f_i = a_i \) for \( i = 1, 2, \ldots \).
- The companion matrix \( \mathbf{A} \) is given by:

\[
\mathbf{A} = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

### Conclusion:

- The companion matrix captures the roots of the characteristic polynomial, which are the solutions to Maxwell's equations in terms of the two potentials (Sommerfeld, 1952).

### Important Terms:
- **Characteristic polynomial**: As discussed on page 53, it represents the equation whose roots are the characteristic coefficients of the differential equation.

---

**References:**

- See pages 162, 167, and 172 for detailed discussions on the application of algebraic equations in physics and engineering.
- Further reading on the topic is recommended for a comprehensive understanding.

---

**Chapter 1. Introduction**

- Fundamental theorems of vector calculus. Images of Helmholtz and Kirchhoff are provided...
6. C = conv(A, B): Vector C ∈ \mathbb{C}^{N+M-1} contains the polynomial coefficients of the convolution of the two vector of coefficients of polynomials A, B ∈ \mathbb{C}^N and B ∈ \mathbb{C}^M. For example [1, 2, 1] = conv([1, 1], [1, 1]).

7. \{C, R\} = deconv(N, D): Vectors C, N, D ∈ \mathbb{C}. This operation uses long division of polynomials to find C(s) = N(s)/D(s) with remainder R(s), where N = conv(D, C) + R, namely
   \[ C = \frac{N}{D} \text{ remainder } R \]  \hspace{1cm} (1.40)

8. A = compan(D): Vector D = [1, d_{N-1}, d_{N-2}, \cdots, d_0]^T contains the coefficients of polynomial
   \[ D(s) = s^N + \sum_{k=1}^{N} d_{N-k}s^k, \]
   and A is the companion matrix, of vector D (Eq. F.3, p. 214). The eigen-values of A are the roots of monic polynomial D(s).

Exercise: Practice the use of Matlab’s/Octave’s related functions, which manipulate roots, polynomials and residues: root(), conv(), deconv(), poly(), polyval(), polyder(), residue(), compan().

Solution: Try Newton’s method for various polynomials. Use \text{N=poly}(R) to provide the coefficients of a polynomial given the roots \text{R}. Then use root() to factor the resulting polynomial. Then use Newton’s method and show that the iteration converges to the nearest root.\footnote{A Matlab/Octave program that does this may be downloaded from http://jontalle.web.engr.illinois.edu/uploads/493/M/NewtonJPD.m.}

1.5. STREAM 3B: VECTOR CALCULUS (10 LECTURES)

1. \vec{V}_2 = \vec{V}_1 + \int_{2}^{1} \nabla \Phi \cdot d\vec{S}

2. \oint_{\partial D} \vec{n} \cdot d|\vec{S}| = \int_{\partial D} \nabla \cdot d|\vec{S}|

3. \oint_{\partial E} \vec{E} \cdot d\vec{l} = \oint_{\partial \vec{E}} (\nabla \times \vec{E}) \cdot \vec{n} \cdot d|\vec{S}|

1.5.14 Lec 40 Second-order operators: Terminology

Besides the above first-order vector derivatives, second-order combinations exist, the most common being the scalar Laplacian \(\nabla^2\) (Table 1.4, p. 129; Appendix 1.5.2, p. 132).

There are other important second-order combinations of \(\nabla\), enough that we need a memory aid to remember them. Thus I define mnemonics DoC, DoG, CoC and GoD as follows:

1. DoG: Divergence of the gradient (\(\nabla \cdot \nabla = \nabla^2\)), i.e., Laplacian,
2. DoC*: Divergence of the curl (\(\nabla \cdot \nabla \times \vec{A}\)),
3. CoG*: Curl of the gradient (\(\nabla \times \nabla \times \vec{A}\)),
4. CoC: Curl of the curl (\(\nabla \times \nabla \times \vec{A}\)), and
5. GoD: Gradient of the Divergence (GoD) \(\nabla \cdot \nabla \times \vec{A}\) and the vector Laplacian GoD* \(\nabla^2\).

DoC* () and CoG* () are special because they are always zero:
\[ \nabla \times \nabla \phi = 0; \quad \nabla \cdot \nabla \times \vec{A} = 0, \]
a property that makes them useful in proving the fundamental Theorem of Vector Calculus (Helmholtz’ decomposition, Eq. 1.183, p. 164). A third key vector identity CoC may be expanded as
\[ \nabla \times \nabla \times \vec{A} = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}, \]  \hspace{1cm} (1.178)
defining the vector Laplacian (i.e., GoD* = GoD·CoC): \(\nabla^2\) = \(\nabla \cdot \nabla (\cdot) - \nabla \times \nabla (\times)\).

When using second-order differential operators one must be careful with the order of operations, which can be subtle in a few cases. Most of this is common sense. For example, don’t operate on a scalar field with \(\nabla \times\), and don’t operate on a vector field with \(\nabla\). The vector Laplacian GoD* must not be thought of as \(\nabla (\nabla \cdot \vec{A})\), rather it acts as the Laplacian on each vector component \(\nabla^2 \vec{A} = \nabla^2 A_1 \vec{e}_1 + \nabla^2 A_2 \vec{e}_2 + \nabla^2 A_3 \vec{e}_3\).

Exercise: Show that GoD and GoD* differ. Solution: Use CoC on a \(\vec{A}\) to explore this relationship.

Helmholtz’ decomposition

We may now restate everything defined above in terms of two types of vector fields that decompose every analytic vector field. The \textit{irrotational field} is defined as one that is \textit{curl free.} An \textit{incompressible field} is one that is \textit{diverge free.} According to Helmholtz’s decomposition, every analytic vector field may be decomposed into independent rotational and a compressible components. Another name for Helmholtz decomposition is the fundamental theorem of vector calculus (FTVC); Gauss’s and Stokes’s laws, along with Helmholtz’s decomposition, form the three key


(1.17)

\[ \frac{d}{dx} \left( \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \right) = \sum_{n=1}^{\infty} \frac{(-x)^{n-1}}{(n-1)!} = e^{-x} \]

(1.19)

\[ \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \]

1.3. STREAM 2: ALGEBRAIC EQUATIONS (12 LECTURES)

The Taylor formula is a prescription for how to uniquely define the coefficients of the series expansion formula, we would have no way of determining the Taylor series formula, we would have no way of determining the coefficient of the nth term.

To fully appreciate the differences between Gauss's and Stokes's laws, these two types of vector products must be mastered.

\[ \nabla \times H \equiv \lim_{|S| \to 0} \left\{ \int_{\partial S} \hat{n} \times H \right\} \]

The trivial (corner) case is the geometric series

\[ x = 1 + x + 1 = x^2 \]

The function \( f(x) \) is called a geometric series if \( x \) is infinite, and the series

\[ 1 + x + x^2 + \cdots = \frac{1}{1-x} \]

is called a geometric series if \( x \) is finite and

\[ \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \]

for \( |x| < 1 \).

The Taylor series plays an important role in mathematics, as the coefficients of the series expansion formula, we would have no way of determining the coefficient of the nth term.

(1.42)

\[ \oint_{\partial S} \hat{n} \times H \equiv \lim_{|S| \to 0} \left\{ \int_{\partial S} \hat{n} \times H \right\} \]

\[ \oint_{\partial S} \hat{n} \times H \]

For avoiding this limitation is to move the expansion point. But this analytic continuation is a non-trivial exercise because it requires working with the derivatives of the function at the new expansion point. If the function fails to have derivatives, due to a possible singularity, the function is not analytic at the new expansion point. The analytic function is the ideal starting point for solving differential equations, which is a limitation of the Taylor series expansion.

There are at least two main theorems related to scalar calculus, and three more for vector calculus.

- **Residue Theorem**
- **Cauchy's Theorem**
- **Leibniz (FTC)**

The key distinction between the two laws naturally follows from the properties of the scalar (1.41) follow from Eq. 1.42.

\[ \nabla \cdot \mathbf{E} = \rho \]

\[ \nabla \times \mathbf{B} = \mathbf{0} \]

The trivial (corner) case is the geometric series

\[ x = 1 + x + x^2 + \cdots = \frac{1}{1-x} \]

for \( |x| < 1 \).

The Taylor series plays an important role in mathematics, as the coefficients of the series expansion formula, we would have no way of determining the coefficient of the nth term.

(1.42)

\[ \oint_{\partial S} \hat{n} \times H \equiv \lim_{|S| \to 0} \left\{ \int_{\partial S} \hat{n} \times H \right\} \]

\[ \oint_{\partial S} \hat{n} \times H \]

For avoiding this limitation is to move the expansion point. But this analytic continuation is a non-trivial exercise because it requires working with the derivatives of the function at the new expansion point. If the function fails to have derivatives, due to a possible singularity, the function is not analytic at the new expansion point. The analytic function is the ideal starting point for solving differential equations, which is a limitation of the Taylor series expansion.

There are at least two main theorems related to scalar calculus, and three more for vector calculus.

- **Residue Theorem**
- **Cauchy's Theorem**
- **Leibniz (FTC)**

The key distinction between the two laws naturally follows from the properties of the scalar (1.41) follow from Eq. 1.42.

\[ \nabla \cdot \mathbf{E} = \rho \]

\[ \nabla \times \mathbf{B} = \mathbf{0} \]
Brune impedance: A third very special family of functions are formed from ratios of polynomials, typically used to define impedances. Impedance functions as a class of functions are special because they must have a positive real part, so as to obey conservation of energy. A physical impedance cannot have a negative resistance (the real part); otherwise it would act like a power source, violating conservation of energy. Most impedances are in the class of Brune impedances, defined by the ratio of two polynomials, of degrees $M$ and $N$

$$\frac{P_n(s)}{P_m(s)} = \frac{s^M + a_1s^{M-1} \cdots a_0}{s^N + b_1s^{N-1} \cdots b_0}$$ (1.45)

where $M = N \pm 1$ (i.e., $N = M \pm 1$). This fraction of polynomials is sometimes known as a “Padé approximation,” but more specifically this ratio is a Brune impedance, with poles and zeros, defined as the complex roots of the two polynomials. The key property of the Brune impedance is that the real part of the impedance is non-negative (positive or zero) in the right $s$ half-plane

$$\Re Z(s) = \Re [R(\sigma, \omega) + jX(\sigma, \omega)] = R(\sigma, \omega) \geq 0 \quad \text{for} \quad \Re s = \sigma \geq 0.$$ (1.46)

Since $s = \sigma + \omega j$, the complex frequency ($s$) right half-plane (RHP) corresponds to $\Re s = \sigma \geq 0$. This condition defines the class of positive-real functions, also known as the the Brune condition, which is frequently written in abbreviated form as $\Re Z(\Re s \geq 0) \geq 0$.

As a result of this positive-real constraint on impedance functions, the subset of Brune impedances (those given by Eq. 1.45 satisfying Eq. 1.46) must be complex analytic in the entire right $s$ half-plane. This is a powerful constraint that places strict limitations on the locations of both the poles and the zeros of every Brune impedance.

**Exercise:** Find the RoC of the following by application of Eq. 1.42.

1. $w(x) = \frac{1}{x^2}$
   **Solution:** From a straightforward expansion we know the coefficients are
   $$\frac{1}{1-x} = 1 + x + (x)^2 + \cdots = 1 + x - x^2 - \cdots$$

   Working this out using Eq. 1.42 is more work:
   $$c_0 = \frac{1}{1-x^2} |_{x=0} = 1; \quad c_1 = \frac{\frac{1}{1-x^2} \bigg|^x |_{x=0} = \frac{1}{(1-x^2) x} = \frac{1}{1-x^2} |_{x=0} = -1; \quad c_2 = \frac{\frac{1}{1-x^2} \bigg|^x |_{x=0} = \frac{1}{2(1-x^2)} |_{x=0} = -1.$$ (1.46)

   However, if we take derivatives of the series expansion it is much easier, and one can even figure out the term for $c_n$:
   $$c_0 = \frac{x}{1-x^2} |_{x=0} = j; \quad c_2 = \frac{\frac{1}{1-x^2} \bigg|^x |_{x=0} = \frac{1}{2x} |_{x=0} = 2(j)^2; \quad c_3 = \frac{\frac{1}{1-x^2} \bigg|^x |_{x=0} = (j)^3 = -j; \quad \cdots; \quad c_n = \frac{1}{n} n! = \frac{1}{n}.$$ (1.47)

2. $w(x) = e^{-x}$
   **Solution:** $c_n = \frac{1}{n} f^n$.

> CHAPTER 1. INTRODUCTION

When the surface integral over the normal component of $D(x)$ is zero, the charge density $\nabla \cdot D = \rho(x)$ and total charge, are both zero.

Taking the derivative with respect to time, Eq. 1.175 evaluates the total component of the current, normal to the surface:

$$I = \int_{|S|} D \cdot n \, d|S| = Q_{\text{enc}} = \int_{|S|} \rho_{\text{enc}} \, d|S|.$$ (1.176)

As summarized by Feynman (1970c, p. 13-2):

The current leaving the closed surface $|S|$ equals the rate of the charge leaving that volume $|S|$, defined by that surface.

Of course the surface must be closed to define the volume, a necessary condition for Gauss’s law. This reduces to a common sense summary that can be grasped intuitively, an example of the beauty in Feynman’s understanding.

**Integral definition of the curl:** $\nabla \times H = C$

As briefly summarized on page 130 (p. 130), the differential definition of the curl maps $\mathbb{R}^3 \rightarrow \mathbb{R}^3$.

For example, the curl of the magnetic field strength $H(x)$ is equal to the total current $C$

$$\nabla \times H \equiv \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = C. \quad [A]$$

As we shall see in Sec. 1.5.15 (p. 166), the curl and the divergence are both key when writing out Maxwell’s four equations. Without a full understanding of these two differential operators ($\nabla$, $\nabla \cdot$), there is no hope of understanding Maxwell’s basic result, typically viewed as the most important equations of mathematical physics, and the starting point for Einstein’s relativity theories. Some will say that quantum mechanics falls outside the realm of MEs, but this is at least open to debate, if not hotly debated.

The curl is a measure of the rotation of a vector field. If this were water, it would correspond to the angular momentum of the water, such as water going down the drain, as in a whirlpool, or with a wind, a tornado. A spinning top is another excellent example, given a spinning solid body. While a top (aka gyroscope) will fall over if not spinning, once it is spinning, it can stably stand on its pointed tip. These systems are stable due to conservation of angular momentum: Once something is spinning, it will continue to spin.

**Example:** When $H = -j \phi + x \hat{y} + z \hat{z}$, $\nabla \times H = 2\hat{x}$ and thus has a constant rotation; when $H = 0\hat{x} + 0\hat{y} + 0\hat{z}$, $\nabla \times H = 0$ has a curl of zero, and thus is irrotational. There are rules that precisely govern when a vector field is rotational versus irrotational, and compressible versus incompressible. These classes are dictated by Helmholtz’s theorem, the third fundamental theorem of vector calculus (Eq. 1.183, p. 164).

**Curl and Stokes’s law:** As in the cases of the gradient and divergence, the curl also may be written in integral form, allowing for the physical interpretation of its meaning.

Surface integral definition of $\nabla \times H = C$ where the current $C$ is $\perp$ to the rotation plane of $H$. Stokes’s law states that the open surface integral over the normal component of the curl of the
Exercise: Find the residue of \( \frac{1}{z} \) at \( z = 0 \).

Solution: Applying the formula for the residue (Eq. 1.109, p. 120) we find

\[
\text{Residue} = \lim_{z \to 0} z \cdot \frac{1}{z} = 1.
\]

Thus the residue is zero.

Example: Consider the function \( f(z) = \frac{1}{z^2 + 1} \). Determine the residue at \( z = i \).

Solution: First, find the roots of the denominator, which are \( z = i, -i \). Since \( z = i \) is inside the circle of convergence, we can use the formula for residues to find the residue at \( z = i \):

\[
\text{Residue} = \frac{1}{2i}.
\]

Exercise: Find the residue of \( \frac{1}{z^2} \) at \( z = 0 \).

Solution: Applying the formula for the residue (Eq. 1.109, p. 120) we find

\[
\text{Residue} = \lim_{z \to 0} z \cdot \frac{1}{z^2} = 0.
\]

Thus the residue is zero.

Example: Consider the function \( f(z) = \frac{1}{z^2 + 1} \). Determine the residue at \( z = i \).

Solution: First, find the roots of the denominator, which are \( z = i, -i \). Since \( z = i \) is inside the circle of convergence, we can use the formula for residues to find the residue at \( z = i \):

\[
\text{Residue} = \frac{1}{2i}.
\]

Exercise: Find the residue of \( \frac{1}{z^2} \) at \( z = 0 \).

Solution: Applying the formula for the residue (Eq. 1.109, p. 120) we find

\[
\text{Residue} = \lim_{z \to 0} z \cdot \frac{1}{z^2} = 0.
\]

Thus the residue is zero.

Example: Consider the function \( f(z) = \frac{1}{z^2 + 1} \). Determine the residue at \( z = i \).

Solution: First, find the roots of the denominator, which are \( z = i, -i \). Since \( z = i \) is inside the circle of convergence, we can use the formula for residues to find the residue at \( z = i \):

\[
\text{Residue} = \frac{1}{2i}.
\]

Exercise: Find the residue of \( \frac{1}{z^2} \) at \( z = 0 \).

Solution: Applying the formula for the residue (Eq. 1.109, p. 120) we find

\[
\text{Residue} = \lim_{z \to 0} z \cdot \frac{1}{z^2} = 0.
\]

Thus the residue is zero.

Example: Consider the function \( f(z) = \frac{1}{z^2 + 1} \). Determine the residue at \( z = i \).

Solution: First, find the roots of the denominator, which are \( z = i, -i \). Since \( z = i \) is inside the circle of convergence, we can use the formula for residues to find the residue at \( z = i \):

\[
\text{Residue} = \frac{1}{2i}.
\]

Exercise: Find the residue of \( \frac{1}{z^2} \) at \( z = 0 \).

Solution: Applying the formula for the residue (Eq. 1.109, p. 120) we find

\[
\text{Residue} = \lim_{z \to 0} z \cdot \frac{1}{z^2} = 0.
\]

Thus the residue is zero.

Example: Consider the function \( f(z) = \frac{1}{z^2 + 1} \). Determine the residue at \( z = i \).

Solution: First, find the roots of the denominator, which are \( z = i, -i \). Since \( z = i \) is inside the circle of convergence, we can use the formula for residues to find the residue at \( z = i \):

\[
\text{Residue} = \frac{1}{2i}.
\]

Exercise: Find the residue of \( \frac{1}{z^2} \) at \( z = 0 \).

Solution: Applying the formula for the residue (Eq. 1.109, p. 120) we find

\[
\text{Residue} = \lim_{z \to 0} z \cdot \frac{1}{z^2} = 0.
\]

Thus the residue is zero.

Example: Consider the function \( f(z) = \frac{1}{z^2 + 1} \). Determine the residue at \( z = i \).

Solution: First, find the roots of the denominator, which are \( z = i, -i \). Since \( z = i \) is inside the circle of convergence, we can use the formula for residues to find the residue at \( z = i \):

\[
\text{Residue} = \frac{1}{2i}.
\]
The complex analytic power series (i.e., complex analytic functions) may also be integrated, term by term, since
\[ \int f(x)dx = \sum_{k=0}^{\infty} \frac{\partial k}{k+1} x^{k+1}. \]  
(1.47)

Newton took full advantage of this property of the analytic function and used the analytic series (Taylor series) to solve analytic problems, especially for working out integrals, allowing him to solve differential equations. To fully understand the theory of differential equations, one must master single-valued analytic functions and their analytic power series.

**Single- vs. multi-valued functions:** Polynomials and their \( \infty \)-degree extensions (analytic functions) are single valued: for each \( x \) there is a single value for \( P_n(x) \). The roles of the domain and codomain may be swapped to obtain an inverse function, with properties that can be quite different from those of the function. For example, \( y(x) = x^2 + 1 \) has the inverse \( x = \sqrt{y - 1} \), which is double valued, and complex when \( y < 1 \). Periodic functions such as \( y(x) = \sin(x) \) are even more "exotic" since \( x(y) = \arcsin(x) = \sin^{-1}(x) \) has an infinite number of \( x(y) \) values for each \( y \). This problem was first addressed in Riemann's 1851 PhD thesis, written while he was working with Gauss.

**Exercise:** Let \( y(x) = \sin(x) \). Then \( dy/dx = \cos(x) \). Show that \( dx/dy = -1/\sqrt{1 - x^2} \).

**Solution:** See the implicit function theorem (D'Angelo, 2017, p. 104). Add solution.

**Exercise:** Let \( y(x) = \sin(x) \). Then \( dy/dx = \cos(x) \). Show that \( dx/dy = -j/\sqrt{1 + x^2} \).

**Solution:** Add solution.

**Exercise:** Find the Taylor series coefficients of \( y = \sin(x) \) and \( x = \sin^{-1}(y) \).

**Solution:** Add solution.

**Complex analytic functions:** When the argument of an analytic function \( F(x) \) is complex, that is, \( x \in \mathbb{R} \) is replaced by \( s = \sigma + \omega \in \mathbb{C} \) (recall that \( \mathbb{R} \subset \mathbb{C} \))

\[ F(s) = \sum_{n=0}^{\infty} c_n (s - s_0)^n, \]  
(1.48)

with \( c_n \in \mathbb{C} \), that function is said to be a complex analytic.

For example, when the argument of the exponential becomes complex, it is periodic on the \( \omega \) axis, since

\[ e^{\omega} = e^{(\sigma + \omega)it} = e^{\omega t} e^{\sigma i t} = e^{\omega t} [\cos(\omega t) + j \sin(\omega t)]. \]  
(1.49)

Taking the real part gives

\[ \Re\{e^{\omega t}\} = e^{\omega t} \frac{\cos(\omega t) + e^{-\omega t}}{2} = e^{\omega t} \cos(\omega t), \]

and \( \Im\{e^{\omega t}\} = e^{\omega t} \sin(\omega t) \). Once the argument is allowed to be complex, it becomes obvious that the exponential and circular functions are fundamentally related. This exposes the family of **entire circular functions** [i.e., \( e^x, \sin(x), \cos(x), \tan(x), \cosh(s), \sinh(s) \)] and their inverses [\( \ln(s), \arcsin(s), \arccos(s), \arctan(s), \cosh^{-1}(s), \sinh^{-1}(s) \)], first fully elucidated by Euler (c. 1750).

1.5. STREAM 3B: VECTOR CALCULUS (10 LECTURES)

**How does this work?** To better understand what Eq. 1.173 means, consider a three-dimensional Taylor series expansion of the potential in \( x \) about the limit point \( x_0 \). To compute the higher order terms (HOT) one needs the Hessian matrix \(^{106}\)

\[ \phi(x) = \phi(x_0) + \nabla \phi(x) \cdot (x - x_0) + \text{HOT.} \]

We could define the gradient using this relationship as

\[ \nabla \phi(x_0) = \lim_{x \to x_0} \frac{\phi(x) - \phi(x_0)}{x - x_0}. \]

For this definition to apply, \( x \) must approach \( x_0 \) along \( n \).

The natural form for the surface \( |S| \) is to lie along the iso-potential surfaces as much as possible, so that the integral is a constant (the potential) times the area. The remainder of the surface must be perpendicular to these iso-potential surfaces, in the direction of the maximum, or change in the potential. The secret to the integral definition is in taking the limit. As the volume \( |S| \) shrinks to zero, the HOT terms are small, and the integral reduces to the first-order term in the Taylor expansion, since the constant term integrates to zero. Such a construction is used in the proof of the Webster horn equation (1.5.6, p. 143; Fig. 1.33, p. 144).

**Divergence:** \( \nabla \cdot D = \rho [\text{Col/m}^3] \)

As briefly summarized on page 130, the differential definition of the gradient which maps \( \mathbb{R}^3 \to \mathbb{R}^1 \)

\[ \nabla \cdot D \equiv \left[ \frac{\partial D_x}{\partial x} - \frac{\partial D_y}{\partial y} - \frac{\partial D_z}{\partial z} \right] = \rho(x, y, z). \]

The divergence is a direct measure of the flux (flow) of the vector field it operates on (\( D \)), coming from \( x \). A vector field is said to be incompressible if the divergence of that field is zero. It is therefore compressible when the divergence is non-zero. Compared to air, water is considered to be incompressible. However at very low frequencies, air can also be considered as incompressible. Thus the definition of compressible depends on the wavelength in the medium, so the terms must be used with some awareness of the circumstances.

**Models of the electron:** It is helpful to consider the physics of the electron, a negatively charged particle that is frequently treated as a single point in space. If the size were truly zero, there could be no magnetic spin moment. One size estimate is the Lorentz radius, \( 2.810^{-15} [\text{m}] \). One could summarize the Lorentz radius as follows: Here lie many unsolved problems in physics. More specifically, at dimensions of the Lorentz radius, what exactly is the structure of the electron?

Ignoring the difficulties, if one integrates the charge density of the electron over the Lorentz radius and places the total charge at a single point, then one may make a grossly oversimplified model of the electron. For example, the electric displacement \( D = \epsilon_E E \) (flux density) around a point charge is

\[ D = -\epsilon_E \nabla \phi(R) = -Q \nabla \left( \frac{1}{R} \right) = -Q \delta (R). \]

This is a formula taught in many classic texts, but one should remember how crude a model of an electron it is. But it does describe the electric flux in an easily remembered form. However, \(^{106}\) \( \nabla_{ij} = \partial^2(x) / \partial x_i \partial x_j \), which will exist if the potential is analytic in \( x \) at \( x_0 \).
The application of complex-analytic functions to physics was dramatic, as may be seen in the examples discussed throughout this chapter. For instance, the electric field strength is the gradient of the voltage around a point charge (Greenberg, 1988, footnote p. 762).

\[ \nabla \phi(x, y, z) = -\nabla \phi(x, y, z) \ [V/m] \]

As briefly summarized on page 129, the differential definition of the gradient maps

\[ \nabla \phi(x, y, z) \equiv \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \]

around a point \( \phi \) at \( (x, y, z) \) in the potential \( \phi(x, y, z) \) is the gradient of the point.

The dimensions of Eq. 1.173 are in the units of the potential times the area, divided by the volume,

\[ \text{[V/m]} \]

Here \( \phi \) is the electric field magnitude \( |E| \) and \( |\phi| \) is the magnitude of the complex variable \( \phi \).

The integral definition of the gradient:

\[ \nabla \phi(x, y, z) = \lim_{S \to 0} \frac{1}{|S|} \int_{S} \nabla \phi(x, y, z) \cdot dA \]

Here \( S \) is a surface \( x, y, z \) having area \( |S| \), and \( \nabla \phi(x, y, z) \cdot dA \) is the dot product of the surface. The extended definition of Eq. 1.173 is

\[ \nabla \phi(x, y, z) \cdot dA = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \cdot \left( dx, dy, dz \right) \]

As the volume goes to zero, so must

\[ \nabla \phi(x, y, z) \]

The gradient is the slope of the tangent plane of the potential \( \phi(x, y, z) \) at \( (x, y, z) \).

If one were free-fall skiing this surface, they would be the first one down the hill. Normally we avoid going downhill as needed for a gradient (e.g., \([V/m]\)).

The gradient of a complex analytic function leaves a major hole in one's understanding of the complex analytic function. For instance, the electric field strength is the gradient of the voltage around a point charge (Greenberg, 1988, footnote p. 762).

Sections 1.3.11-1.3.14 (pp. 93-99).

The natural way to define the surface and volume is to place the surface on the iso-potential \((\phi)\) and the volume \((R)\) so that they are perpendicular to the surface at \( (x, y, z) \). As the volume goes to zero, so must the negative gradient

\[ \nabla \phi(x, y, z) \]

The dimensions of Eq. 1.173 are in the units of the potential times the area, divided by the volume,

\[ \text{[V/m]} \]

Here \( \phi \) is the electric field magnitude \( |E| \) and \( |\phi| \) is the magnitude of the complex variable \( \phi \).

The integral definition of the gradient:

\[ \nabla \phi(x, y, z) = \lim_{S \to 0} \frac{1}{|S|} \int_{S} \nabla \phi(x, y, z) \cdot dA \]

Here \( S \) is a surface \( x, y, z \) having area \( |S| \), and \( \nabla \phi(x, y, z) \cdot dA \) is the dot product of the surface. The extended definition of Eq. 1.173 is

\[ \nabla \phi(x, y, z) \cdot dA = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \cdot \left( dx, dy, dz \right) \]

As the volume goes to zero, so must

\[ \nabla \phi(x, y, z) \]

The gradient is the slope of the tangent plane of the potential \( \phi(x, y, z) \) at \( (x, y, z) \).

If one were free-fall skiing this surface, they would be the first one down the hill. Normally we avoid going downhill as needed for a gradient (e.g., \([V/m]\)).

The gradient of a complex analytic function leaves a major hole in one's understanding of the complex analytic function. For instance, the electric field strength is the gradient of the voltage around a point charge (Greenberg, 1988, footnote p. 762).

Sections 1.3.11-1.3.14 (pp. 93-99).

The natural way to define the surface and volume is to place the surface on the iso-potential \((\phi)\) and the volume \((R)\) so that they are perpendicular to the surface at \( (x, y, z) \). As the volume goes to zero, so must the negative gradient

\[ \nabla \phi(x, y, z) \]

The dimensions of Eq. 1.173 are in the units of the potential times the area, divided by the volume,

\[ \text{[V/m]} \]

Here \( \phi \) is the electric field magnitude \( |E| \) and \( |\phi| \) is the magnitude of the complex variable \( \phi \).

The integral definition of the gradient:

\[ \nabla \phi(x, y, z) = \lim_{S \to 0} \frac{1}{|S|} \int_{S} \nabla \phi(x, y, z) \cdot dA \]

Here \( S \) is a surface \( x, y, z \) having area \( |S| \), and \( \nabla \phi(x, y, z) \cdot dA \) is the dot product of the surface. The extended definition of Eq. 1.173 is

\[ \nabla \phi(x, y, z) \cdot dA = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \cdot \left( dx, dy, dz \right) \]

As the volume goes to zero, so must

\[ \nabla \phi(x, y, z) \]

The gradient is the slope of the tangent plane of the potential \( \phi(x, y, z) \) at \( (x, y, z) \).

If one were free-fall skiing this surface, they would be the first one down the hill. Normally we avoid going downhill as needed for a gradient (e.g., \([V/m]\)).

The gradient of a complex analytic function leaves a major hole in one's understanding of the complex analytic function. For instance, the electric field strength is the gradient of the voltage around a point charge (Greenberg, 1988, footnote p. 762).

Sections 1.3.11-1.3.14 (pp. 93-99).

The natural way to define the surface and volume is to place the surface on the iso-potential \((\phi)\) and the volume \((R)\) so that they are perpendicular to the surface at \( (x, y, z) \). As the volume goes to zero, so must the negative gradient

\[ \nabla \phi(x, y, z) \]

The dimensions of Eq. 1.173 are in the units of the potential times the area, divided by the volume,
1.3.4 Lec 12: Polynomial root classification by convolution

Following the exploration of algebraic relationships by Fermat and Descartes, the first theorem was being formulated by d’Alembert. The idea behind this theorem is that every polynomial of degree $N$ (Eq. 1.27) has at least one root. This may be written as the product of the root and a second polynomial of degree of $N-1$. The recursive application of this concept, it is clear that every polynomial of degree $N$ has $N$ roots. Today this result is known as the fundamental theorem of algebra:

Every polynomial equation $P(z) = 0$ has a solution in the complex numbers. As Descartes observed, a solution $z = a$ implies that $P(z)$ has a factor $z - a$. The quotient

$$Q(z) = \frac{P(z)}{z - a} = \frac{P(z)}{a} \left[1 + \frac{z}{a} + \frac{(z/a)^2}{2} + \frac{(z/a)^3}{3} + \cdots\right]$$

(1.50)

is then a polynomial of one lower degree. . . . We can go on to factorize $P(z)$ into $n$ linear factors.


The ultimate expression of this theorem is given by Eq. 1.27 (p. 59), which indirectly states that an $n^{th}$ degree polynomial has $n$ roots. We shall use the term degree when speaking of polynomials and the term order when speaking of differential equations. A general rule is order applies to the time domain and degree to the frequency domain, since the Laplace transform of a differential equation, having constant coefficients, of order $N$, is a polynomial of degree $N$ in Laplace frequency $s$.

Exercise: Explore expressing Eq. 1.50 in terms of real 2x2 matrices, as described in Section 1.2.1, p. 32.

Today this theorem is so widely accepted we fail to appreciate it. Certainly about the time you learned the quadratic formula, you were prepared to understand the concept of polynomials having roots. The simple quadratic case may be extended a higher degree polynomial. The Matlab/Octave command roots returns $[x_1, x_2, x_3]$ of the cubic equation, defined by the coefficient vector $[a_0, a_1, a_2, a_3]$. The command poly returns the coefficient vector. I don’t know the largest degree that can be accurately factored by Matlab/Octave, but I’m sure its well over $N = 10^3$. Today, finding the roots numerically is a solved problem.

Factorization versus convolution: The best way to gain insight into the polynomial factorization problem is through the inverse operation, multiplication of monomials. Given the roots $x_k$, there is a simple algorithm for computing the coefficients $a_n$ of $P_n(x)$ for any $n$, no matter how
Multiplying the following three factors gives since

When the roots are the coefficients gives the same answer as the product of the polynomials. 

The question is “What is the relationship between the coefficients

As the degree increases, the algebra becomes more difficult. Imagine trying to work out the coefficients for the cubic (x − 1)(x − 2)(x − 3).

To verify, substitute the roots as two vectors:

where

This explains why convolution of monomials

The convolution of two vectors:

is done as follows: reverse one of the two monomials, padding unused elements

outside the range shown, all the elements are zero. In summary,

are Fourier transforms and functions of Laplace frequency

Because the wave equation (Eq. 1.137) is 2nd order in time, there are two causal independent

Eigen-solutions

Because the characteristic equation

of thermodynamics and the concept of an adiabatic process.

The roots of the polynomials are the coefficients and hence the book-keeping, by formalizing the procedure.

The eigen-functions of Eq. 1.140 are complex analytic and thus

are necessarily complex analytic in

The convolution of two functions

is expressed in the frequency-domain, also sometimes called the time-independent representation.

Albeit with an error of over 200 years, following the creation

of Huygens’s principle. While his concept showed a deep insight

into the behavior of waves, it was later realized that his analysis was incomplete.

The convolution of a continuous function of the form

with a discrete impulse train

is equivalent to the integral of the continuous function multiplied

by a pulse train

of equal width.

convolution

Convolution is said to be a

convolution of two vectors:

which must be causal, in terms of their poles and zeros, as complex

algebraic expressions in terms of the coefficients and roots of the polynomials.

As previously discussed, Newton (1687) was the first to calculate the speed of sound

in air. Despite his erroneous approximation, his work laid the foundation for later

studies of acoustics and the development of modern sound engineering.

Algebraic Equations (12 Lectures)
There are two distinct mathematical methods used to describe physical systems: lumped models (i.e., quasi-statics) and differential equations. We shall describe these methods for the case of the scalar wave equation, which describes the evolution of a scalar field, such as a pressure or voltage, or equivalently, the flow.

1. **Lumped-element method**: A system may be represented in terms of lumped elements, such as electrical inductors, capacitors and resistors, or their mechanical counterparts, masses, springs and dashpots. Such systems are represented by transmission-matrices rather than by differential equations, the number of which is equal to the number of elements in the network. When the system of lumped element networks contains only resistors and capacitors, it does not support waves, and is related to the diffusion equation in its solution. Depending on the elements in the system of equations, there can be an overlap between a diffusion process and scalar waves, represented as transmission lines, both modeled as lumped networks of 2x2 matrices (Section 1.3.9, Eq. 1.70, p. 90).

When lumped elements are used, the equations accurately approximate the transmission line equations below a critical frequency \( f_c \), which depends on the density of model-elements. When the wavelength is longer than the physical distance between the elements (one per matrix), the approximation is equivalent to a transmission line. As the frequency increases, the wavelength eventually becomes equal to \( f = f_c \), and then shorter than the element spacing, where the quasi-static (lumped element) model breaks down. This is under the control of the modeling process, as more elements are required to represent higher frequencies (shorter wavelengths). If the nature of the solution at high frequencies is desired, the lumped parameter model fails and one must use the differential equation method. However for many (perhaps most) problems, lumped elements are easy to use, and accurate, for frequencies below the cutoff (where the wavelength approaches the element spacing). These relations are elegantly explained in Brillouin (1953).

2. **Separable coordinate systems**: Classically PDEs are often solved by a technique called separation of variables, which is limited to a few coordinate systems such as rectangular, cylindrical and spherical coordinates (Morse, 1948, p. 296-7). Even a slight deviation from separable specific coordinate systems represents a major barrier to further understanding, blocking insight into more general cases. These few separable coordinate systems are special cases, which have high degrees of symmetry, while the wave equation is not tied to a specific coordinate system. Thus lumped-parameter methods (quasi-statics) provides solutions over a much wider class of geometries.

When the coordinate system is separable the resulting PDE is called a Sturm-Liouville equation, and its eigen-functions are the basis functions for solutions to these equations. Webster horn theory (Webster, 1919; Morse, 1948; Pierce, 1981) is a generalized Sturm-Liouville equation which adds physics to the mathematical 19th century approach of Sturm-Liouville, in the form of the area-function of the horn.

As is common in mathematical physics, it is the physical applications, not the mathematics, that make a theory powerful. Mathematics provides rigor, while physics provides a physical rational. Both are important: however, the relative importance depends on ones view point, and the nature of the problem being solved.

---

Exercise:

1. What are the three nonlinear equations that one would need to solve to find the roots of a cubic? **Solution**: From our formula for the convolution of three monomials we may find the nonlinear “deconvolution” relations between the roots\(^{34}\) \([-a, -b, -c]\) and the cubic’s coefficients \([1, \alpha, \beta, \gamma]\)

\[
(x + a) \ast (x + b) \ast (x + c) = (x + c) \ast (x^2 + (a + b)x + ab) = x \cdot ((x^2 + (a + b)x + ab) + c \cdot (x^2 + (a + b)x + ab)) = x^3 + (a + b + c)x^2 + (ab + ac + cb)x + abc = [1, a + b + c, ab + ac + cb, abc].
\]

It follows that the nonlinear equations must be

\[
\alpha = a + b + c
\]

\[
\beta = ab + ac + bc
\]

\[
\gamma = abc.
\]

Clearly these are solve by the classic cubic solution which appears to be a deconvolution problem, also know as long division of polynomials. It follows that the following long-division of polynomials must be true:

\[
\frac{x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc}{x + a} = x^2 + (b + c)x + bc
\]

The product of monomial \( P_1(x) \) with a polynomial \( P_N(x) \) gives \( P_{N+1}(x) \): This statement is another way of stating the fundamental theorem of algebra. Each time we convolve a monomial with a polynomial of degree \( N \), we obtain a polynomial of degree \( N + 1 \). The convolution of two monomials results in a quadratic (degree 2 polynomial). The convolution of three monomials gives a cubic (degree 3). In general, the degree \( k \) of the product of two polynomials of degree \( n, m \) is the sum of the degrees (\( k = n + m \)). For example, if the degrees are each 5 (\( n = m = 5 \)), then the resulting degree is 10.

\(^{34}\)By working with the negative roots we may avoid an unnecessary and messy alternating sign problem.
This seems in conflict with Residue expansions of rational functions since they may be written as two polynomials, \( N \) and \( D \), having \( m \) and \( n \) roots respectively. In summary, the product of two polynomials of degree \( m \) and \( n \), respectively, is of degree \( m + n \). The term \( \text{residue}() \) converts the ratio of two polynomials into a linear system of equations in the unknown residues. Methods for doing this will be discussed in Appendix F.1 (p. 213).

The angle \( \Theta \) is a measure of the solid (cone) angle. When \( \Theta = \pi \), the entire sphere is included. For the conical horn the radius is proportional to the range variable \( r \), as seen from curves 3 and 4 of Fig. 1.34, since no energy can be reflected back into the horn. The angle \( \Theta = \pi \) is the azimuthal angle, and \( \Theta = \pi/2 \) is the polar angle. When \( \Theta = 0 \), the entire sphere is included. In summary, the product of two polynomials of degree \( m \) and \( n \), respectively, is of degree \( m + n \). The term \( \text{residue}() \) converts the ratio of two polynomials into a linear system of equations in the unknown residues. Methods for doing this will be discussed in Appendix F.1 (p. 213).

The angle \( \Theta \) is a measure of the solid (cone) angle. When \( \Theta = \pi \), the entire sphere is included. For the conical horn the radius is proportional to the range variable \( r \), as seen from curves 3 and 4 of Fig. 1.34, since no energy can be reflected back into the horn. The angle \( \Theta = \pi \) is the azimuthal angle, and \( \Theta = \pi/2 \) is the polar angle. When \( \Theta = 0 \), the entire sphere is included.
the mathematical literature as Möbius transformation, which comes from a corresponding scalar differential equation, of the form
\[ \sum_{k=0}^{K} b_k \frac{d^k}{dt^k}(t) = \sum_{n=0}^{N} a_n s^n(t) \leftrightarrow I(\omega) \sum_{k=0}^{K} b_k s^k = V(\omega) \sum_{n=0}^{N} a_n s^n. \] (1.53)

This equation, as well as 1.52, follow from the Laplace transform (See Section 1.3.14, p. 99) of the differential equation (on left), by forms the impedance \( Z(s) = V/I = A(s)/B(s) \). This form of the differential equation follows from Kirchhoff’s voltage and current laws (KCL, KVL) or from Newton’s laws (for the case of mechanics).

The physical properties of an impedance: Based on d’Alembert’s observation that the solution to the wave equation is the sum of forward and backward traveling waves, the impedance may be rewritten in terms of forward and backward traveling waves
\[ Z(s) = \frac{V}{I} = \frac{V^+ + V^-}{I^+ - I^-} = \frac{1}{\Gamma(s)} \left[ 1 + \Gamma(s) \right]^2, \] (1.54)
where \( r_o = P^+/I^+ \) is called the surge impedance of the transmission line (e.g., wire) connected to the load impedance \( Z(s) \), and \( \Gamma(s) = P^+/P_o = I^+/I^+ \) is the reflection coefficient corresponding to \( Z(s) \). Any impedance of this type is called a Brune impedance due to its special properties (discussed on p. 284) (Brune, 1931a). Like \( Z(s) \), \( \Gamma(s) \) is causal and complex analytic. Note that the impedance and the reflectance function must both be complex analytic, since they are connected by the bilinear transformation, which assures the mutual complex analytic properties.

Due to the bilinear transformation, the physical properties of \( Z(s) \) and \( \Gamma(s) \) are very different. Specifically, the real part of the load impedance must be non-negative (\( \Re\{Z(s)\} \geq 0 \)), and if and only if \( |\Gamma(s)| \leq 1 \). In the time domain, the impedance \( z(t) \leftrightarrow Z(s) \) must have a value of \( r_o \) at \( t = 0 \). Correspondingly, the time domain reflectance \( \gamma(t) \leftrightarrow \Gamma(s) \) must be zero at \( t = 0 \).

This is the basis of conservation of energy, which can be traced back to the properties of the reflectance \( \Gamma(s) \).

Exercise: Show that if \( \Re\{Z(s)\} \geq 0 \) then \( |\Gamma(s)| \leq 1 \). Solution: Their two equivalent proofs, both of which start from the relation between \( Z(s) \) and \( \Gamma(s) \). Taking the real part of Eq. 1.54, which must be \( \geq 0 \), we find
\[ \Re\{Z(s)\} = \frac{r_o}{2} \left[ \frac{1}{1 - \Gamma(s)} \right]^2 \left( 1 + \Gamma(s) \right)^2 = \frac{1}{1 - |\Gamma(s)|^2} \geq 0. \]
Thus \( |\Gamma| \leq 1 \).

1.3.6 Lect 14: Introduction to Analytic Geometry

Analytic geometry came about with the merging of Euclid’s geometry with algebra. The combination of Euclid’s (332 BCE) geometry and al-Khwarizmi’s (830 CE) algebra resulted in a totally new powerful tool, analytic geometry, independently worked out by Descartes and Fermat (Stillwell, 2010). The addition of matrix algebra during the 18th century, allow an analysis in more than 3 dimensions, which today is one of the most powerful tools used in artificial intelligence, data

1.5. STREAM 3B: VECTOR CALCULUS (10 LECTURES) 151

Applying the boundary conditions: The general solution is compactly formulated as an ABCD matrix (i.e., Section 1.3.9, p. 90), starting from
\[ \begin{bmatrix} P(x) \\ V(x) \end{bmatrix} = \begin{bmatrix} e^{-\alpha x} & e^{\alpha x} \\ e^{-\beta x} & e^{\beta x} \end{bmatrix} [\alpha, \beta], \] (1.163)
where \( \alpha, \beta \) are the relative weights on the two unknown eigen-functions, to be determined by the boundary conditions at \( x = 0, L \), and \( s = \pm c, Y = 1/|\Delta|/|r_o| \).

Solving Eq. 1.163 for \( \alpha \) and \( \beta \) (with \( Z = 1/\Delta \) and determinant \( \Delta = -2Y \)), at \( x = L \)
\[ \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_L = \frac{1}{2Y} \begin{bmatrix} -Y e^{cL} - Y e^{-cL} \\ -Y e^{cL} - Y e^{-cL} \end{bmatrix}, \] (1.164)
\[ \begin{bmatrix} P \\ V \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{cL} Z e^{cL} & -Z e^{cL} \\ e^{-cL} - Z e^{-cL} \end{bmatrix} \begin{bmatrix} P \\ V \end{bmatrix}_L. \]

We may uniquely determine these two weights given the pressure and velocity at the boundary \( x = L \), which is typically determined by the load impedance \( (P_L/V_L) \).

Once the weights have been determined, they may be substituted back into Eq. 1.163, to determine the pressure and velocity amplitudes at any point \( 0 \leq x \leq L \).

\[ \begin{bmatrix} P \\ V \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{-s(L-x)} & e^{s(L-x)} \\ e^{s(L-x)} & e^{-s(L-x)} \end{bmatrix} \begin{bmatrix} P \\ V \end{bmatrix}_L, \] (1.165)
\[ \begin{bmatrix} P \\ V \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{c(L-x)} & e^{-c(L-x)} \\ e^{-c(L-x)} + e^{c(L-x)} \end{bmatrix} \begin{bmatrix} P \\ V \end{bmatrix}_L. \] (1.166)

Applying the last boundary condition, we evaluate Eq. 1.164 to obtain the ABCD matrix at the input \( (x = 0) \) (Pipes, 1958)
\[ \begin{bmatrix} P \\ V \end{bmatrix} = \begin{bmatrix} \cosh(cL) & Z \sinh(cL) \\ Z \sinh(cL) & \cosh(cL) \end{bmatrix} \begin{bmatrix} P \\ V \end{bmatrix}_L. \] (1.167)

Exercise: Evaluate this expression in terms of the load impedance. Solution: Since \( Z_{load} = P_L/V_L \),
\[ \begin{bmatrix} P \\ V \end{bmatrix}_0 = \begin{bmatrix} Z_{load} \cosh(cL) & -Z \sinh(cL) \\ Z_{load} \sinh(cL) & -\cosh(cL) \end{bmatrix} \begin{bmatrix} P \\ V \end{bmatrix}_L. \] (1.168)

Impedance matrix: Expressing Eq. 1.168 as an impedance matrix gives (algebra required)
\[ \begin{bmatrix} P \\ P_L \end{bmatrix} = \begin{bmatrix} Z_{load} \cosh(cL) & -Z \sinh(cL) \\ Z_{load} \sinh(cL) & -\cosh(cL) \end{bmatrix} \begin{bmatrix} P \\ V \end{bmatrix}_L. \]

Input admittance \( Y_{in} \): Given the input admittance of the horn, it is possible to determine if it is sinusoidal, without further analysis. Namely, if the horn is uniform and infinite in length, the input admittance at \( x = 0 \) is
\[ Y_{in}(0, s) \equiv \frac{V(0, \omega)}{P(0, \omega)} = Y, \]
since \( \alpha = 1 \) and \( \beta = 0 \). That is, for an infinite uniform horn, there are no reflections.
Three examples of horns vectors, their Pythagorean lengths. An attempt at a detailed comparison is summarized in Table 1. 2. Important similarities include Newton and Johann and his son Daniel Bernoulli, and Euler. Gauss had the advantage of input from Newton, Euler, the time-line for this period is provided in Fig. 1. 15. science and machine learning. The utility and importance of these new tools cannot be overstated. Figure 1.15: STREAM 2: ALGEBRAIC EQUATIONS (12 LECTURES) 1.3. Composition of functions: If \( f(x) = 2x + 1 \) and \( g(x) = x^2 \), then the composition of functions \( (f \circ g)(x) = f(g(x)) = (2x^2 + 1) \). Doing so gives the characteristic admittance (Eq. 1.148) is independent of direction. The signs must be "physically chosen," with the velocity \( V_\pm \) into the port, to assure that \( Y > 0 \), for both waves.

Summary of four classic horns:

1) The uniform horn
2) Conical horn
3) Exponential horns.

There are several new concepts that come with the development of analytic geometry: what algebra also added to geometry was the ability to compute with complex numbers. For example, the length of a line (Eq. 1.55) was measured in Geometry with a compass: numbers \( \sqrt{a^2 + b^2 - 2ab} = \sqrt{(a-b)^2} \).
Table 1.2: An ad-hoc comparison between Euclidean geometry and analytic geometry. I am uncertain as to the classification of the items in the third column.

<table>
<thead>
<tr>
<th>Euclidean geometry: ( \mathbb{R}^3 )</th>
<th>Analytic geometry: ( \mathbb{R}^n )</th>
<th>Uncertain</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Proof</td>
<td>1. Numbers</td>
<td>1. Cross product (( \mathbb{R}^3 ))</td>
</tr>
<tr>
<td>2. Line length</td>
<td>2. Algebra</td>
<td>2. Recursion</td>
</tr>
<tr>
<td>3. Line intersection</td>
<td>3. Power series</td>
<td>3. Iteration ( \in \mathbb{C}^3 ) (e.g., Newton’s method)</td>
</tr>
<tr>
<td>4. Point</td>
<td>4. Analytic functions</td>
<td>4. Iteration ( \in \mathbb{R}^n )</td>
</tr>
<tr>
<td>5. Projection (e.g. scalar product)</td>
<td>5. Complex analytic functions: e.g., inertial body mass, electric field, magnetic field</td>
<td></td>
</tr>
<tr>
<td>7. Vector (sort of)</td>
<td>7. Elimination</td>
<td></td>
</tr>
<tr>
<td>8. Conic section</td>
<td>8. Integration</td>
<td></td>
</tr>
<tr>
<td>9. Square roots (e.g., spiral of Theodorus)</td>
<td>9. Derivatives</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10. Calculus</td>
<td></td>
</tr>
<tr>
<td></td>
<td>11. Polynomial ( \in \mathbb{C} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>12. Fund. thm. algebra</td>
<td></td>
</tr>
<tr>
<td></td>
<td>13. Normed vector spaces</td>
<td></td>
</tr>
<tr>
<td></td>
<td>14. \ldots</td>
<td></td>
</tr>
</tbody>
</table>

played no role. Once algebra was available, the line’s Euclidean length could be computed numerically, directly from the coordinates of the two ends, defined by the \( \mathbf{x} \)-vector

\[
e = x\mathbf{x} + y\mathbf{y} + z\mathbf{z} = [x, y, z]^T,
\]

which represents a point at \((x, y, z) \in \mathbb{R}^3 \subset \mathbb{C}^3\) in three dimensions, having direction, from the origin \((0, 0, 0)\) to \((x, y, z)\). An alternative matrix notation is \(e = [x, y, z]^T\), a column vector of three numbers. These two notations are different ways of representing exactly the same thing. I view them as equivalent notations.

By defining the vector, analytic geometry allows Euclidean geometry to become quantitative, beyond the physical drawing of an object (e.g., a sphere, triangle or line). With analytic geometry we have the Euclidean concept of a vector, a line having a magnitude (length) and direction, but analytic defined in terms of physical coordinates (i.e., numbers). The difference between two vectors defines a third vector, a concept already present in Euclidean geometry. For the first time, complex numbers were allowed into geometry (but rarely used until Cauchy and Riemann).

As shown in Fig. 1.16, there are two types of products, the 1) scalar \( \mathbf{A} \cdot \mathbf{B} \) and 2) vector \( \mathbf{A} \times \mathbf{B} \) products.

**Scalar product of two vectors:** When using algebra, many concepts, obvious with Euclid’s geometry, may be made precise. There are many examples of how algebra extends Euclidean geometry: 1. Cross product (\( \mathbb{R}^3 \)), 2. Recursion, 3. Iteration \( \in \mathbb{C}^3 \) (e.g., Newton’s method). 4. Iteration \( \in \mathbb{R}^n \)

### 1.5.10 Lec 37c Finite length horns

For a horn of finite length \( L \) the acoustic variables \( \mathcal{P}(x, s), \mathcal{V}(x, s) \) may be expressed in terms of pressure eigen-functions. If we define the forward wave \( \mathcal{P}^+(x, \omega) \) as launched from \( x = 0 \) and the retrograde wave \( \mathcal{P}^-(x, \omega) \) as launched from \( x = L \), we may write the pressure and velocity as

\[
\begin{bmatrix}
\mathcal{P}(x) \\
\mathcal{V}(x)
\end{bmatrix}
=
\begin{bmatrix}
\mathcal{P}^+(x) \\
\mathcal{V}(x)\mathcal{P}^+(x) - \mathcal{V}(x)\mathcal{P}^-(x - L)
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix}.
\]

(1.158)

Here \( \alpha(x, \omega) \) scales the forward wave and \( \beta(x, \omega) \) scales the retrograde wave. Thus the reflectance \( \Gamma(L, \omega) = \beta/\alpha \) is defined at the site of reflection \( x = L \). Typically the characteristic admittance \( \mathcal{Y}(x) = A(x)/\rho c_0 \) only depends on both the location \( x \), and not on the Laplace frequency \( s \). This formula may not be correct if the horn has losses (\( \mathcal{Y}_e \in \mathbb{C} \)), as discussed in Kirchhoff (1868); Mason (1927, 1928); Robinson (2017).

To evaluate the coefficients \( \alpha(\omega) \) and \( \beta(\omega) \) we must invert Eq. 1.158. \( \alpha, \beta \) are determined at the site of reflection \( x = L \).

**Notation:** To simplify the notation we adopt subscript notation: \( \mathcal{P}^+_0 \equiv \mathcal{P}^+(x = 0) \), \( \mathcal{P}^+_L \equiv \mathcal{P}(x = L) \) and \( \mathcal{V}_L \equiv \mathcal{V}(x = L) \). The eigen-functions are normalized as \( \mathcal{P}^+_0 = 1 \) and \( \mathcal{P}^+_L = 1 \). The determinant of the matrix is \( \Delta_x \).

The determinant \( \Delta_x \)

\[
\Delta_x = -2\mathcal{Y}(x)\mathcal{P}^+(x)\mathcal{P}^-(x - L).
\]

At \( x = L \)

\[
\Delta_L = -2\mathcal{Y}_L\mathcal{P}^+_L\mathcal{P}^+_L,
\]

where \( \mathcal{P}_L = \mathcal{P}(x = L) \).

**Proof read red**

**Proportional load end**

\[
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix}_L
=
\begin{bmatrix}
\mathcal{Y}_L\mathcal{P}^+_L \\
-\mathcal{Y}_L\mathcal{P}^+_L
\end{bmatrix}
\begin{bmatrix}
\mathcal{P}^+_L \\
\mathcal{V}_L
\end{bmatrix}.
\]

(1.159)

(1.160)

Note that the load impedance at \( x = L \) is \( Z_{load}(x = L, s) = -\mathcal{P}_L/\mathcal{V}_L \).

Substituting Eq. 1.160 into Eq. 1.158 results in the following for the input acoustic variables at \( x = 0 \) in terms of the unknowns at \( x = L \):

\[
\begin{bmatrix}
\mathcal{P}^+_0 \\
\mathcal{V}^+_0
\end{bmatrix}
=
\begin{bmatrix}
1 & \mathcal{P}^-_{L,0} \\
\mathcal{Y}^+(r_0) & \mathcal{Y}^+(r_0)\mathcal{P}^-_{L,0}
\end{bmatrix}
\begin{bmatrix}
\mathcal{P}^+_L \\
\mathcal{V}^+_L
\end{bmatrix}.
\]

(1.161)

Here \( \mathcal{P}^+_0 = \mathcal{P}^+(x = L) \) at \( x = 0 \) and \( \mathcal{P}^-_{L,0} = \mathcal{P}^-(x = x = 0) \).
the area of the trapezoid formed by the two vectors, while the triple product $\text{triple product} = 0$.

The scalar product computes the projection of one vector onto the other. The vector product $\times$ computes the cross product of the formed parallelepiped (i.e., prism). When all the angles are right, the volume becomes a cuboid.

In matrix notation the scalar product is written as $\vec{A} \cdot \vec{B} = ||A|| ||B|| \cos \theta$.

The dot product is equal to the sum of the products of the corresponding components.

To prove this take the real part of $\vec{A} \cdot \vec{B} = (A_x B_x + A_y B_y + A_z B_z)$.

If $\Theta = 90^\circ$, then the dot product is zero. If $\Theta = 0^\circ$, then the dot product is $\Theta$. If $\Theta = 180^\circ$, then the dot product is $-\Theta$. If $\Theta = 360^\circ$, then the dot product is $\Theta$. If $\Theta = 0^\circ$, then the dot product is $\Theta$. If $\Theta = 90^\circ$, then the dot product is zero. If $\Theta = 180^\circ$, then the dot product is $-\Theta$. If $\Theta = 360^\circ$, then the dot product is $\Theta$.

In conclusion:

1. Show that $\Theta = 90^\circ$ if $\Theta = 0^\circ$ if only if $\Theta = 180^\circ$. Hint: Use Eq. 1.153.

2. Showing that the unit circle in the $\vec{A}$ space maps onto the $\vec{B}$ axis in the impedance plane.

Solution: To prove this take the real part of $\vec{A} \cdot \vec{B} = \Theta$. Use Eq. 1.153 (or 1.153).

$$\Theta = \Theta \cos \Theta$$

The dot product gives the character of $\Theta$. For example, if $\Theta = 0^\circ$, then the dot product is $\Theta$. If $\Theta = 90^\circ$, then the dot product is $\Theta$. If $\Theta = 180^\circ$, then the dot product is $\Theta$. If $\Theta = 360^\circ$, then the dot product is $\Theta$.

From this formula we see that the norm of the difference of two vectors is simply a compact expression for the Euclidean length. A zero-length vector, such as a point, is the result of the fact that $||x - x|| = 0$.

In conclusion:

1. Show that $\Theta = 90^\circ$ if $\Theta = 0^\circ$ if only if $\Theta = 180^\circ$. Hint: Use Eq. 1.153.

2. Showing that the unit circle in the $\vec{A}$ space maps onto the $\vec{B}$ axis in the impedance plane.

Solution: To prove this take the real part of $\vec{A} \cdot \vec{B} = \Theta$. Use Eq. 1.153 (or 1.153).

$$\Theta = \Theta \cos \Theta$$

The dot product gives the character of $\Theta$. For example, if $\Theta = 0^\circ$, then the dot product is $\Theta$. If $\Theta = 90^\circ$, then the dot product is $\Theta$. If $\Theta = 180^\circ$, then the dot product is $\Theta$. If $\Theta = 360^\circ$, then the dot product is $\Theta$.
Integral definition of a scalar product: Up to this point, following Euclid, we have only considered a vector to be a set of elements \( \{x_n\} \in \mathbb{R} \), index over \( n \in \mathbb{N} \), as defining a linear vector space with scalar product \( x \cdot y \), with the scalar product defining the norm or length of the vector \( ||x|| = \sqrt{x \cdot x} \). Given the scalar product, the norm naturally follows.

At this point an obvious question presents itself: Can we extend our definition of vectors to differentiable functions (i.e., \( f(t) \) and \( g(t) \)), indexed over \( t \in \mathbb{R} \), with coefficients labeled by \( t \in \mathbb{R} \), rather than by \( n \in \mathbb{N} \)? Clearly, if the functions are analytic, there is no obvious reason why this should be a problem, since analytic functions may be represented by a convergent series having Taylor coefficients, thus integrable term by term.

Specifically, under certain conditions, the function \( f(t) \) may be thought of as a vector, defining a normed vector space. This intuitive and somewhat obvious idea is powerful. In this case the scalar product must be defined in terms of the integral

\[
\int f(t) \cdot g(t) \, dt
\]

summed over \( t \in \mathbb{R} \), rather than a sum over \( n \in \mathbb{N} \).

This definition of the vector scalar product allows for a significant but straightforward generalization of our vector space, which will turn out to be both useful and an important extension of the concept of a normed vector space. In this space we can define the derivative of a norm with respect to \( t \), which is not possible for the case of the discrete norm, indexed over \( n \). The distinction introduces the concept of continuity in the index \( t \), which does not exist for the discrete index \( n \in \mathbb{N} \).

Pythagorean theorem and the Schwarz inequality: Regarding Fig. 1.16, suppose we compute the difference between vector \( A \in \mathbb{R} \) and \( \alpha B \in \mathbb{R} \) as \( L = ||A - \alpha B|| \in \mathbb{R} \), where \( \alpha \in \mathbb{R} \) is a scalar that modifies the length of \( B \). We seek the value of \( \alpha \), which we denote as \( \alpha^* \), that minimizes the length of \( L \). From simple geometrical considerations, \( L(\alpha) \) will be minimum when the difference vector is perpendicular to \( B \), as shown in the figure by the dashed line from the tip of \( A \perp B \).

To show this algebraically we write out the expression for \( L(\alpha) \) and take the derivative with respect to \( \alpha \), and set it to zero, which gives the formula for \( \alpha^* \). The argument does not change, but the algebra greatly simplifies, if we normalize \( A, B \) to be unit vectors \( a = A/||A|| \) and \( b = B/||B|| \), which have norm = 1.

\[
L^2 = (a - \alpha b) \cdot (a - \alpha b) = 1 - 2 \alpha a \cdot b + \alpha^2.
\]

Thus the length is shortest \( L = L_{min} \), as shown in Fig. 1.16 when

\[
\frac{d}{d\alpha} L^2 = -2a \cdot b + 2\alpha = 0.
\]

Solving for \( \alpha^* \) we find \( \alpha^* = a \cdot b \). Since \( L_{min} > 0 \) \((\alpha \neq b) \), Eq. 1.58 becomes

\[
1 - 2(a \cdot b)^2 + (a \cdot b)^2 = 1 - (a \cdot b)^2 > 0.
\]

In conclusion \( \cos \theta = |a \cdot b| < 1 \). In terms of \( A, B \) this is \( |A \cdot B| < ||A|| ||B|| \cos \theta \), as shown next to \( B \) in Fig. 1.16. Thus the scalar product between two vectors is their direction cosine. Furthermore since this forms a right triangle, the Pythagorean theorem must hold. The triangle inequality says that the lengths of the two sides must be greater than the hypotenuse. Note that \( \Theta \in \mathbb{R} \neq \mathbb{C} \).

This derivation is an abbreviated version of a related discussion in Section H.2.1 (p. 86).

1.5. STREAM 3B: VECTOR CALCULUS (10 LECTURES) 147

admittance. When the horn is terminated, reflections occur, resulting in the horn having poles and zeros at frequencies \( s_k \in \mathbb{C} \), where \( \Gamma(r, s_k) = \pm 1 \).

The reflectance is defined as

\[
\Gamma(r, s) \equiv \frac{P^-(r, s)}{P^+(r, s)} \tag{1.154}
\]

which follows by a rearrangement of terms in Eq. 1.148. The magnitude of the reflections depends \( |\Gamma| \), which must be between 0 and 1. Alternatively this equation may be obtained by solving Eq. 1.153 for \( \Gamma(r, s) \).

Horn radiation admittance: A horn’s acoustic radiation admittance \( Y_{rad}(r, s) \) is the input admittance (Eq. 1.153) when there is no terminating load. The horn’s eigen-functions becomes the radiation admittance when the horn is infinite in length, namely it is the input admittance for an eigen-function. A table of properties is given in Table 1.5 for four different simple horns.

<table>
<thead>
<tr>
<th>( A )</th>
<th>Name</th>
<th>radius</th>
<th>Area/( \lambda )</th>
<th>( P(x) )</th>
<th>( P^-(x, s) )</th>
<th>( g^-(x, t) )</th>
<th>( Y_{rad} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1D uniform</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>( e^{-\pi x^2} )</td>
<td>( \delta(t \mp x/c) )</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2D parabolic</td>
<td>( x \sqrt{2/x} )</td>
<td>( x/\sqrt{x} )</td>
<td>1/( \sqrt{x} )</td>
<td>( H_0^2 ) ( -j(x/s) )</td>
<td>( j(x/s) )</td>
<td>( -j x H_0^2 / H_n^2 )</td>
<td></td>
</tr>
<tr>
<td>3D conical</td>
<td>2</td>
<td>( x^2 )</td>
<td>2/( \sqrt{x} )</td>
<td>( e^{2\pi x^2 / \sqrt{x}} )</td>
<td>( \delta(t \mp x/c) )</td>
<td>1/( \pm c/sx )</td>
<td></td>
</tr>
<tr>
<td>EXP exponential</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1.5.9 Lec37b: Complex-adjointic \( \Gamma(s) \) and \( Z_{in}(s) \)

When defining the complex reflectance \( \Gamma(s) \) as a function of the complex frequency \( s = \sigma + j\omega \), a very important assumption has been made: even though \( \Gamma(s) \) is defined by the ratio of two functions of real (radian) frequency \( \omega \), like the impedance, the reflectance must be causal (postulate P1, p. 102). Namely \( \Gamma(s) \leftrightarrow \gamma(\tau) \) is zero for \( \tau < 0 \). The same holds for the time-domain impedance \( \zeta(\tau) \leftrightarrow Z_{in}(s) \). That \( \gamma(\tau) \) and \( \zeta(\tau) \) are causal is required by the physics.

The forward and retrograde waves are functions of frequency \( \omega \), as they depend on the source pressure (or velocity) and the point of horn excitation. The reflectance is a transfer function (thus the source term cancels) that depends only on the Thévenin impedance (or reflectance) looking into the system (at any position \( r \)).
Vector (×) product of two vectors: As shown in Fig. 1.16, the vector product (aka, cross-produced) \( \mathbf{a} \times \mathbf{b} \) is the second type of product between two vectors. The vector product defines the cross product in equations.

For example, if the two vectors are in \( \mathbf{X} \) and \( \mathbf{Y} \), then the cross-product is \( \mathbf{Z} = \mathbf{X} \times \mathbf{Y} \). The vector product of the two vectors is perpendicular to the plane of the two vectors being multiplied. The formula for computing the cross product is:

\[
\mathbf{a} \times \mathbf{b} = (a_x \mathbf{e}_y - a_y \mathbf{e}_x) \times (b_x \mathbf{e}_y - b_y \mathbf{e}_x) = (a_y b_z - a_z b_y) \mathbf{e}_x - (a_x b_z - a_z b_x) \mathbf{e}_y + (a_x b_y - a_y b_x) \mathbf{e}_z
\]

The magnitudes of the two vectors are parallel to each other, i.e., \( |\mathbf{a}| = |\mathbf{b}| \) if the cross product is a zero vector. If the two vectors are perpendicular to each other, i.e., \( \mathbf{a} \perp \mathbf{b} \), the cross product is a vector of magnitude \( |\mathbf{a}| |\mathbf{b}| \) perpendicular to both the vectors.

The vector product of a third vector with the vector product \( \mathbf{a} \times \mathbf{b} \) is called the **triple product**, which represents the volume of a parallelepiped.

Impact of Analytic Geometry: The most obvious impact of analytic geometry, was a detailed analysis of the conic sections using algebra, rather than drawings via a compass and ruler. An important example is the composition of the line and circle, a venerable construction, presumably going back to before Diophantus' time. With this analytic technique, the composition could be reduced to an equation, i.e., the first two mathematicians to appreciate this mixture of Euclidian geometry and the new methods were Fermat and Descartes (Stillwell, 2010, p. 111-115); soon Newton contributed to this approach. In the early 17th century, Fermat and Descartes worked out thousands of results in conic sections, alone and in concert with each other. Newton’s solution to this dilemma was to simply ignore the imaginary cases (Stillwell, 2010, p. 110).

1.5.8 **Lect 3A: d’Alembert’s eigen-vector superposition principle**

Since the wave form equation (Eq. 1.140) is second order in time, it has two unique pressure eigenfunctions \( P^1(r, s) \) and \( P^2(r, s) \). The general solution may always be written as the superposition of pressure eigenfunctions, with amplitudes determined by the boundary conditions. Based on d’Alembert’s superposition principle, the pressure \( P \) and velocity \( v \) may be decomposed in terms of the pressure eigenfunctions \( P^1 \) and \( P^2 \).

This equation has several applications.

**Generalized admittance/impedance:** The generalized admittance/impedance \( Y_{m,n} = Y_{m,n} P^m P^2 \) leading into the horn is

\[
Y = \frac{1}{\Gamma} \left( \frac{P^1}{Y} - \frac{P^2}{Y} \right)
\]

Here we have factored out the forward traveling eigenfunction \( P^1 \) and \( P^2 \) and re-expressed \( Y \) in terms of two ranges, the characteristic admittance \( Y_{m,n} = Y_{m,n} P^m P^2 \) (Eq. 1.148) and the reflectance \( Y_{m,n} = Y_{m,n} P^m P^2 \).

**Admittance/impedance:**

\[
Y_m = Y_m P_m^2
\]

This admittance/impedance is a “generalized” in the sense that it is not a Brune, rational function, impedance.
Composition and Elimination

In algebra there are two contrasting operations on functions: composition and elimination.

Composition:
Composition is the merging of functions, by feeding one into the other. If the two functions are \( f, g \) then their composition is indicated by \( f \circ g \), meaning the function \( y = f(x) \) is substituted into the function \( z = g(y) \), giving \( z = g(f(x)) \).

Composition is not limited to linear equations, even though that is where it is most frequently applied. To compose two functions, one must substitute one equation into the other. That requires solving for that substitution variable, which is not always possible in the case of nonlinear equations. However, many tricks are available that may work around these restrictions. For example, if one equation is in \( x^2 \) and the other in \( \sqrt{x} \), it may be possible to multiply the first by \( x \) or square the second. The point is that one of the variables must be isolated so that when it is substituted into the other equations, the variable is removed from the mix.

Examples:
Let \( y = f(x) = x^2 - 2 \) and \( z = g(y) = y + 1 \). Then
\[
g \circ f = g(f(x)) = (x^2 - 2) + 1 = x^2 - 1.
\]

In general composition does not commute (i.e., \( f \circ g \neq g \circ f \)), as is easily demonstrated. Swapping the order of composition for our example gives
\[
f \circ g = f(g(y)) = (y + 1)^2 - 2 = y^2 + 2y - 1.
\]

Intersection:
Complementary to composition is intersection (i.e., decomposition) (Stillwell, 2010, pp. 119, 149). For example, the intersection of two lines is defined as the point where they meet. This is not to be confused with finding roots. A polynomial of degree \( N \) has \( N \) roots, but the roots where two polynomials intersect has nothing to do with the roots of the polynomials. The intersection is a function (equation) of lower degree, implemented with Gaussian elimination.

Intersection of two lines
Unless they are parallel, two lines meet at a point. In terms of linear algebra this may be written as 2 linear equations\(^{65}\) (on the left), along with the intersection point \( [x_1, x_2]^T \), given by the inverse of the 2x2 set of equations (on the right)
\[
\begin{bmatrix}
    a & b \\
    c & d \\
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
\end{bmatrix}
=
\begin{bmatrix}
    y_1 \\
    y_2 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
    d & b \\
    c & a \\
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
\end{bmatrix}
=
\begin{bmatrix}
    y_1 \\
    y_2 \\
\end{bmatrix}
\]
(1.61)

By substituting the expression for the intersection point \( [x_1, x_2]^T \) into the original equation, we see that it satisfies the equations. Thus the equation on the right is the solution to the equation on the left.

Note the structure of the inverse: 1) The diagonal values \( (a, d) \) are swapped, 2) the off-diagonal values \( (b, c) \) are negated and 3) the 2x2 matrix is divided by the determinant \( \Delta = ad - bc \). If \( \Delta = 0 \), there is no solution. When the determinant is zero \( (\Delta = 0) \), the slopes of the two lines
\[
slope = \frac{dx_2}{dx_1} = \frac{b}{a} = \frac{d}{c}
\]
are equal, thus the lines are parallel. Only if the slopes differ can there be a unique solution.

65When writing the equation \( Ax = y \) in matrix format, the two equations are \( ax_1 + bx_2 = y_1 \) and \( cx_1 + dx_2 = y_2 \) with unknowns \( [x_1, x_2] \), whereas in the original equations \( ay_1 + bx_1 = c \) and \( dy_2 + cx_2 = f \), they were \( y, x \). Thus in matrix format, the names are changed. The first time you see this scrambling of variables, it can be confusing.

1.5. STREAM 3B: VECTOR CALCULUS (10 LECTURES)

Speed of sound \( c_s \):
In terms of \( M(x) \) and \( C(x) \), the speed of sound and the acoustic admittance are
\[
c_s = \frac{1}{\sqrt{C(x)M(x)}} = \sqrt{\frac{\eta_o P_o}{\rho_o}} = \frac{1}{\sqrt{\text{stiffness} \cdot \text{mass}}}
\]
(1.147)

Characteristic admittance \( Y(x) \):
Since the horn equation (Eq. 1.140) is 2d order, it has pressure eigen-function solutions \( P^+ \) and \( P^- \) and their corresponding velocity eigen-functions \( V^+ \) and \( V^- \), related through Eq. 1.144, which defines the characteristic admittance \( Y(x) \)
\[
Y(x) = \sqrt{\frac{\text{stiffness} \cdot \text{mass}}{C(x)M(x)}} = \frac{A(x)}{\rho_o c_s^2} \frac{V^+}{P^+} \frac{V^-}{P^-}
\]
(1.148)
(Campbell, 1903a, 1910, 1922). The characteristic impedance \( Z(x) = \frac{1}{V(x)} \). Based on physical requirements that the admittance must be positive, thus only the positive square root is allowed.

Since the horn (Eq. 1.140) is loss less, \( Y(x) \) must be real and positive. If losses are introduced, the propagation function \( s(s) \) (p. 111) and the characteristic impedance \( Y(x, s) \) would become complex analytic functions of the Laplace frequency \( s \) (Kirchhoff, 1974; Mason, 1928; Ramo et al., 1965; Pierce, 1981, p. 532-4).

One must be carefully in the definition the area \( A(x) \). This area is nor the cross-sectional area of the horn, rather it is the wave-front area, as discussed in Section 1.5.7 (p. 145). Since \( A(x) \) is independent of frequency, it is independent the wave direction.

1.5.7 Matrix formulation of WHEN (III)
Newton’s conservation of momentum law (Eq. 1.135), along with conservation of mass (Eq. 1.136), are modern versions of Newton’s starting point for accurately calculating the horn lowest order plane-wave eigen-mode. Following the simplification of averaging the normal component of the particle velocity over the iso-pressure wave front, Eqs. 1.144, 1.146 may be rewritten as a 2x2 matrix in the acoustic variables, average pressure \( P(r, \omega) \) and volume velocity \( \dot{V}(r, \omega) \) (here we replace the range-variable \( x \) by \( r \))
\[
\frac{\partial \dot{V}}{\partial r} = \frac{\partial \dot{M}(r)}{\partial r} V + sM(r) \frac{\partial V}{\partial r}
\]
(1.149)
where \( M(r) = \rho_o A(r) \) and \( C(r) = A(r)/\eta_o P_o \), are the unit-length mass and compliance of the horn (Ramo et al., 1965, p. ??). The acoustic variables \( P(r, \omega) \) and \( V(r, \omega) \) are sometimes referred to as conjugate variables.\(^{66}\)

To obtain the Webster horn pressure equation Eq. 1.140 (p. 142) from Eq. 1.149 take the partial derivative of the top equation
\[
\frac{\partial^2 P}{\partial r^2} = s \frac{\partial M(r)}{\partial r} V + sM(r) \frac{\partial V}{\partial r}
\]

Use the lower equation to remove \( \partial \dot{V}/\partial r \)
\[
\frac{\partial^2 P}{\partial r^2} + \frac{\partial \dot{M}(r)}{\partial r} V = s^2 M(r) C(r) P = \frac{\partial^2 P}{\partial r^2}
\]

Exercise: Show that the equation on the right is the solution of the equation on the left.

Figure 1.33: Derivation of horn equation using Gauss's law: The divergence of the velocity gives $\nabla \cdot \mathbf{u} = \partial \rho / \partial t + \mathbf{u} \cdot \nabla \mathbf{u}$, or the divergence of the vector is to have a finite length.

Another important example of algebraic expressions in mathematics is Hilbert's generalization of the inner product, as for the lossy vector wave equation, or the lengths of vectors in a Hilbert space. The inner product between two such vectors generalizes the finite-dimensional case with itself, defining the length of a vector. However, working with systems of equations, there are many uses of equations, and we need to become more familiar with this type of relationship when working with Pell's equation (p. 50) and the Fibonacci sequence (p. 53).

Conservation of mass:

\[ \delta x \delta \nu \partial \rho / \partial t + \mathbf{u} \cdot \nabla \mathbf{u} = 0 \]

\[ \text{where } \delta x = \text{thickness}, \quad \delta \nu = \text{volume velocity} \]

\[ \text{and } \mathbf{u} = \text{average pressure} \]

\[ \text{right-hand side we use our definition for the } \delta \nu \text{ of vector } \mathbf{u} \]

\[ \text{and } \delta x \text{ is defined by two iso-pressure surfaces between } x, t \]

\[ \text{is a unit vector perpendicular to the iso-pressure surface} \]

\[ \text{is the input? The obvious answer is that} \]

\[ f > f \]

\[ \text{Eq. 1.144 and 1.146 accurately characterize the Webster plane-wave mode up to the frequency} \]

\[ \text{where } c = \text{propagation}

\[ \text{This results in the difference of the} \]

\[ \text{Next the divergence is converted into the difference between two volume velocities, namely} \]

\[ \nu(x + \delta x) - \nu(x), \quad \text{and } \partial \nu / \partial x \text{ as the limit of this difference over} \delta x, \quad \text{as } \delta x \rightarrow 0. \]

\[ \text{Another important example of algebraic expressions in mathematics is} \]

\[ \text{the functional relation} \]

\[ x = (x') \mathcal{E} \]

\[ \text{where } \mathcal{E} = \text{an algebraic expression in mathematics} \]

\[ z \]

\[ \text{is the norm of vector } c, \text{ akin to a length.} \]

\[ \text{Add Triangle inequality proof} \]

\[ \sum_p \sum_q \sum_r \sum_s \]

\[ \sum_p \sum_q \sum_r \sum_s = \text{not the solution of the equation on the left.} \]

\[ \text{not the solution of the equation on the right.} \]

\[ \text{The obvious answer is that} \]

\[ \text{where } c = \text{magnitude of a vector} \]

\[ \text{and } c = \text{unit vector} \]

\[ \text{Therefore} \]

\[ \text{which gives the identity matrix} \]

\[ \text{right-hand side we use our definition for the } \delta \nu \text{ of vector } \mathbf{u} \]

\[ \text{where } c = \text{unit vector} \]

\[ \text{and } c = \text{unit vector} \]

\[ \text{Add Triangle inequality proof} \]
1.5. STREAM 3B: VECTOR CALCULUS (10 LECTURES)

The limits of the Webster horn equation: It is frequently (i.e., always) stated that the Webster horn equation (WHEN) is fundamentally limited, thus is an approximation that only applies to frequencies much lower than \( f_c \). However in all these discussions it is assumed that the area function \( A(r) \) is the horn’s cross-sectional area, not the area of the iso-pressure wave-front (Morse, 1948; Shaw, 1970; Pierce, 1981).

In the next section it is shown that this “limitation” may be totally avoided (subject to the \( f < f_c \) quasi-static limit (P10, p. 103)), making the Webster horn theory an “exact” solution for the lowest order “plane-wave” eigen-function. The nature of the quasi-static approximation is that it “ignores” higher order evanescent modes, which are naturally small since they are in cutoff (evanescent modes do not propagate). This is the same approximation that is required to define an impedance, since every eigen-mode defines an impedance.

To apply this theory, the acoustic variables (eigen-functions) are redefined for the average pressure and its corresponding volume velocity, each defined on the iso-pressure wave-front boundary (Webster, 1919; Hanna and Slepian, 1924). The resulting impedance is then the ratio of the average pressure over the volume velocity. This approximation is valid up to the frequency where the next mode begins to propagate (\( f > f_c \)), which may be estimated from the roots of the Bessel eigenfunctions (Morse, 1948). Perhaps it should be noted that these ideas, that come from acoustics, apply equally well to electromagnetics, or any other wave phenomena described by eigen-functions.

The best known examples of wave propagation are electrical and acoustic transmission lines. Such systems are loosely referred to as the telegraph or telephone equations, referring back to the early days of their discovery (Brillouin, 1953; Heaviside, 1892; Campbell, 1903b; Feynman, 1970a). In acoustics, guided waves are called horns, such as the horn connected to the first phonograph (Webster, 1919). Thus the names reflect the historical development, to a time when the mathematics and the applications were running in close parallel.

1.5.6 Lea 36b: Webster horn equation (II): Derivation

Here we transform the acoustic equations Eq. 1.135 and 1.136 (p. 141) into their equivalent integral form Eq. 1.140 (p. 142). This derivation is similar (but not identical) to that of Hanna and Slepian (1924) and Pierce (1981, p. 360). Extensive experimental analysis for various types of horns (conical, exponential, parabolic) along with a review of horn theory may be found in Goldsmith and Minton (1924).

where \( \phi(x,t) \leftrightarrow \mathcal{P}(r,s) \) is the average pressure (Hanna and Slepian, 1924; Mawardi, 1949; Morse, 1948), Olson (1947, p. 101), Pierce (1981, p. 360). Extensive experimental analysis for various types of horns (conical, exponential, parabolic) along with a review of horn theory may be found in Goldsmith and Minton (1924).

The limits of the Webster horn equation: It is frequently (i.e., always) stated that the Webster horn equation (WHEN) is fundamentally limited, thus is an approximation that only applies to frequencies much lower than \( f_c \). However in all these discussions it is assumed that the area function \( A(r) \) is the horn’s cross-sectional area, not the area of the iso-pressure wave-front (Morse, 1948; Shaw, 1970; Pierce, 1981).

In the next section it is shown that this “limitation” may be totally avoided (subject to the \( f < f_c \) quasi-static limit (P10, p. 103)), making the Webster horn theory an “exact” solution for the lowest order “plane-wave” eigen-function. The nature of the quasi-static approximation is that it “ignores” higher order evanescent modes, which are naturally small since they are in cutoff (evanescent modes do not propagate). This is the same approximation that is required to define an impedance, since every eigen-mode defines an impedance.

To apply this theory, the acoustic variables (eigen-functions) are redefined for the average pressure and its corresponding volume velocity, each defined on the iso-pressure wave-front boundary (Webster, 1919; Hanna and Slepian, 1924). The resulting impedance is then the ratio of the average pressure over the volume velocity. This approximation is valid up to the frequency where the next mode begins to propagate (\( f > f_c \)), which may be estimated from the roots of the Bessel eigenfunctions (Morse, 1948). Perhaps it should be noted that these ideas, that come from acoustics, apply equally well to electromagnetics, or any other wave phenomena described by eigen-functions.

The best known examples of wave propagation are electrical and acoustic transmission lines. Such systems are loosely referred to as the telegraph or telephone equations, referring back to the early days of their discovery (Brillouin, 1953; Heaviside, 1892; Campbell, 1903b; Feynman, 1970a). In acoustics, guided waves are called horns, such as the horn connected to the first phonographs from around the turn of the century (Webster, 1919). Thus the names reflect the historical development, to a time when the mathematics and the applications were running in close parallel.

1.5.6 Lea 36b: Webster horn equation (II): Derivation

Here we transform the acoustic equations Eq. 1.135 and 1.136 (p. 141) into their equivalent integral form Eq. 1.140 (p. 142). This derivation is similar (but not identical) to that of Hanna and Slepian (1924) and Pierce (1981, p. 360).

Conservation of momentum: The first step is an integration of the normal component of Eq. 1.135 over the iso-pressure surface \( S \), defined by \( \nabla p = 0 \)

\[
- \int_S \nabla p(x,t) \cdot dA = \frac{\partial}{\partial t} \int_S u(x,t) \cdot dA.
\]

The average pressure \( \phi(x,t) \) is defined by dividing the total area

\[
\phi(x,t) = \frac{1}{A(x)} \int_S p(x,t) \hat{n} \cdot dA. \tag{1.141}
\]
Ohm's law and impedance: The ratio of voltage over the current is called the impedance which has units of ohms. For example, a given resistor of $R = 10 \, \text{ohms}$.

Kirchhoff's laws KCL, KVL: The laws of electricity and mechanics may be written down using expressions for the variables of the system under study, with complex coefficients indicating the direction of the flow from the source to the load. The laws may be written several ways, the most common being $\sum_i I_i = 0$ (KCL) and $\sum_i \Delta V_i = 0$ (KVL).

Another place where equations of second degree appear in physical applications is in energy and power calculations. The electrical power is given by the product of the voltage $v(t)$ and current $i(t)$ (or in mechanics as the force times the velocity). For example, if we define $P = v(t) \cdot i(t)$ to be the power $P \, \text{watts}$, then the total energy [$\text{joules}$] at time $t$ is $E(t) = \int_0^t P(t) \, dt$.

Points of major confusion are a number of terms that are misused, and overused, in the fields of mathematics, physics and engineering. Some of the most obviously abused terms are "resistor", "current", and "voltage" ("velocity" and "force") of the system under study, with complex coefficients indicating the direction of the flow from the source to the load. The laws may be written several ways, the most common being $\sum_i I_i = 0$ (KCL) and $\sum_i \Delta V_i = 0$ (KVL).

The term on the right of Eq. 1.139, which is identical to Eq. 1.122 (p. 132), is also the Laplacian $\nabla^2$. This condition may be written several ways, the most common being $\sum_i I_i = 0$ (KCL) and $\sum_i \Delta V_i = 0$ (KVL).

Another place where equations of second degree appear in physical applications is in energy and power calculations. The electrical power is given by the product of the voltage $v(t)$ and current $i(t)$ (or in mechanics as the force times the velocity). For example, if we define $P = v(t) \cdot i(t)$ to be the power $P \, \text{watts}$, then the total energy [$\text{joules}$] at time $t$ is $E(t) = \int_0^t P(t) \, dt$.

Points of major confusion are a number of terms that are misused, and overused, in the fields of mathematics, physics and engineering. Some of the most obviously abused terms are "resistor", "current", and "voltage" ("velocity" and "force") of the system under study, with complex coefficients indicating the direction of the flow from the source to the load. The laws may be written several ways, the most common being $\sum_i I_i = 0$ (KCL) and $\sum_i \Delta V_i = 0$ (KVL).

The term on the right of Eq. 1.139, which is identical to Eq. 1.122 (p. 132), is also the Laplacian $\nabla^2$. This condition may be written several ways, the most common being $\sum_i I_i = 0$ (KCL) and $\sum_i \Delta V_i = 0$ (KVL).

Another place where equations of second degree appear in physical applications is in energy and power calculations. The electrical power is given by the product of the voltage $v(t)$ and current $i(t)$ (or in mechanics as the force times the velocity). For example, if we define $P = v(t) \cdot i(t)$ to be the power $P \, \text{watts}$, then the total energy [$\text{joules}$] at time $t$ is $E(t) = \int_0^t P(t) \, dt$.

Points of major confusion are a number of terms that are misused, and overused, in the fields of mathematics, physics and engineering. Some of the most obviously abused terms are "resistor", "current", and "voltage" ("velocity" and "force") of the system under study, with complex coefficients indicating the direction of the flow from the source to the load. The laws may be written several ways, the most common being $\sum_i I_i = 0$ (KCL) and $\sum_i \Delta V_i = 0$ (KVL).

The term on the right of Eq. 1.139, which is identical to Eq. 1.122 (p. 132), is also the Laplacian $\nabla^2$. This condition may be written several ways, the most common being $\sum_i I_i = 0$ (KCL) and $\sum_i \Delta V_i = 0$ (KVL).
Transfer functions (transfer matrix): The only method that seems to work, to sort this out, is to cite the relevant physical application, in specific contexts. The most common standard reference is a physical system that has an input \( x(t) \) and an output \( y(t) \). If the system is linear, then it may be represented by its impulse response \( h(t) \). In such cases the system equation is

\[
y(t) = h(t) * x(t) \leftrightarrow Y(\omega) = H(\omega) \big|_{\omega=0} X(\omega);
\]

namely, the convolution of the input with the impulse response gives the output. From Fourier analysis this relation may be written in the real frequency domain as a product of the Laplace transform of the impulse response, evaluated on the \( \omega \) axis and the Fourier transform of the input \( X(\omega) \leftrightarrow x(t) \) and output \( Y(\omega) \leftrightarrow y(t) \).

If the system is nonlinear, then the output is not given by a convolution, and the Fourier and Laplace transforms have no obvious meaning.

The question that must be addressed is why is the power said to be nonlinear whereas a power series of \( H(s) \) is said to be linear? Both have powers of the underlying variables. This is massively confusing, and must be addressed. The question will be further addressed in Section H.5.1 in terms of the system postulates of physical systems.

What’s going on? The domain variables must be separated from the codomain variables. In our example, the voltage and current are multiplied together, resulting in a nonlinear output, the power. If the frequency is squared, this is describing the degree of a polynomial. This is not because it does not impact the signal output; it characterizes the Laplace transform of the system response.

### 1.3.8 Lec 15 Gaussian Elimination

The method for finding the intersection of equations is based on the recursive elimination of all the variables but one. This method, known as Gaussian elimination, works across a broad range of cases, but may be defined as a systematic algorithm when the equations are linear in the variables. Rarely do we even attempt to solve problems in several variables of degree greater than 1. But Gaussian eliminations may still work in such cases (Stillwell, 2010, p. 90).

In Appendix B.3 (p. 196) the inverse of a 2x2 linear system of equations is derived. Even for a 2x2 case, the general solution requires a great deal of algebra. Working out a numeric example of Gaussian elimination is more instructive. For example, suppose we wish to find the intersection of the two equations

\[
\begin{align*}
x - y &= 3 \\
2x + y &= 2.
\end{align*}
\]

This 2x2 system of equations is so simple that you may immediately visualize the solution: By adding the two equations, \( y \) is eliminated, leaving \( 3x = 5 \). But doing it this way takes advantage of the specific example, and we need a method for larger systems of equations. We need a generalized (algorithmic) approach. This general approach is called Gaussian elimination.

Start by writing the equations in matrix format (note this is not in the standard form \( Ax = y \))

\[
\begin{bmatrix}
1 & -1 \\
2 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
3 \\
2
\end{bmatrix}.
\]

### 1.5. STREAM 3B: VECTOR CALCULUS (10 LECTURES)

fact that the voltage is not the force. Rather the force is the voltage drop, referenced to the ground, which is defined as zero volts. It seems misleading (more precisely it is wrong) to state Ohm’s law as the voltage over the current, since Ohm’s law actually says that the voltage drop (i.e., voltage gradient) over the current defines the impedance. Like a voltage, the pressure is the potential, the gradient of which is a force density, which drives the flow. More on this in Section 1.5.14 (p. 161), where we introduce the fundamental theorem of vector calculus (aka Helmholtz’ decomposition theorem), which generalizes Ohm’s law to include circulation (e.g., angular momentum, vorticity and the related EM magnetic effects). To understand these generalizations in flow one needs to understand compressible and rotational fields, complex analytic functions, and a lot more history of mathematical-physics (Table 1.7, p. 165).

#### 1.5.4 Lec 35 (II): Scalar Wave Equations (Acoustics)

In this section we discuss the general solution to the wave equation. The wave equation has two forms: scalar waves (acoustics) and vector waves (electromagnetics). These have an important mathematical distinction, but have a similar solution space, one scalar and the other vector. To understand the differences we start with the scalar wave equation.

The scalar wave equation: A good starting point for understanding PDEs is to explore the scalar wave equation. The scalar wave equation:

\[
\nabla^2 \psi(x, t) = \frac{1}{c_0^2} \frac{\partial^2 \psi(x, t)}{\partial t^2} \leftrightarrow \frac{\partial^2 \psi}{\partial t^2} \leftrightarrow \frac{s^2}{c_0^2} P(x, s)
\]

with \( c_0 = \sqrt{\frac{\eta}{\rho_o}} \) is the sound velocity. Because the merged equations describe the pressure, which is a scalar field, this is an example of the scalar wave equation
We shall see that this is always true because the magnetic charge 

\[
\mathbf{A} = \mathbf{A}(x, y, z, t)
\]

is a legal vector field (the components are analytic in 

\( x, xy, xyz \)). Note that it is the difference 

\[
\mathbf{A}(x, y, z, t) - \mathbf{A}(x, y, z, t-\Delta t)
\]

may be defined as

\[
\Theta(x, y, z, t) = \frac{\partial \mathbf{A}}{\partial t}
\]

The above operations may be automated by finding a carefully chosen upper-diagonalization notation is to let

\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\]

be upper triangular (zero below the diagonal) you have the solution. Starting from the bottom equation,

\[
x_3 = \frac{-q - p \cdot 3}{q \cdot 3}
\]

and

\[
x_2 = \frac{-q + p \cdot 3}{q \cdot 3}
\]

In Appendix B.3 the inverse of a general 2x2 matrix is summarized in terms of three steps: 1) Taylor series expansion.

\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix}
  d & -b \\
  -c & a
\end{bmatrix}
\]

Next, eliminate the lower left term (2x) using a scaled version of (1.65). For every potential 

\[
\phi = \phi(x, y, z, t)
\]

there exists a force density 

\[
\mathbf{F} = \nabla \phi
\]

In electrical circuits it is common to define a zero potential 

\[
V = V(x, y, z)
\]

As the potential drops between two nodes, the work done by the force is the product of the force and the flow. Very few impedance relationships involve numerical methods.

\[
\frac{\partial \mathbf{A}}{\partial t}
\]

is a special kind of derivative. The potential 

\[
\phi(x, y, z, t)
\]

is typically involving numerical methods. Ohm's law, Kirchhoff's laws, Laplace's law, or Newton's laws. In the simplest of cases, they are all linearized (proportional) complex relationships between a force and a flow. Very few impedance relationships involve numerical methods.

\[
\begin{bmatrix}
  p & q \\
  r & s
\end{bmatrix}
\]

In electrical circuits it is common to define a zero potential 

\[
V = V(x, y, z)
\]

As the potential drops between two nodes, the work done by the force is the product of the force and the flow. Very few impedance relationships involve numerical methods.

\[
\begin{bmatrix}
  p & q \\
  r & s
\end{bmatrix}
\]

In electrical circuits it is common to define a zero potential 

\[
V = V(x, y, z)
\]

As the potential drops between two nodes, the work done by the force is the product of the force and the flow. Very few impedance relationships involve numerical methods.

\[
\begin{bmatrix}
  p & q \\
  r & s
\end{bmatrix}
\]

In electrical circuits it is common to define a zero potential 

\[
V = V(x, y, z)
\]

As the potential drops between two nodes, the work done by the force is the product of the force and the flow. Very few impedance relationships involve numerical methods.

\[
\begin{bmatrix}
  p & q \\
  r & s
\end{bmatrix}
\]

In electrical circuits it is common to define a zero potential 

\[
V = V(x, y, z)
\]

As the potential drops between two nodes, the work done by the force is the product of the force and the flow. Very few impedance relationships involve numerical methods.
 CHAPTER 1. INTRODUCTION

Thus dividing by the determinant gives the 2x2 identity matrix. A good strategy, when you don’t trust your memory, is to write down the inverse as best you can, and then verify.

Using the 2x2 matrix inverse on our example (Eq. 1.63), we find

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{1+2} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -6 + 2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -4/3 \end{bmatrix}. \quad (1.69)$$

If you use this method, you will rarely (never) make a mistake, and the solution is easily verified. Either you can check the numbers in the inverse, as was done in Eq. 1.68, or you can substitute the solution back into the original equation.

1.3.9  Lec 16: Transmission (ABCD) matrix composition method

Matrix composition: Matrix multiplication represents a composition of 2x2 matrices, because the input to the second matrix is the output of the first (this follows from the definition of composition: \( f(x) \circ g(x) = f(g(x)) \)). Thus the ABCD matrix is also known as the transmission matrix method, or occasionally the chain matrix. The general expression for an transmission matrix \( T(s) \) is

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \quad (1.70)$$

The four coefficients \( A(s), B(s), C(s), D(s) \) are all complex functions of the Laplace frequency \( s = \sigma + j \omega \) (p. 1). The derivation is repeated with more detail in Section H.3.2 (p. 252).

It is a standard convention to always define the current into the node, but since the input current (on the left) is the same as the output current on the right (\( I_L \)), hence the negative sign on \( I_L \) to meet the convention of current into every node. When transmision matrices are cascaded, all the signs then match.

We have already used 2x2 matrix composition in representing complex numbers (p. 32), and for computing the gcd\((m, n)\) of \( m, n \in \mathbb{N} \) (p. 43), Pell’s equation (p. 50) and the Fibonacci sequence (p. 55).

Definitions of \( A, B, C, D \): The definitions of the four functions of Eq. 1.70 are easily read off of the equation, as

$$\begin{aligned} A(s) &= \frac{V_1}{V_2 I_{L=0}}; & B(s) &= \frac{V_1}{I_2 V_{L=0}}; & C(s) &= \frac{I_1}{V_2 I_{L=0}}; & D(s) &= \frac{I_1}{I_2 V_{L=0}}. \quad (1.71) \end{aligned}$$

Solution evolution: The partial differential equation defines the “evolution” of the scalar field (pressure \( p(x,t) \) and temperature \( T(x,t) \), or vector field \( \mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H} \), as functions of space \( x \) and time \( t \). There are two basic categories of field evolution, diffusion and propagation.

1. Diffusion: The simplest and easiest PDE example, easily visualized, is a static (time invariant) scalar temperature field \( T(x) \) [°C]. Just like an impedance or admittance, a field has regions where it is analytic, and for the same reasons, \( T(x,t) \) satisfies Laplace’s equation

$$\nabla^2 T(x,t) = 0.$$ 

Since there is no current when the field is static, such systems are lossless, and thus are conservative.

When \( T(x,t) \) depends on time (is not static), it is described by the diffusion equation

$$\nabla^2 T(x,t) = \frac{\partial}{\partial t} T(x,t), \quad (1.133)$$

a rule for how \( T(x,t) \) evolves with time from its initial state \( T(x,0) \). Constant \( \kappa \) is called the thermal conductivity which depends on the properties of the fluid in the container, with \( s \kappa \) being the admittance per unit area. The conductivity is a measure of how the heat gradients induce heat currents \( J = \kappa \nabla T \), analogous to Ohm’s law for electricity.

Note that if \( T(x, \infty) \) reaches steady state \( J = 0 \) as \( t \to \infty \), it evolves into a static state, thus \( \nabla^2 T = 0 \). This depends on what is happening at the boundaries. When the wall temperature of a container is a function of time, then so will the internal temperature continue to change, but with a delay, that depends on the thermal conductivity \( \kappa \).

Such a system is analogous to an electrical resistor-capacitor series circuit, connected to a battery. The wall temperature and the voltage on the battery represent the potential driving the system, the thermal conductivity \( \kappa \) and the electrical resistor are analogous, and the fluid (like the electrical capacitor), are being heated (charged) by the heat (charge) flux. In both cases Ohm’s law defines the ratio of the potential and the flux. How this happens can only be understood once the solution to the equations has been established.

2. Propagation: Pressure and electromagnetic waves are described by a scalar potential (pressure) (Eq. 1.23, p. 57) and a vector potential (electromagnets) (Eq. 1.188, p. 168) resulting in scalar and vector wave equations.

All these partial differential equations, scalar and vector wave equations, and the diffusion equation, depend on the Laplacian \( \nabla^2 \), which we first saw with the Cauchy-Riemann conditions (Eq. 1.96, p. 109).

The vector Taylor series: Next we shall expand the concept of the one-dimensional Taylor series, a function of one variable, to \( x \in \mathbb{R}^3 \). Just as we generalized the derivative with respect to the real frequency variable \( \omega \in \mathbb{R} \) to the complex analytic frequency \( s = \sigma + \omega j \in \mathbb{C} \), here we generalize the derivative with respect to \( x \in \mathbb{R} \), to the vector \( x \in \mathbb{R}^3 \).

\(^{86}\text{Postulate (P3), p. 102.}\)
The transmission matrix is always constructed from the product of constant matrices. Thus for the case of rectangular systems, the transmission matrix is given by:

\[
[ T \mid S ] = [ I \mid 0 ] \cdot [ s \mid s ]
\]

where \( s \) and \( s \) are the component matrices of the source and load, respectively.

Exercise: Show that the Thévenin source impedance is

\[
Z_{\text{Thévenin}} = \frac{V_{\text{in}}}{I_{\text{short}}}
\]

where \( V_{\text{in}} \) is the Thévenin voltage and \( I_{\text{short}} \) is the short-circuit current.

The Thévenin parameters of a source:

- **Source impedance**: \( Z_{\text{Thévenin}} = \frac{V_{\text{in}}}{I_{\text{short}}} \)
- **Thévenin voltage**: \( V_{\text{Thévenin}} = V_{\text{in}} \)
- **Thévenin current**: \( I_{\text{Thévenin}} = I_{\text{short}} \)

The Thévenin parameters are useful in reducing complex systems to simpler equivalents.

### 1.3. STREAM 2: ALGEBRAIC EQUATIONS (12 LECTURES)

- **Laplace's equation**: \( \nabla^2 \Phi = 0 \)
- **Poisson's equation**: \( \nabla^2 \Phi = \rho \)
- **Fourier diffusion equation**: \( \frac{\partial^2 \Phi}{\partial t^2} = \nabla^2 \Phi \)

These definitions follow trivially from Eq. 1.3. These definitions have general names:

- **Laplacian**: \( \nabla^2 \)
- **Transverse**: \( \nabla^2 \Phi = 0 \)
- **Divergence**: \( \nabla \cdot \mathbf{V} \)
- **Gradient**: \( \nabla \Phi \)
- **Helmholtz**: \( \nabla^2 \Phi = \rho \)

### 1.4. STREAM 3: FOURIER SERIES AND TRANSFORMS (10 LECTURES)

- **Fourier series**: \( f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \)
- **Fourier transform**: \( F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \)

These definitions follow from the work of Fourier and others. The Fourier transform is useful in analyzing periodic signals.

### 1.5. STREAM 4: PARTIAL DIFFERENTIAL EQUATIONS (16 LECTURES)

- **Wave equation**: \( \frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \)
- **Laplace's equation**: \( \nabla^2 u = 0 \)

These definitions follow from the work of Laplace and others. The wave equation is useful in analyzing wave phenomena.
1.3.10 The impedance matrix

With a bit of algebra, one may find the impedance matrix in terms of \( A, B, C, D \) (Van Valkenburg, 1964a, p. 310):

\[
\begin{bmatrix}
V_1 \\
V_2 \\
\end{bmatrix} = \begin{bmatrix}
z_{11} & z_{12} \\
z_{21} & z_{22} \\
\end{bmatrix} \begin{bmatrix}
I_1 \\
I_2 \\
\end{bmatrix} - \frac{1}{\sigma} \begin{bmatrix}
A & \Delta_T \\
1 & D \\
\end{bmatrix} \begin{bmatrix}
I_1 \\
I_2 \\
\end{bmatrix}
\]

(1.72)

For reciprocal systems (P6, p. 102) \( z_{12} = z_{21} \), since \( \Delta_T = 1 \). For anti-reciprocal systems, such as dynamic loudspeakers and microphones (Kim and Allen, 2013), \( z_{21} = -z_{12} = 1/C, \) since \( \Delta_T = -1 \).

Impedance is a very general concept, closely tied to the definition of power \( P(t) \) (and energy). Power is defined as the product of the effort (force) and the flow (current). As described in Fig. 1.3, these concepts are very general, applying to mechanics, electrical circuits, acoustics, thermal circuits, or any other case where conservation of energy applies. Two basic variables are defined, generalized force and generalized flow, also called conjugate variables. The product of the conjugate variables is the power, and the ratio is the impedance. For example, for the case of voltage and current,

\[ P(t) = \int v(t) i(t) dt, \quad Z(s) = \frac{V(\omega)}{I(\omega)} \]

Ohm’s law In general, impedance is defined as the ratio of a force over a flow. For electrical circuits, the voltage is the ‘force’ and the current is the ‘flow.’ Ohm’s law states that the voltage across and the current through a circuit element are related by the impedance of that element (which is typically a function of the Laplace frequency \( s = \sigma + \omega j \)). For resistors, the voltage over the current is called the resistance, and is a constant (e.g. the simplest case, \( V/I = R \)). For inductors and capacitors, the impedance depends on the Laplace frequency \( s \) (e.g. \( V/I = Z(s) \)).

Table 1.3: Impedance is defined as the ratio of a force over a flow, a concept that also holds in mechanics and acoustics. In mechanics, the ‘force’ is equal to the mechanical force on an element (e.g. a mass, dashpot, or spring), and the ‘flow’ is the velocity. In acoustics, the ‘force’ is pressure, and the ‘flow’ is the volume velocity or particle velocity of air molecules.

<table>
<thead>
<tr>
<th>Case</th>
<th>Force</th>
<th>Flow</th>
<th>Impedance</th>
<th>units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electrical</td>
<td>voltage (V)</td>
<td>current (I)</td>
<td>( Z = V/I )</td>
<td>ohms [Ω]</td>
</tr>
<tr>
<td>Mechanics</td>
<td>force (F)</td>
<td>velocity (U)</td>
<td>( Z = F/U )</td>
<td>mechanical ohms [Ω]</td>
</tr>
<tr>
<td>Acoustics</td>
<td>pressure (P)</td>
<td>particle velocity (V)</td>
<td>( Z = P/V )</td>
<td>acoustic ohms [Ω]</td>
</tr>
<tr>
<td>Thermal</td>
<td>temperature (T)</td>
<td>heat-flux (J)</td>
<td>( Z = T/J )</td>
<td>thermal ohms [Ω]</td>
</tr>
</tbody>
</table>

As discussed in Fig. 1.3, the impedance concept also holds for mechanics and acoustics. In mechanics, the ‘force’ is equal to the mechanical force on an element (e.g. a mass, dashpot, or spring), and the ‘flow’ is the velocity. In acoustics, the ‘force’ is pressure, and the ‘flow’ is the volume velocity or particle velocity of air molecules.

In this section we shall derive the method of linear composition of systems, also known as the \( ABCD \) transmission matrix method, or in the mathematical literature as the Möbius (bilinear) transformation. Using the method of matrix composition, a linear system of 2x2 matrices can represent a large and important family of networks. By the application of Ohm’s law to the circuit shown in Fig. 1.18, we can model a cascade of such cells.

1.5. STREAM 3B: VECTOR CALCULUS (10 LECTURES)

The conservative field: An important question is: “When is a field conservative?” A field is conservative when the work done by the motion is independent of the path of motion. Thus the conservative field is related to the FTC, which states that the integral of the work only depends on the end points.

A more complete answer must await the introduction of the fundamental theorem of vector calculus, discussed in Sect. 1.5.14 (Eq. 1.183, p. 164). A few specific examples provide insight:

**Example:** The gradient of a scalar potential, such as the voltage (Eq. 1.112), defines the electric field, which drives a current (flow) across a resistor (impedance). When the impedance is infinite, the flow will be zero, leading to zero power dissipation. When the impedance is lossless, the system is conservative.

**Example:** At audio frequencies the viscosity of air is quite small and thus, for simplicity, it may be taken as zero. However when the wavelength is small (e.g. at 100 [kHz]) \( \lambda = v/c_f = 345/10^3 = 3.45 \) [mm] the lossless assumption breaks down, resulting in a significant propagation loss. When the viscosity is taken into account, the field is lossy, thus the field is no longer conservative.

**Example:** If a temperature field is a time-varying constant (i.e., \( T(x, t) = T_s(t) \)), there is no “heat flux,” since \( \nabla T_s(t) = 0 \). When there is no heat flux (i.e., flux, or flow), there is no heat power, since the power is the product of the force times the flow.

**Example:** The force of gravity is given by the gradient of Newton’s gravitational potential (Eq. 1.22, p. 56)

\[ F = -\nabla \phi_g(r) \]

Historically this was the first conservative field, used to explain the elliptic orbits of the planets around the sun.

1.5.3 Lec 35 (I): Partial differential equations and field evolution:

In all cases the space operator is the Laplacian \( \nabla^2 \), the definition of which depends on the dimensionality of the waves, thus on the coordinate system being used. There are three main categories of partial differential equations (PDEs): parabolic, elliptic and hyperbolic, distinguished by the order of the derivative with respect to time:

1. **Diffusion equations** (Eq. 1.133), describe the evolution of the scalar temperature \( T(x, t) \) (a scalar potential), gradients of solution concentrations (i.e., ink in water) and Brownian motion. Diffusion is first-order in time, which is categorized as parabolic (first-order in time, second-order in space). When these equations are Laplace transformed, diffusion has a single real root, resulting in a real solution (e.g., \( T \in \mathbb{R} \)). There is no wave-front for the case of the diffusion equation. As soon as the source is turned on, the field is non-zero at every point in the bounded container.

2. **Poisson’s equation:** In the steady state the diffusion equation degenerates to either Poisson’s or Laplace’s equation, which are classified as elliptic equations (2nd order in space, 0th order in time). Like the diffusion equation, the evolution has a wave velocity that is functionally infinite.

3. **Wave equations**

https://en.wikipedia.org/wiki/Laminar_flow#Examples
Exercises
Exercise 3.4. Example 2: The gradient of the pressure gives rise to a force density in the fluid (Feynman, 1970c, p. 3-7).

Example 3.5. Example 2: The gradient of the pressure gives rise to a force density in the fluid (Feynman, 1970c, p. 3-7).

Example 3.6. Example 2: The gradient of the pressure gives rise to a force density in the fluid (Feynman, 1970c, p. 3-7).

Example 3.7. Example 2: The gradient of the pressure gives rise to a force density in the fluid (Feynman, 1970c, p. 3-7).

Example 3.8. Example 2: The gradient of the pressure gives rise to a force density in the fluid (Feynman, 1970c, p. 3-7).
addition of the language of algebra changed everything. The analytic function was a key development, heavily used by both Newton and Euler. Also the investigations of Cauchy made important headway with his work on complex variables. Of special note was integration and differentiation in the complex plane of complex analytic functions, which is the topic of stream 3.

It was Riemann, working with Gauss in the final years of Gauss’s life, who made the breakthrough, with the concept of the extended complex plane. This concept was based on the composition of a line with the sphere, similar to the derivation of Euclid’s formula for Pythagorean triplets (Fig. G.3, p. 235). While the importance of the extended complex plane was unforeseen, it changed analytic mathematics forever, along with the physics it supported. It unified and thus simplified many important integrals, to the extreme. This idea is captured by the fundamental theorem of complex integral calculus (Table 1.6 p. 160) and 1.4, p. 104.

Exercise: Compute \( \hat{n} \) for \( \phi(x, t) \) (Eq. 1.127). Solution: \( \hat{n} = \kappa/||\kappa|| \). This represents a unit vector in the direction of the current.

Exercise: If the sign of \( \kappa \) is negative, what are the eigen-vectors and eigen-values of \( \nabla \phi(x, t) \)? Solution:
\[
\nabla e^{-\kappa \tau}u(t) = -\kappa \cdot \nabla(x)e^{-\kappa \tau}u(t) = -\kappa e^{-\kappa \tau}u(t).
\]
No thing changes other than the sign of \( \kappa \). Physically this means the wave is traveling in the opposite direction, corresponding to the forward and retrograde d’Alembert waves.

Prior to this section, we have only considered the Taylor series in one variable, such as for polynomials \( P_n(x), x \in \mathbb{R} \) (Sect. 1.3.1, Eq. 1.27 p. 59) and \( P_n(s), s \in C \) (Sect. 1.4.2, Eq. 1.48 p. 72). The generalization from real to complex analytic functions led to the Laplace transform, an integral transform, heavily used by both Newton and Euler. Also the investigations of Cauchy made important contributions, particularly in the development of complex integral calculus (Table 1.6 p. 160) and 1.4, p. 104.

Exercise: If \( E(x, t) = E_0 \hat{\kappa} \), express \( E(x, t) \) in terms of the voltage potential \( \phi(x, t) \) [V]. Solution: The electric field strength may be found from the voltage as
\[
E(x, t) = -\nabla \phi(x, t) = -\nabla \phi(x, t). ~[V/m]
\]

Exercise: Find the velocity \( v(t) \) of an electron in a field \( E \). Solution: From Newton’s 2nd law, \(-qE = m_e v(t) \) [Nt], where \( m_e \) is the mass of the electron. Thus we must solve this first-order differential equation to find \( v(t) \). This is easily done in the frequency domain \( v(t) \leftrightarrow V(\omega) \).

Role of Potentials: Note that the scalar fields (e.g., temperature, pressure, voltage) are all scalar potentials, summarized in Fig. 1.3 (p. 92). In each case the gradient of the potential results in a vector field, just as in the electric case above (Eq. 1.112).

It is important to understand the physical meaning of the gradient of a potential, which is typically a generalized force (electric field, acoustic force density, temperature flux), which in turn generates a flow (current, velocity, heat flux). The ratio of the gradient over the flow determines the impedance.

1. Example 1: The voltage drop across a resistor causes a current to flow, as described by Ohm’s law. Taking the difference in voltage between two points is a crude form of gradient when the frequency \( f \) [Hz] is low, such that the wavelength is much larger than the distance between the two points. This is the essence of the quasi-static approximation P10 (103).
In mathematics the bilinear transformation in both the input and output variables. Since we are engineers we shall stick with the engineering formalism.\[ \frac{(q-p) + z(q-p)}{p + z} = \frac{m}{n} \]

The bilinear transformation on the complex plane is defined. The distinction between an open and closed set is important, because the closed set on the complex plane (Riemann sphere) is analytic at \( z = \infty \), and one may take the derivatives \( \frac{dw}{dz} \) not yet defined.

The transformation from a complex set to another complex set is a cascade of four independent compositions:

1. Translation
2. Rotation
3. Scaling
4. Inversion

\[ w' = \frac{a(z - b)}{cz + d} \]

Note that I neglected a factor of 4, but I have included a factor of \( \pi \). We have used the fact that the bilinear transformation is conformal, i.e., it preserves angles and shapes. Also, we utilized the fact that the complex plane is analytic at \( z = \infty \), one may take the derivatives \( \frac{dw}{dz} \) not yet defined.

The function of \( w'(\infty) \) or \( w'(0) \) is an analytic function of the bilinear operator. Solution

Exercise: Show that Eq. (1.17) is an analytic function of the bilinear operator.

Diffusion process up a balloon, describes a wave equation (Eq. 1.24, p. 57), as well as an eigen-function of the gradient operator \( \phi(x) \). It is an important case since it represents one of d'Alembert's two solutions (Eq. 1.97, p. 110) of the wave equation.

\[ \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0 \]

\[ \phi(x, t) = \frac{1}{\sqrt{4\pi \sigma^2 t}} e^{-x^2/4\sigma^2 t} \]

\( \sigma \) is the Gaussian parameter. The solution to the heat equation is then:

\[ u(x, t) = \frac{1}{\sqrt{4\pi \sigma^2 t}} e^{-x^2/4\sigma^2 t} \]

\( u(x, t) \) is the heat function. The heat equation is so important that it is used in many different fields.

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \]

\( u(x, t) \) is the temperature function. The heat equation is so important that it is used in many different fields.

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \]

\( u(x, t) \) is the temperature function. The heat equation is so important that it is used in many different fields.

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \]

\( u(x, t) \) is the temperature function. The heat equation is so important that it is used in many different fields.
there, defining the Taylor series with the expansion point at ∞. When the bilinear transformation rotates the Riemann sphere, the point at infinity is translated to a finite point on the complex plane, revealing the analytic nature at infinity. A second way to transform the point at infinity is by the bilinear transformation \( \zeta = 1/z \), mapping a zero (or pole) at \( z = \infty \) to a pole (or zero) at \( \zeta = 0 \). Thus this construction of the Riemann sphere and the Möbius (bilinear) transformation allows us to understand the point at infinity, and treat it like any other point. If you feel that you never understood the meaning of the point at ∞ (likely), this should help.

1.3.12 Lec 18: Complex analytic mappings (Domain-coloring)

One of the most difficult aspects of complex functions of a complex variable is understanding the mapping from \( z = x + yj \) to \( w(z) = u + vj \). For example, \( w(z) = \sin(z) \) is trivial when \( z = x + yj \) is real (i.e., \( y = 0 \)), because \( \sin(z) \) is real. Likewise for the case where \( x = 0 \), where

\[
\sin(y) = \frac{e^{-jy} - e^{jy}}{2j} = j \sin(y)
\]

is purely imaginary. But the general case, \( w(z) = \sin(z) \in \mathbb{C} \),

\[
\sin(z) = \sin(x + yj) = j \sin(z),
\]

is not easily visualized. Thus when \( u(x, y) \) and \( v(x, y) \) are not well known functions, \( w(z) \) can be much more difficult to visualize.

Fortunately with computer software today, this problem can be solved by adding color to the chart. A Matlab/Octave script \( zviz.m \) has been used to make the charts shown here.\(^\text{62}\) This tool is also known as domain-coloring.\(^\text{63}\) Rather than plotting \( u(x, y) \) and \( v(x, y) \) separately, domain-coloring allows us to display the entire function on one chart.

Figure 1.20: Left: Domain-colorized map showing the complex mapping from the \( x + yj \) plane to the \( w(z) = u(x) + v(x)j \) plane. This mapping may be visualized by the use of intensity (light/dark) to indicate magnitude, and color (hue) to indicate angle (phase) of the mapping. Right: This shows the \( w(z) = z - \sqrt{2}j \) plane, shifted to the right and up by \( \sqrt{2}j \). The white and black lines are the iso-real and iso-imaginary contours of the mapping.

Visualizing complex functions: The mapping from \( z = x + yj \) to \( w(z) = u(x, y) + iv(x, y) \) is a \( 2 \times 2 \) dimensional graph. This is difficult to visualize, because for each point in the domain \( z \), we would like to represent both the magnitude and phase (or real and imaginary parts) of \( w(z) \). A good way to visualize these mappings is to use color (hue) to represent the phase and intensity

\[ u+jv = \sin(0.5\pi((x+jy)-i)) \]

for example an automobile tire. On the far left let’s assume there is a pump injecting the fluid into the hose. Consider two different fluids: air and water. Air may be treated as a compressible fluid, whereas water is incompressible. However such a classification is relative, being determined by the relative compliance of the balloon (i.e., tire) at the relatively rigid pump and hose.

This is a special case of a more general situation: When the fluid is treated as incompressible (rigid) the speed of sounds becomes infinite, and the wave equation is not the best describing equation, and the motion is best approximated using Laplace’s equation. This is the transition from short to long wavelengths, from wave propagation, with delay, to quasi-statics, having no apparent delay.

This example may be modeled as either an electrical or mechanical system. If we take the electrical analog, the pump is a current source, injecting charge (\( Q \)) into the hose, which being rigid cannot expand (has a fixed volume). The hose may be modeled as a resistor, and the tire as a capacitor \( C \), which fills with charge as it is delivered via the resistor, from the pump. A capacitor obeys the same law as a spring \( F = KV \), or in electrical terms, \( Q = CV \). Here \( V \) is the voltage, which acts as a force \( F \), \( Q \) is the charge, which acts like the mass of the fluid. The charge is conserved, just as the mass of the fluid is conserved, meaning they cannot be created or destroyed.

The flow of the fluid is called the flux, which is the general term for the mass or charge current. The two equations may be rewritten directly in terms of the force \( F \), \( V \) and flow, as an electrical current \( I = dQ/dt \).\(^\text{64}\) of mass flux \( J = dM/dt \), giving two impedance relations:

\[ I = \frac{d}{dt}CV \quad \text{[A]} \]  
\[ J = \frac{d}{dt}CF \quad \text{[kgm/m^2]} \]

It is common to treat the stiffness of the balloon, which acts as a spring with compliance \( 1/C \), in the frequency domain Ohm’s law becomes Eq. 1.126 for the case of a spring and Eq. 1.125 for the capacitor.

The final solution of this system is solved in the frequency domain. The impedance seen by the source is the sum of the resistance \( R \) added to the impedance of the load, giving

\[ Z = R + \frac{1}{sC} \]

The solution is simply the relation between the force and the flow, as determined by the action of the source on the load \( Z(s) \). The final answer is given in terms of the voltage across the compliance in

\[ \mu + jv = \sin(0.5(\pi((x+jy)-i)) \]
In general it may be shown that in $N = 1, 2, 3 \ldots$ the case of a rigid hose, a rigid tube, terminated on the right in an elastic medium (the above example of a balloon), a hydrostatic pressure exists in the fluid, which is the sum of the external hydrostatic and the internal osmotic pressure. The internal osmotic pressure is the pressure that the particle in question exerts on the fluid in the direction of the osmotic potential.

An example of such a pressure is the osmotic pressure of a solution of a solute in a solvent. The osmotic pressure is given by the van't Hoff equation:

$$ P = nRT $$

where $P$ is the osmotic pressure, $n$ is the number of moles of solute, $R$ is the ideal gas constant, and $T$ is the temperature in kelvin.

For example, if we have a solution of 1 molar potassium chloride ($\text{KCl}$) in water at 298 K, the osmotic pressure is:

$$ P = \frac{nRT}{V} $$. 

The osmotic pressure is a measure of the tendency of the solute to osmotically expand the solvent, and is proportional to the concentration of the solute.

In the case of a rigid tube, terminated on the right in an elastic medium, the hydrostatic pressure is given by the pressure at the right end of the tube, which is the sum of the external hydrostatic and the internal osmotic pressure.

The solution to the boundary value problem for the hydrostatic pressure is given by the Laplace equation:

$$ \nabla^2 P = 0 $$

where $\nabla^2$ is the Laplacian operator.

The solution to this equation is given by the pressure distribution in the tube, which is the sum of the external hydrostatic and the internal osmotic pressure.

In general, the solution to the boundary value problem for the hydrostatic pressure is given by the pressure distribution in the tube, which is the sum of the external hydrostatic and the internal osmotic pressure.
The function \( w = s \) has a dark spot (a zero) at \( s = 0 \), and becomes brighter away from the origin. On the right it is \( w(z) = z - \sqrt{z} \), which shifts the zero to \( z = \sqrt{z} \). Thus domain-coloring gives the full picture of the complex analytic function mappings \( w(x, y) = u(x, y) + iv(x, y) \) in colored polar coordinates.

Two additional examples are given in Fig. 1.22 to help you interpret the two complex mappings \( w = e^w \) (left) and its inverse \( s = \ln(w) \). The exponential is relatively easy to understand because \( s(w) = |e^w|^2 = e^w \). The red region is where \( \omega \approx 0 \) in which case \( \omega \approx e^\omega \). As \( \omega \) becomes large and negative, \( \omega \approx -1 \), thus the entire field becomes dark on the left. The field is becoming light on the right where \( \omega = e^\omega \to \infty \). If we let \( \sigma = 0 \) and look along the \( \omega \) axis, we see that the function is changing phase, sea-green (90°) at the top and violet (-90°) at the bottom.

In the right panel note the zero for \( \ln(w) = \ln|w| + i\phi = 0 \), \( \phi = 0 \), since \( \ln(1) = 0 \). More generally, the log of \( \omega = |w|e^{i\phi} \) is \( s = \ln|w| + i\phi \). Thus \( s(w) \) can be zero only when the angle of \( \omega \) is zero.

The \( \ln(w) \) function has a branch cut along the \( \phi = 180^\circ \) axis. As one crosses over the cut, the phase goes above 180°, and the plane changes to the next sheet of the log function. The only sheet with a zero is the principle value, as shown. For all others, the log function is either increasing or decreasing monotonically, and there is no zero, as seen for sheet 0 (the one showing in Fig. 1.22).

### 1.3.13 Lec 19: Signals: Fourier transforms

Two basic transformations in engineering mathematics are the Fourier and the Laplace transforms, which deal with time–frequency analysis.

**Notation:** The Fourier transform takes a time domain signal \( f(t) \in \mathbb{R} \) and transforms it to the frequency domain \( \omega \in \mathbb{R} \), where it is complex \( (F(\omega) \in \mathbb{C}) \). For the Fourier transform, both the time \( -\infty < t < \infty \) and frequency \( -\infty < \omega < \infty \) are strictly real. The relationship between \( f(t) \) and its transform \( F(\omega) \) is indicated by the double arrow symbol

\[
f(t) \leftrightarrow F(\omega).
\]

Since the FT obeys superposition, it is possible to define a FT of a complex time function \( f(t) \in \mathbb{C} \), \( t \in \mathbb{R} \). This is useful in certain engineering applications (i.e., Hilbert envelope, Hilbert transforms). It is accepted in the engineering and physics literature to use the case of the variable to indicate the type of argument. A time function is \( f(t) \), where \( t \) has units of seconds \([s]\) and is in lower case, whereas its Fourier transform, a function of frequency, having units of either hertz \([Hz]\) or radians \([2\pi Hz]\) is written using upper case \( F(\omega) \). This helps the reader parse the type of variable under consideration. This is a helpful piece of notation, but not entirely in agreement with notation used in mathematics.

**Definition of the Fourier transform:** The forward transform takes \( f(t) \) to \( F(\omega) \) while the inverse transform takes \( F(\omega) \) to \( f(t) \). The tilde symbol indicates that in general the recovered inverse transform signal can be slightly different from \( f(t) \). We give examples of this in Table F.2.

\[
F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt
\]
\[
\tilde{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega
\]
\[
F(\omega) \leftrightarrow \tilde{f}(t) \leftrightarrow F(\omega).
\]

**Example:** A second classic example is an acoustic pressure field \( \varphi(x, t) \) \([\text{Pa}]\), which defines a vector force density \( F(x, t) = -\nabla \varphi(x, t) \) \([\text{N/m}^2]\) (Eq. 1.135, p. 141). When this force density \([\text{N/m}^2]\) is integrated over an area, the net radial force \([\text{N}]\) is

\[
F_r = -\int_S \nabla \varphi(x)|d|S||. \quad [\text{N}]
\]

An inflated balloon with a static internal pressure of 3 \([\text{atm}]\) in an ambient pressure of 1 \([\text{atm}]\) (sea level), forms a sphere due to the elastic nature of the rubber, which acts as a stretched spring under tension. The net force on the surface of the balloon is its area times the pressure drop of 2 \([\text{atm}]\) across the surface. Thus the static pressure

\[
\varphi(x) = 3u(r_o - r) + 1. \quad [\text{Pa}]
\]

where \( u(r) \) is a step function of the radius \( r = ||x|| > 0 \), centered at the center of the balloon, having radius \( r_o \).

Taking the gradient gives the negative\(^{11} \) of the radial force density (i.e., perpendicular to the surface of the balloon)

\[
-f_r = \nabla \varphi(x) = \frac{\partial}{\partial r}3u(r_o - r) + 1 = -2b(r_o - r). \quad [\text{Pa}]
\]

This describes a static pressure that is 3 atmospheres inside the balloon, and 1 atmosphere \([\text{atm}]\) \((1 \text{ atm} \approx 10^5 \text{ [Pa]}\) outside. Note that the net positive force density is the negative of the gradient of the static pressure.

Taking the divergence of this radial force gives the Laplacian of the scalar pressure field

\[
\nabla^2 \varphi(x) = \nabla \cdot \nabla \varphi(x) = -\nabla F(x).
\]

\(^{11}\)The force is pointing out, stretching the balloon.
Consider the exercise:

When a function is both discrete in time and frequency, it is necessarily periodic in time and frequency. The best known example is the class of signals that have a period of one. Important examples of the use of the gradient include the electric field vector \( \mathbf{E} \), the force density \( \mathbf{F} \), and the magnetic field vector \( \mathbf{B} \). The curl of the electric field vector results in a scalar field. For example, the divergence of the electric field vector \( \mathbf{D} \) (Coulomb's law) is the scalar field charge density \( \rho / \varepsilon_0 \). The divergence is

\[ \nabla \cdot \mathbf{D} = \nabla \cdot (|\nabla \Phi| \mathbf{n}) = \frac{\partial |\nabla \Phi|}{\partial x} \mathbf{n}_x + \frac{\partial |\nabla \Phi|}{\partial y} \mathbf{n}_y + \frac{\partial |\nabla \Phi|}{\partial z} \mathbf{n}_z = \frac{\partial |\nabla \Phi|}{\partial x} + \frac{\partial |\nabla \Phi|}{\partial y} + \frac{\partial |\nabla \Phi|}{\partial z} = \nabla \cdot |\nabla \Phi| \mathbf{n} \]

Thus it is analogous to the dot product of two vectors. When working with guided waves (narrow tubes of fluid), when the diameter is small compared with the wavelength \( \lambda \), the divergence is

\[ \nabla \cdot \mathbf{D} = \nabla \cdot (|\nabla \Phi| \mathbf{n}) = \frac{\partial |\nabla \Phi|}{\partial x} \mathbf{n}_x + \frac{\partial |\nabla \Phi|}{\partial y} \mathbf{n}_y + \frac{\partial |\nabla \Phi|}{\partial z} \mathbf{n}_z = \frac{\partial |\nabla \Phi|}{\partial x} + \frac{\partial |\nabla \Phi|}{\partial y} + \frac{\partial |\nabla \Phi|}{\partial z} = \nabla \cdot |\nabla \Phi| \mathbf{n} \]

The electric field vector is given by \( \mathbf{E} = \nabla \Phi \). The area element in rectangular coordinates is then operated on by the divergence to take the output of the gradient back to a scalar field. One of the classic equations is

\[ \nabla \cdot \mathbf{D} = \nabla \cdot (|\nabla \Phi| \mathbf{n}) = \frac{\partial |\nabla \Phi|}{\partial x} \mathbf{n}_x + \frac{\partial |\nabla \Phi|}{\partial y} \mathbf{n}_y + \frac{\partial |\nabla \Phi|}{\partial z} \mathbf{n}_z = \frac{\partial |\nabla \Phi|}{\partial x} + \frac{\partial |\nabla \Phi|}{\partial y} + \frac{\partial |\nabla \Phi|}{\partial z} = \nabla \cdot |\nabla \Phi| \mathbf{n} \]

The notation \( \nabla \cdot \mathbf{D} \) indicates the divergence (Appendix A, p. 185). \( \nabla \) is shorthand for \( \partial / \partial x \) and \( \partial / \partial y \) and \( \partial / \partial z \). The Laplacian field is given by

\[ \nabla \cdot \mathbf{D} = \nabla \cdot (|\nabla \Phi| \mathbf{n}) = \frac{\partial |\nabla \Phi|}{\partial x} \mathbf{n}_x + \frac{\partial |\nabla \Phi|}{\partial y} \mathbf{n}_y + \frac{\partial |\nabla \Phi|}{\partial z} \mathbf{n}_z = \frac{\partial |\nabla \Phi|}{\partial x} + \frac{\partial |\nabla \Phi|}{\partial y} + \frac{\partial |\nabla \Phi|}{\partial z} = \nabla \cdot |\nabla \Phi| \mathbf{n} \]

The notation \( \nabla \cdot \mathbf{D} \) indicates the divergence (Appendix A, p. 185). \( \nabla \) is shorthand for \( \partial / \partial x \) and \( \partial / \partial y \) and \( \partial / \partial z \). The Laplacian field is given by

\[ \nabla \cdot \mathbf{D} = \nabla \cdot (|\nabla \Phi| \mathbf{n}) = \frac{\partial |\nabla \Phi|}{\partial x} \mathbf{n}_x + \frac{\partial |\nabla \Phi|}{\partial y} \mathbf{n}_y + \frac{\partial |\nabla \Phi|}{\partial z} \mathbf{n}_z = \frac{\partial |\nabla \Phi|}{\partial x} + \frac{\partial |\nabla \Phi|}{\partial y} + \frac{\partial |\nabla \Phi|}{\partial z} = \nabla \cdot |\nabla \Phi| \mathbf{n} \]

The notation \( \nabla \cdot \mathbf{D} \) indicates the divergence (Appendix A, p. 185). \( \nabla \) is shorthand for \( \partial / \partial x \) and \( \partial / \partial y \) and \( \partial / \partial z \). The Laplacian field is given by

\[ \nabla \cdot \mathbf{D} = \nabla \cdot (|\nabla \Phi| \mathbf{n}) = \frac{\partial |\nabla \Phi|}{\partial x} \mathbf{n}_x + \frac{\partial |\nabla \Phi|}{\partial y} \mathbf{n}_y + \frac{\partial |\nabla \Phi|}{\partial z} \mathbf{n}_z = \frac{\partial |\nabla \Phi|}{\partial x} + \frac{\partial |\nabla \Phi|}{\partial y} + \frac{\partial |\nabla \Phi|}{\partial z} = \nabla \cdot |\nabla \Phi| \mathbf{n} \]
1.5. STREAM 3B: VECTOR CALCULUS (10 LECTURES)


Definition of the Laplace transform: The forward and inverse Laplace transforms are

\[
F(s) = \int_0^{\infty} f(t)e^{-st}dt
\]

\[
f(t) = \frac{1}{2\pi j} \int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds
\]

Here \( s = \sigma + j\omega \in \mathbb{C} \) [2\pi Hz] is the complex Laplace frequency in radians and \( t \in \mathbb{R} \) [s] is the time in seconds.

When dealing with engineering problems it is convenient to separate the signals we use from the systems that process them. We do this by treating signals, such as a musical signal, differently from a system, such as a filter. In general signals may start and end at any time. The concept of causality has no mathematical meaning in signal space. Systems, on the other hand, obey very rigid rules (to assure that they remain physical). These physical restrictions are described in terms of the network postulates, which are first discussed in Sect. H.5.1. There is a question as to why postulates are need, and which are the best postulates.\(^{65}\)

As discussed in Section 1.4.8 (p. 123), we must use the Cauchy residue theorem (CRT), requiring closure of the contour \( C \) at \( \omega \to \pm j\infty \)

\[
\int_C F(s)ds = \int_{C_{\infty}} + \int_{C_{-\infty}}
\]

where the path represented by \( 'C_{\infty}' \) is a semicircle of infinite radius. For a causal, ‘stable’ (e.g. doesn’t “blow up” in time) signal, all of the poles of \( F(s) \) must be inside of the Laplace contour, in the left half \( s \)-plane.

Figure 1.23: This three-element mechanical resonant circuit consisting of a spring, mass and dashpot (e.g., viscous fluid).

Hooke’s law for a spring states that the force \( f(t) \) is proportional to the displacement \( x(t) \), i.e., \( f(t) = Kx(t) \). The formula for a dashpot is \( f(t) = Rv(t) \), and Newton’s famous formula for mass is \( f(t) = d[Mv(t)]/dt \), which for constant \( M \) is \( f(t) = Mdv/dt \). The equation of motion for the mechanical oscillator in Fig. 1.23 is given by Newton’s second law; the sum of the forces must balance to zero

\[
M\frac{d^2x}{dt^2} + R\frac{dx}{dt} + Kx(t) = f(t).
\]

These three constants, the mass \( M \), resistance \( R \) and stiffness \( K \) are all real and positive. The dynamical variables are the driving force \( f(t) \leftrightarrow F(s) \), the position of the mass \( x(t) \leftrightarrow X(s) \) and its velocity \( v(t) \leftrightarrow V(s) \), with \( v(t) = dx(t)/dt \leftrightarrow V(s) = sX(s) \).

1.5.2 Lec 34: Gradient \( \nabla, \) divergence \( \nabla_\cdot \), curl \( \nabla \times \), and Laplacian \( \nabla^2 \)

There are three key vector differential operators that are required for understanding linear partial differential equations, such as the wave and diffusion equations. All of these begin with the \( \nabla \) operator:

\[
\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}
\]

The official name of this operator is nabla. It has three basic uses: 1) the gradient of a scalar field, the 2) divergence of a vector field, and 3) the curl of a vector field. If properly noted, the shorthand notation \( \nabla \phi(x, t) = (\partial \phi/\partial x, \partial \phi/\partial y, \partial \phi/\partial z) \phi(x, t) \) is convenient.

\[
\phi(x) = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z}
\]

Gradient: The gradient transforms a scalar field into vector field. In \( \mathbb{R}^3 \) the gradient of a scalar field \( \nabla \phi(x) \) is defined as

\[
\nabla \phi(x) = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)
\]

The gradient may be factored into a unit vector \( \hat{n} \) as defined in Fig. 1.31, defining the direction of the gradient, and the gradient’s length \( |\nabla \phi| \), defined in terms of the norm. Thus the gradient of

\[
\nabla \phi(x, t)
\]

Basic differential vector operator definitions: The basic definitions of each of the vector operators are summarized in Table 1.4.

Table 1.4: The three vector operators manipulate scalar and vector fields, as indicated here. The divergence maps vector fields to scalar fields. Finally the curl maps vector fields to vector fields. It is helpful to have a name for second-order operators (e.g., DoG, GoD: mnemonics defined in Sect. 1.5.13, p. 161).

<table>
<thead>
<tr>
<th>Name</th>
<th>Input</th>
<th>Output</th>
<th>Mnemonic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gradient</td>
<td>Scalar</td>
<td>Vector</td>
<td>( \nabla )</td>
</tr>
<tr>
<td>Divergence</td>
<td>Vector</td>
<td>Scalar</td>
<td>( \nabla_\cdot )</td>
</tr>
<tr>
<td>Curl</td>
<td>Vector</td>
<td>Vector</td>
<td>( \nabla \times )</td>
</tr>
<tr>
<td>Laplacian</td>
<td>Scalar</td>
<td>Vector</td>
<td>( \nabla^2 )</td>
</tr>
<tr>
<td>Vector Laplacian</td>
<td>Vector</td>
<td>Vector</td>
<td>( \nabla \nabla \phi )</td>
</tr>
</tbody>
</table>

\(^{65}\)https://www.youtube.com/watch?v=JMAcEBbaz_4

\(^{66}\)https://www.youtube.com/watch?v=YuUlgXRFMMY
The Fourier transform is real, thus typically not analytic. These are not superficial differences. The concept of 
causality, leading to a complex analytic frequency response. The frequency response of the Fourier 
transform is real, thus typically not analytic. These are not superficial differences. The concept of 
causality, leading to a complex analytic frequency response. The frequency response of the Fourier 
transform is real, thus typically not analytic. These are not superficial differences. The concept of 
causality, leading to a complex analytic frequency response. The frequency response of the Fourier 
transform is real, thus typically not analytic. These are not superficial differences. The concept of 
causality, leading to a complex analytic frequency response. The frequency response of the Fourier 
transform is real, thus typically not analytic. These are not superficial differences. The concept of 
causality, leading to a complex analytic frequency response. The frequency response of the Fourier 
transform is real, thus typically not analytic. These are not superficial differences. The concept of 
causality, leading to a complex analytic frequency response. The frequency response of the Fourier
(P1) causality (non-causal/acausal): Causal systems respond when acted upon. Virtually all physical systems obey causality. An example of a causal system is an integrator, which has a response of a step function. Filters are also examples of causal systems. Signals represent acausal responses. They do not have a clear beginning or end, such as the sound of the wind or traffic noise.

(P2) linearity (nonlinear): Linear systems obey superposition. Two signals \( x(t) \) and \( y(t) \) are the inputs to a linear system, producing outputs \( x'(t) \) and \( y'(t) \). When the inputs are presented together as \( ax(t) + by(t) \) with weights \( a, b \in \mathbb{C} \), the output will be \( ax'(t) + by'(t) \). If either \( a \) or \( b \) is zero, the corresponding signal is removed from the output.

Nonlinear systems mix the two inputs, thereby producing signals not present in the input. For example, if the inputs to a nonlinear system are two sine waves, the output will contain distortion components, having frequencies not present at the input. An example of a nonlinear system is one that multiplies the two inputs. A second is a diode, which rectifies a signal, letting current flow only in one direction. Most physical systems have some degree of nonlinear response, but this is not always desired. Other systems are designed to be nonlinear, such as the diode example.

(P3) passive (active): An active system has a power source, such as a battery, while a passive system has no power source. While you may consider a transistor amplifier to be active, it is only so when connected to a power source.

(P4) real (complex) time response: Typically systems are “real in, real out.” They do not naturally have complex responses (real and imaginary parts). While a Fourier transform takes real inputs and produces complex outputs, this is not an example of a complex time response. P4 is a characterization of the input signal, not its Fourier transform.

(P5) time-invariant (time varying): For a system to be a time varying system the output must depend on when the input signal starts or stops. If the output, relative to the input, is independent of the starting time, then the system is said to be time-invariant (static).

(P6) reciprocal (non- or anti-reciprocal): In many ways this is the most difficult property to characterize and thus understand. It is characterized by the ABCD matrix. If \( B = C \), the system is said to be reciprocal. If \( B = -C \), it is said to be anti-reciprocal. The impedance matrix is reciprocal while a loudspeaker is anti-reciprocal and modeled by the gyrator rather than a transformer. All non-reciprocal systems are modeled by gyrators, which swap the force and flow variables.

(P7) reversibility (non-reversible): If swapping the input and output of a system leaves the system invariant, it is said to be reversible. When \( A = D \) the system is reversible. Note the similarity and differences between reversible and reciprocal.

(P8) space-invariant (space-variant): If a system operates independently as a function of where it physically is in space, then it is space-invariant. When the parameters that characterize the system depend on position, it is space-variant.

(P9) Deterministic (random): Given the wave equation, along with the boundary conditions, the system’s solution may be deterministic, or not, depending on its extent. Consider a radar or...
1.3. STREAM 2: ALGEBRAIC EQUATIONS (12 LECTURES)

A sonar wave propagating out into uncharted territory. When the wave hits an object, the reflection can return waves that are not predicted, due to unknown objects. This is an example where the boundary condition is not known in advance.

Quasi-static \((ka < 1)\) Quasi-statics follows from systems that have dimensions that are small compared to the local wavelength. This assumption fails when the frequency is raised (the wavelength becomes short). Thus this is also known as the long-wavelength approximation. Quasi-statics is typically stated as \(ka < 1\), where \(k = \frac{2\pi}{\lambda} = \frac{\omega}{c}\) and \(a\) is the smallest dimension of the system. See p. 170 for a detailed discussion of the role of quasi-statics in acoustic horn wave propagation.

Postulate (P10) is closely related to the Feynman (1970c, Ch. 12-7) titled “The underlying unity” of nature, where Feynman asks Why do we need to treat the fields as smooth? The answer is related to the wavelength of the probing signal relative to the dimensions of the object being probed. This raises the fundamental question: Are Maxwell’s equations a band-limited approximation to reality? I have no idea what the answer is.

Summary discussion of the 10 network postulates:

Each postulate has two (or more) categories. For example, (P1) is either causal, non-causal or acausal while (P2) is either linear or non-linear. (P6) and (P9) only apply to 2-port algebraic networks (those having an input and an output). The others apply to both 2- or 1-port networks (e.g., an impedance is a 1-port). An interesting example is the anti-reciprocal transmission matrix of a loudspeaker, shown in Fig. H.7 (p. 268).

Related forms of these postulates may be found in the network theory literature (Van Valkenburg, 1964a,b; Ramo et al., 1965). Postulates (P1-P6) were introduced by Carlin and Giordano (1964) and (P7-P9) were added by Kim et al. (2016). As noted by Carlin and Giordano (1964, p. 5), while linearity (P2), passivity (P3), realness (P4) and time-invariance (P5) are independent, causality (P1) is a consequence of linearity (P2) and passivity (P3).

1.4.10 Solving differential equations: Method of Frobenius

Many differential equations may be solved by assuming a power series (i.e., Taylor series) solution of the form

\[
y(x) = \sum_{n=0}^{\infty} c_n x^n
\]

with coefficients \(c_n \in \mathbb{C}\). The method of Frobenius is quite general (Greenberg, 1988, p. 193).

Example: If we assume a complex analytic solution of the form \(y(x) = e^{\lambda x}\), where \(\lambda\) is the root of the differential equation, we find the recursion relation

\[
c_n = c_{n-1} / n!
\]

which provides a closed form solution to the recursion relation for coefficients \(c_k\) to be

\[
c_k = -\frac{1}{k(k+2\nu)}c_{k-2}
\]

(Greenberg, 1988, p. 231).

1.4.11 Lec 32: Review for Exam III

Solving differential equations to find the recursion relation for coefficients \(c_k\) to be

\[
\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + \left(\frac{x^2}{\nu^2} - \frac{1}{x^2}\right)\right)y(x) = 0
\]

Exercise: Find the recursion relation for Bessel’s equation of order \(\nu\)

\[
x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \nu^2\right)y = 0
\]

Solution: If we assume a complex analytic solution of the form \(y(x) = e^{\lambda x}\), we find the recursion relation

\[
\lambda^2 + \frac{1}{x}\lambda + \left(\frac{x^2}{\nu^2} - \frac{1}{x^2}\right) = 0
\]

which provides a closed form solution to the recursion relation for coefficients \(c_k\) to be

\[
c_k = -\frac{1}{k(k+2\nu)}c_{k-2}
\]

(Greenberg, 1988, p. 231).

1.4.11 Lec 32: Review for Exam III
1.4 Stream 3a: Scalar (i.e., Ordinary) Differential Equations

Stream 3 is \(\infty\), a concept which typically means unbounded (immeasurably large), but in the case of calculus, \(\infty\) means infinitesimal (immeasurably small), since taking a limit requires small numbers. Taking a limit means you may never reach the target, a concept that the Greeks called Zeno’s paradox (Stillwell, 2010, p. 76).

When speaking of the class of ordinary (versus vector) differential equations, the term scalar is preferable, since the term “ordinary” is vague, if not a meaningless label. There are a special subset of fundamental theorems for scalar calculus, all of which are about integration, as summarized in Table 1.6 (p. 160), starting with Leibniz’s theorem. These will be discussed below, and more extensively in Lec. 1.4.1 (p. 106) and Chapters I (p. 275) and J (p. 275).

Following the integral theorems on scalar calculus, are those on vector calculus, without which there could be no understanding of Maxwell’s equations. Of these, the fundamental theorem of complex calculus (aka, Helmholtz decomposition), Gauss’s law and Stokes’s theorem, form the cornerstone of modern vector field analysis. These theorems allow one to connect the differential (point) and macroscopic (integral) relationships. For example, Maxwell’s equations may be written as either vector differential equations, as shown by Heaviside (along with Gibbs and Hertz), or in integral form. It is helpful to place these two forms side-by-side, to fully appreciate their significance. To understand the differential (microscopic) view, one must understand the integral (macroscopic) view. These are presented in Sections 1.5.13 (p. 155) and Fig. 1.36 (p. 160).

Chronological history post 16\(^{th}\) century

\(^{16^{th}}\) Bonbelli 1526–1572

\(^{17^{th}}\) Galileo 1564–1642, Kepler 1571–1630, Newton 1642–1727; Principia 1687; Mersenne, Huygen; Pascal; Fermat, Descartes (analytic geometry); Bernoullis Jakob, Johann & son Daniel

\(^{18^{th}}\) Euler 1707–1783; Student of Johann Bernoulli; d’Alembert 1717–1783; Kirchhoff; Lagrange; Laplace; Gauss 1777–1855


\(^{20^{th}}\) Hilbert; Einstein; …

Time-Line

<table>
<thead>
<tr>
<th>1525</th>
<th>1600</th>
<th>1700</th>
<th>1800</th>
<th>1875</th>
<th>1925</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bombelli</td>
<td>Newton</td>
<td>Descartes</td>
<td>Mersenne</td>
<td>Gauss</td>
<td>Maxwell</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Johann Bernoulli</td>
<td></td>
<td>Hilbert</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Riemann</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Rayleigh</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Cauchy</td>
<td>Helmholtz</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Fermat</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>d’Alembert</td>
<td>US Civil War</td>
</tr>
</tbody>
</table>

Figure 1.24: Final overview of the time-line for the four centuries from the 16\(^{th}\) and 20\(^{th}\) CE covering Bombelli to Einstein. Mozart and the US Civil War are indicated along the bottom, for orientation.

\(^{67}\)https://en.wikipedia.org/wiki/History_of_Maxwell\%27s_equations

1.4. STREAM 3A: SCALAR CALCULUS (11 LECTURES)

By taking the LT of the convolution we can derive this relationship

\[
\int_0^\infty [f(t) \ast g(t)] e^{-st} dt = \int_0^\infty \int_0^\infty f(\tau) g(t-\tau) e^{-st} d\tau d\tau \\
= \int_0^\infty f(\tau) \left( \int_0^\infty g(t-\tau) e^{-st} dt \right) d\tau \\
= \int_0^\infty f(\tau) e^{-s\tau} \left( \int_0^\infty g(t) e^{-st} dt \right) d\tau \\
= G(s) \int_0^\infty f(\tau) e^{-s\tau} d\tau \\
= G(s) F(s).
\]

We first encountered this relationship in Section 1.3.5 (p. 77)) in the context of multiplying polynomials, which was the same as convolving their coefficients. Hopefully the parallel is obvious. In the case of polynomials, the convolution was discrete in the coefficients, and here it is continuous in time. But the relationships are the same.

Time-shift property: When a function is time-shifted by time \(T_o\), the LT is modified by \(e^{-sT_o}\), leading to the property

\[f(t - T_o) \leftrightarrow e^{-sT_o} F(s).\]

This is easily shown by applying the definition of the LT to a delayed time function.

Time derivative: The key to the eigen-function analysis provided by the LT is the transformation of a time derivative on a time function, that is,

\[\frac{d}{dt} f(t) \leftrightarrow sF(s).\]

Here \(s\) is the eigen-value corresponding to the time derivative of \(e^{st}\). Given the definition of the derivative of \(e^{st}\) with respect to time, this definition seems trivial. Yet that definition was not obvious to Euler. It needed to be extended to the space of complex analytic function \(e^{st}\), which did not happen until at least Riemann (1851).

Given a differential equation of order \(K\), the LT results in a polynomial in \(s\), of degree \(K\). It follows that this LT property is the cornerstone of why the LT is so important to scalar differential equations, as it was to the early analysis of Pell’s equation and the Fibonacci sequence, as presented in earlier chapters. This property was first uncovered by Euler. It is not clear if he fully appreciated its significance, but by the time of his death, it certainly would have been clear to him. Who first coined the terms eigen-value and eigen-function? The word eigen is a German word meaning of one.

Initial and final value theorems: There are much more subtle relations between \(f(t)\) and \(F(s)\) that characterize \(f(t)\) and \(f(t \to \infty)\). While these properties can be very important in certain application, they are beyond the scope of the present treatment. These relate to so-called initial value theorems. If the system under investigation has potential energy at \(t = 0\), then the voltage (velocity) need not be zero for negative time. An example is a charged capacitor or a moving mass. These are important situations, but better explored in a more in-depth treatment.
The properties of the \( \mathcal{L} \) trans, for example, linearity, convolution, and time-shift, have profound implications on the nature of the transformed functions. These properties are crucial in many fields, including signal processing and control systems.

One of the most basic and useful properties is the convolution property:

\[
\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}
\]

This property allows us to simplify complex systems into smaller, more manageable components. By understanding the convolution property, we can better design and analyze systems in various applications.

The beginning of modern mathematics was a pivotal moment in the history of science. The establishment of modern mathematics as we know it today began in the 16th to 18th centuries, as described in Fig. 1.24. The oldest brother, Jacob, taught his creativity in scientific circles was certainly well known due to his many skills and contributions.

As outlined in Fig. 1.24, mathematics as we know it today began in the 16th to 18th centuries, ar-

Put this notional property in Appendix A.

The properties of the \( \mathcal{L} \) trans, for example, linearity, convolution, and time-shift, have profound implications on the nature of the transformed functions. These properties are crucial in many fields, including signal processing and control systems.

One of the most basic and useful properties is the convolution property:

\[
\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}
\]

This property allows us to simplify complex systems into smaller, more manageable components. By understanding the convolution property, we can better design and analyze systems in various applications.

The beginning of modern mathematics was a pivotal moment in the history of science. The establishment of modern mathematics as we know it today began in the 16th to 18th centuries, as described in Fig. 1.24. The oldest brother, Jacob, taught his creativity in scientific circles was certainly well known due to his many skills and contributions.

As outlined in Fig. 1.24, mathematics as we know it today began in the 16th to 18th centuries, ar-

Put this notional property in Appendix A.
Taking the log, and using the definition of $\omega(z) = \tan^{-1}(z)$, we obtain Eq. 1.83.

These equations are the basis of transmission lines (TL) and the Smith chart. Here $z(\omega)$ is the TL’s input impedance and Eq. 1.85 is the reflectance.

While many high school student memorize Euler’s relation, it seems unlikely they appreciate the significance of complex analytic functions (Eq. 1.49, p. 72).

**A brief history of complex analytic functions:** Newton famously ignored imaginary numbers, and called them imaginary in a disparaging (pejorative) way. Given Newton’s prominence, his view certainly must have attenuated interest in complex algebra, even though it had been previously quantified by Bombelli in 1525, likely based on his serendipitous finding of Diophantus’s book *Aritmethic* in the Vatican library.

Euler derived his relationships using real power series (i.e., real analytic functions). While Euler was fluent with $j = \sqrt{-1}$, he did not consider functions to be complex analytic. That concept was first explored by Cauchy almost a century later. The missing link to the concept of complex analytic is the definition of the derivative with respect to the complex argument

$$F'(s) = \frac{dF(s)}{ds} \quad (1.86)$$

where $s = \sigma + \omega j$, without which the complex analytic Taylor coefficients may not be defined.

Euler did not appreciate the role of complex analytic functions, because these were first discovered well after his death (1783), by Augustin-Louis Cauchy (1789–1857).

### 1.4.1 Lec 23: Fundamental theorems of calculus

**History of the fundamental theorem of calculus:** In some sense, the story of calculus begins with the fundamental theorem of calculus (FTC), also known generically as *Leibniz’s formula*. The simplest integral is the length of a line $L = \int_{x}^{x+dx}dx$. If we label a point on a line as $x = 0$ and wish to measure the distance to any other point $x$, we form the line integral between the two points. If the line is straight, this integral is simply the Euclidean length given by the difference between the two ends (Eq. 1.3.6, p. 81).

**Case for zero time ($t = 0$):** When time is zero, the integral does not, in general, converge, leaving $f(t)$ undefined. This is most clear in the case of the step function $u(t) \leftrightarrow 1/s$, where the integral may not be closed, because the convergence factor $e^{st} = 1$ is lost for $t = 0$.

The fact that $u(t)$ does not exist at $t = 0$ explains the Gibbs phenomenon in the inverse Fourier transform. At times where a jump occurs, the derivative of the function does not exist, and thus the time response function is not analytic. The Fourier expansion cannot converge at places where the function is not analytic. A low pass filter may be used to smooth the function, but at the cost of temporal resolution. Forcing the function to be analytic at the discontinuity, by smoothing the jumps, is an important computational method.

### 1.4.8 Lec 30: Inverse Laplace transform ($t > 0$)

**Case of $t > 0$:** Next we investigate the convergence of the integral for positive time $t > 0$. In this case we must close the integral in the LHP ($\sigma < 0$) for convergence, so that $st < 0$ ($\sigma < 0$ and $t > 0$). When there are poles on the $\omega_j = 0$ axis, $\sigma > 0$ assures convergence by keeping the on-axis poles inside the contour. At this point the Cauchy residue theorem (Eq. 1.108) is relevant. If we restrict ourselves to simple poles (as required for a Brune impedance), the residue theorem may be directly applied.

The most simple example is the step function, for which $F(s) = 1/s$ and thus

$$u(t) = \oint_{\text{LHP}} \frac{e^{st} ds}{2\pi j} \leftrightarrow \frac{1}{s} \quad (1.120),$$

which is a direct application of the Cauchy Residue theorem, Eq. 1.108 (p. 120). The forward transform of $u(t)$ is straight forward, as discussed in Section 1.3.14 (p. 99). This is true of most if not all of the elementary forward Laplace transforms. In these cases, causality is built into the integral by the limits, so is not a result, as it must be in the inverse transform. An interesting problem is proving that $u(t)$ is not defined at $t = 0$.  

Figure 1.30: Left: Colorized plot of $u(z) = J_{0}(x z)$. The first zero is at $2.405$, and thus appears at $0.7655 = 2.405/\pi$, somewhat larger than the root of $\cos(x z)$. Right: Note the similarity to $u(z) = \sin(x z)$. The LT’s have similar characteristics, as documented in Table F.3 (p. 212).

The inverse Laplace transform of $F(s) = 1/(s + 1)$ has a residue of $1$ at $s = -1$, thus that is
1.4.1. Introduction

The key point is that this theorem applies when \( n \in \mathbb{R} \). For these cases the result is always zero, since by definition, the residue is given by Eq. 1.109. This is known as the fundamental theorem of complex calculus (FTCC), and Eq. 1.88 may be viewed as the Cauchy residue theorem, and how the causal inverse transform comes about. 

The key point is that this theorem applies when \( n \in \mathbb{R} \). For these cases the result is always zero, since by definition, the residue is given by Eq. 1.109. This is known as the fundamental theorem of complex calculus (FTCC), and Eq. 1.88 may be viewed as the Cauchy residue theorem, and how the causal inverse transform comes about. 

Equations 1.87 and 1.89 differ because the path of the integral is complex. Thus the line integral \( F \) is analytic in both \( \sigma > 0 \) and \( \sigma < 0 \). In this case as \( \sigma \to \infty \), the closure integral \( F \) will diverge. Thus we may not close the curve, at infinity, and show that the integral at \( \sigma > 0 \) in the RHP. In the RHP, \( \sigma > 0 \) the integral \( F \) is analytic, and the Cauchy residue theorem must be used. This argument holds for any \( F \) that is analytic in the RHP, \( \sigma > 0 \). 

Compared to the real Laplace transform, the complex Laplace transform is much more general. While the real Laplace transform is defined only for functions that are analytic in the right half-plane, the complex Laplace transform is defined for functions that are analytic in any region of the complex plane. This makes the complex Laplace transform much more useful in applications, as it can be used to analyze a much wider range of systems. 

1.4.7. Inverse Laplace Transform & Cauchy Residue Theorem

The inverse Laplace transform is defined as 

\[
F(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{C_T} f(s) e^{st} ds,
\]  

where \( C_T \) is any contour in the complex plane that encloses the pole at \( s = \sigma \). 

The Cauchy residue theorem states that the integral of a function \( f(s) \) around a closed contour \( C \) in the complex plane is given by 

\[
\oint_C f(s) ds = 2\pi i \sum_{n=1}^{N} \text{Res}(f, z_n),
\]  

where \( z_n \) are the poles of \( f(s) \) inside the contour. 

The key point is that this theorem applies when \( n \in \mathbb{R} \). For these cases the result is always zero, since by definition, the residue is given by Eq. 1.109. This is known as the fundamental theorem of complex calculus (FTCC), and Eq. 1.88 may be viewed as the Cauchy residue theorem, and how the causal inverse transform comes about.
Complex analytic functions: The definition of a complex analytic function \( F(s) \) of \( s \in \mathbb{C} \) is that the function may be expanded in a Taylor series (Eq. 1.48, p. 72) about an expansion point \( s_0 \in \mathbb{C} \). This definition follows the same logic as the FTC. Thus we need a definition for the coefficients \( c_n \in \mathbb{C} \), which most naturally follow from Taylor’s formula
\[
 c_n = \frac{1}{n!} \frac{d^n}{ds^n} F(s) \Bigg|_{s=s_0} .
\] (1.91)

The requirement that \( F(s) \) have a Taylor series naturally follows by taking derivatives with respect to \( s \) at \( s_0 \). The problem is that both integration and differentiation of functions of complex Laplace frequency \( s = \sigma + j\omega \) have not yet been defined.

Thus the question is: What does it mean to take the derivative of a function \( F(s) \in \mathbb{C}, s = \sigma + j\omega \), with respect to \( s \), where \( s \) defines a plane rather than a real line? We learned how to form the derivative on the real line. Can the same derivative concept be extended to the complex plane?

The Cauchy-Riemann conditions: The FTC defines the area as an integral over a real differential \((dx \in \mathbb{R})\), while the FTC relates an integral over a complex function \( F(s) \in \mathbb{C} \), along a complex integral (i.e., path) \((ds \in \mathbb{C})\). For the FTC the area under the curve only depends on the end points of the anti-derivative \( f(x) \). But what is the meaning of an “area” along a complex path?

The Cauchy-Riemann conditions provide the answer.

1.4.2 Lec 24: Cauchy-Riemann conditions

For the integral of \( Z(s) = R(\sigma, \omega) + jX(\sigma, \omega) \) to be independent of the path, the derivative of \( Z(s) \) must be independent of the direction of the derivative. As we show next, this leads to a pair of equations known as the Cauchy-Riemann conditions. This is an important generalization of Eq. 1.1, p. 18 which goes from real integration \((x \in \mathbb{R})\) to complex integration \((s \in \mathbb{C})\), based on lengths, thus on area.

To define
\[
dds Z(s) = \frac{d}{ds} [R(\sigma, \omega) + jX(\sigma, \omega)]
\]

take partial derivatives of \( Z(s) \) with respect to \( \sigma \) and \( j\omega \), and equate them:
\[
\frac{\partial Z}{\partial \sigma} = \frac{\partial R}{\partial \sigma} + j \frac{\partial X}{\partial \sigma} \quad \text{and} \quad \frac{\partial Z}{j\omega} = \frac{\partial R}{j\omega} + j \frac{\partial X}{j\omega} .
\]

This says that a horizontal derivative, with respect to \( \sigma \), is equivalent to a vertical derivative, with respect to \( j\omega \). Taking the real and imaginary parts gives the two equations
\[
\text{CR-1: } \frac{\partial R(\sigma, \omega)}{\partial \sigma} = \frac{\partial X(\sigma, \omega)}{\partial j\omega} \quad \text{and} \quad \text{CR-2: } \frac{\partial R(\sigma, \omega)}{\partial j\omega} = -j \frac{\partial X(\sigma, \omega)}{\partial \sigma} ,
\] (1.92)

1. In general it makes no sense to integrate through a pole, thus the poles (or other singularities) must not lie on \( C \).

2. The Cauchy integral theorem (Eq. 1.106), follows trivially from the fundamental theorem of complex calculus (Eq. 1.189, p. 107), since if the integral is independent of the path, and the path returns to the starting point, the closed integral must be zero. Thus Eq. 1.106 holds when \( F(s) \) is complex analytic within \( C \).

3. Since the real and imaginary parts of every complex analytic function obey Laplace’s equation (Eq. 1.196, p. 109), it follows that every closed integral over a Laplace field, i.e., one defined by Laplace’s equation, must be zero. In fact this is the property of a conservative system, corresponding to many physical systems. If a closed box has fixed potentials on the walls, with any distribution whatsoever, and a point charge (i.e., an electron) is placed in the box, then a force equal to \( F = qE \) is required to move that charge, and thus work is done. However if the point is returned to its starting location, the net work done is zero.

4. Work is done in charging a capacitor, and energy is stored. However when the capacitor is discharged, all of the energy is returned to the load.

5. Soap bubbles and rubber sheets on a wire frame obey Laplace’s equation.

6. These are all cases where the fields are Laplacian, thus closed line integrals must be zero. Laplacian fields are commonly observed because they are so basic.

7. We have presented the impedance as the primary example of a complex analytic function. Physically, every impedance has an associated stored energy, and every system having stored energy has an associated impedance. This impedance is usually defined in the frequency \( s \) domain, as a force over a flow (i.e., voltage over current). The power \( P(t) \) is defined as the force times the flow and the energy \( E(t) \) as the time integral of the power
\[
E(t) = \int_{-\infty}^{t} P(t)dt ,
\] (1.110)

which is similar to Eq. 1.87 (p. 107) [see Section H.2.1, Eq. 1.62 (p. 87)]. In summary, impedance and power and energy are all fundamentally related.

1.4.6 Lec 28: Cauchy Integral Formula & Residue Theorem

The Cauchy integral formula (Eq. 1.107) is an important extension of the Cauchy integral theorem (Eq. 1.106) in that a pole has been explicitly injected into the integrand at \( s = s_0 \). If the pole location is outside of the curve \( C \), the result of the integral is zero, in keeping with Eq. 1.106. When the pole is inside of \( C \), the integrand is no longer complex analytic at the enclosed pole. When this pole is simple, the residue theorem applies. By a manipulation of the contour in Eq. 1.107, the pole can be isolated with a circle around the pole, and then taking the limit, the radius may be taken to zero, in the limit, isolating the pole.

For the related Cauchy residue theorem (Eq. 1.108) the same result holds, except it is assumed that there are \( K \) simple poles in the function \( F(s) \). This requires the repeated application of Eq. 1.107, \( K \) times, so it represents a minor extension of Eq. 1.107. The function \( F(s) \) may be written as \( f(s)/P_R(s) \), where \( f(s) \) is analytic in \( C \) and \( P_R(s) \) is a polynomial of degree \( K \), with all of its roots \( s_k \in C \).
1. Cauchy’s (Integral) Theorem: 
\[ \int_{c} F(s)ds = 0, \]
if and only if \( F(s) \) is complex-analytic inside of a simple closed curve \( C \) (Boas, 1987, p. 45). The FTCC (Eq. 1.89) says that the integral only depends on the end points if \( F(s) \) is complex-analytic. By closing the path (contour \( C \)) the end points are the same, thus the integral must be zero, as long as \( F(s) \) is complex analytic.

2. Cauchy’s Integral Formula: 
\[ \frac{1}{2\pi i} \int_{c} \frac{F(s)}{s-s_{o}}ds = \begin{cases} F(s_{o}), & s_{o} \in C \text{ (inside)} \\ 0, & s_{o} \notin C \text{ (outside)} \end{cases} \]

Here \( F(s) \) is required to be analytic everywhere within (and on) the contour \( C \) (Greenberg, 1988, p. 1200). (Boas, 1987, p. 51). The value \( F(s_{o}) \in C \) is the \text{residue} of the pole \( s_{o} \) of \( F(s)/(s-s_{o}) \).

3. (Cauchy) Residue Theorem (Greenberg, 1988, p. 1241). (Boas, 1987, p. 73)
\[ \int_{c} F(s)ds = 2\pi i \sum_{k=1}^{K} c_{k} = 2\pi i \sum_{k=1}^{K} \int_{c} \frac{F(s)}{s-s_{o}}ds, \]
where the residues \( c_{k} \in C \), corresponding to the \( k \)th poles of \( f(s) \), enclosed by the contour \( C \). By the use of Cauchy’s integral formula, the last form of the residue theorem is equivalent to the residue form.\(^{81}\)

How to calculate the residue: The case of first degree polars, while special, is important, since the Brune impedance only allows simple poles and zeros, increasing the utility of this special case. The residues for simple polars are \( F(s_{o}) \), which is complex analytic in the neighborhood of the pole, but not at the pole.

Consider the function \( f(s) = F(s)/(s-s_{o}) \), where we have factored \( f(s) \) to isolate the first-order pole at \( s = s_{o} \), with \( F(s) \) analytic at \( s_{o} \). Then the residue of the poles at \( c_{k} = F(s_{o}) \). This coefficient is computed by removing the singularity, by placing a zero at the pole frequency, and taking the limit as \( s \to s_{o} \), namely
\[ c_{k} = \lim_{s \to s_{o}} [(s-s_{o})F(s)] \]


When the pole is an \( N^{th} \) degree, the procedure is much more complicated, and requires taking \( N-1 \) order derivatives of \( f(s) \), followed by the limit process (Greenberg, 1988, p. 1242). Higher degree polars are rarely encountered: thus it is good to know that this formula exists, but perhaps it is not worth the effort to memorize it.

Summary and examples: These three theorems, all attributed to Cauchy, collectively are related to the fundamental theorems of calculus. Because the names of the three theorems are so similar, they are easily confused.

\(^{81}\)This theorem is the same as a 2D version of Stokes’s thm (citations).
its direction, and fact 2) the real and imaginary parts of the function each obey Laplace’s equation.

Such relationships are known as harmonic functions.88

As we shall see in the next few lectures, complex analytic functions must be smooth since every analytic function may be differentiated an infinite number of times, within the ROC. The magnitude must attain its maximum and minimum on the boundary. For example, when you stretch a rubber sheet over a jagged frame, the height of the rubber sheet obeys Laplace’s equation. Nowhere can the height of the sheet rise above or below its value at the boundary.

Harmonic functions define conservative fields, which means that energy (like a volume or area) is conserved. The work done in moving a mass from $a$ to $b$ in such a field is conserved. If you return the mass from $b$ back to $a$, the energy is retrieved, and zero net work has been done.

1.4.3 Lec 25: Complex Analytic functions and Brune Impedance

It is rarely stated that the variable that we are integrating over, either $x$ (space) or $t$ (time), is real $(x, t \in \mathbb{R})$, since that fact is implicit, due to the physical nature of the formulation of the integral. But this intuition must be refined once complex numbers are included with $s \in \mathbb{C}$, where $s = \sigma + \omega j$.

That time and space are real variables is more than an assumption: it is a requirement, that follows from the real order property. Real numbers have order. For example, if $t = 0$ is now (the present), then $t < 0$ is the past and $t > 0$ is the future. The order property of time and space allows one to order these along a real axis. To have time travel, time and space would need to be complex (they are not), since if the space axis were complex, as in frequency $\omega$, the order property would be invalid. It follows that if we require order, time and space must be real $(x, t \in \mathbb{R})$. Interestingly, it was shown by d’Alembert (1747) that time and space are related by the pure delay due to the wave speed $c_w$. To obtain a solution to the governing wave equation, that d’Alembert first proposed for sound waves, $x, t \in \mathbb{R}^1$ may be combined as functions of

$$\zeta = t \pm x/c_w,$$

where $c$ [m/s] is the phase velocity of the waves. The d’Alembert solution to the wave equation, describing waves on a string under tension, is

$$u(x, t) = F(t - x/c_w) + G(t + x/c_w),$$  \hspace{1cm} (1.97)

which describes the transverse velocity (or displacement) of two independent waves $F(\zeta), G(\zeta)$ on the string, which represent forward and backward traveling waves.69 For example, starting with a string at rest, if one displaces the left end, at $x = 0$, by a step function $u(t)$, then that step displacement will propagate to the right as $u(t - x/c_w)$, arriving at location $x_w$ [m], at time $x_w/c_w$ [s]. Before this time, the string will not move to the right of the wave-front, at $x_o$ [m], and after $t_o$ [s] it will have displacement 1. Since the wave equation obeys superposition (postulate P1, p. 101), it follows that the “plane-wave” eigen-functions of the wave equation for $x_k \in \mathbb{R}^1$ are given by

$$\psi_k(x, t) = \delta(t \pm |k| \cdot x) \leftrightarrow e^{j\omega t + k \cdot x},$$  \hspace{1cm} (1.98)

where $|k| = 2\pi / |\lambda| = \omega / c_w$, is the wave number, $|\lambda|$ is the wavelength, and $s = \sigma + \omega j$.

88When the function is the ratio of two polynomials, as in the cases of the Brune impedance, they are also related to Mobius transformations, also known as bi-harmonic operators.

69D’Alembert’s solution is valid for functions that are not differentiable, such as $\delta(t - c_w x)$.

1.4. STREAM 3A: SCALAR CALCULUS (11 LECTURES)

will not give a residue. But there remains open the problem of generalizing the concept of the Riemann integral theorem, to include $k \in \mathbb{F}$. One way to do this is to use the logarithmic derivative which renders fractional poles to simple poles with fractional residues.

If the singularity had an irrational degree $(k \in \mathbb{R})$, the branch cut has the same “irrational degree.” Accordingly there would be an infinite number of Riemann sheets, as in the case of the log function. An example is $F(s) = e^{-s \log(s)} = e^{-\pi \log(s)} = e^{-\pi \log(\rho) e^{-\pi \theta j}}$

where the domain is expressed in polar coordinates $s = \rho e^{\theta j}$. When $k \in \mathbb{F}$ it may be maximally close (e.g. $\pi/12 = 881$ and $\pi/13 = 883$)67 the branch cut could be very subtle (it could even been unnoticed), but it would have a significant impact on the nature of the function, and of course, on the inverse Laplace transform.

Multivalued functions: The two basic functions we review, to answer the questions about multi-valued functions and branch cuts, are $w(s) = s^\sigma$ and $w(s) = e^s$, along with their inverse functions $w(s) = \sqrt{s}$ and $w(s) = \log(s)$. For uniformity we shall refer to the complex abscissa ($s = \sigma + \omega j$) and the complex ordinate $w(s) = u + v j$. When the complex abscissa and domain are swapped, by taking the inverse of a function, multi-valued functions are a common consequence. For example, $f(t) = \sin(t)$ is single valued, and analytic in $t$, thus has a Taylor series. The inverse function $t(f)$ is not so fortunate as it is multi-valued.

The modern terminology is the domain and range, or alternatively the co-domain.80

Log function: Next we discuss the multi-valued nature of the log function. In this case there are an infinite number of Riemann sheets, not well captured by Fig. 1.22 (p. 97), which only displays the principal sheet. However if we look at the formula for the log function, the nature is easily discerned. The abscissa $s$ may be defined as multi-valued since

$$s_k = r e^{\pi i k l} t / \theta.$$

Here we have extended the angle of $s$ by $2\pi k$, where $k$ is the sheet index $\in \mathbb{Z}$. Taking the log

$$\log(s) = \log(\rho) + (\theta + 2\pi k) j.$$  \hspace{1cm} (1.99)

When $k = 0$ we have the principal value sheet, which is zero when $s = 1$. For any other value of $k$ $w(s) \neq 0$, even when $r = 1$, since the angle is not zero, except for the $k = 0$ sheet.

1.4.5 Lec 27: Three Cauchy Integral Theorems

Cauchy’s theorems for integration in the complex plane

There are three basic definitions related to Cauchy’s integral formula. They are closely related, and can greatly simplify integration in the complex plane.

87Since there are no even primes other than $\pi_1 = 2$, the minimum difference is 2. Out of $10^9$ primes, 5 have a spacing of 80, with a uniform distribution on a log scale.

88The best way to create confusion is to rename something. The confusion grows geometrically with each renaming. I suspect that everyone who cares knows the terms abscissa and ordinate, and some fraction know the equivalent terms domain and range.
Conservation of energy (or power) is a cornerstone of modern physics. It may have first been under-
\[ w(t) = u(t) \star (v(t)) \]
\[ = u(t) \star i(t) \star z(t) \star i(t) \]
\[ = \int_0^t i(t) \left( \int_0^t z(t)(t - \tau) d\tau \right) dt \]
\[ = \int_0^t z(t) \left( \int_0^t i(t)(t - \tau) d\tau \right) dt \]
\[ \leq \int_0^t z(t)|w|^2(t) d\tau \]
\[ + \frac{1}{s} Z(s)|I(\omega)|^2 \geq 0. \]

The step from time to frequency follows from the fact that
\[ |w|^2(t) = i(t) \star i(t) = \int_0^t i(t)(t - \tau) \rightarrow |I(\tau, \omega)|^2 > 0 \]
always has a positive Fourier transform for every possible \( i(t) \).

**Example:** Let \( i(t) = \delta(t) \). Then \( |w|^2(\tau) = i(t) \star i(t) = \delta(\tau) \). Thus
\[ w(t) = \int_0^\infty z(\tau)|w|^2(\tau) d\tau = \int_0^\infty z(\tau)\delta(\tau) d\tau = \int_0^\infty z(\tau) d\tau. \]
The Brune impedance always has the form \( z(t) = r_0(\delta(t) + \zeta(t)) \). The surge impedance is defined as
\[ r_o = \int_0^\infty z(\tau) d\tau. \]
The integral of the reactive part (\( \zeta(t) \)) is always zero, since the reactive part cannot store energy.

Perhaps easier to visualize is when working in the frequency domain where the total energy, equal to the integral of the real part of the power, is
\[ \frac{1}{s} \mathcal{R} V = \frac{1}{2s} \left(V^* I + V I^* \right) = \frac{1}{2s} \left(Z^* I I + Z I I^* \right) = \frac{1}{s} \mathcal{R} Z(s)|I|^2 \geq 0. \]

Formally this is related to a positive definite operator where the positive resistance forces the definiteness, which is sandwiched between the current.

In conclusion conservation of energy is totally dependent on the properties of the impedance. Thus of the most important and obvious applications of complex functions of a complex variable is the impedance function. This seems to be the ultimate example of the FTC, applied to \( z(t) \), in the name of conservation of energy.

**Poles and zeros of PR functions must be first degree:** We conjecture that this proof also requires that the poles and the zeros of the impedance function be simple (only first degree). Second degree poles would have a reactive “secular” response of the form \( h(t) = t \sin(\omega t) + \phi)tu(t) \), and these terms would not average to zero, depending on the phase, as is required of an impedance. As a result, only single degree poles would be possible.\(^{71}\) Furthermore, when the impedance is the ratio of two polynomials, where the lower degree polynomial is the derivative of the higher degree one, then the poles and zeros must alternate. This is a well-known property of the Brune\(^{72}\) poles of degree two since \( w(t) \star u(t) = tu(t) \).

\(^{71}\)Secular terms result from second degree poles since \( u(t) \star u(t) = tu(t) \).

\(^{72}\)This presumes that poles appear in pairs, one of which may be at \( \infty \).
Here we see the mapping for the square root function $f(z) = \sqrt{z}$, which has two single-valued parts. From the causality postulate (P1) of Section 1.1.2, one of the two delivered functions $f(z)$ is selected. The condition on the simple poles is necessary but not sufficient. When $\sigma > 0$, this condition would not meet. Therefore, for the condition to hold, $\sigma$ must be negative.

The concepts of the branch cut, the sheets, and the Riemann surface allow us to interpret these complex functions. The branch cut is a line that separates the various single-valued parts of the function, valid in its local region of convergence. This figure has been taken from Stillwell (2010, p. 303). A more sophisticated version can be found in Boas (1987, Section 29, pp. 221-225).

As an example, a series resistor $R$ and an inductor $L$ in parallel gives an impedance given by (Table F.4, p. 213)

$$Z(s) = \frac{1}{sC}$$

where the poles and zeros might lie in the right half plane, with $\sigma > 0$. The second condition requires that the impedance has simple poles. If there were a pole in the region $\sigma > 0$, then the first condition would not be met. Therefore, there can only be simple poles. Of course, this condition on the simple poles is necessary but not sufficient. When $\sigma > 0$, this condition would not meet. Therefore, for the condition to hold, $\sigma$ must be negative.

The concepts of the branch cut, the sheets, and the Riemann surface allow us to interpret these complex functions. The branch cut is a line that separates the various single-valued parts of the function, valid in its local region of convergence. This figure has been taken from Stillwell (2010, p. 303). A more sophisticated version can be found in Boas (1987, Section 29, pp. 221-225).

As an example, a series resistor $R$ and an inductor $L$ in parallel gives an impedance given by (Table F.4, p. 213)

$$Z(s) = \frac{1}{sC}$$

where the poles and zeros might lie in the right half plane, with $\sigma > 0$. The second condition requires that the impedance has simple poles. If there were a pole in the region $\sigma > 0$, then the first condition would not be met. Therefore, there can only be simple poles. Of course, this condition on the simple poles is necessary but not sufficient. When $\sigma > 0$, this condition would not meet. Therefore, for the condition to hold, $\sigma$ must be negative.

The concepts of the branch cut, the sheets, and the Riemann surface allow us to interpret these complex functions. The branch cut is a line that separates the various single-valued parts of the function, valid in its local region of convergence. This figure has been taken from Stillwell (2010, p. 303). A more sophisticated version can be found in Boas (1987, Section 29, pp. 221-225).

As an example, a series resistor $R$ and an inductor $L$ in parallel gives an impedance given by (Table F.4, p. 213)

$$Z(s) = \frac{1}{sC}$$

where the poles and zeros might lie in the right half plane, with $\sigma > 0$. The second condition requires that the impedance has simple poles. If there were a pole in the region $\sigma > 0$, then the first condition would not be met. Therefore, there can only be simple poles. Of course, this condition on the simple poles is necessary but not sufficient. When $\sigma > 0$, this condition would not meet. Therefore, for the condition to hold, $\sigma$ must be negative.

The concepts of the branch cut, the sheets, and the Riemann surface allow us to interpret these complex functions. The branch cut is a line that separates the various single-valued parts of the function, valid in its local region of convergence. This figure has been taken from Stillwell (2010, p. 303). A more sophisticated version can be found in Boas (1987, Section 29, pp. 221-225).

As an example, a series resistor $R$ and an inductor $L$ in parallel gives an impedance given by (Table F.4, p. 213)

$$Z(s) = \frac{1}{sC}$$

where the poles and zeros might lie in the right half plane, with $\sigma > 0$. The second condition requires that the impedance has simple poles. If there were a pole in the region $\sigma > 0$, then the first condition would not be met. Therefore, there can only be simple poles. Of course, this condition on the simple poles is necessary but not sufficient. When $\sigma > 0$, this condition would not meet. Therefore, for the condition to hold, $\sigma$ must be negative.
Complex analytic functions: To solve a differential equation, or integrate a function, Newton used the Taylor series to integrate one term at a time. However, he only used real functions of a real variable, due to the fundamental lack of appreciation of the complex analytic series. This same method is how one finds solutions to scalar differential equations today, but using an approach that makes the solution method less obvious. Rather than working directly with the Taylor series, today we use the complex exponential, since the complex exponential is an eigen-function of the derivative

$$\frac{d}{dt}e^{s t} = s e^{s t}.$$  

Since $e^{s t}$ may be expressed as a Taylor series, having coefficients $c_k = 1/n!$, in some real sense the modern approach is a compact way of doing what Newton did. Thus every linear constant coefficient differential equation in time may be simply transformed into a polynomial in complex Laplace frequency $s$, by looking for solutions of the form $A(s)e^{s t}$, transforming the differential equation into a polynomial $A(s)$ in complex frequency. For example

$$\frac{d}{dt} f(t) + a f(t) \leftrightarrow (s + a)F(s).$$

The root of $A(s) = s + a$ is the eigen-value of the differential equation. A powerful tool for understanding the solutions of differential equations, both scalar and vector, is to work in the Laplace frequency domain. The Taylor series has been replaced by $e^{s t}$, transforming Newton’s real Taylor series into the complex exponential eigen-function. In some sense, these are the same method, since

$$e^{s t} = \sum_{n=0}^{\infty} \frac{(s t)^n}{n!}.$$  

Taking the derivative with respect to time gives

$$\frac{d}{dt}e^{s t} = s \sum_{n=0}^{\infty} \frac{(s t)^n}{n!},$$

which is also complex analytic. Thus if the series for $F(s)$ is valid (i.e., it converges), then its derivative is also valid. This was a very powerful concept, exploited by Newton for real functions of a real variable, and later by Cauchy and Riemann for complex functions of a complex variable. The key here is “Where does the series fail to converge?” in which case, the entire representation fails. This is the main message behind the FTCC (Eq. 1.89).

The FTCC (Eq. 1.90) is formally the same as the FTC (Eq. 1.88) (Leibniz formula), the key (and significant) difference being that the argument of the integrand $s \in \mathbb{C}$. Thus this integration is a line integral in the complex plane. One would naturally assume that the value of the integral depends on the path of integration.

But, according to FTCC, it does not. In fact they are clearly distinguishable from the FTC. And the reasoning is the same. If $F(s) = \frac{df(s)}{ds}$ is complex analytic (i.e., has a power series $f(s) = \sum c_k s^k$, with $f(s), c_k \in \mathbb{C}$), then it may be integrated, and the integral does not depend on the path. At first blush, this is sort of amazing. The key is that $F(s)$ and $f(s)$ must be complex analytic, which means they are differentiable. This all follows from the Taylor series formula Eq. 1.91 (p. 108) for the coefficients of the complex analytic series. For Eq. 1.89 to hold, the derivatives must be independent of the direction, as discussed in Section 1.4.2. The concept of a complex analytic function therefore has eminent consequences, in the form of several key theorems on complex integration discovered by Cauchy (c1820).

1.4. STREAM 3A: SCALAR CALCULUS (11 LECTURES)

The use of the complex Taylor series generalize the functions it describes, with unpredictable consequences, as nicely shown by the domain coloring diagrams presented in Section 1.3.12 (p. 96). Cauchy’s tools of complex integration were first exploited in physics by Sommerfeld (1952), to explain the onset transients in waves, as explained in detail in Brillouin (1960, Chap. 3).

Up to 1910, when Sommerfeld first published his results using complex analytic signals and saddle point integration in the complex plane, there was a poor understanding of the implications of the causal wave-front. It would be reasonable to say that his insights changed our understanding of wave propagation, for both light and sound. Sadly this insight has not been fully appreciated, even to this day. If you question my summary, please read Brillouin (1960, Chap. 1).

The full power of the complex analytic function was first appreciated by Bernard Riemann (1826–1866), in his University of Göttingen PhD Thesis of 1851, under the tutelage of Carl Friedrich Gauss (1777–1855), and drawing heavily on the work of Cauchy.

The key definition of a complex analytic function is that it has a Taylor series representation over a region of the complex frequency plane $s = a + j\omega$, that converges in a region of convergence (RoC) about the expansion point, with a radius determined by the nearest pole of the function. A further surprising feature of all analytic functions is that within the RoC, the inverse of that function also has a complex analytic expansion. Thus given $w(s)$, one may also determine $s(w)$ to any desired accuracy, critically depending on the RoC.

1.4.4 Lec 26: Multi-valued functions

In the field of mathematics there seems to have been a tug-of-war regarding the basic definition of the concept of function. The accepted definition today seems to be a single-valued mapping from the domain to the codomain (or range). This makes the discussion of multi-valued functions somewhat tedious. In 1851 Riemann (working with Gauss) seems to have resolved this problem for the natural set of multi-valued functions by introducing the concept of the branch-cut and sheets.

Two simple examples of multi-valued functions are the circle $z^2 = x^2 + y^2$ and $w = \log(z)$. For example, assume $z$ is the radius of the circle, solving for $y(x)$ gives the multi-valued function

$$y(x) = \pm \sqrt{z^2 - x^2}.$$  

If we accept the modern definition of a function as the mapping from one set to a second, then $y(x)$ is not a function, or even two functions. For example, what if $x > z$? Or worse, what if $z = 2j$ with $|x| < 1$? Riemann’s construction, using branch cuts for multi-valued function, resolves all these difficulties (as best I know).

To proceed we need definitions and classifications of the various types of complex singularities:

1. Poles of degree 1 are called simple poles. Their amplitude called the residue (e.g. $\alpha/s$ has residue $\alpha$). Simple poles are special (Eq. 1.108, p. 120)7 and play a key role in mathematical physics. Consider the function $y(s) = \sqrt{z^2 - s^2}$ with $s \in \mathbb{C}$.

2. When the numerator and denominator of a rational function have a common root (i.e., factor), that root is said to be removable.

3. A singularity that is not 1) removable, 2) a pole or 3) a branch point, is called essential.

7https://en.wikipedia.org/wiki/Pole_(complex_analysis)