Chapter 5
Stream 3B: Vector Calculus

5.1 Properties of fields and potentials
Before we can define the vector operations \( \nabla \), \( \nabla \cdot \), \( \nabla \times \), and \( \nabla^2 \), we must define the objects they operate on: scalar and vector fields.

The word field has two very different meanings: a mathematical one, which defines an algebraic structure, and a physical one, discussed next.

Ultimately we wish to integrate in \( \mathbb{R}^3 \), \( \mathbb{R}^n \) and \( \mathbb{C}^n \). Integration is quantified by several fundamental theorems of calculus, each about integration (see pp. 179–181).

5.1.1 Scalar and vector fields: notation

Scalar fields:
We use the term scalar field interchangeably with analytic in a connected region of the spatial vector \( \mathbf{x} = [x, y, z]^T \in \mathbb{R}^3 \). In mathematics, functions that are piecewise differentiable are called smooth, which is different from analytic. A smooth function has at least one or more derivatives. Every analytic function is single-valued and is an infinitely differentiable power series (Sec. 3.2.4, p. 100).
Vector fields: A vector field is composed of three scalar fields. For example, the electric field used in Maxwell’s equations, \( \mathbf{E}(x, t) = [E_x, E_y, E_z]^T \) [V/m], has three components, each a scalar field. When the magnetic flux vector \( \mathbf{B}(x) \) is static (Postulate P5, p. 163), the potential \( \phi(x) \) [V] uniquely defines \( \mathbf{E}(x, t) \) via the gradient,
\[
\mathbf{E}(x, t) = -\nabla \phi(x, t) \quad [\text{V/m}].
\] (5.1.1)
The electric force on a charge \( q \) is \( \mathbf{F} = q \mathbf{E} \); thus \( \mathbf{E} \) is proportional to the force, and when the medium is conductive, the current density (a flow) is \( \mathbf{J} = \sigma \mathbf{E} \) [A/m²]. The ratio of the potential to the flow is an impedance, so \( \sigma \) is a conductance.

Example: Suppose we are given the vector field in \( \mathbb{R}^3 \)
\[
\mathbf{a}(x) = [\phi(x), \psi(x), \theta(x)]^T \quad [\text{Wh/m}],
\] where each of the three functions is a scalar field. Then \( \mathbf{a}(x) = [x, y, z]^T \) is a legal vector field that has components analytic in \( x \).

Example: From Maxwell’s equations, the magnetic flux vector is given by
\[
\mathbf{b}(x, t) = \nabla \times \mathbf{a}(x, t) \quad [\text{Wb/m²}].
\] (5.1.2)
We shall see that this is always true because the magnetic charge \( \nabla \cdot \mathbf{b}(x, t) \) must be 0, which is always true in-vacuo. Feynman (1970b, pp. 14-1 to 14-3) provides an extended and helpful tutorial on the vector potential, with many examples.

To verify that a field is a potential, we may check the units. However, a proper mathematical definition is that the potential must be an analytic function of \( x \) and \( t \), so that we can operate on it with \( \nabla(\cdot) \) and \( \nabla \times (\cdot) \). Note that the divergence of a scalar field is not a legal vector operation.

Scalar potentials: The above discussion describes the utility of potentials for defining vector fields (e.g., Eqs. 5.1.1 and 5.1.2). The key distinction between a potential and a scalar field is that potentials have units and thus have a physical meaning. Scalar potentials (i.e., voltage \( \phi(x, t) \) [V], temperature \( T(x, t) \) [°C], and pressure \( \psi(x, t) \) [pascals]) are physical scalar fields. All potentials are composed of scalar fields, but not all scalar fields are potentials.

For example, the \( y \) component of \( \mathbf{E} \), \( E_y(x, t) \), is not a potential. While \( \mathbf{E}_y \) is mathematically defined as the gradient of one component of a vector field, it has no physical meaning (as best I know).

Vector potentials: Vector potentials, like scalar potentials, are vector fields with physically meaningful units. They are more complicated than scalar potentials because they are composed of three scalar fields. Vector fields are composed of laminar and rotational flow, which are mathematically described by the fundamental theorem of vector calculus (also called Helmholtz’s decomposition theorem). One superficial but helpful comparison is the momentum of a mass, which may be decomposed into its forward (linear) and rotational momentum.

Since we find it is helpful to analyze problems using potentials (e.g., voltage) and then take the gradient (i.e., voltage difference) to find the flow (i.e., current density \( j = \sigma \mathbf{E}(x, t) \) [A/m²]), where \( \sigma \) [A/m²] is the conductivity, the same logic and utility apply when we use the vector potential to describe the magnetic flux (flow) \( \mathbf{b}(x, t) \) (Feynman, 1970c). When operating on a scalar potential, we use a gradient, whereas for the vector potential, we operate with the curl (Eq. 5.1.2).
In Eq. 5.1.1 we assumed that the magnetic flux vector $\mathbf{b}(x)$ was static, and thus $\mathbf{e}(x, t)$ is the gradient of the time-dependent voltage $\phi(x, t)$. However, when the magnetic field is dynamic (not static), Eq. 5.1.1 is not valid, due to magnetic induction: A voltage induced into a loop of wire is proportional to the time-varying flux cutting across that loop of wire. This is known as the Ampere-Maxwell law. In the static case the induced voltage is zero.

In summary, the electric field strength includes both scalar potential $\phi(x, t)$ and magnetic flux vector potential $\mathbf{a}(x, t)$ components, while the magnetic field strength depends only on the magnetic potential.

5.1.2 Gradient $\nabla$, divergence $\nabla \cdot$, curl $\nabla \times$, and Laplacian $\nabla^2$

Three key vector differential operators are used in linear partial differential equations, such as the wave and diffusion equations. All of these begin with the $\nabla$ operator:

$$\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}.$$ 

As outlined in Table 5.1, the official name of this operator is nabla. It has three basic uses:

1. the gradient of a scalar field, 
2. the divergence of a vector field, and
3. the curl of a vector field. The shorthand notation $\nabla \phi(x, t) = (\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z})\phi(x, t)$ is convenient. 1

<table>
<thead>
<tr>
<th>Name</th>
<th>Input</th>
<th>Output</th>
<th>Operator</th>
<th>Mnemonic</th>
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<tbody>
<tr>
<td>Gradient</td>
<td>Scalar</td>
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<td>Divergence</td>
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<td>$\nabla \cdot (\nabla \times)$</td>
<td>$\text{DoC}$</td>
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<td>Curl of Grad</td>
<td>Scalar</td>
<td>0</td>
<td>$\nabla \times \nabla$</td>
<td>$\text{CoG}$</td>
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<td>Curl of Wedge</td>
<td>Vector</td>
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Gradient \( \nabla() \):  
As shown in Fig. 5.1, the gradient transforms a complex scalar field \( \Phi(x, s) \in \mathbb{C} \) into a vector field (\( \mathbb{C}^3 \))  
\[
\nabla \Phi(x, s) = \left( \frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z} \right) \Phi(x, s)
\]
\[
= \frac{\partial \Phi}{\partial x} \hat{x} + \frac{\partial \Phi}{\partial y} \hat{y} + \frac{\partial \Phi}{\partial z} \hat{z}.
\]

The gradient may also be factored into a unit vector \( \hat{n} \), as defined in Fig. 5.1, that gives the direction of the gradient, and the gradient’s length \( ||\nabla \Phi|| \), defined in terms of the norm of the gradient. Thus the gradient of \( \Phi(x) \) may be written in “polar coordinates” as \( \nabla \Phi(x) = ||\nabla \Phi|| \hat{n} \), which leads to the unit vector  
\[
\hat{n} = \frac{\nabla \Phi(x)}{||\nabla \Phi||}.
\]

Consider the paraboloid \( z = 1 - (x^2 + y^2) \) as the potential, with isopotential circles of constant \( z \) that have a radius of zero at \( z = 1 \) and unit radius at \( z = 0 \). The negative gradient  
\[
e(x) = -\nabla z(x, y) = 2(x \hat{x} + y \hat{y} + 0 \hat{z})
\]
is \( \perp \) to the circles of constant radius (constant \( z \)) and thus points in the direction of the radius.

A skier in free fall on this surface would be the first one down the hill. Normally skiers try to stay close to the isoclines (not in the direction of the gradient) so they can stay in control. If you ski on an isocline, you must walk, since there is no pull due to gravity. The gravitational potential at the surface of the earth is  
\[
\phi = \frac{GM}{r},
\]

Divergence \( \nabla(\cdot) \):

The divergence of a vector field results in a scalar field. For example, the divergence of the electric field flux vector \( d(x) \) [C/m²] equals the scalar field charge density \( \rho(x) \) [C/m³]:  
\[
\nabla \cdot d(x) = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot d(x) = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho(x).
\]

Thus the divergence is analogous to the scalar (dot) product (e.g., \( \mathbf{a} \cdot \mathbf{b} \)) between two vectors. 

Recall that the voltage is the line integral of the electric field,  
\[
V(a) - V(b) = \int_a^b \mathbf{E}(x) \cdot dx = -\int_a^b \nabla V(x) \cdot dx = -\int_a^b \frac{dV}{dx} dx.
\]
\[ \frac{\partial}{\partial x} \mu \cdot \mathbf{u} + \nabla \cdot \mathbf{f} = 0 \]

which is simply the fundamental theorem of calculus (see p. 179). In a conservative system, the work done by the system is independent of the path taken in reaching the final state. In a conservative system, the integral is independent of the path from the initial to the final state. In a conservative system, the integral is independent of the path from the initial to the final state.

Properties of the divergence:

\[ \nabla \cdot \mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \]

The divergence is a direct measure of the source or sink of a vector field. A vector field is said to be incompressible if the divergence of that field is zero. It is said to be incompressible. A vector field is said to be incompressible. A vector field is said to be incompressible. A vector field is said to be incompressible.

\[ \nabla \times \mathbf{u} = \left( \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) \mathbf{i} + \left( \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) \mathbf{j} + \left( \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \mathbf{k} \]

The curl of a vector field is a measure of the rotation of the vector field in a plane about the axis perpendicular to that plane. In the case of fluids, the vorticity is defined as the curl of the velocity field. The curl is a measure of the rotation of a vector field in a plane about the axis perpendicular to that plane. In the case of fluids, the vorticity is defined as the curl of the velocity field. The curl is a measure of the rotation of a vector field in a plane about the axis perpendicular to that plane. In the case of fluids, the vorticity is defined as the curl of the velocity field. The curl is a measure of the rotation of a vector field in a plane about the axis perpendicular to that plane. In the case of fluids, the vorticity is defined as the curl of the velocity field.
Exercise #1
If we let \( h = -y\hat{x} + x\hat{y} + 0\hat{z}, \) \( \nabla \times h = 2\hat{z}, \) then \( h \) has a constant rotation; when \( h = 0\hat{x} + 0\hat{y} + z\hat{z}, \) \( \nabla \times h = 0 \) has a curl of zero and thus is irrotational.

There are simple rules that precisely govern when a vector field is rotational versus irrotational, and compressible versus incompressible. These classifications are dictated by Helmholtz’s theorem, the fundamental theorem of vector calculus (Eq. 5.6.11, p. 261).

Wedge \( \nabla \wedge (\cdot) \)
A special case of the curl is the two-dimensional differential wedge products
\[
\nabla \times h(x, t) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} - C_s(x, s) \quad [\text{A/m}^2].
\]

The curl is made up of three such differential wedge products.\(^3\)

Laplacian \( \nabla^2 \)
The Laplacian operator \( \nabla^2 = \nabla \cdot \nabla \) (Table 5.1, page 225) is defined as the divergence of the gradient
\[
\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \quad (5.1.7)
\]

Since the Laplacian does so much common work, we nickname it DoG (Div of Grad).

Starting from a scalar field, the gradient produces a vector, which is then operated on by the divergence to take the output of the gradient back to a scalar field. Thus the Laplacian transforms a scalar field back to a scalar field. We have seen the Laplacian before when we defined complex-analytic functions (Eq. 4.2.8, p. 182).

A classic example of the Laplacian is a voltage scalar field \( \Phi(x) \) [V], which results in the electric field vector
\[
e(x) = [E_x(x), E_y(x), E_z(x)]^T = -\nabla \Phi(x) \quad \text{(V/m)}.
\]

When this is scaled by the permittivity, we obtain the electric flux \( \mathbf{d} = \varepsilon_0 \mathbf{e} \) [C/m\(^2\)], the charge density per unit area. Here \( \varepsilon_0 \) [F/m] is the vacuum permittivity \( \varepsilon_0 = \frac{1}{\varepsilon_r c_0} \approx 8.85 \times 10^{-12} \) [F/m].

Taking the divergence of \( \mathbf{d} \) results in the charge density \( \mathbf{\rho}(x) \) [C/m\(^2\)] at \( x \):
\[
\nabla \cdot \mathbf{d} = \nabla \cdot \mathbf{\rho}(x) = \mathbf{\rho}(x).
\]

Thus the Laplacian of the voltage, scaled by \( \varepsilon_0 \), results in the local charge density.

Another classic example of the Laplacian is an acoustic pressure field \( \rho(x, t) \) [Pa], which defines a vector force density \( \mathbf{f}(x, t) = -\nabla \rho(x, t) \) [N/m\(^2\)] (Eq. 5.2.5, p. 239). When this force density [N/m\(^2\)] is integrated over an area, the net radial force \( [\text{N}] \) is
\[
F_r = -\int_S \mathbf{\rho}(x) dx \quad [\text{N}]. \quad (5.1.8)
\]

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\(^1\)https://en.wikipedia.org/wiki/Triple_product#A_x_an_exterior_product

\(^2\)This notation suggests that \( ||\nabla \times \mathbf{e} + \mathbf{j} \wedge \mathbf{e}||^2 = ||\nabla \mathbf{e}||^2 + ||\mathbf{j} \wedge \mathbf{e}||^2 \), which is related to Helmholtz’s theorem.
\[ \nabla^2 \rho = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \rho}{\partial r} \right) \]

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An inflated balloon with a static internal pressure of 3 \( \text{atm} \), in an ambient pressure of 1 \( \text{atm} \), is a step function of the radius \( u > r \).

The net normal force on the surface of the balloon is its area times the pressure drop across the balloon.

Finally, taking the divergence of the force produces a double delta function at the balloon's surface:

\[ \nabla \cdot \mathbf{F} = 4 \delta(r) \delta(u-r) \]

Thus, the force is pointing out, stretching the balloon.

A second form of the Laplacian is the vector Laplacian:

\[ \nabla^2 \mathbf{f} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \mathbf{f}}{\partial r} \right) \]

For each value of \( r \), \( \mathbf{f} \) is a function of \( u \).

This result will turn out to be useful when dealing with the properties of fields and potentials.
resulting in the general d’Alembert solutions (Eq. 4.4.1 p. 188) for the spherical wave equation,
\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial P}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial P}{\partial \theta} \right) = \frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} \]
where \( s = r/c \).

**Exercise 82**
Prove the result of the previous example by expanding Eqs. 5.1.10 and 5.1.11 using the chain rule.

**Solution:** Expanding Eq. 5.1.10:
\[ \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial P}{\partial \theta} \right) + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial P}{\partial r} \right) = \frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} \]
Expanding Eq. 5.1.11, we obtain
\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial P}{\partial r} \right) = \frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} \]
Thus the two are equivalent.

**Summary**
The radial component of the Laplacian in spherical coordinates (Eq. 5.1.10) simplifies to
\[ \nabla^2 \psi(r) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} \]
Since DoG is \( \nabla^2 = \nabla \cdot \nabla \), it follows that the net force \( F(x) = |F_r, 0, 0|^2 \), Eq. 5.1.8 in spherical coordinates has a radial component \( F_r \) and angular component of zero. Thus the force across a balloon may be approximated by a delta function across the thin sheet of stretched rubber.

We can extended the preceding example in an interesting way to the case of a rigid hose, a rigid tube, that terminates on the right in an elastic medium (the above example of a balloon), such as an automobile tire. On the far left let’s assume there is a pump injecting the fluid into the rigid hose. Consider two different fluids: air and water. Air is treated as a compressible fluid, whereas water is incompressible. However, such a classification is relative, determined by the relative compliance of the balloon (i.e., tire) at the relatively rigid pump and hose.

This is a special case of a more general situation: When a fluid is treated as incompressible (rigid), the speed of sound becomes infinite, and the wave equation is an invalid description. In this case the motion is best approximated by Laplace’s equation. This represents the transition from short to long wavelengths, from wave propagation having delay, to quasistatics, having no delay.

This example may be modeled as either an electrical or a mechanical system. While the two systems are very different in their physical realization, they are mathematically equivalent, forming a perfect analog. If we take the electrical analog, the pump is a current source, injecting...
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charge $Q_o$ into the hose, which being rigid cannot expand (has a fixed volume). The hose, which is the tube, acts as a capacitor $C$, which fills with charge as it is connected to the source. The force applied by the source to the hose is given by $F = \frac{Q_o}{C} \frac{dQ_o}{dt}$, where $\frac{dQ_o}{dt}$ is the change in the charge of the hose. The force is proportional to the change in the charge, and is the change in the capacitance, which is the change in the charge from the hose to the tube.

The flow of the hose is called the flux, which is the general rate of the mass flow. The flux is given by $\Phi = \frac{Q_o}{C} \frac{dQ_o}{dt}$, where $\Phi$ is the flux, $Q_o$ is the charge, and $C$ is the capacitance. The flux is proportional to the change in the charge, and is the change in the capacitance, which is the change in the charge from the hose to the tube.

In thermodynamics the thermal compliance is $C = \frac{1}{K}$, where $K$ is the thermal resistance, and is the change in the temperature with the change in the temperature. The thermal compliance is a measure of the change in the temperature with the change in the temperature, and is the change in the temperature from the initial temperature to the final temperature.

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Cauchy’s residue theorem (p. 201) gives the final answer, which describes how the process of blowing up a balloon, charging a capacitor, or heating water on a stove, the difference being only the physical notation, as the mathematics is identical.

Note that this differential equation is first-order in time, which in frequency means the impedance has a single pole. Thus the equation for charging a capacitor or pumping up a balloon describes a diffusion process. If we had taken the impedance of the mass of the fluid in the hose into account, we would have a lumped-parameter model of the wave equation with a second-order system. This is mathematically related to the homework assignment about train cars (masses) connected by springs (Fig. 5.4, Homework DE-3, problem 2).

Example: The voltage
\[ v(t) = e^{-\alpha t} u(t - \tau) + \frac{1}{\alpha} e^{-\alpha t} \]  
represents one of d’Alembert’s solution (Eq. 4.4.1, p. 188) of the wave equation (Eq. 3.1.5, p. 75) as well as an eigenfunction of the gradient operator \( \nabla \). From the definition of the scalar (dot) product of two vectors (Fig. 3.4, p. 119),
\[ \mathbf{k} \cdot \mathbf{x} = k_x x_1 + k_y y_1 + k_z z_1 = ||\mathbf{k}|| ||\mathbf{x}|| \cos \theta_{\mathbf{k}x}, \]
where \( ||\mathbf{k}|| = \sqrt{k_1^2 + k_y^2 + k_z^2} \) and \( ||\mathbf{x}|| = \sqrt{x_1^2 + y_1^2 + z_1^2} \) are the lengths of vectors \( \mathbf{k} \) and \( \mathbf{x} \), and \( \theta_{\mathbf{k}x} \) is the angle between them. As before, \( s = \sigma + \omega \) is the Laplace frequency.

To keep things simple, we let \( \mathbf{k} = [k_x, 0, 0]^T \) so that \( \mathbf{k} \cdot \mathbf{x} = k_x x_1 \). We shall soon see that \( ||\mathbf{k}|| = 2\pi/\lambda \) follows from the basic relationship between a wave’s radian frequency \( \omega = 2\pi f \) and its wavelength \( \lambda \):
\[ \omega \lambda = c. \]

As the frequency increases, the wavelength becomes shorter. This key relationship may have been first researched by Galileo in about 1564, followed by Mersenne in about 1627 (Fig. 1.1, p. 8).

Exercise 83
Show that Eq. 5.1.14 is an eigenfunction of the gradient operator \( \nabla \).

Solution: Taking the gradient of \( \phi(x,t) \) gives
\[ \nabla e^{-\alpha t} u(t - \tau) = -\mathbf{k} \cdot \mathbf{x} e^{-\alpha t} u(t) \]
or in terms of \( \phi(x,t) \),
\[ \nabla \phi(x,t) = -\mathbf{k} \phi(x,t) e^{-\alpha t} \]
Thus \( \phi(x,t) \) is an eigenfunction of \( \nabla \), having the vector eigenvalue \( \mathbf{k} \). As before, \( \nabla \phi \) is proportional to the current, since \( \phi \) is a voltage, and the ratio (i.e., the eigenvalue) may be thought of as a mass, analogous to the impedance of a mass (or inductor). In general, the units provide the physical interpretation of the eigenvalues and their spectra. \[ \text{See [https://www-history.mcs.st-and.ac.uk/PrintBiographies/Mersenne.html](https://www-history.mcs.st-and.ac.uk/PrintBiographies/Mersenne.html).} \]

\[ 83 \text{ See the early [1920s, Mersenne listed Galileo among the innovators in natural philosophy whose views should be rejected.} \]
\[ \text{However, by the early 1630s, less than a decade later, Mersenne had become one of Galileo’s most ardent supporters.} \]

Garber, 2004

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5.1. PROPERTIES OF FIELDS AND POTENTIALS

Exercise #4
Compute \( \hat{n} \) for \( \phi(x, s) \) as given by Eq. 5.1.14.

Solution: \( \hat{n} \) is the gradient of the potential at the points in question, defining a virtual force on the ions, resulting in an ionic current.

The gradient of the pressure gives rise to a force density in the \( x \) direction, as described by Ohm's law.

The driving-point impedance is typically a generalized force (electric field, acoustic force density, temperature field, the gradient of which defines a virtual force on the ions, resulting in an ionic current.)

The ratio of the potential flow (current, velocity, heat flux, entropy flow) between two points is called the driving-point impedance. Four examples are:

- As before, this results in a vector force on the ions.
- Note that the scalar fields (e.g., temperature, pressure, voltage) are all potential functions, summarized in Table 3.2 (p. 149). In each case the gradient of the potential defines a force density.
- Forward transformation of the pressure at a point gives rise to a force density at that point.
- Each potential function is given by a coordinate function, which is typically a generalized force (electric field, acoustic force density, temperature field, the gradient of which defines a virtual force on the ions, resulting in an ionic current.)
Thus in the above examples there is a potential, the gradient of which is a force, that when applied to the medium (an impedance) causes a flow (flux or current) proportional to that impedance due to the medium, which is the ratio of the gradient of the potential to the current. The product of the force and flow is a power.

Exercise 5.7

Show that the integral of Eq. 5.1.1 is an antiderivative.

Solution: We use the definition of the antiderivative given by the FTC (Eq. 4.2.2, p. 180):

\[
\phi(x, t) - \phi(x_0, t) = \int_{x_0}^{x} \mathbf{e}(x, t) \cdot dx
\]

\[
= \int_{x_0}^{x} \nabla \phi(x, t) \cdot dx
\]

\[
= -\iint_{\Gamma_{\text{closed}}} \mathbf{g} \cdot \mathbf{n} ds.
\]

If we apply the FTC, the antiderivative must be \( \phi(x, t) - E_0 z \mathbf{k} + \phi_0 \mathbf{k} \). This same point is made by Feynman (1970b, p. 4-1, Eq. 4.28).

Given that the force on a charge is proportional to the gradient of the potential, this exercise shows that the integral of the gradient depends on only the end points, the work done in moving a charge depends on only the limits of the integral, which is the definition of a conservative field but holds only in the ideal case where \( \mathbf{e} \) is determined by Eq. 5.1.1—that is, the medium has no friction (there are no other forces on the charge).

The conservative field: An important question is: When is a field conservative? A field is conservative when the work done by the motion is independent of the path of the motion. Thus the conservative field is related to the FTC, which states that the integral of the work depends on only the end points.

A more complete answer must await the introduction of the fundamental theorem of vector calculus (Eq. 5.6.11, p. 261). A few examples provide insight:

Example: The gradient of a scalar potential, such as the voltage (Eq. 5.1.1), defines the electric field, which drives a current (flow) across a resistor (impedance). When the impedance is infinite, the flow is zero, leading to zero power dissipation. When the impedance is lossless, the system is conservative.
Example: At audible frequencies the viscosity of air is quite small and thus, for simplicity, may be taken as zero. However, when the wavelength is small (e.g., at 100 [kHz]) $\lambda \approx \frac{c}{f}$, in which $c$ is the sound speed, the viscous force plays a large role and the sound is no longer conservative. In narrow tubes, for example, a flute, thermal losses play a much larger role due to the wall (Appendix D).

Exercise #8: Find the time that it takes to fall from a distance $h$.

Solution: Given Galileo's potential $\phi = \frac{GMm}{r}$, the gravitational constant is

$$
\phi(\mathbf{r}) = \frac{GMm}{r}
$$

where $m$ is the object's mass and $G$ is the gravitational constant.

Show that $h(t) = \frac{GMm}{2}$. This formula applies if you know the viscous force is $f_{\text{viscous}} = \mu \frac{dv}{dt}$ and the acceleration is $a = \frac{dv}{dt}$.

Example: In the adiabatic process, the temperature of a gas changes reversibly without heat exchange. Then, $dQ = 0$, $dW = 0$, and $dE = 0$. Therefore, $dE = -dU$ and $dE = -dU = \int_{V_i}^{V_f} P dV$, where $P$ is the pressure and $V$ is the volume. Thus, the work done by the gas is

$$
W = \int_{V_i}^{V_f} P dV
$$

which is the change in internal energy $\Delta U = U_f - U_i$.

Example: The force of gravity is given by the gradient of Newton's gravitational potential.

$$
F = \nabla \phi = -G \frac{Mm}{r^2} \hat{r}
$$

where $G$ is Newton's gravitational constant, $M$ and $m$ are the masses, and $r$ is the distance between the masses.

Solution: The force is directed towards the center of mass and has magnitude $F = \frac{GMm}{r^2}$.

Exercise #9: Find the time that it takes to fall from a distance $L$.

Solution: Setting $t = 0$ gives $\frac{1}{2} \frac{d^2 s}{dt^2} = g \frac{ds}{dt}$, where $s$ is the distance.

Thus the time to fall is $T(L) = \sqrt{\frac{2L}{g}}$.

Example: The force of gravity is given by the gradient of Newton's gravitational potential.

$$
F = \nabla \phi = -G \frac{Mm}{r^2} \hat{r}
$$

where $G$ is Newton's gravitational constant, $M$ and $m$ are the masses, and $r$ is the distance between the masses.

Solution: The force is directed towards the center of mass and has magnitude $F = \frac{GMm}{r^2}$.

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where $G$ is Newton's gravitational constant, $M$ and $m$ are the masses, and $r$ is the distance between the masses.

Solution: The force is directed towards the center of mass and has magnitude $F = \frac{GMm}{r^2}$.

Exercise #10: Show that the force of gravity is given by the gradient of Newton's gravitational potential.

Solution: The force is directed towards the center of mass and has magnitude $F = \frac{GMm}{r^2}$.

Example: The force of gravity is given by the gradient of Newton's gravitational potential.

$$
F = \nabla \phi = -G \frac{Mm}{r^2} \hat{r}
$$

where $G$ is Newton's gravitational constant, $M$ and $m$ are the masses, and $r$ is the distance between the masses.

Solution: The force is directed towards the center of mass and has magnitude $F = \frac{GMm}{r^2}$.

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Example: The force of gravity is given by the gradient of Newton's gravitational potential.

$$
F = \nabla \phi = -G \frac{Mm}{r^2} \hat{r}
$$

where $G$ is Newton's gravitational constant, $M$ and $m$ are the masses, and $r$ is the distance between the masses.

Solution: The force is directed towards the center of mass and has magnitude $F = \frac{GMm}{r^2}$.

Exercise #11: Show that the force of gravity is given by the gradient of Newton's gravitational potential.

Solution: The force is directed towards the center of mass and has magnitude $F = \frac{GMm}{r^2}$.

Example: The force of gravity is given by the gradient of Newton's gravitational potential.

$$
F = \nabla \phi = -G \frac{Mm}{r^2} \hat{r}
$$

where $G$ is Newton's gravitational constant, $M$ and $m$ are the masses, and $r$ is the distance between the masses.

Solution: The force is directed towards the center of mass and has magnitude $F = \frac{GMm}{r^2}$.

Example: The force of gravity is given by the gradient of Newton's gravitational potential.

$$
F = \nabla \phi = -G \frac{Mm}{r^2} \hat{r}
$$

where $G$ is Newton's gravitational constant, $M$ and $m$ are the masses, and $r$ is the distance between the masses.

Solution: The force is directed towards the center of mass and has magnitude $F = \frac{GMm}{r^2}$.

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where $G$ is Newton's gravitational constant, $M$ and $m$ are the masses, and $r$ is the distance between the masses.

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$$

where $G$ is Newton's gravitational constant, $M$ and $m$ are the masses, and $r$ is the distance between the masses.

Solution: The force is directed towards the center of mass and has magnitude $F = \frac{GMm}{r^2}$.
5.2 Partial differential equations and field evolution

The three main classes of partial differential equations (PDEs) are: elliptic, parabolic, and hyperbolic, distinguished by the order of the time derivative. These categories seem to have little mathematical utility (the categories are labels).

5.2.1 The Laplacian $\nabla^2$:

In the most important case the space operator is the Laplacian $\nabla^2$, the definition of which depends on the dimensionality of the waves—that is, the coordinate system being used. We first discussed the Laplacian as a 2D operator on p. 182 where we studied complex-analytic functions, and again on p. 225. An expression for $\nabla^2$ for one, two, and three-dimensions was provided as Eq. 5.1.9 (p. 229). In three dimensional rectangular coordinates, it is defined as (see p. 228)

$$\nabla^2 T(x) = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) T(x)$$  \hspace{1cm} (5.2.1)

The Laplacian operator is ubiquitous in mathematical physics, starting with simple complex-analytic functions (Laplace’s equation) and progressing to Poisson’s equation, the diffusion equation, and finally the wave equation. Only the wave equation results in a delay. The diffusion equation “wave” has an instantaneous spread (the effective “wavefront” velocity is infinite, yet the wavelength is long; it’s not a traveling wave).

Examples of elliptic, parabolic, and hyperbolic equations are

1. Laplace’s equation: The equation

$$\nabla^2 \Phi(x) = 0$$  \hspace{1cm} (5.2.2)

describes, for example, the voltage inside a closed chamber that has a given voltage on the walls or the steady-state temperature within a closed container having a specified temperature distribution on the walls. There are no dynamics to the potential, even when it is changing, since the potential instantaneously follows the potential on the walls.

2. Poisson’s equation: In the steady state, the diffusion equation degenerates to either Poisson’s or Laplace’s equation; both are classified as elliptic equations (second-order in space, zero-order in time). As in the diffusion equation, the evolution has a wave velocity that is functionally infinite. For example,

$$\nabla^2 \Phi(x, t) = \rho(x, t)$$

holds for gravitational fields or the voltage around a charge. It does not describe gravity waves, which travel at the speed of light.

3. Fourier diffusion equation: Equation 5.2.3 describes the evolution of the scalar temperature $T(x, t)$ (a scalar potential), gradients of solution concentrations (i.e., ink in water), and Brownian motion. Diffusion is first-order in time, which is categorized as parabolic (first-order in time, second-order in space). When these equations are Laplace transformed, diffusion has a single real root, resulting in a real solution (e.g., $T \in \mathbb{R}$). There is no wavefront in the case of the diffusion equation. As soon as the source is turned on, the field is nonzero at every point in the bounded container. As an example,

$$\nabla^2 T(x, t) = \kappa_o \frac{\partial T(x, t)}{\partial t} \rightarrow \kappa_o T(x, s)$$  \hspace{1cm} (5.2.3)
describes the temperature $T(x, t) \rightarrow T(x, \omega)$, as proposed by Fourier in 1822, or the $T(x, t)$ field is static, such systems are lossless and thus are analogous. The fluid (the electrical capacitor) is being heated (charged) by the postulate $P3$, p. 163.

When the temperature reaches a steady state, $\nabla_\omega \rightarrow \infty = 0$ the temperature wavefront propagates instantaneously. The diffusion equation does a poor job of representing the velocity of diffusion of two miscible liquids (Fick, 1855) or Brownian motion (Einstein, 1905). The diffusion equation is not a wave equation, since the temperature wavefront propagates infinitely fast. The diffusion equation is a scalar version of the heat equation, with a frequency-dependent delay that depends on the thermal conductivity of the material.

There are scalar and vector forms of wave equations. Wave equations:

\[ \rho \frac{\partial^2 u}{\partial t^2} - \nabla \cdot (\kappa \nabla u) = 0 \]

is the wave equation for a scalar field $u$, where $\rho$ is the density and $\kappa$ is the thermal conductivity. The wave equation is second-order in time. When transformed into the frequency domain, the solution has pairs of complex conjugate roots, leading to two real solutions. The wave equation is classified as hyperbolic, since it describes the propagation of waves in space and time.

When the boundary conditions are time-dependent, the wave equation becomes a partial differential equation in time and space:

\[ \rho \frac{\partial^2 u}{\partial t^2} - \nabla \cdot (\kappa \nabla u) = f(t, x) \]

is the wave equation with a source term $f(t, x)$.

5.2. PARTIAL DIFFERENTIAL EQUATIONS AND FIELD EVOLUTION


heat (charge) flux. In all cases Ohm’s law defines the ratio of the potential (voltage) to the flux (current). How this happens can be understood only once the solution to the equations has been established. The fluid has a heat capacity analogous to that of an electrical capacitor (Kirchhoff, 1869, 1974). Sometimes the diffusion equation is called the telegraph equation.

2. Propagation: Pressure and electromagnetic waves are described by a scalar potential (pressure) (Eq. 3.1.3, p. 75) and a vector potential (Eq. 5.7.4, p. 268), leading to scalar and vector wave equations. Sometimes the wave equation is called the telephone equation.

The Taylor series of \( f(x) \): Next we extend the concept of the Taylor series of one variable to \( x \in \mathbb{R}^d \). Just as we generalized the derivative with respect to a real frequency variable \( \omega \in \mathbb{R} \) to complex frequency \( s = \sigma + \omega j \in \mathbb{C} \), here we generalize the derivative with respect to \( s \in \mathbb{R} \) to the vector \( x \in \mathbb{R}^d \).

Since the scalar field is analytic in \( x \), it is a good place to start. Assuming we have carefully defined the Taylor series (Eq. 3.2.9, p. 95) in one and two (Eq. 4.2.7, p. 182) variables, the Taylor series of \( f(x) \) in \( x \in \mathbb{R}^d \) about \( x = 0 \) may be defined as

\[
\begin{align*}
    f(x + \delta x) &= f(x) + \nabla f(x) \cdot \delta x + \frac{1}{2} \sum_{i, j = 1}^d \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \delta x_i \delta x_j + \text{HOT},
\end{align*}
\]

where \( \text{HOT} \) stands for Higher Order Terms (Greenberg, 1988, p. 639). From this definition, it is clear that the gradient is the generalization of the second term in the one-dimensional Taylor series expansion.

Summary: For every potential \( \phi(x,t) \) there exists a force density \( f(x,t) = -\nabla \phi(x,t) \), proportional to the potential, that drives a generalized flow \( \mathbf{w}(x,t) \). If the normal components of the force and flow are averaged over a surface, the mean force and volume flow (i.e., volume velocity for the acoustic case) are defined. In such cases the impedance \( Z(s) \) is the net force through the surface divided by the net flow, and Gauss’s law and quasi-static (Postulate P10, p. 163) come into play (Feynman, 1970a). We call this the generalized impedance. An example is \( Z(s) = \sqrt{s} \).

Assuming linearity (Postulate P2, p. 162), the product of the force and flow is the power, and the ratio (force/flow) is an impedance (Table 3.2, p. 149). This impedance statement is called Ohm’s law, Kirchhoff’s law, Laplace’s law, or Newton’s laws. In the simplest cases, they are all linearized (proportional) complex relationships between a force and a flow. Very few impedance relationships are inherently linear over a large range of force or current, but for physically useful levels, they are treated as linear. Nonlinear interactions require a more sophisticated approach, typically involving numerical methods, working in the time domain.

In electrical circuits it is traditional to define a zero potential ground point that all voltages use as the reference potential. The ground is a useful convention as a simplifying rule, but it obscures the physics and obscures the fact that the voltage is not the force. Rather, the force is the voltage difference, referenced to the ground, which is defined as zero volts. This results in abstracting away (i.e., hiding) the difference in voltage. It seems misleading (more precisely, it is wrong) to state Ohm’s law as the voltage over the current, since Ohm’s law actually says that the voltage difference (i.e., voltage gradient) over the current defines an impedance (Kennelly, 1893).

**BIBLIOGRAPHY**


When we measure the voltage between two points, it is a crude approximation to the gradient relationship, justifying the property (5.2.5) and the vector case:

\[
(\nabla \cdot \mathbf{U})(\nabla \cdot \mathbf{U}^\ast) = (\nabla \cdot \mathbf{U})^\ast (\nabla \cdot \mathbf{U}^\ast) - \nabla \cdot (\nabla \cdot \mathbf{U}^\ast)
\]

and the triple vector:

\[
(\nabla \times \mathbf{U})(\nabla \times \mathbf{U}^\ast) = (\nabla \times \mathbf{U})^\ast (\nabla \times \mathbf{U}^\ast) + \nabla \times (\nabla \times \mathbf{U}^\ast)
\]

Pressures (acoustics) and vector waves (electromagnetics). These have an important mathematical relationship, justifying the property (5.2.5) and the vector case:

\[
\nabla^2 \rho(x, t) = \frac{1}{c_0^2} \frac{\partial^2 \rho(x, t)}{\partial t^2} \leftrightarrow s^2 \frac{1}{c_0^2} P(x, s)
\]

Powers (acoustics) and vector waves (electromagnetic). These have an important mathematical relationship, justifying the property (5.2.5) and the vector case:

\[
\nabla^2 \rho(x, t) = \frac{1}{c_0^2} \frac{\partial^2 \rho(x, t)}{\partial t^2} \leftrightarrow s^2 \frac{1}{c_0^2} P(x, s)
\]
Exercise 6. This is called the half-wavelength assumption, a synonym for the quasistatic approximation, and the Nyquist theorem (See DE-3, #2).

with \( c_\infty = \sqrt{\mu_0/\rho_0} \) being the air sound velocity, and \( \nu = \sqrt{\mu_0/\rho_0} \) being the characteristic resistance of air, assuming no visco-thermal losses.

5.2.3 The Webster horn equation (WHEN)

An important generalization of the problem of lossless plane-wave propagation in one-dimensional uniform tubes is known as the WEBER line theory. As depicted in Fig. 5.2, by allowing the area \( A(r) \) of an acoustic waveguide (horn) to vary along the radius \( r \) (the direction of wave propagation), we can explore general solutions to the wave equation. Classic applications of horns include vocal tract acoustics, loudspeaker design, cochlear mechanics, quantum mechanics (e.g., the hydrogen atom), and wave propagation in periodic media (Brillouin, 1953).

We must be more precise when defining the area \( A(r) \). The area is not the cross-sectional area of the horn, rather it is the wavefront (or pressure) area, which is related to Gauss’s law, since the gradient of the pressure defines the force that drives the mass flow (also called volume velocity).

For the scalar wave equation (Eq. 5.1.9, p. 229), the Webster Laplacian is

\[
\nabla^2 p(r, t) = \frac{1}{A(r)} \frac{\partial}{\partial r} \left[ A(r) \frac{\partial}{\partial r} p(r, t) \right].
\]

The Webster Laplacian is based on the quasistatic approximation (Postulate P10, p. 163), which requires that the frequency lie below the critical value \( f_c = c_\infty/2d \) of a half-wavelength greater than the horn diameter \( d \) (i.e., \( d < \lambda/2 \)). For the adult human ear canal,
\[
\frac{\partial}{\partial t} + \nabla \cdot \mathbf{v} = 0
\]

The initial condition is that the vector potential is constant and the electric field

\[
\mathbf{E}(x, t) = -\nabla \phi(x, t)
\]

The electric field is orthogonal to the magnetic field and is given by the gradient of the scalar potential. The magnetic field is then determined from the curl of the vector potential.

\[
\mathbf{B}(x, t) = \nabla \times \mathbf{A}(x, t)
\]

The magnetic field is orthogonal to the electric field and is also determined from the curl of the vector potential.

\[
\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}
\]

The Maxwell stress tensor is given by

\[
T_{ij} = \varepsilon \varepsilon_0 \left[ \frac{1}{2} \left( \frac{\partial \mathbf{E}}{\partial x_i} \cdot \mathbf{E} \right) + \frac{1}{2} \left( \frac{\partial \mathbf{E}}{\partial x_j} \cdot \mathbf{E} \right) - \frac{1}{2} \left( \frac{\partial \mathbf{E}}{\partial x_i} \cdot \frac{\partial \mathbf{E}}{\partial x_j} \right) \right]
\]

The energy-momentum tensor is given by

\[
T_{ij} = \rho \left( \frac{\partial \mathbf{v}}{\partial x_i} \right) \cdot \mathbf{v} + \varepsilon \varepsilon_0 \left[ \frac{1}{2} \left( \frac{\partial \mathbf{E}}{\partial x_i} \cdot \mathbf{E} \right) + \frac{1}{2} \left( \frac{\partial \mathbf{E}}{\partial x_j} \cdot \mathbf{E} \right) - \frac{1}{2} \left( \frac{\partial \mathbf{E}}{\partial x_i} \cdot \frac{\partial \mathbf{E}}{\partial x_j} \right) \right]
\]

The stress-energy tensor is related to the energy-momentum tensor by

\[
T_{ij} = \rho \left( \frac{\partial \mathbf{v}}{\partial x_i} \right) \cdot \mathbf{v} + \varepsilon \varepsilon_0 \left[ \frac{1}{2} \left( \frac{\partial \mathbf{E}}{\partial x_i} \cdot \mathbf{E} \right) + \frac{1}{2} \left( \frac{\partial \mathbf{E}}{\partial x_j} \cdot \mathbf{E} \right) - \frac{1}{2} \left( \frac{\partial \mathbf{E}}{\partial x_i} \cdot \frac{\partial \mathbf{E}}{\partial x_j} \right) \right]
\]

The pressure is given by

\[
p = \frac{\partial E}{\partial V}
\]

The volume is given by

\[
V = \int d^3 x
\]

The entropy is given by

\[
S = \frac{\partial E}{\partial T}
\]

The temperature is given by

\[
T = \frac{\partial E}{\partial S}
\]

The entropy production is given by

\[
\dot{S} = \frac{\partial E}{\partial T} \frac{\partial T}{\partial S} - \frac{\partial E}{\partial S} \frac{\partial S}{\partial T}
\]

The entropy production is related to the heat flux by

\[
\dot{S} = \mathbf{q} \cdot \nabla T
\]

The heat flux is given by

\[
\mathbf{q} = -k \nabla T
\]

The thermal conductivity is given by

\[
k = \frac{\partial E}{\partial T} \frac{\partial T}{\partial S} - \frac{\partial E}{\partial S} \frac{\partial S}{\partial T}
\]

The thermal diffusion is given by

\[
D = \frac{\partial E}{\partial T} \frac{\partial T}{\partial S} - \frac{\partial E}{\partial S} \frac{\partial S}{\partial T}
\]

The thermal diffusion is related to the diffusivity by

\[
D = \frac{\partial E}{\partial T} \frac{\partial T}{\partial S} - \frac{\partial E}{\partial S} \frac{\partial S}{\partial T}
\]

The heat capacity is given by

\[
C_v = \frac{\partial E}{\partial T}
\]

The specific heat is given by

\[
C_p = \frac{\partial E}{\partial T} + \frac{\partial E}{\partial P}
\]

The specific heat is related to the heat capacity by

\[
C_p = C_v + \frac{\partial E}{\partial P}
\]

The pressure is related to the specific volume by

\[
p = \frac{\partial E}{\partial V}
\]

The specific volume is given by

\[
V = \frac{\partial E}{\partial p}
\]

The specific volume is related to the pressure by

\[
V = \frac{\partial E}{\partial p}
\]

The pressure is related to the specific volume by

\[
p = \frac{\partial E}{\partial V}
\]

The specific volume is given by

\[
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The pressure is related to the specific volume by

\[
p = \frac{\partial E}{\partial V}
\]
taken as the second conductor, to save wire). Thus the telegraph equation is best modeled by a diffusion line. The telephone equation included loading coils, consisting of inductors placed periodically in the wire, to increase the circuits inducance. This converted the circuit into a true transmission line. The loading coils were introduced by the AT&T engineer and mathematician George Ashley Campbell (Campbell, 1903, 1937), however they were first proposed and promoted by Heaviside.

In acoustics, waveguides are known as horns, such as the horn connected to the first phonographs from around the turn of the century (Webster, 1919). Thus the names reflect the historical development, back to a time when mathematics and its applications were related.

### 5.2.4 Matrix formulation of the WHEN

Newton’s laws of conservation of momentum (Eq. 5.2.5) and mass (Eq. 5.2.6) are modern versions of Newton’s starting point for calculating the horn lowest-order plane-wave eigenmode wave speed.

The acoustic equations for the average pressure $P(r, \omega)$ and the volume velocity are derived in Appendix G, where the pressure and particle velocity equations (Eqs. G.1.4 and G.1.6) are transformed into a $2 \times 2$ matrix of acoustical variables, average pressure $P(r, \omega)$ and volume velocity $\nabla r \omega P$: 

$$
\frac{d}{dr} P(r, \omega) = \begin{bmatrix} \frac{\eta_o}{\rho_o c_o^2} \ 0 \\
0 \ \frac{\eta_o}{\rho_o c_o^2} \end{bmatrix} \begin{bmatrix} P(r, \omega) \\
\nabla r \omega P \end{bmatrix}.
$$

(5.2.11)

The equations $M(r) = \rho_o/A(r)$ and $C(r) = A(r)/\eta_o P_o$ define the per-unit-length mass and compliance of the horn (Ramo et al., 1966, p 213). The product of $P(r, s)$ and $V(r, s)$ defines the acoustic power [W/m²], while their ratio defines the horn’s admittance $Y(r, s) = V(r, s)/P_o$, looking in the two directions (Pierce, 1981, p 37–41).

To obtain the Webster horn pressure equation Eq. 5.2.10 from Eq. 5.2.11, we take the partial derivative of the top equation

$$
\frac{\partial^2 P}{\partial r^2} = \frac{sM(r)}{\rho_o c_o^2} V + sM(r) \frac{\partial V}{\partial r}
$$

and then use the lower equation to remove $\partial V/\partial r$,

$$
\frac{\partial^2 P}{\partial r^2} = \frac{sM(r)}{\rho_o c_o^2} V - \frac{s^2 M(r)c_o^2}{\rho_o c_o^2} P = \frac{x^2}{c_o^2} P.
$$

Note that $c_o^2 = MC = \left(\frac{\rho_o c_o^2}{\eta_o}\right) \left(\frac{\partial P_o}{\partial P}\right)$. In air $c_o = \sqrt{\eta_o P_o/\rho_o}$.

We must then use the upper equation a second time to remove $V$:

$$
\frac{\partial^2 P}{\partial r^2} + \frac{1}{A(r)} \frac{\partial A(r)}{\partial r} \frac{\partial P}{\partial r} = \frac{x^2}{c_o^2} P(r, s).
$$

(5.2.13)

By use of the chain rule, equations of this form may be directly integrated, since

$$
\nabla r \omega P = \frac{1}{A(r)} \frac{\partial}{\partial r} \left[ A(r) \frac{\partial}{\partial r} P(r, s) \right] = \frac{\partial^2}{\partial r^2} P(r, s) + \frac{1}{A(r)} \frac{\partial A(r)}{\partial r} P(r, s).
$$

(5.2.14)
This is equivalent to integration by parts, with integration factor $r \equiv \kappa(r,s)$. Finally we set $\kappa(r,s)$ to to Eq. 5.2.15 in a horn.

The Webster horn equation is closely related to Schrödinger's equation (Salmon, 1946).
5.3 Problems VC-1

5.3.1 Topics of this homework:
Vector algebra and fields in \( \mathbb{R}^3 \), gradient and scalar Laplacian operators, definitions of divergence and curl, Gauss’s (divergence) and Stokes’s (curl) laws, system classification (postulates).

5.3.2 Scalar fields and the \( \nabla \) operator

Problem # 1: Let \( T(x, y) = x^2 + y \) be an analytic scalar temperature field in two dimensions (single-valued \( \in \mathbb{R}^2 \)).

- 1.1: Find the gradient of \( T(x) \) and make a sketch of \( T \) and the gradient.
- 1.2: Compute \( \nabla^2 T(x) \) to determine whether \( T(x) \) satisfies Laplace’s equation.
- 1.3: Sketch the iso-temperature contours at \( T = -10, 0, 10 \) degrees.
- 1.4: The heat flux is defined as \( J(x, y) = -\kappa(x, y)\nabla T \), where \( \kappa(x, y) \) denotes the thermal conductivity at the point \( (x, y) \). Given that \( \kappa = 1 \) everywhere (the medium is homogeneous), plot the vector \( J(x, y) = -\nabla T \) at \( x = 2, y = 1 \). Be clear about the origin, direction, and length of your result.
- 1.5: Find the vector \( \nabla \perp T(x, y) \)—that is, tangent to the iso-temperature contours. Hint: Sketch it for one \( (x, y) \) point (e.g., 2, 1) and then generalize.
- 1.6: The thermal resistance \( R_T \) is defined as the potential drop \( \Delta T \) over the magnitude of the heat flux \( |J| \). At a single point the thermal resistance is \( R_T(x, y) = \frac{-\Delta T}{|J|} \).

How is \( R_T(x, y) \) related to the thermal conductivity \( \kappa(x, y) \)?

Problem # 2: Acoustic wave equation

Note: In this problem, we will work in the frequency domain.

8The heat flux is proportional to the change in temperature times the thermal conductivity \( \kappa \) of the medium.

Bibliography


5.3. PROBLEMS VC-1

2.1: The basic equations of acoustics in one dimension are
\[- \frac{\partial}{\partial x} P = \rho_o \vec{V} \]
\[- \frac{\partial}{\partial x} \vec{V} = s \eta_o P \]

Here \( P(x, \omega) \) is the pressure (in the frequency domain), \( \vec{V}(x, \omega) \) is the volume velocity (the integral of the velocity over the wavefront with area \( A \)), \( s = \sigma + \omega \eta \), \( \rho_o = \frac{1}{2} \) is the specific density of air, \( \eta_o = \frac{1}{4} \), and \( P_o \) is the atmospheric pressure (i.e., \( 10^5 \) Pa). Note that the pressure field \( P \) is a scalar (pressure does not have direction), while the volume velocity field \( \vec{V} \) is a vector (velocity has direction).

We can generalize these equations to three dimensions using the \( \nabla \) operator
\[- \nabla P = \rho_o \vec{V} \]
\[- \nabla \cdot \vec{V} = s \eta_o P \]

2.2: Starting from these two basic equations, derive the scalar wave equation in terms of the pressure \( P \),
\[ \nabla^2 P = s^2 \frac{c_0^2}{\rho_o} P, \]
where \( c_0 \) is a constant representing the speed of sound.

2.3: What is \( c_0 \) in terms of \( \eta_0 \), \( \rho_0 \), and \( P_0 \)?

2.4: Rewrite the pressure wave equation in the time domain using the time derivative property of the Laplace transform \([e.g., \frac{dx}{dt} \leftrightarrow sX(s)]\). For your notation, define the time-domain signal using a lowercase letter, \( p(x, y, z, t) \leftrightarrow P(t) \).

5.3.3 Vector fields and the \( \nabla \) operator

5.3.4 Vector algebra

Problem #3: Let \( R(x, y, z) \equiv x \hat{x} + y \hat{y} + z \hat{z} \).

3.1: If \( a \), \( b \), and \( c \) are constants, what is \( R(x, y, z) \cdot R(a, b, c) \)?

3.2: If \( a \), \( b \), and \( c \) are constants, what is \( \frac{d}{dt}(R(x, y, z) \cdot R(a, b, c)) \)?

Problem #4: Find the divergence and curl of the following vector fields:

4.1: \( \mathbf{v} = \hat{x} + \hat{y} + 2 \hat{z} \)

4.2: \( \mathbf{v}(x, y, z) = x \hat{x} + xy \hat{y} + z^2 \hat{z} \)

4.3: \( \mathbf{v}(x, y, z) = x \hat{x} + xy \hat{y} + \log(z) \hat{z} \)

4.4: \( \mathbf{v}(x, y, z) = \nabla \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \)
5.3.5 Vector and scalar field identities

Problem #5: Find the divergence and curl of the following vector fields:

– 5.1: \( \mathbf{v} = \nabla \phi \), where \( \phi(x, y) = xe^y \)
– 5.2: \( \mathbf{v} = \nabla \times \mathbf{A} \), where \( \mathbf{A} = x\mathbf{\hat{k}} + y\mathbf{\hat{y}} + z\mathbf{\hat{z}} \)
– 5.3: \( \mathbf{v} = \nabla \times \mathbf{A} \), where \( \mathbf{A} = y\mathbf{\hat{k}} + x^2\mathbf{\hat{y}} + z\mathbf{\hat{z}} \)
– 5.4: For any differentiable vector field \( \mathbf{v} \), write two vector calculus identities that are equal to zero.
– 5.5: What is the most general form a vector field may be expressed in, in terms of scalar \( \Phi \) and vector \( \mathbf{A} \) potentials?

Problem #6: Perform the following calculations. If you can state the answer without doing the calculation, explain why.

– 6.1: Let \( \mathbf{v} = \sin(x)\mathbf{\hat{k}} + y\mathbf{\hat{y}} + z\mathbf{\hat{z}} \). Find \( \nabla \cdot (\nabla \times \mathbf{v}) \).
– 6.2: Let \( \mathbf{v} = \sin(x)\mathbf{\hat{k}} + y\mathbf{\hat{y}} + z\mathbf{\hat{z}} \). Find \( \nabla \times (\nabla \cdot \mathbf{v}) \)
– 6.3: Let \( \mathbf{v}(x, y, z) = \nabla(x + y^2 + \sin(\log(z))) \). Find \( \nabla \times \mathbf{v}(x, y, z) \).

5.3.6 Integral theorems

Problem #7: For each of the following problems, in a few words, identify either Gauss’s or Stokes’s law, define what it means, and explain the formula that follows the question.

– 7.1: What is the name of this formula?
\[
\int_S \mathbf{n} \cdot \mathbf{v} \, dA = \int_V \nabla \cdot \mathbf{v} \, dV.
\]

– 7.2: What is the name of this formula?
\[
\int_S (\nabla \times \mathbf{V}) \cdot d\mathbf{S} = \int_C \mathbf{V} \cdot d\mathbf{R}
\]
Give one important application.

– 7.3: Describe a key application of the vector identity
\[
\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}.
\]
I.4.2 Monics having coefficients taken from \{1, 0\} (Bernoulli trials)

A second interesting question is what are the roots of polynomials having coefficients from \{1, 0\} (Bernoulli trials). For the purposes of this chapter, we consider a Bernoulli trial as a random experiment with two possible outcomes, often referred to as „success“ and „failure“.

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Table 5.2 summarizes the properties of four of these horns: uniform (cylindrical) \((A(r) = A_0)\), conical (spherical) \((A(r) = A_0 r^2)\), and exponential \((A(r) = A_0 e^{2mr})\), where \(m\) is called the flair parameter.

Table 5.2. Some properties of four of these horns: uniform (cylindrical) \((A(r) = A_0)\), conical (spherical) \((A(r) = A_0 r^2)\), and exponential \((A(r) = A_0 e^{2mr})\), where \(m\) is called the flair parameter.

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5.4. Examples of finite-length horns

Figure 5.3: Throat acoustical resistancer \(A\) and acoustical reactancex \(A\), ... (A(r) = A_0 r^2), and exponential (A(r) = A_0 e^{2mr}), where \(m\) is called the flair parameter.
5.4.1 Uniform Horn #5  

The one-dimensional wave equation \[ A(x) = A_0 = 1 \text{[cm]} \]

\[ \frac{d^2}{dx^2} P = \kappa^2(s) P, \]

thus the roots (eigenvalues) are \( \kappa^2(s) = \pm \sqrt{L} \).

**Solution:** The two eigenfunctions of this equation are the two d'Alembert waves (Eq. 4.4.1, p. 188):

\[ \phi(x, t) = P_0^+ e^{\kappa L} + e^{-\kappa L} \]

where \( P_0^+ \in \mathbb{C} \) are wave amplitudes and \( \kappa(s) = \sqrt{L}, \kappa = \omega/c \) is called the propagation function (also known as the wave-evolution function, propagation constant, and wave number).

Note that for the uniform lossless horn, \( \omega/c = 2\pi/\lambda \geq 0 \). It is convenient to normalize \( P_0^+ = 1 \) and \( P_0^- = 1 \). The characteristic admittance \( Y(x) \) (Table 5.2) is independent of direction. The signs must be physically chosen, with the velocity \( V^+ \) into the port, to ensure that the real part of \( Y \) is positive, to assure the system remains stable.

**Applying the boundary conditions:** The general solution in terms of the eigenvector matrix, evaluated at \( s = L \), is

\[
\begin{bmatrix}
P(L) \\
V(L)
\end{bmatrix} =
\begin{bmatrix}
e^{-x} & e^{-(x-L)} \\
e^{-x} & e^{-(x-L)}
\end{bmatrix}
\begin{bmatrix}
P(0) \\
V(0)
\end{bmatrix}
\begin{bmatrix}
1 & \kappa \\
1 & -\kappa
\end{bmatrix}
\begin{bmatrix}
P_0^+ \\
P_0^-
\end{bmatrix}.
\]

**(5.4.1)**

where \( P_0^+ \) and \( P_0^- \) are the relative amplitudes of the two unknown eigenfunctions to be determined by the boundary conditions at \( s = 0, L, \kappa = s/c, \) and \( \lambda = 1/Z, \lambda = -1/Z, \lambda = 1/Z \).

Solving Eq. 5.4.1 for \( P_0^+ \) and \( P_0^- \) with determinant \( \Delta = -2\kappa e^{-L} \), we get

\[
\begin{bmatrix}
P_0^+ \\
P_0^-
\end{bmatrix} =
\begin{bmatrix}
-1 & Z e^{-x} \\
1 & \kappa
\end{bmatrix}
\begin{bmatrix}
1 & 2 & Z e^L \\
-1 & 1 & Z
\end{bmatrix}
\begin{bmatrix}
P(L) \\
V(L)
\end{bmatrix}.
\]

In the final step we swapped all the signs, including those on \( V \), and moved \( Z = 1/Y \) inside the matrix.

We can uniquely determine these two weights \( P_0^+ \), \( P_0^- \) given the pressure and velocity at the boundary \( s = L \), which is typically determined by the load impedance \( Z_L(s) = P_L/V_L \).

The weights may now be substituted back into Eq. 5.4.1 to determine the pressure and velocity amplitudes at any point \( 0 \leq s \leq L \):

\[
\begin{bmatrix}
P \\
V
\end{bmatrix} =
\begin{bmatrix}
e^{-x} & e^{-(x-L)} \\
e^{-x} & e^{-(x-L)}
\end{bmatrix}
\begin{bmatrix}
e^{x} & e^{-(x-L)} \\
e^{x} & e^{-(x-L)}
\end{bmatrix}
\begin{bmatrix}
P(L) \\
V(L)
\end{bmatrix}.
\]

**(5.4.3)**

Setting \( \kappa = 0 \) and multiplying these out give the final transmission matrix:

\[
\begin{bmatrix}
P \\
V
\end{bmatrix} =
\begin{bmatrix}
e^{x} & e^{-l} \\
e^{x} & e^{-l}
\end{bmatrix}
\begin{bmatrix}
e^{x} & e^{-(x-L)} \\
e^{x} & e^{-(x-L)}
\end{bmatrix}
\begin{bmatrix}
P(L) \\
V(L)
\end{bmatrix}.
\]

**(5.4.4)**

Note that the diagonal terms are \( \cosh \kappa L \) and the off-diagonal terms are \( \sinh \kappa L \).

1.4 Properties of Diophantine Polynomials

1.4.1 Polynomials having prime roots

There is an interesting debate regarding the convergence of Newton’s method. My view is that it always converges (Sec. 3.1.2).

The following are the coefficients of a polynomial built from prime numbers

\[
C = [950488110, -1696163097, 1253300345, -500888300, 123054800, -18928908, 1879134, -119820, 473]
\]

Since we know the roots are prime, we may find them by factoring \( C(1) \), which gives

\[
R = [2, 3, 5, 11, 13, 13, 19, 23]
\]

where \( \text{factor}(950488110) \).

Directly finding the roots of the polynomial using Matlab/Octave’s \( \text{roots}(C) \) gives

\[
[0.5, 0.33333, 0.33333, 0.2, 0.090909, 0.076923, 0.076923, 0.076923, 0.052632, 0.043478]
\]

due to floating point numerical issues.

There are several alternatives to this. One is to use the continued fraction algorithm (CFA) to find rational expansions of the roots. In Matlab the CFA is implemented with the command \( \text{rat}(\cdot) \). For example

\[
\text{rat}(1/0.5) = 1/2
\]

and the fifth root gives

\[
\text{rat}(1/0.090909) = 11.
\]

However the 6th root fails the rats expansion

\[
\text{rat}(1/0.076923) = 13 + 1/(-596)
\]

and for the 7th root,

\[
\text{rat}(1/0.076923) = 0 + 1/(13 + 1/(1165)).
\]

An interesting question remains: “Can these errors be removed by factoring out a few of the smaller primes \((2, 3, 5)\), or the larger primes \((19, 23)\)?” Since there are 3 factors of 13, what is the effect of trying to remove that factor. Namely is there a remainder due to the near miss, or will it factor 13 with no remainder?

This question is interesting because Newton’s method could be used to find roots, which could then be removed using Matlab/Octave’s long division routine (\( \text{deconv}(\cdot) \)). Other potential options are a residue expansion, or its more power sister the \( \text{vector fitting} \) algorithm (Matlab’s \( \text{rootsfit3}(\cdot) \)).
I.3 Taking the inverse to get the gcd

Variable p and q are the remainders r mod n−1. The Euclidean algorithm gives Eq. I.2.1. Inverting this gives the formula for the GCD:

\[ \gcd(p, q) = \frac{\gcd(q, r)}{\gcd(p, r)} \]

where p = r mod n−1 and the remainders are calculated using the Euclidean algorithm. These equations show that the Euclidean algorithm is reversible (P7) and reciprocal (P6). Note that the uniform horn is reversible (P7) and reciprocal (P6).

When the uniform horn is terminated with a fixed impedance \( Z_{\text{in}} \), we can substitute (5.4.5) into Eq. 5.4.2 to obtain the input admittance at \( x = 0 \) is

\[ Z_{\text{in}}(x) = \frac{\rho_0 c}{A_0} \]

This gives Eq. I.2.1. Inverting this gives the formula for the GCD:

\[ \gcd(p, q) = \frac{\gcd(q, r)}{\gcd(p, r)} \]

where p = r mod n−1 and the remainders are calculated using the Euclidean algorithm. These equations show that the Euclidean algorithm is reversible (P7) and reciprocal (P6). Note that the uniform horn is reversible (P7) and reciprocal (P6).

When the uniform horn is terminated with a fixed impedance \( Z_{\text{in}} \), we can substitute (5.4.5) into Eq. 5.4.2 to obtain the input admittance at \( x = 0 \) is

\[ Z_{\text{in}}(x) = \frac{\rho_0 c}{A_0} \]
5.4.2 Conical horn #2

Using the conical horn area $A(r) \propto r^2$ in Eq. 5.2.10, on...$S(x) = \frac{\rho c}{\omega}$.

Radiation admittance for the conical horn: The conical horn’s acoustic input admittance $Y_m(r,s)$ at any location $r$ is found by dividing $V(r,s)$ by $P(r,s)$:

$$Y_m(r,s) = \frac{V_m}{P_m} = -\frac{A(r)}{\rho c} \ln P_m(r,s)$$

$$Y_m(r,s) = \frac{V_m}{P_m} = -\frac{A(r)}{\rho c} \ln \left(1 + \frac{c_s^2}{c^2} s^2 \right) + \frac{c_s^2}{c^2} s^2 \ln (r - r/c)$$

The pressure pulse is delayed by $r/c$, due to conservation of mass, and the characteristic admittance $Y_m(r)$ variation with $r$.

5.4.3 Exponential horn #3

If we define the area as $A(r) = A_0 e^{2mr}$, the eigenfunctions of the horn are

$$P_m(r,s) = e^{-mr} e^{2mr} s^2$$

which may be shown by the substitution of $P_m(r,s)$ into Eq. 5.2.10 (p. 241).

This case is of special interest because the radiation impedance is purely reactive below the horn’s cutoff frequency ($\omega < \omega_c = mc_0$), as may be seen from curves 3 and 4 of Fig. 5.3 (p. 247). As a result, no energy can radiate from an open horn for $\omega < \omega_c$, because the eigenvalues

$$\kappa(s) = -m \pm \frac{3}{2} \sqrt{\omega^2 - \omega_c^2} = -m \pm \sqrt{\omega^2 - \omega_c^2}$$

are purely real (this is the case of nonpropagating evanescent waves). If we use Eq. 4.4.3 (p. 190), the input admittance is

$$Y_m(r,s) = -\frac{A(x)}{\rho c} \left( m \pm \sqrt{m^2 + \kappa^2} \right) s,$$

(5.4.11)

Kleiner (2013) gives an equivalent expression for $V_m(r,s)$ given area $S(x) = e^{mr}$,

$$V_m(r,s) = S(x) \frac{m}{2} \left[ \sqrt{\omega^2 - (\omega_c)^2} \right].$$

and impedance

$$Z_m(r,s) = \frac{\rho c}{S(x)} \left[ \frac{\omega}{\omega} + \sqrt{1 - \left( \frac{\omega}{\omega_c} \right)^2} \right].$$

I.2. DERIVATION OF THE CFA MATRIX

From the remainder formula, $u_1 = \lfloor p/q \rfloor$. Continuing with $n = 2$:

$$\begin{bmatrix} r_2 \\ r_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r_2 \\ r_1 \end{bmatrix},$$

where $u_3 = \lfloor r_0/r_1 \rfloor$. Continuing with $n = 3$:

$$\begin{bmatrix} r_3 \\ r_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r_3 \\ r_2 \end{bmatrix},$$

where $u_4 = \lfloor r_1/r_2 \rfloor$. For arbitrary $n$ we find

$$\begin{bmatrix} r_n \\ r_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r_n \\ r_{n-1} \end{bmatrix},$$

(5.1.2)

Example: We let $p = 355$ and $q = 113$, which are coprime, and set $n = 1$. Then Eq. 5.1.2 becomes

$$\begin{bmatrix} 355 \\ 113 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix},$$

since $u_1 = \lfloor 355/113 \rfloor = 3$. Solving for the RHS gives $[r_0, r_1] = [113; 16]$. To find $[r_0, r_1]$, we take the inverse:

$$\begin{bmatrix} 113 \\ 16 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix},$$

since $u_2 = \lfloor 113/16 \rfloor = 7$. Solving for the RHS gives $[r_1, r_2] = [16; 1]$. Let’s go one step further.

For $n = 2$, we have

$$\begin{bmatrix} 16 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix},$$

since $u_3 = \lfloor 16/1 \rfloor = 16$. Solving for the RHS gives $[r_1, r_2] = [1; 0]$. This confirms that we have done, since $r_3 = 0$.

Derivation of Eq. 1.2.1: Equation 1.2.1 is derived as follows: Starting from the target $x \in \mathbb{R}$, we define

$$p = [x]$$

and $q = \frac{1}{x-p} \in \mathbb{R}$.

These two relationships for truncation and remainder allow us to write the general matrix recursion relation for the CFA (Eq. 1.2.1). Given $[p, q]$, we can write the above CFA method.

One slight problem with the above is that the output is on the right and the input on the left. Thus we need to take the inverse of these relationships to turn this into a composition.
We can also define the number field $\mathbb{Q}(i)$ as the set of all numbers of the form $a + bi$ where $a, b \in \mathbb{Q}$.

### Example

Consider the set $S = \{2, 3, 5, 7, 11\}$. We can find the greatest common divisor (GCD) of any two numbers in this set.

We can use the Euclidean algorithm to find the GCD of any two numbers in this set.

#### Euclidean Algorithm

1. If the numbers are equal, the GCD is the number itself.
2. If one number is greater than the other, divide the larger number by the smaller number. The remainder is the new smaller number.
3. Repeat the process until the remainder is 0. The GCD is the last non-zero remainder.

#### Example

Find the GCD of 48 and 18.

- $48 \div 18 = 2$ remainder 12
- $18 \div 12 = 1$ remainder 6
- $12 \div 6 = 2$ remainder 0

The GCD of 48 and 18 is 6.

### Sturm-Liouville Methods and Eigenvectors

When the coordinate system is separable, it is useful to use Sturm-Liouville methods and eigenvectors.

#### Separable Coordinate Systems

1. Rectangular coordinates
2. Cylindrical coordinates
3. Spherical coordinates

#### Separated Variables

When the coordinate system is separable, the wave equation is separated into ordinary differential equations in the separated coordinates. The resulting PDEs are always reduced to a system of Sturm-Liouville equations. The solutions to these equations are the eigenvalues and eigenvectors.

#### Eigenvalue Problems

The eigenvalues are the solutions to the Sturm-Liouville equation. The eigenvectors are the functions corresponding to the eigenvalues.

### Solution Methods

#### 5.5 Solution Methods

There are two main solution methods for the wave equation.

1. Separation of variables
2. Fourier methods

#### Separation of Variables

When the coordinate system is separable, the wave equation is separable. This method is limited to a few ortho-normal coordinate systems, such as rectangular, cylindrical, and spherical coordinates.

#### Fourier Methods

The Fourier method involves expressing the solution as a series of eigenfunctions. This method is useful for problems with periodic boundary conditions.

### Additional Notes

1. Mathematics provides rigor.
2. Physics provides understanding.
3. The wave equation is a partial differential equation.
4. Solutions to the wave equation can be found using separation of variables or Fourier methods.
5. The accuracy of the solution depends on the choice of basis functions.

We can use numerical methods to find solutions to the wave equation. These methods include finite difference, finite element, and spectral methods.
When a system of lumped-element networks contains only resistors and capacitors or resistors and inductors, the solution is a diffusion equation, which does not support propagated waves. Depending on the elements in the system of equations, there can be an overlap between a diffusion process and scalar waves, represented as transmission lines, both modeled as lumped-element networks of $2 \times 2$ matrices (Eq. 3.8.1, p. 144) (Campbell, 1922; Brillouin, 1953; Ramo et al., 1965).

Nyquist sampling and quaistatic: Quaistatic methods provide band-limited solutions below a critical frequency $f_\text{c}$ for a much wider class of geometries, by avoiding high-frequency cross-modes. The model of a train is depicted in Fig. 5.4, which is the same as Fig. 4.8 (p. 221).

Example: Train-mission-line problem. The mechanical mass-spring system of Fig. 5.4 is the electrical equivalent circuit. The mass is modeled as an inductor and the springs as capacitors to ground. The velocity is analogous to a current and the force $f_j(t)$ to the voltage $\phi_j(t)$. The length of each cell is $\Delta$ [m].

![Figure 5.4](image)

Figure 5.4: Depiction of a train consisting of cars, treated as a mass $M$ and leakages, treated as springs of stiffness $K_b$ or compliance $C = 1/K_b$. The equivalent electrical circuit is shown below the mass-spring system, with the masses modeled by inductors ($M$) and springs modeled as capacitors ($C$). For this model to accurately represent a transmission line the frequency must be less than the equivalent Nyquist frequency $f_\text{c}$. The delay of one cell is $D = \Delta/c$. One can measure either $f_\text{c}$ or $\Delta$ and $c = \sqrt{\frac{K_b}{M}}$.

Since $\lambda = 2\Delta$, it follows that the cutoff frequency is $f_\text{c} = c / 2\Delta = 1/2T$. For $f < f_\text{c}$, the frequency response is independent of frequency, thus acting as a pure delay. As the frequency increases above $f_\text{c}$, the wavelength becomes shorter than the critical wavelength $\lambda = 2\Delta$, and the delay line becomes a lowpass filter, strongly departing from transmission line properties. For more detailed information see the Figure caption on p. 221.

When the wavelength $\lambda = c / f$ is greater than twice the physical distance $\Delta$ between the elements

$$\lambda > \lambda_\text{c} = 2\Delta \quad \text{[m]};$$

the approximation is mathematically equivalent to a transmission line. As described in DE-3, problem #2, the velocity is $v = \sqrt{1/C}$ [m/s]. As the frequency increases, the wavelength becomes shorter. When the frequency is equal to the critical frequency $f_\text{c}$, the critical wavelength $\lambda_\text{c} = c / f = 2\Delta$. Above the critical frequency the quaistatic (lumped-element) model breaks down and transitions from a delay line to a lowpass filter, as discussed in DE-3, problem #2.

The frequency is under the control of the modeling process, since more elements may be added to allow for higher frequencies (shorter wavelengths). If the nature of the solution at high frequencies ($f > f_\text{c}$) is desired, we may add more sections (make $\Delta$ smaller).

Appendix I
Number theory applications

I.1 Division with rounding method

We want to show that the GCD form, $n, k \in \mathbb{N}$ (Eq. 2.4.2, p. 47) may be written in matrix form

$$\begin{pmatrix} m \\ n \end{pmatrix} \big|_{k+1} = \begin{pmatrix} 0 & 1 \\ 1 & -\frac{n}{m} \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}.$$  \hfill (I.1.1)

Eq.I.1.1 implements the Euclidean algorithm, for solving $m, n \in \mathbb{N}$, for $m > n$.

This starts with $k = 0, m_k = a$, and $n_k = b$. With this method there is no need to test whether $n_k < m_k$ as it is built into the procedure. The method uses the floor function $\lfloor x \rfloor$, which finds the integer part of $x$ ($\lfloor x \rfloor$) rounds toward $-\infty$. After each step we will see that the value $n_{k+1} < m_{k+1}$. The method terminates when $n_{k+1} = 0$ with $\gcd(a,b) = m_{k+1}$.

The following vectorized code is more efficient than the direct matrix method:

```matlab
function n=gcd2(a,b)
M=[abs(a);abs(b)]; %Save (a,b) in array M(2,1)
while M(1) ˜= 0
    if M(1) > M(2)
        M(1)=M(1)-M(2);
    end
    M(2)=M(2); %Save (a,b) in array M(2,2)
end
n=M(1);
```

% Save (a,b) in array M(2,1)

With a minor extension in the test for “end,” this code can be made to work with irrational inputs (e.g., $(\sqrt{2}, \pi)$).

This method calculates the number of times $n < m$ must subtract from $m$ using the floor function. This operation is the same as the mod function. Specifically,

$$n_{k+1} = m_k - \left\lfloor \frac{m_k}{n_k} \right\rfloor n_k.$$  \hfill (I.1.2)

so that the output is the definition of the remainder of modular arithmetic. This would have been obvious to anyone using an abacus, which explains why it was discovered so early.

1https://en.wikipedia.org/wiki/Modulo_operation
many (perhaps most) problems, lumped elements are easy to use and accurate, as long as we don't violate the Nyquist condition (Nyquist, 1924; Brillouin, 1953; Ramo et al., 1965).

5.5.1 Eigenfunctions

Because the wave equation (Eq. 5.2.7) is second-order in time, there are two causal independent eigenfunction solutions: an outbound (right–traveling) \( \varrho^+ (r, t) \) wave, and an inbound (left–traveling) \( \varrho^- (r, t) \) wave.

Every eigenfunction depends on an area function \( A(r) \) (Eq. 5.2.10, p. 241). In theory then, given an eigenfunction, it should be possible to find the area \( A(r) \). This is known as the inverse problem, which is generally believed to be a difficult problem. Specifically, given the eigenvalues \( \lambda_k \), how does one determine the corresponding area function \( A(r) \)?

Because the characteristic impedance \( Y_r(r) \) of the wave in the horn changes with location, there are local reflections due to these area variations. Thus there are fundamental relationships between the area change \( dA(r)/dr \), the horn's eigenfunctions \( P_{\pm} (r, s) \), the eigenmodes, and the input impedance.

Complex vs. real frequency: We shall continue to maintain the distinction that functions of \( \omega \) are Fourier transforms and causal functions of Laplace frequency \( s \) correspond to Laplace transforms, which are necessarily complex–analytic in the right half-plane (RHP) region of convergence (RoC). This distinction is critical, since we typically describe impedance \( Z(s) \) and admittance \( Y(s) \) as complex–analytic functions in \( s \) in terms of their poles and zeros. The eigenfunctions \( P_{\pm} (r, s) \) of Eq. 5.2.10 are also causal complex–analytic functions of \( s \).

Plane-wave eigenfunction solutions: In 1690, nine years before Newton's publication of "Principia", Christiaan Huygens was the first to gain insight into wave propagation, today known as Huygens's principle. While his concept showed a deep insight, we now know it was flawed, as it ignored the backward-traveling wave (Miller, 1991). In 1747 d'Alembert published the first correct solution for the plane-wave scalar wave equation,

\[
\rho(x, t) = f(t-x/c_0) + g(t+x/c_0)
\]

(5.5.1)

where \( f(\cdot) \) and \( g(\cdot) \) are general functions of their argument. That this is the solution may be shown by use of the chain rule, by taking partials with respect to \( x \) and \( t \).

In terms of physics, d'Alembert's general solution describes two arbitrary waveforms \( f(\cdot) \) and \( g(\cdot) \) traveling at a speed \( c_0 \), one forward and one reversed. This solution is quite easily visualized.

Exercise #13

By the use of the chain rule, prove that d'Alembert's formula satisfies the one-dimensional wave equation.

Solution:

Taking a derivative with respect to \( t \) and \( r \) gives

\[
\frac{\partial}{\partial t}\rho(r, t) = -c_0 f'(r-c_0 t) + c_0 g'(r+c_0 t)
\]

\[
\frac{\partial}{\partial r}\rho(r, t) = f'(r-c_0 t) + g'(r+c_0 t),
\]

where \( f' \) and \( g' \) are the derivatives of \( f \) and \( g \) with respect to their arguments.
and a second derivative gives
\[ \partial^{2t} \rho(r, t) = c^2 \phi''(r - ct) + c^2 \phi''(r + ct). \]

From these last two equations we have the one-dimensional wave equation
\[ \partial^{2t} \rho(r, t) = \frac{1}{c^2} \partial^2 \rho(r, t), \]
which has solutions in Eq. 5.5.1.

**Exercise #14**
Assuming \( f(t) \) and \( g(t) \) are \( \delta \) functions, find the Laplace transform of the solution corresponding to the uniform horn \( A(t) = 1 \).

**Solution:** Using Table 3.9 (p. 160) of Laplace transforms on Eq. 5.5.1 gives
\[ \phi(x, t) = \delta(t - x/c) + \delta(t + x/c) \leftrightarrow e^{-x/vc}, \quad e^{x/vc}. \] (5.5.2)

Note that the delay \( T_0 = \pm x/c \) depends on the range \( x \). 

Three-dimensional d’Alembert spherical eigenfunctions: We can generalize the d’Alembert solution to spherical waves by changing the area function of Eq. 5.2.10 to \( A(r) = A_0r^{-3} \) (see Eq. 5.1.9, p. 229 and Table 5.2, p. 247). The radial wave equation then becomes
\[ \nabla^2 \phi(r, t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \phi(r, t), \] (5.5.3)
where
\[ \nabla^2 \phi(r, t) = \nabla \cdot \nabla \phi(r, t). \] (5.5.4)

Multiplying by \( r \) (see the discussion of the conical horn in Sec. 5.4.7, p. 254) provides the d’Alembert wave equation for a conical horn \( f(t) = r(t - i/\omega) \), or \( g(t + i/\omega) \).

Finally dividing by \( r \) results in the well-known spherical (3D) d’Alembert wave equation eigenfunctions (Eq. 5.5.3)
\[ \phi(r, t) = f(t - i/\omega) + g(t + i/\omega). \] (5.5.5)

This dependence on \( r \) is required for the conservation of energy for spherical waves (Allen and Berkey, 1979).

**5.6 Integral definitions of \( \nabla \), \( \nabla \cdot \), \( \nabla \times \), and \( \nabla \wedge \)**

In Sec. 5.2, page 236, we described to two forms of wave equations, scalar and vector. Up to now we have only discussed the scalar case. The vector wave equation describes the evolution of a vector field, such as Maxwell’s electric field vector \( \mathbf{E}(r, t) \).

There are two equivalent definitions for each of the four operators: differential and integral. The integral form provides a more intuitive view of the operator, which in the limit converges to the differential form. Following a discussion of the gradient, divergence, and curl integral operators, we discuss these two forms.

In addition there are three fundamental vector theorems: Gauss’s law (divergence theorem), Stokes’s law (curl theorem), and Helmholtz’s decomposition theorem. Without the use of these fundamental vector calculus theorems, we could not understand Maxwell’s equations.

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**Solution:** Starting from Eq. H.1.2 with area \( A(r) = A_0 \lambda^2r^2 \)
\[ \frac{1}{A_0} \partial \left( \frac{A_0}{A} \partial \right) \phi(r, t) = \frac{1}{c^2} \partial^2 \phi(r, t), \]
\[ \phi(r, t) + 2m \phi(r, t) = \frac{1}{c^2} \partial^2 \phi(r, \omega), \]
which is the time-domain version of Eq. H.4.6.

Since this equation is second-order in time with constant coefficients, it has two closed-form solutions:
\[ P^+(\omega) = \frac{e^{-\omega \sqrt{(2m + \omega^2)}}}{\sqrt{2\omega}} \]
\[ P^-(\omega) = \frac{e^{\omega \sqrt{(2m + \omega^2)}}}{\sqrt{2\omega}}, \]
with \( \omega = m/c \). The two wave amplitudes \( P^+_m(\omega) \) must be determined from the boundary conditions.

**Exercise #2**
Show that \( P^+(\omega, \omega) \) satisfy Eq. H.4.6.

**Solution:** Taking partials with respect to \( r \),
\[ \partial_r P^+(\omega, r) = \left( -m \mp \sqrt{m^2 + \omega^2} \right) \frac{P^+(\omega, r)}{r}, \]
\[ \partial_r P^-(\omega, r) = \left( -m \pm \sqrt{m^2 + \omega^2} \right) \frac{P^-(\omega, r)}{r}, \]
\[ = 2m^2 \pm \frac{2m \sqrt{m^2 + \omega^2}}{2} \frac{P^+(\omega, r)}{r}. \]
Thus Eq. H.4.6 reduces to
\[ \left( 2m^2 + \omega^2 + 2m \sqrt{m^2 + \omega^2} \right) + 2m \left( -m \mp \sqrt{m^2 + \omega^2} \right) = 0, \]
which is an identity.

Next consider the Fourier series (or Fourier transform) of the area function,
\[ A(r) = \sum_k a_k e^{-2\pi kr}. \]
It follows from the linearity of the wave equation that the general solution of Eq. H.4.6 is
\[ P^+(\omega, r) = \sum_k a_k^+ e^{-\omega k \sqrt{(2m + \omega^2)}}, \]
Here we have combined \( P^+(\omega) \) and \( a_k \) as coefficients \( a_k^+ \) (\( \omega \)).
Show that Eq. H.4.6 follows from Eq. H.1.2.
CHAPTER 5. STREAM 3B: VECTOR CALCULUS

The natural way to define the surface and volume is to place the surface on the isopotential surfaces, forming either a cube or a pill-box-shaped volume. As the volume \(|S|\) goes to zero, so must the area \(|\partial S|\). One must avoid irregular volumes where the area is finite as the volume goes to zero (Greenberg, 1988, footnote p 762).

A well-known example is the potential
\[
\phi(x, y, z) = \frac{Q}{\varepsilon_0 \sqrt{x^2 + y^2 + z^2}} - \frac{Q}{\varepsilon_0 R} \quad [\text{V}]
\]
around a point charge \( Q \) [SI units of coulombs]. The constant \( \varepsilon_0 \) is the permittivity [F/m].

A second well-known example is the acoustic pressure potential around an oscillating sphere, which has the same form (see Table 5.2, p. 247).

How does this work? To better understand what Eq. 5.6.5 means, consider a three-dimensional Taylor series expansion (See Eq. 5.2.4, p. 238) of the potential in \( x \) about the limit point \( x_o \):
\[
\phi(x) \approx \phi(x_o) + \nabla \phi(x_o) \cdot (x - x_o) + \text{HOT}.
\]
We could define the gradient using this relationship as
\[
\nabla \phi(x_o) = \lim_{x \to x_o} \frac{\phi(x) - \phi(x_o)}{x - x_o}.
\]
For this definition to apply, \( x \) must approach \( x_o \) along \( \hat{n} \). To compute the higher-order terms (HOT), we need the Hessian matrix.\(^1\)

The natural way to define a surface \(|S|\) is to find the isopotential contours. The gradient is in the direction of maximum change in the potential, thus perpendicular to the isopotential surface. The key to the integral definition is in taking the limit. As the volume \(|S|\) shrinks to zero, the HOT are small and the integral reduces to the first-order term in the Taylor expansion, since the constant term integrates to zero. Such a construction was used in the proof of the Webbon horn equation (Appendix G, p. 331; Fig. G.1, p. 332).

One major problem with Eq. 5.6.5 is that this definition is self-referencing, since \( \hat{n} \) is based on the gradient. Thus the integral definition of the gradient is based on the gradient. Equation 5.6.5 is similar to the mean value theorem for the gradient.

### 5.6.4 Divergence: \( \nabla \cdot \mathbf{d} = \rho \) [C/m\(^2\)]

The definition of the divergence at \( x = (x, y, z) \) is (see Eq. 5.1.3)
\[
\nabla \cdot \mathbf{d}(x, t) \equiv \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \left( \frac{\partial \mathbf{d}}{\partial x} + \frac{\partial \mathbf{d}}{\partial y} + \frac{\partial \mathbf{d}}{\partial z} \right) \mathbf{d}(x, t) = \rho(x, t),
\]
which maps \( \mathbb{R}^3 \times \mathbb{R}^1 \).

### 5.6.5 Divergence and Gauss’s law

Like the gradient, the divergence of a vector field may be defined as the surface integral of a compressible vector field in the limit as the volume enclosed by the surface goes to zero. As in

\[^1\]H.1, \( \nabla \cdot = \partial^2 \phi / \partial x_1 / \partial x_2 / \partial x_3 \), which exists if the potential is analytic in \( x \) at \( x_o \).

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H.4. RELATIONS BETWEEN STURM-LIOUVILLE AND QUANTUM MECHANICS

#### H.3.1 Eigenmodes of the Rydberg atom

One way to think of eigenmodes is to make an analogy to a piano string or an organ pipe. In these much simpler systems, there is an almost constant delay, say \( \tau \), due to a characteristic length, say \( L = \tau c_0 \), such that the eigenmodes of a string are given by integer multiples of a half wavelength \( \nu = n \omega / 2L \), while the eigenmodes of the organ pipe are multiples of a quarter wavelength. The distinction is the boundary conditions. For the string the boundary conditions are pinned displacement (i.e., zero velocity). The organ pipe is closed at one end and open at the other, resulting in multiples of a quarter wavelength \( \nu = n \omega / 4L \). In each case \( \nu = n \omega / \tau \), where \( \tau = 2L / c_0 \) is the round-trip delay; thus \( \nu = n \omega / 4L \).

We suggest looking at the Rydberg formula in the same way, but with very different eigenfrequencies (Eq. H.3.1). Sommerfeld (1949, p. 201) makes a very interesting comment regarding Eq. H.3.1:

This equation reduces to a simple mathematical formula the eigenvalues of the spectral lines, with their finite cumulation point, the behavior of which differs so fundamentally from that of all mechanical systems.

#### H.3.2 Discussion

The Rydberg frequencies \( f_{n,l} \) \((n = 1, l = 1, \ldots, \infty)\) has poles in the radiation impedance (Eq. H.1.3) when \( \phi(x_o) \in \mathbb{N} \). Working backwards from the Rydberg formula (Eq. H.2.1), we have solved for \( \phi(x_o) \) indicating where this condition is valid (Eq. H.2.3). Since the reflectance and the impedance must be complex-analytic functions of Laplace frequency \( s \), we must replace the discrete frequency \( f_{n,l} \) with \( s \):

\[ e^{2\pi s} f_{n,l} \rightarrow s = \sigma + \omega j, \]

thereby forcing \( \Gamma(s) \) to be a complex-analytic function of \( s \). Then the poles of the radiation impedance must satisfy

\[ \Gamma(s_{nl}) = e^{2\pi s_{nl} f_{nl}} = 1, \]

resulting in eigenfrequencies at \( f_{n,l} \).

The next step in this analysis is to determine the area function \( A(r) \) given \( Z_{nl} \) (Eq. H.1.3). To do this we must solve an integral equation for \( A(r) \), as discussed by Sondhi and Gopinath (1971) and by Yool (1964).

Perhaps this could be done using an analytic representation for the area function,

\[ A(r) = \sum_{k} \alpha_{k} s^{k}. \]

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H.4. Relations between Sturm-Liouville and quantum mechanics

If we compare the Schrödinger equation (SE) for hydrogen with the corresponding Sturm-Liouville equation we can begin to appreciate their differences. The QM equation for hydrogen
APPENDIX H. QUANTUM MECHANICS AND THE QUANTUM THEORY OF ATOMS

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The Rydberg formula is expressed as:

\[ \nu_{n,m} = \frac{1}{4} \left( \frac{\theta}{\rho} \right)^2 \]

where \( \nu_{n,m} \) are the possible eigenfrequencies, \( c_0 \) is the speed of light, \( R \) is the radius of the electron orbit, and \( a = 4 \tan^2 \theta \).

---

5.6. INTEGRAL DEFINITIONS OF \( \nabla(), \nabla \cdot(), \nabla \times(), \text{ AND } \nabla \wedge() \)

- **S Surface (closed)**
- **Tangent plane**
- **\( \rho_{enc} \)** Charge
- **\( \hat{n}_\perp dS \)**
- **\( V \)** Volume

The integral definitions are used to define the electric field and magnetic field in terms of the charge distribution within a volume. The definitions reduce to a common-sense summary that can be grasped intuitively.

The vector calculus integral definitions are:

1. **Divergence**
   
   \[ \nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \rho_{enc} / \varepsilon_0 \]

2. **Curl**
   
   \[ \nabla \times \mathbf{E} = \hat{z} \frac{\partial E_z}{\partial y} - \hat{y} \frac{\partial E_z}{\partial x} - \hat{x} \frac{\partial E_z}{\partial z} = 0 \]

3. **Gauss's Law**
   
   \[ \nabla \cdot \mathbf{E} = \rho_{enc} / \varepsilon_0 \]

4. **Faraday's Law**
   
   \[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \]

---

The series is the Lyman series corresponding to hydrogen. The amplitudes are zero when \( m = n \). The amplitudes are either empty or filled. Why are states either empty or filled?
5.6.6 Magnetic fields \( h, b \)

5.6.7 Integral definition of the curl: \( \nabla \times h = c \)

The second of Maxwell’s equations is as briefly summarized on page 227, the differential definition of the curl maps \( \mathbb{R}^3 \rightarrow \mathbb{R} \). The curl of the magnetic field strength \( h(x) \) is the current density \( c = \sigma e + d \):

\[
\nabla \times h = \left[ \frac{\partial h_y}{\partial x} - \frac{\partial h_z}{\partial y}, \frac{\partial h_z}{\partial x} - \frac{\partial h_x}{\partial z}, \frac{\partial h_x}{\partial y} - \frac{\partial h_y}{\partial x} \right] \text{ [A/m²]}
\]

The definition must hold even in the limit where the surface does not exist (when we define a volume, the surface must be closed).

**Figure 5.6:** Left: The integral definition of the curl is related to that of the divergence (Eq. 5.6.7), as an integration over the tangent to the surface; except: (1) the curl is defined as the cross-product \( h \times b \) [A/m²] of unit vectors \( h \) with the current density \( b \); (2) the magnetic vector \( h \) is open, leaving a boundary \( S \) along the open edge. As with the divergence, which leads to Gauss’s law, this definition leads to a second identity for the magnetic vector \( h \). Stokes’ law (also called the curl theorem for plane (law):

**Curl and Stokes’ law:** Like the gradient and divergence, the curl may be written in integral form, allowing for the physical interpretation of its meaning:

The surface integral definition of \( \nabla \times h = c \) [A/m²], where the current density \( c = \sigma e + d \) is perpendicular to the rotation plane of \( h \).

Stokes’ law states that the open surface integral over the normal component of the curl of the magnetic field strength \( \hat{n} \cdot (\nabla \times h) \text{ [A/m²]} \) is equal to the line integral along the boundary \( \partial S \). As summarized in Fig. 5.6, Stokes’ law is

\[
\mathcal{I}_{\,\text{nc}} = \int_S (\nabla \times h) \cdot \hat{n} \, dS = \oint_{\partial S} h \, d\ell \text{ [A].}
\]

That is: The line integral of \( H \) along the open surface’s boundary \( S \) is equal to the total current enclosed \( \mathcal{I}_{\,\text{nc}} \).

In many texts the normalization (denominator under the integral) is a volume \( V \) (Greenberg, 1988, p. 778, 823–4). However, because the surface is open, this volume does not exist (when we define a volume, the surface must be closed). The definition must hold even in the limit

**H.3. Euclid’s formula and the Rydberg atom model**

Typically one uses the reflectance phase \( 2\pi \phi(f) \); thus the group delay is \( \tau(f) = -\frac{\partial \phi(f)}{\partial f} \), which is physically interpreted here as the frequency-dependent delay from the proton to the radius of the electron’s orbit. Thus this delay is given by

\[
\tau(f) = -\frac{\partial \phi(f)}{\partial f} = \frac{1}{c} \left( \frac{n^2}{c_R \omega} \right)^{1/2} \left( \frac{n^2}{c_R \omega} - \frac{1}{c_R \omega} \right)^{-1/2},
\]

which is constant for low frequencies and then rises to \( \infty \) as frequency approaches the Rydberg frequency \( f \rightarrow c_R/n^2 \).

One may solve Eq. H.1.1 for \( n \), for the case of the Lyman series \( (n_\ell = 1) \), by the use of the following identity for the Rydberg eigenfrequencies \( f_{n}\ell,m \), which follow directly from Eq. H.1.1, with \( m = n_\ell + (\ell + 1)/n \in \mathbb{N} \):

\[
f_{n}\ell,m = \frac{c}{n_\ell} - \frac{c_R}{n_\ell} \left( \frac{1}{n_\ell^2} - \frac{1}{(1 + 1/n_\ell)^2} \right) \text{. (H.2.2)}
\]

Note that as \( f \to \infty \), \( f_{n}\ell,m \to c_R/n_\ell^2 \), which is Sommerfeld’s “finite cumulation point” (Hz).

We can solve Eq. H.2.2 for the mode number \( l/n < 1 \) as a function of mode frequency:

\[
l/n = \frac{1}{\left(1 + \left|\frac{c_R}{c}\right|\right)^2} - 1 \quad \text{Starting from Eq. H.2.2}
\]

\[
\frac{1}{(1 + l/n)^2} = 1 - \frac{n_\ell^2 c_R^2}{c^2} \quad \text{Solving for} \ l
\]

\[
\frac{1}{(1 + l/n)^2} = 1 - \frac{n_\ell^2 c_R^2}{c^2} \quad \text{as summarized in the lower panel of Fig. H.4.}
\]

**H.2.2 Finding the area function**

Once the phase has been determined, we can compute the impedance using Eq. H.1.3. We may then decompose the impedance by using the analytic continued fraction algorithm (or Cauer synthesis), discussed in Sec. 3.8, p. 144.

**H.3 Euclid’s formula and the Rydberg atom model**

Fundamental to quantum mechanics is the Rydberg formula, which describes the quantized energy levels of atoms:

\[
\nu_{n\ell,m} = c_R Z^2 \left( \frac{1}{n^2} - \frac{1}{m^2} \right) \text{. (H.3.1)}
\]

---

3https://www.youtube.com/watch?v=e06WtHsWTo
The group delay, 
\[ \Gamma(s) = e^{-\frac{\omega}{R} s \int_0^t \tau(s) ds} \]

It follows that these relationships are related by a Hilbert transform.

These theorems are mathematical relationships that follow from physical principles.

The proof of Stokes's law.

The integral formulations of Gauss's and Stokes's laws use the concept of the outward normal component. Namely, it must be of the form 
\[ \int_{\partial S} \Phi \cdot \hat{n} = \int_S \nabla \cdot \Phi \, dV \]

This is the time to rethink everything we have de-
CHAPTER 5. STREAM 3B: VECTOR CALCULUS

Theorem, are the three fundamental theorems of vector calculus. Portraits of Helmholtz and Kirchhoff are provided in Fig. 5.8 (p. 261).

Table 5.3: The four possible classifications of scalar and vector potential fields: rotational/irrotational and compressible/incompressible. Rotational fields are generated by the vector potential (e.g., \( A(\mathbf{x}, t) \)), while compressible fields are generated by the scalar potentials (e.g., velocity \( v(\mathbf{x}, t) \), pressure \( p(\mathbf{x}, t) \), or temperature \( T(\mathbf{x}, t) \)).

<table>
<thead>
<tr>
<th>Field</th>
<th>Compressible</th>
<th>Incompressible</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotational</td>
<td>( \nabla \times \mathbf{\nu} \neq 0 )</td>
<td>( \nabla \mathbf{\nu} = 0 )</td>
</tr>
<tr>
<td>Vector wave Eq</td>
<td>( \mathbf{\nabla} \times \mathbf{\nu} = 0 )</td>
<td>Lubrication theory</td>
</tr>
<tr>
<td>Boundary layers</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Irrotational conservative</td>
<td>( \nabla \times \mathbf{\nu} = 0 )</td>
<td>( \mathbf{\nabla} \phi = 0 )</td>
</tr>
</tbody>
</table>
| Acoustics              | \( \mathbf{\nabla} \phi(\mathbf{x}, t) = \frac{2}{\lambda} \phi(\mathbf{x}, t) \) | Laplace’s Eq. (\( \lambda \to \infty \)).

The four categories of linear fluid flow: The following is a summary of the four cases for fluid flow, as shown in Table 5.3:

1. Compressible, rotational fluid (general case): \( \nabla \psi \neq 0, \nabla \times \mathbf{\nu} \neq 0 \). This is wave propagation in a medium where viscosity cannot be ignored, as in acoustics close to the boundaries, where viscosity contributes to losses (Batchelor, 1967).

2. Incompressible, rotational fluid (lubrication theory): \( \mathbf{\nabla} \psi \neq 0, \nabla \mathbf{\nu} = 0, \nabla \times \mathbf{\nu} = 0 \). In this case, the flow is dominated by the walls, while the viscosity and heat transfer introduce shear. This is typical of lubrication theory (solenoidal fields).

3. Compressible, irrotational fluid (acoustics): \( \nu = \nabla \psi, \nabla \times \mathbf{\nu} = 0 \). Here losses (viscosity and thermal diffusion) are small (assumed to be zero). We can define a velocity potential \( \psi \), the gradient of which gives the air particle velocity, thus \( \mathbf{\nu} = -\nabla \psi \). For an irrotational fluid, \( \nabla \times \mathbf{\nu} = 0 \) (Greenberg, 1988, p 826). This is the case for the conservative field, where \( j = \nabla \times \mathbf{E} \) depends on only the end points and \( j = \nabla \times \mathbf{B} = 0 \). When a fluid may be treated as having no viscosity, it is typically assumed to be irrotational, since it is viscosity and shear that lead to rotation (Greenberg, 1988, p 814). A fluid’s angular velocity is \( \Omega = \frac{1}{\rho} \nabla \times \mathbf{\nu} = 0; \) thus irrotational fluids have zero angular velocity (\( \Omega = 0 \)).

4. Incompressible, irrotational fluid (statics): \( \nabla \mathbf{\nu} = 0 \) and \( \nabla \times \mathbf{\nu} = 0 \); thus \( \nu = \nabla \psi \) and \( \nabla \times \mathbf{\nu} = 0 \). An example is water in a small space at low frequencies, where the wavelength is long compared to the size of the container; the fluid may be treated as incompressible. When \( \nabla \times \mathbf{\nu} = 0 \), the effects of viscosity may be ignored, as it is the viscosity that creates the shear that leads to rotation. This is the case in modeling the cochlea, where fluid losses are ignored and the quasistatic limit is justified.

In summary, each of these cases is an approximation that best applies in the low-frequency limit. This is why it is called quasistatic, meaning low but not zero frequency, where the wavelength is more than twice the diameter.

A magnetic solenoidal field is a uniform-flux field \( b_z(\mathbf{x}) \) that is generated by a solenoidal coil and, to an excellent approximation, is uniform inside the coil, making it similar to that of a Rydberg atom, by the use of the Webster horn equation

\[
\frac{1}{A(r)} \frac{\partial}{\partial r} \left( r \frac{\partial A(r)}{\partial r} \right) = \frac{1}{\rho} \frac{\partial}{\partial r} \left( r \rho \frac{\partial A(r)}{\partial r} \right). \tag{H.1.2}
\]

which is a one-dimensional wave equation for the electric potential \( A(r,t) \) propagating in a wave guide having area \( A(r) \) as a function of the range, which \( r \) is the range variable (the axis of wave propagation).

We shall show that given the Rydberg spectrum (Eq. H.1.1), we may accurately estimate the electric reflectance \( \Gamma(s) \) looking out from the origin (i.e., the proton location, as indicated by the small red dot in Fig. H.3). The radiation impedance \( Z_{\text{rad}}(s) \) seen by the proton is related to the reflectance \( \Gamma(s) \) by the relation

\[
Z_{\text{rad}}(s) = \frac{1 + 1/\Gamma(s)}{1 - 1/\Gamma(s)}. \tag{H.1.3}
\]

This formula is the basis of the Smith chart used in both physics and engineering studies. It follows that once \( \Gamma(s) \) is known (i.e., evaluated given Eq. H.1.1), the radiation impedance may be computed. It has been shown that the area function \( A(r) \) may be found given the radiation impedance (Sodhi and Gopinath, 1971; Youla, 1964).
Figure H.1: The spectrum of hydrogen. The 19th century was dominated by theoretical models of hydrogen. This gave rise to the birth of quantum mechanics.

Equation H.1.1: The Rydberg atom model

\[ R = \frac{1}{\lambda^2} \]

The speed of light is given by Eq. H.1.1, having frequencies \( f_n \) and \( \lambda_n \).

\[ f_n = \frac{1}{\lambda_n} \]

The notation "Ly-\( \alpha \)" describes the transition line (see Fig. H.3). For example, according to the work of Tran-Cong, 1993, English translation, losses in the acoustic propagation of airborne sound, as first experimentally described by Mason (1928) p. 311.

\[ f_{\text{Tran-Cong}} = \text{English translation} \]

The Rydberg atom model was supported by the spectrum of the atom, which allows for a radiation spectrum caused by electrons jumping from one energy level to another. It was noted by Bohr in 1913 (Bohr, 1954) and others that the wavelengths of hydrogen, as described in the caption of Fig. H.3, are consistent with Fig. H.2, where the reciprocal wavelength \( \lambda \) is given by Eq. H.1.1, having frequencies \( f_n \) and \( \lambda_n \).

\[ \lambda_n = \frac{1}{f_n} \]

The Rydberg atom model was described by Eq. H.1.1, which is consistent with Fig. H.2, where the reciprocal wavelength \( \lambda \) is given by Eq. H.1.1, having frequencies \( f_n \) and \( \lambda_n \).

\[ \lambda_n = \frac{1}{f_n} \]

For example, think of linear versus angular momentum, which are independent in that they are perpendicular (Fig. 3.4, p. 119).

\[ \mathbf{p} \times \mathbf{v} = \mathbf{s} \times \mathbf{b} \]

The uniform current inside the coil is called the magnetic field produced by a solenoid. A finite solenoidal coil is the north pole of the magnet, and the other end is the south pole. The uniform field inside the coil is created. Such a coil is indistinguishable from a permanent bar magnet with its north and south poles. Depending on the direction of the current, the magnetic field produced by a solenoid is either increasing or decreasing. The uniform field inside the coil is called the magnetic field produced by a solenoid. A finite solenoidal coil is the north pole of the magnet, and the other end is the south pole.

\[ \mathbf{B} = \nabla \times \mathbf{A} \]

The uniform field inside the coil is created. Such a coil is indistinguishable from a permanent bar magnet with its north and south poles. Depending on the direction of the current, the magnetic field produced by a solenoid is either increasing or decreasing.

\[ \mathbf{B} = \nabla \times \mathbf{A} \]

Helmholtz's decomposition theorem is expressed as the linear sum of a scalar potential \( \phi \) and a vector potential \( \mathbf{A} \).

\[ \mathbf{E} = \nabla \phi \]

\[ \mathbf{B} = \nabla \times \mathbf{A} \]

Specifically, \( \mathbf{E} = \nabla \phi \) and \( \mathbf{B} = \nabla \times \mathbf{A} \).

\[ \mathbf{E} = \nabla \phi \]

Strictly speaking, the term "vector potential" applies to only a potential field, as described in the caption of Fig. H.3.

\[ \mathbf{E} = \nabla \phi \]

Of course, this decomposition is general (not limited to the electromagnetic case). It applies to only a potential field, as described in the caption of Fig. H.3.

\[ \mathbf{E} = \nabla \phi \]

This theorem is easily stated (and proved) but less easily appreciated (Heras, 2016). A specific example is the linear sum of a scalar potential \( \phi \) and a vector potential \( \mathbf{A} \).

\[ \mathbf{E} = \nabla \phi \]

In the electric force field, the work done is a constant of four (degrees of freedom).

\[ \mathbf{E} = \nabla \phi \]

Linear and rotational motions are a common theme in physics, rooted in geometry.

\[ \mathbf{E} = \nabla \phi \]

The uniform field inside the coil is created. Such a coil is indistinguishable from a permanent bar magnet with its north and south poles. Depending on the direction of the current, the magnetic field produced by a solenoid is either increasing or decreasing.

\[ \mathbf{E} = \nabla \phi \]
Thus it seems a natural extension to split a vector field into independent translation and rotation parts.

**Example:** A fluid with mass and momentum can be moving along a path and independently be rotating. These independent modes of motion correspond to different types of kinetic energy (modes), such as translational, compressional, and rotational. Each eigenmode of vibration can be viewed as an independent degree of freedom (DoF).

Helmholtz’s decomposition theorem (FTVC) quantifies these degrees of freedom. To prove the construction, second-order vector identities DoC: \( \nabla \nabla \phi = 0 \) and CoG: \( \nabla \times \nabla \phi = 0 \) may be used to verify the FTVC. The FTVC also plays an important role when applied to Maxwell’s equations.

**Helmholtz’s decomposition theorem proof:** To prove Eq. 5.6.11 we must understand how it splits the vector field into parts. The identities have a physical meaning. Every vector field may be split into its translational and rotational parts. If \( e \) is the electric field [V/m], \( \phi \) is the voltage, and \( \alpha \) is the current induced rotational part.

To do this we need the two key vector identities that are always zero for analytic fields: the curl of the gradient (\( \nabla \times \nabla \phi \)),

\[
\nabla \times \nabla \phi (x) = 0, \quad (5.6.12)
\]

and the divergence of the curl (\( \nabla \cdot (\nabla \times \phi) \))

\[
\nabla \cdot (\nabla \times \phi) = 0. \quad (5.6.13)
\]

These identities are easily verified by working out a few specific examples based on the definitions of the three operators, gradient, divergence, and curl, or in terms of the operator’s integral definitions (see page 254).

By applying these two identities to Helmholtz’s decomposition, we can appreciate the theorem’s significance. We can work backward via a physical argument that rotational momentum (rotational energy) is independent of translational momentum. Once these forces are made clear, the vector operations take on meaning. One might conclude that the physics is simply related to geometry via the scalar and vector products.

Specifically, if we take the divergence of Eq. 5.6.11 and use the DoG, then

\[
\frac{1}{\rho} = \nabla \cdot e = \nabla \cdot [\nabla \phi + \nabla \times (\mathbf{A})] - \nabla \cdot \nabla \phi = -\nabla \cdot \alpha.
\]

If we take the curl, then

\[
\mathbf{b} = \nabla \times e = \nabla \times [\nabla \phi + \nabla \times (\mathbf{A})] - \nabla \times \nabla \alpha = \nabla \times A - \nabla \alpha,
\]

since the \( \nabla \times \nabla \) zeros the scalar field \( \phi(x, y, z) \).

**5.6.9 Second-order operators**

In addition to the first-order vector derivatives are second-order operators, the most important being the scalar Laplacian \( \nabla^2 \phi = \nabla \cdot \nabla \phi \) and vector Laplacian \( \nabla^2 \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) \), which operates on vectors.\(^{13}\)

\(^{13}\)Helmholtz was the first person to apply mathematics in modeling the eye and the ear (Helmholtz, 1863a).

**Appendix H**

**Quantum Mechanics and the WHEN**

While it is clear that both Schrödinger’s equation and Dirac’s equations are highly accurate, after about 100 years, it is not clear why. Both of these theories seem to violate classical electromagnetics (EM), such as Ohm’s law, since they are built on energy principles rather than electric and magnetic fields. One point I find disturbing is that QM defines a probabilistic wave function, and appears to not be causal. These two properties are in conflict. How can the past be probabilistic, and how can the future not be?

Here we delve into this question, by providing a classical (i.e., EM-based) derivation for the hydrogen atom, one of the most important and obvious successes of quantum mechanics (QM). The problem with QM is not that it fails—rather, it succeeds, without obvious basis. The problem is that we cannot understand the basic principles, and it seems to be in contradiction with any principles of a physical theory.

Based on the Rydberg formula, we determine the reflection coefficient, and thus the radiation impedance seen by the electron, in a radial coordinate system centered on the proton. Since the electron and proton both have spin \( \frac{1}{2} \), their magnetic fields must attractively align, accounting for the near-field vector potential, and complementing the far-field attraction due to their opposite signs. As the electron and proton approach each other, their far-field potential attraction, the magnetic near field becomes more attractive at close range, due to the magnetic dipoles of the two “particles,” causing them to merge with neutral net magnetic moment and neutral charge, giving a highly stable hydrogen atom. However, given a sufficiently strong distorting field, this highly symmetric state could be disturbed, leading to photon radiation, constrained by the radial eigenstates. It seems more clear than ever that photons and electrons are in a state of equilibrium at the outskirts of very large Rydberg atoms.\(^{14}\)

\(^{14}\)https://physics.aps.org/synopsis-for/10.1103/PhysRevLett.121.193401

**H.1 Equation for Rydberg eigenmodes**

Like every tuned resonant circuit, atoms have well-defined resonant frequencies, or eigenmodes. Figure H.1 shows the observed radiation spectra for hydrogen. From the very beginning, it has been clear that there is a pattern to these spectral lines. In 1880 Rydberg easily fitted a formula that quantifies the observed eigen spectral lines of hydrogen in terms of the reciprocals of the radiated wavelengths:

\[
\frac{1}{\lambda_{\text{min}}} = \frac{R_{\infty}}{c^2} \left( 1 - \frac{1}{n^2} \right), \quad \frac{f_{\text{max}}}{c R_{\infty}} = \frac{1}{m^2} - \frac{1}{n^2} \quad (H.1.1)
\]
There are six second-order combinations of $\nabla$, requiring six mnemonics (Table 5.1, p. 225):

1. **DoG**: Divergence of the gradient (scalar Laplacian operates on scalar potentials (Greenberg, 1988, p. 779)):

   $\nabla^2 \phi = \left( \nabla \cdot \nabla \phi \right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$ (5.6.14)

2. **DoG**: Divergence of the Gradient (Bull-Dog, the vector Laplacian $\nabla^2$ (Sommerfeld, 1952, p. 33)):

   $\nabla^2 a = \left( \nabla \cdot \nabla a \right) = \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} + \frac{\partial^2 a}{\partial z^2} = \nabla^2 a - \nabla \times \nabla \times a$ (5.6.15)

3. **gOd**: Gradient of the Divergence ($\nabla^2 a = \nabla \left( \nabla \cdot a \right)$):

   $\nabla^2 a = \nabla \left( \nabla \cdot a \right) = \hat{x} \frac{\partial}{\partial x} \nabla \cdot a + \hat{y} \frac{\partial}{\partial y} \nabla \cdot a + \hat{z} \frac{\partial}{\partial z} \nabla \cdot a$ (5.6.16)

4. **CoC**: Curl of the curl ($\nabla \times \nabla \times a = \nabla \phi$ - DoG)

   $\nabla \times \nabla \times a = \nabla \left( \nabla \cdot a \right) - \left( \nabla \cdot \nabla \right) a$ which is simply a reorganization of Eq. 5.6.15 (Sommerfeld, 1952, p. 33, Eq. 2b).

5. **DoC**: Divergence of the curl ($\nabla \cdot \nabla \times a = 0$)

6. **CoG**: Curl of the gradient ($\nabla \times \nabla = 0$)

DoC and CoG are special because they are always zero:

$\nabla \cdot \nabla \times a = 0$, $\nabla \times \nabla \phi = 0$, (5.6.17) making them useful in proving the FTVC (Eq. 5.6.11, p. 261).
The third special vector identity
\[ \nabla \times \nabla \times \mathbf{a} = \nabla^2 \mathbf{a} - \nabla \nabla \nabla \mathbf{a}, \quad (5.6.18) \]
as the difference between \( \text{gOd} \) and \( \text{CoC} \) \((\text{i.e., } \text{DoG} = \nabla \nabla \nabla \mathbf{a} - \nabla^2 \mathbf{a}\()\):
\[ \nabla \mathbf{g}(t) = \nabla \nabla \times \mathbf{g}(t), \quad \text{(5.6.19)} \]
The role of \( \text{gOd} \) \((\nabla \nabla \mathbf{a})\) is commonly ignored because it is zero for the magnetic wave equation, due to there being no magnetic charge \( \nabla \mathbf{b}(x, t) = 0 \); thus \( \nabla \mathbf{b}(x, t) \equiv 0 \). However for the electric vector wave equation it plays a role as a source term:
\[ \nabla \mathbf{E}(x, t) = -\nabla \mathbf{E}(x, t) - \frac{1}{\epsilon_0} \nabla \mathbf{E}(x, t), \]
or since \( \nabla \mathbf{d} = \rho \)
\[ \nabla \mathbf{d}(x, t) = \nabla \nabla \mathbf{d} = -\nabla \mathbf{E}(x, t). \]

**Exercise #16**
Show that \( \text{DoG} \): \( \nabla \times \mathbf{g}(t) \) and \( \text{gOd} \): \( \nabla \mathbf{g}(t) \) differ.

**Solution:** Use \( \nabla \times \nabla \mathbf{g}(t) \) to explore this relationship. If \( \text{DoG} \) and \( \text{gOd} \) were the same, \( \nabla \times \nabla \mathbf{g}(t) \) would be null.

**Exercise #17**
What is the difference between \( \text{DoG} \): \( \nabla \times \mathbf{g}(t) \) and \( \text{Bull-DoG} \): \( \nabla \mathbf{g}(t) \)?

**Solution:** \( \text{DoG} \) operates on scalar functions while \( \text{Bull-DoG} \) operates on vector functions.

**Discussion:** It is helpful to view these two groups as playing fundamentally different roles: \( \text{utility operators } \nabla \times \nabla \mathbf{g}(t) \): \( \text{DoG} \), \( \nabla \mathbf{g}(t) \), and \( \text{gOd} \): \( \nabla \mathbf{g}(t) \), and

identity operators \( \text{DoC}: \nabla^2 = 0 \) and \( \text{CoC}: \nabla \times \nabla = 0 \) \((\text{Eq. 5.6.17})\).

When using second-order differential operators, we must be careful with the order of operations, which can be subtle. Most of this is common sense. For example, don’t operate on a scalar field with \( \nabla \) and don’t operate on a vector field with \( \nabla \). \( \text{DoG} \) acts on each vector component \( \nabla \mathbf{a} = \nabla A_\mathbf{k} + \nabla A_\mathbf{k} \hat{\mathbf{r}} + \nabla A_\mathbf{k} \hat{\mathbf{r}} \) \((\text{Eq. 5.6.15})\), which is very different from the action of the Laplacian \( \nabla \times \mathbf{g}(t) \).

5.7 The unification of electricity and magnetism

Once you have mastered the three basic vector operations—gradient, divergence, and curl—we are ready to appreciate Maxwell’s equations. Like the vector operations, these equations may be written in integral or differential form. An important difference is that with Maxwell’s equations, we are dealing with well-defined physical quantities. The scalar and vector fields take on meaning and units. Thus, to understand these important equations, we must master

G.1. OVERVIEW

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**M(x).** The ratio \( C/M \) determines the area-dependent characteristic admittance \( Y_x(x) \in \mathbb{R} \):
\[ Y_x(x) = \sqrt{\text{stiffness}} \frac{1}{\text{mass}} \frac{\sqrt{|u_x(x) s_x|}}{Z(x)} = \frac{C(x)}{M(x)} \frac{Y_x(x)}{\sqrt{|u_x(x) s_x|}} = A(x) \frac{M(x)}{\sqrt{|u_x(x) s_x|}} > 0 \quad (G.1.9) \]
(Campbell, 1903, 1910, 1922). The characteristic impedance is \( Z(x) = 1/\sqrt{Y_x(x)} \). Based on a physical argument, \( Y_x(x) \) must be positive and real; thus only the positive square root is allowed. As long as \( A(x) \) has no jumps (is continuous), \( Y_x(x) \) must be the same in both directions. It is locally determined by the isopressure surface and its volume velocity.

**Radiation admittance**

The radiation admittance is defined looking into a horn with no termination (infinitely long) from the input at \( x = 0 \):
\[ Y^\text{rad}_\omega(x) = \frac{Y^\text{rad}}{Z^\text{rad}} \subset \mathbb{C}. \quad (G.1.10) \]
The impedance depends on the direction, with \( \text{+ looking to the right} \) and \( \text{− to the left} \).

The input admittance \( Y^\text{rad}_\omega(x, s) \) is computed using the upper equation of Eq. 5.2.11 \((p. 242)\) for \( \tilde{V}(x, s) \) and then dividing by the pressure eigenfunction \( P^\text{rad} \). This results in the logarithmic derivative of \( P^\text{rad}(x, s) \):
\[ Y^\text{rad}_\omega(x, s) \equiv \frac{Y^\text{rad}}{Z^\text{rad}} = -\frac{1}{s[M(x)]} \ln P^\text{rad}(x, s). \]

For example, for the conical horn (last column of Table 5.2, p. 247)
\[ Y^\text{rad}_\omega(x, s) = Y_\omega(x, s) = \frac{1}{Z^\text{rad}_\omega(x, s)} \quad (G.1.11) \]
Note that \( Y^\text{rad}_\omega(x, s) + Y^\text{rad}_\omega(x, s) = 2Y^\text{rad}_\omega(x, s) = 2c_0^2 \sqrt{|u_x(x) s_x|} \subset \mathbb{R} \), which shows that the frequency-dependent parts of the two admittances, being equal and opposite in sign, exactly cancel.

As the wavefront travels down the variable-area horn, there is a mismatch in the characteristic admittance due to the change in area. This mismatch creates a reflected wave, which in the case of the conical horn is \( s^{-c_0^2}/s^x \). Due to conservation of volume, there is a corresponding identical forward component that travels forward, equal to \( s^{-c_0^2}/s^x \). The sum of these two responses to the change in area must be zero in order to conserve volume velocity.

The resulting equation for the velocity eigenfunctions is therefore
\[ V^\text{rad}(x, s) = Y^\text{rad}_\omega(x, s) P^\text{rad}(x, s). \]

**Propagator function** \( e(s) \)

The eigenfunctions of the lossless wave equation propagate as
\[ P^\text{rad}(x, s) = e^{s \pm s^x (x)} \sqrt{|u_x(x)|} \]
where \( s = \sqrt{2 Z(x, s)^2} \omega \). The velocity eigenfunctions \( V^\text{rad}(x, s) \) may be computed from Eq. G.1.4.

From the above definitions,
\[ s^x(s) = \sqrt{\frac{\sigma_o s^x(x) H(x)}{\frac{\beta^x(s)}{\text{rad}_\omega(x, s)}}} = \frac{\omega}{\text{rad}_\omega(x, s)} \]
Thus \( s^x(s) \) and \( s^x \) are the eigenvalues of the differential operators \( \partial / \partial x \) and \( \partial / \partial s \) on the pressure \( P(x, s) \). See Appendix D for the inclusion of visco-thermal losses.
Figure G.1: Derivation of the horn equation using Gauss's law: The divergence of...Fig. G.1, taking the limit of the difference between the two volume velocities $\nu(x + \delta x) - \nu(x)$ divided by $\delta x$ results in $\partial \nu / \partial x$.

5.7. THE UNIFICATION OF ELECTRICITY AND MAGNETISM

Table 5.4: This table defines the four electromagnetic variables...are easily understood by noting their units. A speed of light in-vacuo $c$ has units [V/m]. By integrating $\mathbf{E} \cdot d\mathbf{A}$, $\mathbf{E}$ and $d\mathbf{A}$ over a sphere and using the definition of $\mathbf{E}$ on the right in (6.4.16), we obtain (6.4.17), where $-\nabla \cdot \mathbf{D} = \rho / \varepsilon_0 = \sigma / \varepsilon_0$.

Field strength $\mathbf{E}$, $\mathbf{D}$: As summarized by Eq. 5.7.1, there are two field strengths: the electric field strength $\mathbf{E}$ with units of [V/m] and the magnetic field strength $\mathbf{B}$ with units of [T]. The ratio $E / B$ or $B / E$ is called the magnetic field strength in the electric field.

To help us understand the meaning of $\mathbf{E}$, consider the electrically induced magnetic field $\mathbf{B}$ inside a surface $\mathbf{E}$ by a voltage drop. Once you grasp the concept of Ohm's Law, you're there.

5.7.1 Maxwell's equations

Maxwell's equations consist of two curl equations (Eqs. 5.7.11) operating on the field strength $\mathbf{E}$ and $\mathbf{B}$, and two divergence relations (Eqs. 5.7.3) operating on the field strength $\mathbf{D}$ and the electric flux density $\mathbf{B}$.

5.7.1 Maxwell's equations

Maxwell's equations consist of two curl equations, (Eqs. 5.7.11) operating on the field strength $\mathbf{E}$ and $\mathbf{B}$, and two divergence relations (Eqs. 5.7.3) operating on the field strength $\mathbf{D}$ and magnetic flux density $\mathbf{B}$.
on the field fluxes $E$ and $M$. In matrix format, the ME are
\[
\nabla \times \begin{bmatrix} e(x, t) \\ h(x, t) \end{bmatrix} = \hat{\sigma} \begin{bmatrix} 0 & -\mu_0 \\ \mu_0 & 0 \end{bmatrix} \begin{bmatrix} e(x, t) \\ h(x, t) \end{bmatrix} + \begin{bmatrix} -\sigma_c e(x, t) \\ -\mu_0 \omega h(x, t) \end{bmatrix},
\]
(5.7.1)

When the medium is conducting, $\partial / \partial t d$ must be replaced by $e = \sigma_c E + \partial / \partial t \epsilon_0 \omega e(x, t)$, where $\epsilon_0 < \sigma_c$ is an admittance density [S/m²].

There are also two auxiliary equations:
\[
\nabla \cdot \begin{bmatrix} E(x, t) \\ M(x, t) \end{bmatrix} = \hat{\rho} \begin{bmatrix} \rho \equiv \frac{1}{\epsilon_0 c} \\ \mu_0 \omega \end{bmatrix} \begin{bmatrix} E(x, t) \\ M(x, t) \end{bmatrix}.
\]

Exercise #19

Exercise #18

The top equation states conservation of charge, while the lower states that there is no magnetic charge. When expressed in integral format, Stokes’s law follows from the curl equations and Gauss’s law from the divergence equations.

**Exercise #18**

When a static current is flowing in a wire in the $\hat{k}$ direction, the magnetic flux is determined by Stokes’s theorem (Fig. 5.6). Thus, just outside the wire we have
\[
\mathcal{E}_{\text{ext}} = \oint_S \nabla \times h \cdot \hat{n} dS = \oint_B h \cdot d\ell \quad [A].
\]

For this simple geometry, the current in a wire is related to $h(x, t)$ by
\[
\mathcal{E}_{\text{ext}} = \oint_B h \cdot d\ell = H_0 \pi r.
\]

Here $H_0$ is perpendicular to both the radius $r$ and the direction of the current $\hat{k}$. Thus
\[
H_0 = \frac{\mathcal{E}_{\text{ext}}}{2\pi r},
\]
where $\mathcal{E}_{\text{ext}}$ is attenuated by $1/r$ (Ramo et al., 1965, Eq. 9, page 244).

**Exercise #19**

Explain how Stokes’s theorem may be applied to $\nabla \times e = -b$, and explain what it means. Hint: This is the same argument given above for the current in a wire, but for the electric case.

**Solution:** Integrating the left side of equation Eq. 5.7.3 over an open surface results in a voltage (emf) induced in the loop closing the boundary $B$ of the surface
\[
\phi_{\text{eddy}} = \oint_B \nabla \times e \cdot \hat{n} dS = \oint_B e \cdot d\ell \quad [V].
\]

Integrating the left side of equation Eq. 5.7.3 over an open surface results in a voltage (emf) induced in the loop closing the boundary $B$ of the surface
\[
\phi_{\text{eddy}} = \oint_B \nabla \times e \cdot \hat{n} dS = \oint_B e \cdot d\ell \quad [V].
\]

The emf (electromagnetic force) is the same as the Thévenin source voltage induced by the rate of change of the flux. Integrating Eq. 5.7.3 over the same open surface $S$ results in the source
\[
\oint_B \nabla \times e \cdot \hat{n} dS = \oint_B e \cdot d\ell \quad [V].
\]

**Appendix G**

**Webster horn equation derivation**

**G.1 Overview**

In this appendix we transform the acoustic equations, Eqs. 5.2.5 and 5.2.6 (p. 239), into their equivalent integral form, Eq. 5.2.10 (p. 244). This derivation is similar (but not identical) to that of Hanna and Slepian (1924) and Pierce (1981, p. 360).

**G.1.1 Conservation of momentum**

The first step is to integrate the normal component of Eq. 5.2.5 (p. 239) over the isopressure surface $S$, defined by $\nabla p = 0$,
\[
-\int_S \nabla p(x, t) \cdot dA - \rho_s \frac{\partial}{\partial t} \int_S u(x, t) \cdot dA.
\]

The average pressure $p(x, t)$ is defined by dividing by the total area,
\[
p(x, t) = \frac{1}{A(x)} \int_S p(x, t) \hat{n} \cdot dA.
\]

From the definition of the gradient operator, we have
\[
\nabla p = \frac{\partial p}{\partial x},
\]
(5.1.2)

where $\hat{n}$ is a unit vector perpendicular to the isopressure surface $S$. Thus the left side of Eq. 5.2.5 reduces to $(\partial p(x, t)) / \partial x$.

The integral on the right side defines the volume velocity,
\[
u(x, t) = \int_S u(x, t) \cdot dA.
\]

Thus the integral form of Eq. 5.2.5 becomes
\[
-\frac{\partial}{\partial x} p(x, t) = \frac{\rho_s}{A(x)} \frac{\partial}{\partial x} \int_S u(x, t) \hat{n} \cdot dA,
\]
(5.1.3)

where
\[
\int_S u(x, t) \hat{n} \cdot dA = \rho_s (x, t) \int_S u(x, t) \cdot dA
\]
and $M(x) = \rho_s / A(x)$ is the per-unit-length mass density of air.
5.7. THE UNIFICATION OF ELECTRICITY AND MAGNETISM

The induced voltage \( \phi_{\text{induced}} \) is proportional to the rate of change of the flux [weber]:

\[
\phi_{\text{induced}} = - \frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \hat{n} \, dA = L \dot{\psi} \quad \text{[Wb/s] or [V]},
\]

where \( L \) [H] is the inductance of the wire, with impedance \( Z_L(s) = sL \). The area integral on the right is the total flux crossing normal to the surface \( \psi \) [Wb]. The rate of change of the total flux [Wb/s] is the induced (Thévenin) voltage [V].

If we apply Gauss's theorem to the divergence equations, we find the total flux that crosses the closed surface.

Exercise #20

Apply Gauss's theorem to equation ED and explain what it means in physical terms.

Solution:
The area of the normal component of \( \mathbf{D} \) is equal to the volume integral over the charge density. Thus Gauss's theorem says that the total charge within the volume \( Q_{\text{enc}} \) found by integrating the charge density \( \rho(x) \) over the volume \( V \), is equal to the normal component of the flux \( \mathbf{D} \) that crosses the surface \( S \):

\[
Q_{\text{enc}} = \iiint_V \nabla \cdot \mathbf{D} \, dV = \iint_S \mathbf{D} \cdot \hat{n} \, dA.
\]

When equal amounts of positive and negative charge exist within the volume, regardless of its distribution, the integral is zero.

Summary:
Maxwell's four equations relate the field strengths to the flux densities. There are two types of variables: field strengths (\( \mathbf{E}, \mathbf{H} \)) and flux densities (\( \mathbf{D}, \mathbf{B} \)). There are two classes: electric (\( \mathbf{E}, \mathbf{D} \)) and magnetic (\( \mathbf{H}, \mathbf{B} \)). This is a \( 2 \times 2 \) matrix, with column being field strength and flux densities and rows being electric and magnetic variables.

5.7.2 Derivation of the vector wave equation

Next we provide the derivation of the vector wave equation starting from Maxwell's equations (Eq. 5.7.1), which is reminiscent of the derivation of the Webster horn equation (Eq. 5.2.2, p. 239).

Here we use the same matrix method we used in Eq 5.7.1, to both swap \( \mathbf{B} = \mu_0 \mathbf{H} \) and \( \mathbf{D} = \epsilon_0 \mathbf{E} \) and transform \( \frac{\partial}{\partial t} \leftrightarrow s \), due to the Laplace transform relation \( \frac{\partial}{\partial t} \leftrightarrow s = \sigma + j\omega \).
Working in the frequency domain and taking the curl of both sides gives

\[ \nabla \times \nabla \times \left[ \begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right] (x, t) = - \frac{1}{s \epsilon_o} \nabla \times \left[ \begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right] (x, t) = - \frac{1}{s \epsilon_o} \left[ \begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right] \]

Note that \( s \epsilon_o \nabla \times = s^2 \mu_o \). If \( \mu \) and \( \epsilon \) were either functions of space or time, this transformation would not apply.

We must deal with the left hand side, which is \( \nabla \times \nabla \times (\mathbf{E}) = \nabla \times (\mathbf{E}) - \nabla \times (\mathbf{H}) \), which results in

\[ \left[ \begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right] = \left[ \begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right] - \frac{s^2 \mu_o}{c_o^2} \left[ \begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right] \]

Rearranging terms results in driven vector wave equations, for both \( \mathbf{e}(x, t) \) and \( \mathbf{h}(x, t) \)

\[ \left[ \begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right] = \left[ \begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right] - \frac{s^2 \mu_o}{c_o^2} \left[ \begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right], \quad \nabla \times \left[ \begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right] = s^2 \mu_o \left[ \begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right] \]

with the charge term \( \nabla \psi(x, s) \) being the sole source of excitation. Note that if \( \mu \) and \( \epsilon \) depend on \( x \), the terms on the right would not be zero.

In remote outer space, where black holes and hot plasma frequencies exist (e.g., inside the sun), this seems likely.

Recall the d’Alembert solutions of the scalar wave equation (Eq. 4.4.1, p. 188)

\[ \epsilon(x, t) = f(x - ct) + g(x + ct), \]

where \( f \) and \( g \) are arbitrary vector fields and \( c \) is the wave speed. As expected, the solutions are delayed by the speed of light.

**Poynting vector:** The EM power flux density \( \mathbf{P} \) [W/m²] is perpendicular to \( \mathbf{E} \) and \( \mathbf{h} \), denoted as

\[ \mathbf{P} = \mathbf{E} \times \mathbf{h} \quad [\text{W/m}^2] \]

The corresponding EM momentum flux density, \( \mathbf{M} \) [N/m²], is

\[ \mathbf{M} = \epsilon \mathbf{E} \times \mathbf{h} \quad [\text{N/m}^2] \]

Since the speed of light is \( c_o = 1/\sqrt{\epsilon_o \mu_o} \), we see that

\[ \mathbf{P} = \mathbf{E} \mathbf{M} \quad [\text{W/m}^2] \]

which is related to Einstein’s energy–mass equivalence formula \( E = mc_o^2 \), as derived in a more general complex-analytic form on p. 122.

For example, the power emitted by the sun is about 1360 [W/m²], with a radiation pressure of 4 [µN/m²] (i.e., 4 [µPa]) (Fitzpatrick, 2006). By way of comparison, the threshold audible acoustic pressure at the human eardrum at 1 [kHz] is 20 [µPa], which is 14 [dB] (a factor of 5) the solar radiation pressure. Also:

\[ F.3 \quad \text{TAXONOMY OF ALGEBRAIC NETWORKS} \]

Sergei Schelkunoff may have been the first to formalize this concept (Schelkunoff, 1943), but he was not the first to use it, as exemplified by the Helmholz resonator. George Ashley Campbell was the first to use the concept in the important application of a wave filter, some 30 years before Schelkunoff (Campbell, 1903). These two men were 40 years apart and both worked for the telephone company (after 1929, called AT&T Bell Labs) (Fagen, 1975).

There are alternative definitions of the QS approximation, depending on the geometric cell structure. The alternatives are listed in Table F.1.

**The quasistatic approximation:** Since the velocity perpendicular to the walls of a horn must be zero, any radial wave propagation is exponentially attenuated (\( \epsilon(x) \) is real and negative, i.e., the propagation function \( \epsilon(x) \) (Sec. 4.4, p. 189) will not describe radial wave propagation), with a space constant of about 1 diameter. The assumption that these radial waves can be ignored (i.e., more than 1 diameter from their source) is called the quasistatic approximation. As the frequency is increased and once \( f > f_r = 2c_o/\lambda \), the radial wave can satisfy the zero normal velocity wall boundary condition and therefore will not be attenuated. Thus above this critical frequency, radial waves (also known as higher-order modes) are supported (\( \epsilon \) becomes imaginary). Thus for Eq. 5.2.10 (p. 241) to describe guided wave propagation, \( f < f_r \). But even under this condition, the solution is not precise within a diameter (or so) of any discontinuities (i.e., rapid variations) in the area.

Each horn, as determined by the area function \( A(r) \), has a distinct wave equation and thus a distinct solution. Note that the area function determines the upper cutoff frequency via the quasistatic approximation, since \( f_r = c_o/\lambda \), \( \lambda > d \) and \( A(r) = \pi d^2/4 \). Thus to satisfy the quasistatic approximation, the frequency \( f \) must be less than the cutoff frequency:

\[ f < f_r(r) = \frac{c_o}{2} \sqrt{\frac{\pi}{A(r)}} \quad (F.3.4) \]

We have discussed two alternative matrix formulations of these equations: the ABCD transmission matrix, used for computation, and the impedance matrix, used when working with experimental measurements (Pierce, 1981, Chap. 7). For each formulation, reciprocity and reversibility show up as different matrix symmetries, as addressed in Sec. 3.10 (p. 155) (Pierce, 1981, pp. 195–203).

**Periodic ↔ discrete:** P11 As has been shown in the discussion on the Fourier transform, when the time (or frequency) domain response is periodic, the frequency (or time) domain is discrete. This is a fundamental symmetry property that must always be obeyed. This is closely related to the causal ↔ complex-analytic property of the Laplace and \( z \)-transforms.

**Summary**

A transducer converts between modalities. We propose the general definitions of the eleven system postulates that include all transduction modalities, such as electrical, mechanical, and acoustical. It is necessary to generalize the concept of the QS approximation (Postulate P9) to allow for guided waves.

Given the combination of the important QS approximation and these space-time, linearity, and reciprocity properties, a rigorous definition and characterization of a system can thus be established. It is based on a taxonomy of such materials formulated in terms of material and physical properties and extended network postulates.
The lasers used in Inertial Confinement Fusion (e.g., the Nova experiment in Lawrence Livermore National Laboratory) typically have energy fluxes of $2 \times 10^{12}$ W/m$^2$. For comparison, the Bohr radius (hydrogen) is estimated to be $\mu_0 = 2.8 \times 10^{-15}$ m (2.8 femtometers).

For copper $\sigma = 1.6 	imes 10^{7} \Omega^{-1} \text{m}^{-1}$, where $\sigma$ is the conductivity and $\lambda$ is the cellular dimension $= 2 \lambda_0$. For example, $\sigma = 6 \times 10^{7}$ W/m$^2$ for diffusion, as in the diffusion of heat. There is also the diffusion of electrical and magnetic fluxes of $q$. The diffusion of heat is called the skin effect.

The QS property is not mentioned in the six postulates of Carlin and Giordano. Although I have never seen the idea discussed in the literature, the QS approximation is defined as

$$\mu(x, t) = \sigma \phi(x, t)$$

which is the force on a charge (e.g., electron) due to the electric and magnetic fields. The force $\mu$ is the electric potential term and depends on the wave speed. When the wavelength $\lambda$ is much greater than the cell size $\lambda_0$, the required condition is

$$\lambda > \lambda_0$$

Up to now we have considered only the Brune

## 5.8. POTENTIAL SOLUTIONS OF MAXWELL'S EQUATIONS

Although I have never seen the idea discussed in the literature, the QS approximation is defined as

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Up to now we have considered only the Brune...
normally taken to be zero because taking the curl of $h(t)$ naturally removes any electrical potential term due to CoG.\textsuperscript{17} The extension makes the potential solutions symmetric so that $e$ and $h$ each have electrical and magnetic excitation.

**Exercise 2.1**

Explain why some dependence on $\phi(x, t)$ do not appear in Eq. 5.8.2 but do in 5.8.1.

**Solution:** For $h(x, t)$ to depend on $\phi(x, t)$ it must appear through the electric strength, as $e(x, t) = -\nabla \phi(x, t)$. But then $\nabla \times h(x, t)$ would mean applying CoG (i.e., $\nabla \times \nabla \phi = 0$) on the right side of the equation. Since this term would be zero, it is assumed to be zero, thus $h(x, t)$ depends on only $\alpha(x, t)$. To fill out the symmetry, we have added $\partial h(x, t)$ to Eq. 5.8.2, to see what might happen in a more general case.

**Use of Helmholtz’s theorem on potential solutions:** The generalized solutions to Maxwell’s equations (Eqs. 5.8.1 and 5.8.2) have been expressed in terms of EM potentials $\phi(x, t)$ and $\alpha(x, t)$ and Helmholtz’s theorem. These are solutions to Maxwell’s equations expressed in terms of the potentials $\phi(x, t)$ and $\alpha(x, t)$, as determined at the boundaries (Sommerfeld, 1952, p 146).

These relationships are invariant to certain functions added to each potential, as shown below. They are equivalent to Maxwell’s equations following the application of CoG for $x_i f_i$.

When these off-diagonal elements are equal $\{z_{ij}(x) = z_{ji}(x)\}$, the system is said to obey Rayleigh reciprocity. If they are opposite in sign $\{z_{ij}(x) = -z_{ji}(x)\}$, the system is said to be antireciprocal. If a network has neither reciprocal nor antireciprocal characteristics, then we denote it nonreciprocal (McMillan, 1946). The most comprehensive discussion of reciprocity, even to this day, is that of Rayleigh (1896, Vol. 1). The reciprocal case may be modeled as an ideal transformer (Van Valkenburg, 1964a), while for the antireciprocal case the generalized force and flow are swapped across the two-port. An electromagnetic transducer (e.g., a moving-coil loudspeaker or electrical motor) is antireciprocal (Kim and Allen, 2013; Beranek and Mellow, 2012); it requires a gyrator rather than a transformer, as shown in Fig. F.1.

**Reversibility:** P7 A second two-port property is the reversible/nonreversible postulate. A reversible system is invariant to the input and output impedances being swapped. This property is defined by the input and output impedances being equal.

Referring to Eq. F.3.3, when the system is reversible, $z_{ij}(x) = z_{ji}(x)$ or, in terms of the transmission matrix variables, $\Delta s_{ij} = 0$ or simply $A(s) = D(s)$, assuming $C(s) \neq 0$.

An example of a nonreversible system is a transformer with a turns ratio that is not 1. Also, an ideal operational amplifier (when the power is turned on) is nonreversible due to the large impedance difference between the input and output. Furthermore, it is active; it has a power gain due to the current gain at constant voltage (Van Valkenburg, 1964b).

Generalizations of this lead to group theory and Noether’s theorem. These generalizations apply to systems that have many modes, whereas quasistatics holds when they operate below a cutoff frequency (Table F.1), meaning that, as in the case of the transmission line, there are no propagating transverse modes. While this assumption is never exact, it leads to highly accurate results because the nonpropagating evanescent transverse modes are attenuated over a short distance and thus, in practice, may be ignored (Montgomery et al., 1948; Schwinger and Saxen, 1968, Chaps. 9-11).

We extend the Carlin and Giordano postulate set to include Postulate P7, reversibility, which was refined by Van Valkenburg (1964a). To satisfy the reversibility condition, the diagonal components in a system’s impedance matrix must be equal. In other words, the input force and flow are proportional to the output force and flow, respectively (i.e., $Z_{ii} = z_{ii}$).

**Spatial invariance:** P8 The characteristic impedance and wave number $k(x, s, t)$ may be strongly frequency- and/or spatially dependent or even be negative over some limited frequency ranges. Due to causality, the concept of a negative group velocity must be restricted to a limited bandwidth (Brillouin, 1960). As Einstein’s theory of relativity makes clear, all materials must be strictly causal (Postulate P1), a view that must therefore apply to acoustics but at a very

\textsuperscript{17} In vacuo $\epsilon_r = 8.85 \times 10^{-12}$ [F/m$^2$] is the capacitance, and $\epsilon_r$ is the electric compliance-density of light. The related magnetic mass-density is the permeability $\mu_r = 4 \pi \times 10^{-7}$ [H/m$^2$] having an inductive impedance of $\mu_{r0}$ [Ohm]. It is helpful to think of $\epsilon_r$ as a capacitance per unit area and $\mu_r$ as an inductance per unit area (consistent with their units). The in vacuo speed of light is $c_0 = 1/\sqrt{\epsilon_o \mu_o} = 3 \times 10^8$ [m/s], but is slower when traveling in matter (Brillouin, 1969).
APPENDIX F. ELEVEN POSTULATES OF SYSTEMS OF ALGEBRAIC NETWORKS

2. If $Z(s)$ is PR, then its reciprocal $Y(s) = 1/Z(s)$ is PR. This means that the system coefficients of the differential equation describing the system are constant (independent of time).

5.8. POTENTIAL SOLUTIONS OF MAXWELL'S EQUATIONS

This last equation may be split into two independent equations by the use of Helmholtz's theorem:

$$\nabla \cdot \nabla \times h(x, t) = \nabla \cdot j_m(x, t) + \frac{\partial \rho(x, t)}{\partial t} = 0,$$

which is conservation of charge (i.e., Gauss's theorem). ■

Exercise #

Exercise #

obeyed (Schwinger and Saxon, 1968, p.17). This means that the total energy absorbed by any system does not change over time. This requires that the system coefficients of the differential equation describing the system are constant (independent of time).
5.9 Problems VC-2

5.9.1 Topics of this homework:

Partial differential equations: fundamental theorem of vector calculus (Helmholtz’s theorem); wave equation; Maxwell’s equations (ME) and variables (E, D, B, H); Second-order vector differentials; Webster horn equation.

Hint: Expand both sides of the equation.

5.9.2 Partial differential equations (PDEs): Wave equation

Problem # 1: Solve the wave equation in one dimension by defining $\xi = (t-x)/c$.

\[ -1.1: \text{Show that d’Alembert’s solution, } g(x,t) = f(t-x/c) + g(t+x/c), \]
\[ \text{is a solution to the acoustic pressure wave equation in one dimension:} \]
\[ \frac{\partial^2 g(x,t)}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 g(x,t)}{\partial t^2}, \]
\[ \text{where } f(\xi) \text{ and } g(\xi) \text{ are arbitrary functions.} \]

Problem # 2: Solving the wave equation in spherical coordinates (i.e., three dimensions)

\[ -2.1: \text{Write the wave equation in spherical coordinates } g(r,\theta,\phi). \]
\[ \text{Consider only the radial term } r \text{ (i.e., dependence on angles } \theta \text{ and } \phi \text{ is assumed to be zero).} \]
\[ \text{Hint: The form of the Laplacian as a function of the number of dimensions is given in Eq. 5.1.9 (page 229).} \]
\[ \text{Alternatively, look it up on the internet or in a calculus book.} \]

\[ -2.2: \text{Show that this equation is true:} \]
\[ \nabla^2 g(r) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial g(r)}{\partial r} \right) = \frac{1}{r^2} \frac{\partial^2 g(r)}{\partial \theta^2}. \]

(Hint: Expand both sides of the equation.)
what is needed to fully characterize its operation, and that the delay is proportional to the distance. Thus we
generalize Postulate P1 to include the space-dependent delay. When we wish to discuss this
5.9. PROBLEMS VC-2
– 2.3: Use the results from Eq. VC-2.1 to show that the solution to the spher-
ical wave equation, where
\[ \frac{\partial^2 u}{\partial t^2} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) \]
is
\[ u(r, t) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n \pi}{a} r \right) \sin \left( \frac{n \pi}{a} c t \right) \]
where \( a \) is the radius of the sphere, \( c \) is the speed of sound, and \( A_n \) are constants.

\[ \nabla \times \mathbf{v} = \mathbf{0} \]
requires that \( \mathbf{v} \) be irrotational. We say this irrotational field is generated by
\[ \nabla \phi = \nabla (\phi_0 + \phi_1) \]
where \( \phi_0 \) is another scalar field, such that
\[ \phi_1 \propto \| \mathbf{b} \| \cos(\mathbf{b} \cdot \mathbf{a}) \]
and
\[ \phi_0 \propto \| \mathbf{b} \| \sin(\mathbf{b} \cdot \mathbf{a}) \]

\[ \frac{1}{\rho c^2} \frac{\partial^2 \rho c^2}{\partial r^2} + \frac{1}{r} \frac{\partial \rho c^2}{\partial r} + \frac{\partial^2 \rho c^2}{\partial z^2} = \frac{\partial^2 \rho c^2}{\partial t^2} \]
is the spherical wave equation, where \( \rho \) is the density of the medium, \( c \) is the speed of sound, and \( \phi \) is the pressure.

\[ \mathbf{v} \times \nabla = \mathbf{0} \]
and \( \mathbf{v} \cdot \nabla = \mathbf{0} \)
are the incompressible and rotational parts, respectively, of \( \mathbf{v} \).

5.9. PROBLEMS VC-2
– 2.4: Using Eq. (F.2.1) and \( \xi \), show that the solution to the spherical wave equation is
\[ u(r, t) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n \pi}{a} r \right) \sin \left( \frac{n \pi}{a} c t \right) \]
where \( a \) is the radius of the sphere, \( c \) is the speed of sound, and \( A_n \) are constants.

\[ \mathbf{v} = \nabla \phi + \mathbf{w} \]
and
\[ \mathbf{w} = \nabla \times \mathbf{v} \]
are the irrotational and antisymmetric parts, respectively, of \( \mathbf{v} \).

Causality: P1

\[ \mathbf{v} \cdot \mathbf{w} = 0 \]
requires that \( \mathbf{v} \) be divergence-free. We say this conservative field is generated by
\[ \mathbf{v} = \nabla \phi \]
where \( \phi \) is the scalar potential.

\[ \mathbf{v} \cdot \nabla \mathbf{v} = 0 \]
requires that \( \mathbf{v} \) be irrotational. We say this irrotational field is generated by
\[ \mathbf{v} = \nabla \phi \]
where \( \phi \) is the scalar potential.

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requires that \( \mathbf{w} \) be divergence-free. We say this conservative field is generated by
\[ \mathbf{w} = \nabla \phi \]
where \( \phi \) is the scalar potential.

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requires that \( \mathbf{w} \) be divergence-free. We say this conservative field is generated by
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where \( \phi \) is the scalar potential.

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requires that \( \mathbf{w} \) be divergence-free. We say this conservative field is generated by
\[ \mathbf{w} = \nabla \phi \]
where \( \phi \) is the scalar potential.

\[ \mathbf{v} \times \nabla \mathbf{w} = 0 \]
requires that \( \mathbf{w} \) be divergence-free. We say this conservative field is generated by
\[ \mathbf{w} = \nabla \phi \]
where \( \phi \) is the scalar potential.

\[ \mathbf{v} \cdot \nabla \mathbf{v} = 0 \]
requires that \( \mathbf{v} \) be irrotational. We say this irrotational field is generated by
\[ \mathbf{v} = \nabla \phi \]
where \( \phi \) is the scalar potential.
The definitions and generating potential functions of irrotational (conservative) and incompressible (solenoidal) fields naturally follow from two key vector identities: (1) \( \nabla \times (\nabla \times \mathbf{u}) = 0 \) and (2) \( \nabla \times (\nabla \phi) = 0 \).

**Problem # 3:** Define the following:

- 3.1: A conservative vector field
- 3.2: An irrotational vector field
- 3.3: An incompressible vector field
- 3.4: A solenoidal vector field
- 3.5: When is a conservative field irrotational?
- 3.6: When is an incompressible field irrotational?

**Problem # 4:** For each of the following, (i) compute \( \nabla \cdot \mathbf{v} \), (ii) compute \( \nabla \times \mathbf{v} \), and (iii) classify the vector field (e.g., conservative, irrotational, incompressible, etc.).

1. \( \mathbf{v}(x, y, z) = -\nabla(3x^2 + y \log(xy)) \)
2. \( \mathbf{v}(x, y, z) = xy\hat{k} - z\hat{y} \hat{k} \)
3. \( \mathbf{v}(x, y, z) = \nabla \times (x\hat{y} - z\hat{y}) \)

### 5.9.4 Maxwell’s Equations

The variables have the following names and defining equations (see Table 5.4, p. 265):

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Equation</th>
<th>Name</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{E} )</td>
<td>( \nabla \cdot \mathbf{E} = \mathbf{D} )</td>
<td>Electric field strength</td>
<td>[volts/m]</td>
</tr>
<tr>
<td>( \mathbf{D} )</td>
<td>( \nabla \cdot \mathbf{D} = \rho )</td>
<td>Electric displacement (flux density)</td>
<td>[coul/m²]</td>
</tr>
<tr>
<td>( \mathbf{H} )</td>
<td>( \nabla \times \mathbf{H} = \mathbf{J} + \mathbf{D} )</td>
<td>Magnetic field strength</td>
<td>[amps/m]</td>
</tr>
<tr>
<td>( \mathbf{B} )</td>
<td>( \nabla \cdot \mathbf{B} = 0 )</td>
<td>Magnetic induction (flux density)</td>
<td>[webers/m²]</td>
</tr>
</tbody>
</table>

Note that \( \mathbf{J} = \sigma \mathbf{E} \) is the current density (which has units of [amps/m²]). Furthermore, the speed of light in vacuo is \( c_0 = 3 \times 10^8 \) \([\text{m/s}]\), and the characteristic resistance of light \( c_0 = \sqrt{\mu_0/\epsilon_0} [\text{ohms}] \).

**Appendix F**

### Eleven postulates of systems of algebraic networks

Physical systems obey basic rules that follow from the physics. It is helpful to summarize these restrictions as postulates presented in terms of a taxonomy, or categorization method, of the fundamental properties of physical systems. Eleven of these are listed in this appendix from an article by Kim and Allen (2013).

#### F.1 Representative system

A taxonomy of physical systems comes from a systematic summary of the laws of physics, which includes at least the eleven basic network postulates, described in Sec. 3.10.

To describe the network postulates, it is helpful to start from a two-port matrix representation as discussed in Sec. 3.8 (p. 144).

![Figure F.1: The schematic representation of an algebraic network, defined by its two-port ABCD transmission, which has three elements called the Hunt parameters (Hunt, 1982): \( Z_e(s) \), the electrical impedance, \( s \omega_0 \), the mechanical impedance, and \( F(s) \), the transduction coefficient matrix of an electromechanical transducer network. The port variables are \( \hat{V}(s) \) and \( \hat{I}(s) \): the frequency domain voltage and current, and \( F(s) \) and \( U(s) \): the force and velocity (Hunt, 1982; Kim and Allen, 2013). This matrix factors the two-port model into three \( 2 \times 2 \) matrices, separating the three physical elements as matrix algebra. It is a standard impedance convention that the flows through each \( \text{port} \) are defined into each port. Thus it is necessary to apply a negative sign on the velocity \( \hat{U}(s) \) so that it has an outward flow, as required to match the next cell with its inward flow.]

As shown in Fig. F.1, the two-port transmission matrix for an acoustic transducer (loudspeaker) is characterized by the equation

\[
\begin{bmatrix}
\Phi \\
I
\end{bmatrix} = \begin{bmatrix}
A(s) & B(s) \\
C(s) & D(s)
\end{bmatrix} \begin{bmatrix}
\hat{E}_I \\
\hat{I}_I
\end{bmatrix} = \begin{bmatrix}
\omega_0(s) & Z_e(s)\omega_0(s) + T^T \\
0 & \omega_0 \end{bmatrix} \begin{bmatrix}
\hat{E}_I \\
\hat{I}_I
\end{bmatrix}
\]

shown as a product of three \( 2 \times 2 \) matrices in the figure, with each factor representing one of the three Hunt parameters of the loudspeaker.

This figure represents the electromechanical motor of the loudspeaker and consists of three elements: the electrical input impedance \( Z_e(s) \), a gyrator, which is similar to a transformer that relates current to force, and the output mechanical impedance \( \omega_0(s) \). This circuit describes

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5.9.5 Speed of light

Problem #5: The speed of light in vacuo is $c_0 = 1/\sqrt{\mu_0 \varepsilon_0} \approx 3 \times 10^8 \text{[m/s]}$.

The characteristic resistance in vacuo is $r_0 = \mu_0 / \varepsilon_0 \approx 377 \text{[}\Omega\text{]}$.

– 5.1: Find a formula for the in-vacuo permittivity $\varepsilon_0$ and permeability $\mu_0$ in terms of $c_0$ and $r_0$. Based on your formula, what are the numeric values of $\varepsilon_0$ and $\mu_0$?

– 5.2: In a few words, identify the law given by this equation, define what it means, and explain the formula:

$$Z_S \hat{n} \cdot \mathbf{v} \, dA = Z_V \nabla \cdot \mathbf{v} \, dV.$$
Problem # 8: Consider the next equation in three dimensions. Take the divergence of both sides and integrate over a volume \( V \) (closed surface \( S \)):

\[
\iiint_V \nabla \cdot (\nabla \times \mathbf{H}) \, dV = \iiint_V \nabla \cdot \mathbf{C} \, dV.
\]

- 8.1: What happens to the left-hand side of this equation? Hint: Can you apply a vector identity? Apply the divergence theorem (sometimes known as Gauss’s theorem) to the right-hand side of the equation, and interpret your result. Hint: Can you relate your result to one of Kirchhoff’s laws?

5.9.7 Second-order differentials

Problem # 9: This problem is about second-order vector differentials.

- 9.1: If \( \mathbf{v}(x, y, z) = \nabla \phi(x, y, z) \), then what is \( \nabla \cdot \mathbf{v}(x, y, z) \)?

- 9.2: Evaluate \( \nabla^2 \phi \) and \( \nabla \times \nabla \phi \) for \( \phi(x, y) = xe^y \).

- 9.3: Evaluate \( \nabla \cdot (\nabla \times \mathbf{v}) \) and \( \nabla \times (\nabla \times \mathbf{v}) \) for \( \mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \).

- 9.4: When \( \mathbf{V}(x, y, z) = \nabla(1/x + 1/y + 1/z) \), what is \( \nabla \times \mathbf{V}(x, y, z) \)?

- 9.5: When was Maxwell born and when did he die? How long did he live (within \( \pm 10 \) years)?

5.9.8 Capacitor analysis

Problem # 10: Find the solution to the Laplace equation between two infinite\(^\text{19}\) parallel plates separated by a distance \( d \). Assume that the left plate at \( x = 0 \) is at voltage \( V(0) = 0 \) and the right plate at \( x = d \) is at voltage \( V_d \equiv V(d) \).

- 10.1: Write Laplace’s equation in one dimension for \( V(x) \).

- 10.2: Write the general solution to your differential equation for \( V(x) \).

- 10.3: Apply the boundary conditions \( V(0) = 0 \) and \( V(d) = V_d \) to determine the constants in your equation from question 10.2.

\(^{19}\text{We study plates that are infinite because this means the electric field lines are perpendicular to the plates, running directly from one plate to the other. However, we solve for per-unit-area characteristics of the capacitor.}\)
APPENDIX E. THERMODYNAMIC SYSTEMS

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5.10 Webber horn equation

The transmission matrix plays a theoretical and modeling role. Thus $Z$ is used for experimental and $A$ is used for the surface of each plate. The thermodynamic constant $c_p$ is defined as $\frac{\partial Q}{\partial T}|_{P_0}$ (constant $P$) and $c_v$ is defined as $\frac{\partial Q}{\partial T}|_{V_0}$ (constant $V$).

5.10. FURTHER READINGS

- 10.4: Find the charge density per unit area ($\sigma = \frac{Q}{A}$, where $Q$ is charge and $A$ is area) on the surface of each plate. Hint: $E = \nabla \cdot \mathbf{D}$, where $D = Q/A = \varepsilon_0 \varepsilon_r$.

5.10. Further readings

- 20.1: Find the charge density per unit area on the surface of each plate.

- 94: Few of which are:

There are simple relationships between all these matrices (Van Valkenburg, 1964a, p. 310), a few of which are:

- The admittance matrix $Y$ for calculations. Other important matrices are the admittance matrix $Y$ and any case that has wave propagation. Write the formula for the Webster horn equation, and explain the variables.

Electronics is a good match for these acoustic electromechanical gadgets.

From Eq. E.1.1, $w = \frac{\partial u}{\partial t}$ is the reciprocal of the bulk modulus, defined as the ratio of the volume of the pressure compliance for constant entropy (acoustic)-modulus. Likewise, $Z_{22}$ is the ratio of the volume of over the pressure compliance for constant entropy (acoustic). Solving for $T$ or $\frac{1}{S}$ for constant volume (i.e., isobars), $T S$ is the thermodynamic potential, which is the heat of the system. If $T S$ is the thermodynamic potential, then $T S = \frac{\partial L}{\partial T}$, where $L$ is the potential energy of the system. In the electrical system, $L = \frac{1}{2} \mathbf{V}^2$, where $\mathbf{V}$ is the potential. The thermodynamic potential is given by $\Delta S = \frac{\partial L}{\partial T}$, where the heat is absorbed by a small amount, then $T S = \frac{\partial L}{\partial T}$.
Linear $2 \times 2$ systems of equations: The LCV thermodynamic variables are the mechanical (acoustical) pressure $P$, volume velocity $V$, and the heat variables, temperature $T$, and entropy rate $\dot{S}$.

The linear transmission and impedance matrices $T$ and $Z$ relate the acoustic variables $P$, $V$ to the thermodynamic variables $T$, $\dot{S}$ in two very different ways:

$$
\begin{bmatrix}
\dot{T} \\
\dot{S}
\end{bmatrix}
= T \begin{bmatrix}
P \\
V
\end{bmatrix}
- \begin{bmatrix}
A & S \\
C & D
\end{bmatrix}
\begin{bmatrix}
P \\
V
\end{bmatrix}
$$

and

$$
\begin{bmatrix}
P \\
\dot{S}
\end{bmatrix}
= Z \begin{bmatrix}
V \\
\dot{V}
\end{bmatrix}
= \begin{bmatrix}
z_{11} & z_{12} \\
z_{21} & z_{22}
\end{bmatrix}
\begin{bmatrix}
V \\
\dot{V}
\end{bmatrix}.
$$

On the left is the transmission matrix $T$ and on the right is the corresponding impedance matrix $Z$. By inspection, the $T$ and $Z$ matrix elements are

$$
T = \begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix},
Z = \begin{bmatrix}
z_{11} & z_{12} \\
z_{21} & z_{22}
\end{bmatrix}.
$$

Due to reciprocity, the determinant of $T = 1$ and $z_{12} = z_{21}$ (Postulate P8). When using this matrix representation, any algebra is explicit. The transmission matrix $T$ and impedance matrix $Z$ are related by Eq. 3.8.6, as discussed on page 146.

Summary of Maxwell relations

There are six variants on the transmission matrix (Van Valkenburg, 1964a). The most common are the transmission $T$ and the impedance $Z$ matrices. Others, such as the admittance matrix $Y$ is simply the inverse of $Z$. But other variants are also used for special applications, such scattering matrices.

All the matrices used in thermodynamics are reciprocal, meaning $z_{12} = z_{21}$ or $\Delta T = 1$. Each matrix form show reciprocity in a different way. These expressions for reciprocity were first worked out by Maxwell, where they are called Maxwell relations (Ambaum, 2010). From the Maxwell relations, it may be shown that the specific heat capacities are equal to $c_v = y_{22} = z_{11}/\Delta z$ and $c_p = y_{22} = 1/|y_{22}| = \Delta z/z_{11}$.

Physical meaning of the matrix elements

The utility of the $T$ and $Z$ representations are very different. The impedance matrix $Z$ always has a physically measurable interpretation. The diagonal elements $z_{11}, z_{22}$ represent the input impedance looking into the two ports, while the off-diagonal elements are transfer impedances, related to a transfer function between input and output fluxes and potentials. If the system is reciprocal they are equal ($z_{21} = z_{12}$).
2. Specific enthalpy \( h = u + PV \),
3. Helmholtz free energy \( f = u - TS \), and
4. Gibbs function \( g = u - TS + PV \).

The Gibbs function is a linear combination of the other three (Ambaum, 2010, Sec. 3.1). As explained by Tolman (1948), Gibbs viewed a phase transition as a membrane separating two states of a gas. These four relations could be combined into a \( 3 \times 3 \) nonlinear energy matrix, the entries of which would be either 0, 1, or -1.

When many variables are used to describe a system, as in the case of thermodynamics, a matrix formulation is more compact representation. Here, traditional thermodynamic relations are reformulated using linear \( 2 \times 2 \) matrix algebra. Starting from the state equation and the Helmholtz expression for the free energy, we rewrite the system as a linear two-port network. By evaluating the impedance matrix elements, we may obtain the standard thermodynamic relations in terms of the two heat capacities \( c_p \) and \( c_v \). Due to this unification of variables, the \( 2 \times 2 \) formulation, such a formulation manipulation of thermodynamic relations becomes transparent.

An alternative approach is to use a \( 3 \times 3 \) scattering matrix in more than 2 variables (Bilbao, 2004).

In acoustics, the gradient of the pressure \(-\nabla P\) is the acoustic force density, which is proportional to the acoustic impedance, which for air is \( \rho c_o \). Working with impedances it is standard practice to work with the Laplace complex frequency domain \( s \). The \( LT \) transforms differential equations into algebraic relations (i.e., polynomials). In thermodynamics the term potential has a very different meaning. Here we shall not refer to energies as potentials. When speaking of conjugate variables, pressure, voltage and temperature are potentials, and velocity, current and entropy are flows. Using modern engineering terminology, the gradient of every potential is a generalized force \( F \), which according to Ohm’s law, is proportional to a generalized flow \( J \). Working with \( LCV \) and the volume velocity \( V_g \) gives

\[
-\nabla P = \rho c_o V_g.
\]

Here \( \rho c_o \) is the impedance of air, defined as the ratio of the generalized force \( F = -\nabla P \) over the volume velocity \( V_g \). Ohm’s law defines a linear relation, with the impedance \( \rho c_o \) being the ratio of the voltage \( V \) over the current \( I \). When speaking of conjugate variables, pressure, voltage and temperature are potentials, and velocity, current and entropy are flows. Using modern engineering terminology, the gradient of every potential is a generalized force \( F \), which according to Ohm’s law, is proportional to a generalized flow \( J \). Working with \( LCV \) and the volume velocity \( V_g \) gives

\[
-\nabla P = \rho c_o V_g.
\]
Appendix E

Thermodynamic systems

Many people find thermodynamics difficult to understand. Here we explore the reasons behind this lack of transparency, and propose a solution. To understand the problem it is helpful to compare the nonlinear energy-equilibrium methods used in thermodynamic, to linear impedance methods, used in acoustics, mechanics and electrical circuit theory.

Acoustics, mechanics and electrical circuits are explained in terms of linear systems of equations, such as as Kirchhoff’s and Ohm’s laws, as discussed in Appendix D. The linear impedance formulation leads to a system of equations that is easily solved, using standard methods of linear algebra. Thermodynamics on the other hand is formulated in terms of equilibrium energies, resulting in a nonlinear systems of equations. This explains the difficulty in understanding the relationships. A second issue is that these equations are frequently over-specified.

Traditionally thermodynamics has been formulated with two types of variables, those that are proportional to the mass, called extensive such as volume and mass, and the those that are independent of mass, called intensive variables, such as temperature and density (Ambaum, 2010, p. 3).

A more modern and transparent notation is to work with two conjugate variables (CV), $1$ force density and flux. Examples of the CV force density include pressure $P$, temperature $T$, the Nernst potential, and electrical voltage. Fluxes CVs included mass flux and its area integral, volume velocity $V$, heat flux, called entropy $S$, and electrical current. Impedance is the ratio of CV, and the power is their product. For example, in electrical network theory, the impedance is the voltage over the current and the power is the product. Working with impedance always results in systems of linear equations. linear conjugate variables (LCV) are a generalization of force, the gradient of a potential, and flow. Namely the formulation in terms of LCV simplifies the thermodynamic system of nonlinear equations.

E.1 Summary of the thermodynamic relations

To reduce the number of variables we remove mass from the system by working with specific variables (those that are normalized by the mass). The ratio of the volume $V$ and mass $M$ defines the specific volume $\nu \equiv V/M = 1/\rho$, where $\rho$ is the density. Throughout this discussion we shall use SI units, with $V$ [m$^3$], $M$ [kgm], $\rho = 1/\nu$ [kg/m$^3$].

Four classical thermodynamic energies are defined:

1. internal energy $u$,

https://en.wikipedia.org/wiki/Conjugate_variables_(thermodynamics)
Appendix A
Notation

A.1 Number systems

The notation used in this book is defined in this appendix so that it may be quickly accessed. Where the definition is sketchy, page numbers are provided where these concepts are fully explained, along with many other important and useful definitions. For example, a discussion of $N$ may be found on page 24. Math symbols such as $N$ may be found at the top of the index, since they are difficult to alphabetize.

A.1.1 Units

Strangely, or not, classical mathematics, as taught today in schools, does not seem to acknowledge the concept of physical units. Units seem to have been abstracted away. This makes mathematics distinct from physics, where almost everything has units. Presumably this makes mathematics more general (i.e., abstract). But for the engineering mind, this is not ideal, or worse, as it necessarily means that important physical meaning, by design, has been surgically removed. We shall use SI units whenever possible, which means this book is not a typical book on mathematics. Spatial coordinates are quoted in meters [m], and time in seconds [s]. Angles in degrees have no units, whereas radians have units of inverse-seconds [s$^{-1}$]. A complete list of SI units may be found at https://physics.nist.gov/cuu/pdf/sp811.pdf and in Graham et al. (1994), for a discussion of basic math notation.

When writing a complex number, we shall adopt $1\sqrt{-1}$ to indicate $\sqrt{-1}$. Matlab/Octave allows either $1i$ or $1\sqrt{-1}$.

Units are SI; angles are in degrees [deg] unless otherwise noted. The units for $\pi$ are always radians [rad]. For example, $\sin(\pi)$, $e^{i\pi/2}$.

A.1.2 Notation

We use $\ln$ as the log function base $e$, $\log$ as base 2, and $\pi_k$ to indicate the $k$th prime (e.g., $\pi_1 = 2$, $\pi_2 = 3$). It is helpful to know where the letters of the alphabet are. Everyone knows the first letter is /a/ and the last /z/, but what is the 10th or 20th letter? The table below shows that /j/ is the 10th letter, /t/ the 20th, /o/ the 15th, and /z/ the 26th.

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<th>Letter</th>
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This is helpful for quickly moving around the alphabet, when looking up words in the dictionary or an alphabetized list. If you forget how many letters there are in the English alphabet, this will help you recall it is 26.

When working with Fourier $\mathcal{F}$ and Laplace $\mathcal{L}$ transforms, lower case symbols are in the time domain while upper case indicates the frequency domain, as $f(t) \rightarrow \mathcal{F}(\omega)$. An important exception is Maxwell’s equations because they are so widely used as upper-case bold letters (e.g., $\mathbf{E}(x, \omega)$). It would seem logical to change this to $\mathbf{e}(x, \omega)$, to conform.

### A.1.3 Common mathematical symbols

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### Pressure Eigen-solutions

The forwarded $P^-$ and backward $P^+$ pressure waves propagate as

$$P^\pm(s, x) = e^{\mp s \beta}, \quad e^{-i \omega t},$$

where $s(x)$ is the complex conjugate of $s(x)$, such that $\Re(s(x)) > 0$. The term $\beta \sqrt{s}$ affects both the real and imaginary parts of $s(x)$. The real part is a frequency-dependent loss, and the imaginary part introduces a frequency-dependent speed of sound (Mason, 1928).

#### D.1.2 Impact of viscous and thermal losses

Equation D.1.2 and the measured data are compared in Fig. D.1, as reproduced from Mason’s Fig. 4, which shows that the wave speed drops from 344 m/s at 2.6 kHz to 339 m/s at 0.4 kHz, a 1.5% reduction. At 1 kHz the loss is 1 dB/m for a 7.5-mm tube. Note that the loss and the speed of sound vary inversely with the radius. As the radius approaches the boundary layer thickness (i.e., the radial distance such that the loss is $e^{-1}$), the effect of the damping dominates the propagation.

#### D.1.1 Cut-off frequency $s_c$:

The frequency where the lossless part equals the lossy part is defined as $s(s_c) = 0$, namely, $\sqrt{s_c} = -\beta_c$, or $s_c = \beta_c^2$.

To get a feeling for the magnitude of $s_c$, let $R = 0.75/2$ cm (i.e., the average radius of the adult ear canal). Then for $R = 3.75 \times 10^{-3}$ cm

$$s_c = (1.9 \times 10^{-3})^2 / (3.75 \times 10^{-3})^2 = 1/4.$$

We conclude that the losses are insignificant in the audio range, since for the human ear canal, $R_c = 37.5 / 2 = 0.25/\pi = 0.08$ Hz. This frequency represents the lower bound of the transition from adiabatic to iso-thermal equilibrium. It should be clear that acoustic frequencies do not actually obey the adiabatic approximation, due to the thin boundary layer. Both the real and imaginary part of propagation function $s(x)$, the characteristic impedance $\zeta_s(x)$ and the speed of sound $c_s(x)$ all depend on frequency in the auditory range of human hearing.

#### Summary:

The Helmholtz-Kirchhoff theory of viscous and thermal losses results in a frequency-dependent speed of sound that has a frequency dependence proportional to $1/\sqrt{s}$ rather than $1/s$ (Mason, 1928, Eq. 4). This corresponds to a 2% change in the sound velocity over the decade from 0.2 to 2 kHz (Mason, 1928, Fig. 5), in agreement with Mason’s experimental results.
Mason (1928) assumed the viscosity to be \( \mu_0 = 18.6 \times 10^{-6} \) [Pa-s] (viscosity), thus the real and imaginary parts. Thus both the speed of sound and the damping are dependent on frequency, and in a similar way.

### A.1. NUMBER SYSTEMS

#### Greek letters

Frequently Greek letters, as provided in Fig. A.1, are associated in engineering calculations. They may be defined, using set notation (a very sloppy notation with an incomprehensible syntax), as

\[ Q = \{ p/q : q \neq 0 \quad \text{and} \quad p, q \in \mathbb{Z} \} \]

\[ \mathbb{R} \]

are formed from the union of integers and fractions. The rationals may be defined in Eq. D.1.7.

\[ \mathbb{F} \]

In general, the more interesting class are fractionals. The rationals are defined in Eq. D.1.7.

\[ \mathbb{P} \]

is the set of counting numbers, which in turn is in the set of real numbers, or in vernacular language a real counting number.

\[ \mathbb{I} \]

is the set of counting numbers, which in turn is in the set of real numbers, or in vernacular language a real counting number.
which may be read as “the set ‘{⋯}’ of all $p/q$ such that ‘$q$̸= 0,’ and $p, q$⊂$\mathbb{Z}$. The translation of the symbols is in single (‘⋯’) quotes.

Irrational numbers $I$ are very special: They are formed by taking a limit of fractionals, as the numerator and denominator $→ \infty$, and approach a limit point. It follows that irrational numbers must be approximated by fractionals.

The reals $\mathbb{R}$ include complex numbers $\mathbb{C}$ having a zero imaginary part (i.e., $\mathbb{R} \subset \mathbb{C}$).

The size of a set is denoted by $\{· · ·\}$ of all $p/q$ such that “$q$̸= 0,’ and $p, q$⊂$\mathbb{Z}$. Normally in mathematics this symbol indicates the cardinality, so we are defining it differently from the standard notation.

A.1.4 Classification of numbers:

From the above definitions there exists a natural hierarchical structure of numbers:

$$\mathbb{P} \subset \mathbb{N}, \quad \mathbb{Z} : \{\mathbb{N}, 0, -\mathbb{N}\}, \quad \mathbb{F} = \mathbb{Q} \cup \mathbb{F}, \quad \mathbb{R} \cup \mathbb{I} \subset \mathbb{C}$$

1. The primes are a subset of the counting numbers: $\mathbb{P} \subset \mathbb{N}$.
2. The signed integers $\mathbb{Z}$ are composed of $\pm \mathbb{N}$ and 0, thus $\mathbb{N} \subset \mathbb{Z}$.
3. The fractionals $\mathbb{F}$ do not include the signed integers $\mathbb{Z}$.
4. The rationals $\mathbb{Q} = \mathbb{Z} \cup \mathbb{F}$ are the union of the signed integers and fractionals.
5. Irrational numbers $I$ have the special properties $I \notin \mathbb{Q}$.
6. The reals $\mathbb{R}$: $\mathbb{Q}$, $\mathbb{I}$ are the union of rationals and irrationals $I$.
7. Reals $\mathbb{R}$ may be defined as a subset of those complex numbers $\mathbb{C}$ having zero imaginary part.

A.1.5 Rounding schemes

In Matlab/Octave there are five different rounding schemes (i.e., mappings): round($x$), fix($x$), floor($x$), ceil($x$), roundb($x$), with input $x$∈$\mathbb{R}$ and output $k$∈$\mathbb{N}$. For example $5.5 - \lfloor 5.5 \rfloor = 0.5$ rounds to the nearest integer, whereas $3.5 = \lfloor 3.5 \rfloor = 3$ rounds down while $3.5 = \lceil 3.5 \rceil = 4$ rounds up. Rounding schemes are used for quantizing a number and generating a remainder. For example: $y$-round($x$) is equivalent to $y = x - \lfloor x \rfloor$.

Note round($\pi$) $\equiv \lceil \pi \rceil$ introduces negative remainders when ever a number rounds up ($\pi - \lceil \pi \rceil = 0.8543$).

The continued fraction algorithm (CFA), Sec. 2.4.4 (p. 50) is a recursive rounding scheme, operating on the reciprocal of the remainder. For example:

$$\exp(1) = 3 + \frac{1}{(-4/2 + 1/5)} = -[3, -4, 2, 5, -2, -7] - (-1.75 \times 10^{-4})$$

The expressions in square brackets is a notation for the CFA integer coefficients. The Octave-Matlab function having $x$∈$\mathbb{R}$, is either rats($x$) with output $\in \mathbb{N}$, or roundb($x$), with output $\in \mathbb{F}$.

D.1. ADIABATIC APPROXIMATION AT LOW FREQUENCIES

Reduction of Kirchhoff’s equations to complex–analytic form: We may rewrite $\kappa$ and $\rho$, in terms of $\beta_i$, $c_r\rho_i(s) = (\beta_i \pm \sqrt{\beta_i^2 - 1})^2$ (Eq. D.1.5).

The two boxed equations below provide the derivation for the reduction of Mason’s formula, for $c_r\beta(s)$, to its complex–analytic form (Eq. D.1.5). The first step is to define $2\beta_i$ and $c_r\kappa(\omega)$. The second box shows it is equal to $2\beta_i\sqrt{s}$. Thus $\kappa(s)$ is complex–analytic in the Laplace frequency $s$, and following a completion of squares in $\sqrt{s}$, is given by Eq. D.1.5. The inverse Laplace transforms of $\sqrt{s}$ and $1/\sqrt{s}$ are provided in Table C.2 of Appendix C and Table 3.9 of Sect. 3.10.1.

\begin{align}
\text{Starting from Mason (1928) } c_r\beta(s) &= \frac{P_0\beta}{2\sqrt{P_0\beta}} + \left(1 - \frac{P_0\beta}{2\sqrt{P_0\beta}}\right) \\
\text{Define variables } z &\equiv \frac{\sqrt{P_0\beta}}{\sqrt{s}}, \quad s &\equiv \frac{\sqrt{P_0\beta}}{\sqrt{s}} \\
\text{Thus } c_r\beta(s) &\equiv s^2 - \frac{P_0\beta}{2\sqrt{P_0\beta}} \\
\text{Factor out } s &\equiv \frac{\sqrt{P_0\beta}}{\sqrt{s}} \quad \text{REPLACE } s^2 = \frac{P_0\beta}{2\sqrt{P_0\beta}} \\
\text{To show } 1 + j = \sqrt{2}(\text{square both sides: } 1 - \sqrt{2} = 0) \;
\end{align}

Acoustic constants for air: Assuming $\rho_i = 1.4$ (ratio of specific heats), $\gamma = 1.2$ (kg/m$^2$) (density), a temperature of $23.5$ [°C], and $P_0 = 10^5$ [Pa] (atmospheric pressure), the lossless sound velocity is $c_0 = \sqrt{\gamma P_0 / \rho_i} = 341.57$ [m/s]. By a comparison of this value of $c_0$ to Fig. D.2, it is clear that this value does not agree with Mason’s measurements. Thus to agree with his experimental results, either the ratio $P_i / \rho_i$ or $\gamma$ must be decreased. Since 2.4, depends on $\gamma$, $P_i$, and $\rho_i$, this is a good place to look for the discrepancy.

Correction for $\gamma$: The dimensionless constant $\gamma / \sqrt{\gamma}$ is defined as the composite thermodynamic constant (Kirchhoff, 1868; Rayleigh, 1896),

$$\frac{\gamma}{\sqrt{\gamma}} = \left[1 + \sqrt{\frac{\gamma}{2} \left(\eta_0^{\gamma/2} - \eta_0^{-1/2}\right)}\right] = 1.5345.$$
Figure D.1: This figure, taken from Mason (1928), compares the Helmholtz-Kirchhoff theory for visco-thermal losses from the imaginary part of Eq. D.1.2. Figure D.2 directly compares Mason’s measured sound speed with this equation.

A.2. DIFFERENTIAL EQUATIONS VS. POLYNOMIALS

A.1.6 Periodic functions

Fourier series tells us that periodic functions are represented as a sum of sine and cosine functions. For a function f(t) that is periodic with period T, we can write:

\[ f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{2\pi n}{T} t} \]

where \( c_n \) are given by

\[ c_n = \frac{1}{T} \int_{0}^{T} f(t) e^{-i \frac{2\pi n}{T} t} dt \]

This is the general form of a Fourier series. For a function that is periodic with period 2π, we can simplify this to:

\[ f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i n \omega_0 t} \]

where \( \omega_0 = \frac{2\pi}{T} \) is the fundamental frequency.

A polynomial of degree n is defined as:

\[ P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \]

where \( a_n \neq 0 \) and the coefficients are real numbers.

A.2. DIFFERENTIAL EQUATIONS VS. POLYNOMIALS

We now turn to differential equations. A differential equation is an equation that relates a function and its derivatives. For example, a second-order differential equation is:

\[ \frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 6y = 0 \]

This is a linear differential equation with constant coefficients. The general solution to such an equation depends on the roots of the characteristic equation, which is derived from the differential equation.

The complex propagation function used by Mason (1928), as taken from (Rayleigh, 1896, 1922; Stewart and Lindsay, 1930), was acoustic version of the work of George Campbell (1904-). This was an early example of the use of complex numbers in physics. The complex propagation function indicates how sound waves are propagated in space.

The complex propagation function is a frequency expansion of a periodic function. This concept is quite general. Periodic in frequency implies discrete in time. Periodic in time implies sampled in frequency (the case of the DFT). The DFT, the sample period is limited to \( N \) in number of points.

The complex propagation function is given by:

\[ \Gamma(\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{N} \delta(\omega - k\omega_0) \]

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which is a frequency expansion of a periodic function. This concept is quite general. Periodic in frequency implies discrete in time. Periodic in time implies sampled in frequency (the case of the DFT). The DFT, the sample period is limited to \( N \) in number of points.
As with monics, the lead coefficient must always be 1. The complex variable \( s \in \mathbb{C} \) is the Laplace frequency.

The ratio of the output \( Y(s) \) over the input \( X(s) \) is called the system transfer function \( H(s) \).

The coefficient \( a \in \mathbb{R} \) is called the gain. The roots of the numerator are called the zeros and those of the denominator, the poles. When \( H(s) \) is the ratio of two degree 1 monics, the transfer function is said to be \textit{bilinear}, since it is linear in both the input and output. In such cases \( H(s) \) has only one pole and one zero. The inverse Laplace transform of the transfer function is called the \textit{system impulse response}, which describes the system’s output signal \( y(t) \) for any given input signal \( x(t) \), via convolution (i.e., \( y(t) = h(t) \ast x(t) \)).

A.3 Matrix algebra: Systems

A.3.1 Vectors

Vectors are columns of ordered sets of scalars \( \in \mathbb{C} \). When we write them out in text, we typically use row notation, with the transpose symbol:

\[
[a, b, c]^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.
\]

This is strictly to save space on the page. The notation for conjugate transpose is \( \dagger \), for example

\[
[a \ b \ c]^\dagger = \begin{bmatrix} a^* \\ b^* \\ c^* \end{bmatrix}.
\]

The above example is said to be a 3-dimensional vector because it has three components.

A.3.2 Vector products

A scalar product (aka dot product) is defined to “weight” vector elements before summing them, resulting in a scalar. The transpose of a vector (a row-vector) is typically used as a scale factor (i.e., weights) on the elements of a vector. For example,

\[
\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [1 \ 2 -1] [1 \\ 2 \\ 3] = 1 + 2 \cdot 2 - 3 = 2.
\]

A more interesting example is

\[
\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}^T [1 \ 2 \ 4] = 1 + 2 \cdot 2 + 4 \cdot 3 = 14.
\]

Polar scalar product: The vector scalar product in polar coordinates is (Fig. 3.4, p. 119)

\[
B \cdot C = ||B|| ||C|| \cos \theta \in \mathbb{R},
\]

where \( \cos \theta \in \mathbb{R} \) is called the direction-cosine between \( B \) and \( C \).

Appendix D

Visco-thermal losses

D.1 Adiabatic approximation at low frequencies

Newton’s early development understandably ignored viscous and thermal losses, by assuming iso-thermal conditions. But starting at very low frequencies, the isothermal assumption breaks down. Modern theory, for audio frequencies, assumes the adiabatic approximation, and is thus described by the scalar wave equation (Pierce, 1981). But it turns out that even at audio frequencies, the adiabatic approximation is invalid. This was first shown by Kirchhoff, but was not fully appreciated for more than a century, due to mathematical difficulties, which we now believe can be explained, as discussed next.

Following Helmholtz (1858), as extended by Kirchhoff (1868), visco-thermal loss mechanisms are related. The full theory was first worked out by Kirchhoff (1868, 1974). To understand how the are related is complicated, due to both the history and the mathematics, as briefly discussed by Pierce (1981). Both forms of damping are caused by two different, but coupled, diffusion effects: (1) viscous effects, due to shear at the container walls, and (2) thermal effects, due to deviations from adiabatic expansion (Kirchhoff, 1868, 1974). I believe that Einstein was eventually involved, following his studies on Brownian motion (Einstein, 1905)\(^1\). \(^2\)

These two loss mechanism are restricted to a thin region called the boundary layer, which critically depends on the square root of the Laplace frequency. A key transition region is where the acoustic wavelength approaches the complex boundary layer thickness. When the radius of the container (a horn) approaches the viscous boundary layer, the theory breaks down.

D.1.1 Lossy wave-guide propagation

The formulation of visco-thermal loss in air transmission was first worked out by Helmholtz (1863a) and then extended by Kirchhoff (1868) to include thermal damping (Rayleigh, 1896, Vol. II, p. 319). These losses are accurately represented by the complex-analytic propagation function \( n(s) \) (Eq. D.1.5). Following his review of these theories, Crandall (1926, Appendix A), the head of the 1926 Acoustic Research Department at that time, noted that the “Helmholtz-Kirchhoff” theory had never been experimentally verified. Acting on Crandall’s suggestion, Phycisist Warren Mason set out to experimentally verify Kirchhoff’s 60 year old theory. Mason’s analysis consumed several years (Mason, 1928).

This was a continued effort. Mason (1927) extended earlier work of Stewart’s on acoustic transmission lines, by including viscous and thermal losses. Stewart’s acoustic theory (Stewart,
Polar wedge product: The vector wedge product in polar coordinates is (Fig. 3.4, p. 119)

\[
B \wedge C = \hat{\iota} \|B\| \|C\| \sin \theta \in \mathbb{R},
\]

where \(\sin \theta \in \mathbb{R}\) is therefore the direction-sine between \(B\) and \(C\).

Complex polar vector product: From these two polar definitions and

\[
\hat{\iota} \theta = \cos \theta + \hat{\iota} \sin \theta,
\]


\[
B \cdot C + \hat{\iota} B \wedge C = \|B\| \cdot \|C\| \text{est},
\]

Hence

\[
|B \cdot C|^2 + |B \wedge C|^2 = \|B\|^2 \cdot \|C\|^2 \cos 2\theta + \|B\|^2 \cdot \|C\|^2 \sin 2\theta = \|B\|^2 \cdot \|C\|^2.
\]

As described in Fig. 3.4, this relationship holds true in any vector space, of any number of
dimensions, containing vectors \(B\) and \(C\).

\[
\|B\cdot C\| = \|B\| \cdot \|C\| \cos \theta.
\]

Triangular inequality

\[
\|B - C\| \leq \|B\| + \|C\|.
\]

In terms of a right triangle this says the sum of the lengths of the two sides is greater to the
length of the hypotenuse, and equal when the triangle degenerates into a line.

Vector cross product: The vector product (aka cross product) \(A \times B\) is
defined between the two vectors \(A\) and \(B\). In Cartesian coordinates

\[
A \times B = \det \begin{pmatrix}
\hat{x} & \hat{y} & \hat{z} \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3
\end{pmatrix}.
\]

\[
\text{det} \begin{pmatrix}
\hat{x} & \hat{y} & \hat{z} \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3
\end{pmatrix}.
\]

### A.3. Norms of vectors

The norm of a vector is the scalar product of the vector with itself

\[
\|A\| = \sqrt{A \cdot A} \geq 0,
\]

forming the Euclidean length of the vector.

Euclidean distance between two points in \(\mathbb{R}^3\):

The scalar product of the difference between two vectors \((A - B) \cdot (A - B)\) is the Euclidean distance between the points they define

\[
\|A - B\| = \sqrt{\sum (a_i - b_i)^2}.
\]

Triangle inequality

\[
\|A + B\| \leq \|A\| + \|B\|.
\]

In terms of a right triangle this says the sum of the lengths of the two sides is greater to the
length of the hypotenuse, and equal when the triangle degenerates into a line.

### A.3.3 Matrix Algebra of Systems

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The triple product: This is defined between three vectors as
\[ A \cdot (B \times C) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \]
This may be indicated without the use of parentheses, since there can be no other meaningful interpretation. However for clarity, parentheses should be used. The triple product is the volume of the parallelepiped (3D-crystal shape) outlined by the three vectors, as shown in Fig. 3.4 p. 119.

Dialects of vector notation: Physical fields are, by definition, functions of space \( x \) [m], and in the most general case, time \( t \) [s]. When Laplace transformed, the fields become functions of space and complex frequency \( e(x, t) \leftrightarrow \mathcal{L}(s) \). As before, there are several equivalent vector notations. For example, \( e(x, t) = [E_x, E_y, E_z]^T = E_x(x, t)\hat{x} + E_y(x, t)\hat{y} + E_z(x, t)\hat{z} \) is “in-line,” to save space. The same equation may written in “displayed” notation
\[
\begin{bmatrix}
E_x(x, t) \\
E_y(x, t) \\
E_z(x, t)
\end{bmatrix} =
\begin{bmatrix}
\hat{x} \\
\hat{y} \\
\hat{z}
\end{bmatrix} E_x +
\begin{bmatrix}
\hat{x} \\
\hat{y} \\
\hat{z}
\end{bmatrix} E_y +
\begin{bmatrix}
\hat{x} \\
\hat{y} \\
\hat{z}
\end{bmatrix} E_z
\]

Note the three notations for vectors, bold font, element-wise columns, element-wise transpose rows and dyadic format. These are all shorthand notations for expressing the vector. Such usage is similar to a dialect in a language.

Complex elements: When the elements are complex \( \mathbb{C} \), the transpose is defined as the complex conjugate of the elements. In such complex cases the transpose conjugate may be denoted with \( \dagger \) rather than \( T \)
\[
\begin{bmatrix}
-2j \\
3j \\
1
\end{bmatrix}^\dagger =
\begin{bmatrix}
2j \\
-3j \\
1
\end{bmatrix} \in \mathbb{C}
\]
For this case when the elements are complex, the dot product is a real number
\[ a \cdot b = \bar{a}^\dagger b = \begin{bmatrix} a_1^\ast & a_2^\ast & a_3^\ast \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1\bar{b}_1 + a_2\bar{b}_2 + a_3\bar{b}_3 \in \mathbb{R} \]

Norm of a complex vector: The dot product of a vector with itself is called the norm of \( a \)
\[ \|a\| = \sqrt{\overline{a}^\dagger a} \geq 0. \]
which is always non-negative.

Such a construction is useful when \( a \) and \( b \) are related by an impedance matrix
\[ V(s) = Z(s)I(s) \]
and we wish to compute the power. For example, the impedance of a mass is \( m/s \) and a capacitor is \( 1/C \). When given a system of equations (a mechanical or electrical circuit) one may define an impedance matrix.

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Example:
Filter properties: Given the function
\[ F(s) = \frac{(s+1)(s-1)}{(s+2)} \]
1. Find the minimum phase \( M(s) \) and all-pass \( A(s) \) parts. The minimum phase part has all of its poles and zeros in the left half-plane (LHP), while the all-pass part has its poles in the LHP and mirrored zeros in the RHP. Thus we place a removable pole zero pair symmetrically across from the RHP zero, and then write the expression as the product, that is \( F(s) = M(s) \cdot A(s) \):
\[ F(s) = \frac{(s+1)(s-1)}{(s+2)} = \frac{(s+1)^2 - 1}{s+2} = \frac{(s+1)^2 - 1}{s+2} \]
Thus \( M(s) = \frac{s+1}{s+2} \) and \( A(s) = \frac{1}{s+2} \)
2. Find the magnitude of \( M(s) \). Take the real part of the log of \( M \) and then the anti-log. Thus \( |M| = e^{\text{Re}(\text{ln}(M))} \)
3. Find the phase of \( M(s) \). In this case we use the imaginary part: \( \angle M = \Im(\text{ln}(M)) \)
4. Find the magnitude of \( A(s) \) 1, by definition.
5. Find the phase of \( A(s) \) by definition.

More questions
There are a number of question to be addressed:
1. Can we interpret the zeta function as a frequency domain quantity, and then inverse transform it into the time domain?
   The answer to this is yes, and the results are quite interesting.
2. Make a histogram of the entropy for the first million integers.
   This is a 5 minute job in Matlab/Octave. For example:
   ```matlab
   K=1e5; N=1:K; F=zeros(K,10);
   for n=1:K;
       f=factor(n);
       F(n,1:length(f))=f;
   end;
   hist(F);
   ```
This equation is of the form $F(s) = 1 - s^2 \zeta(s)$, where $F(s)$ is defined, in the complex frequency domain $\Omega(s)$, as $F(s) = \int_0^\infty \mathcal{L}(s) dt$. The imaginary part component is available, since only the real part is considered. In the case of the complex frequency domain, a vector is defined by the coordinates of that vector.

**A.3.4 Matrices**

When working with matrices, the role of the weights and vectors can change, depending on the context. A useful way to view a matrix is as a set of column vectors, weighted by the elements of the column vector of weights multiplied from the right. For example, $\mathbf{A} \cdot \mathbf{a} = \mathbf{a}_1 a_1 + \mathbf{a}_2 a_2 + \mathbf{a}_3 a_3$, where the weights are $\{a_1, a_2, a_3\}$.

The determinant of a matrix is defined in either $\mathbb{R}$ or $\mathbb{C}$. In the absolute value, the determinant of a square matrix is $|A|$ or $A$. The inverse of a matrix is not defined, unless $A$ is an invertible matrix.

**A.3.5 2 x 2 complex matrices**

Here are some definitions to keep in mind:

1. **Scalar**: A number for example $(a_1, a_2, a_3, \ldots)$.
2. **Vector**: A quantity having direction as well as magnitude, often depicted by a bold or arrow over the top $\mathbf{a}$.
3. **Matrix**: A rectangular array of numbers, often depicted by a bold letter $\mathbf{A}$.

 Matlab/Octave allows the use of complex numbers, which are typically represented as a single symbol with an imaginary part component.
row \([x_1, x_2, x_3, \ldots]\) or single column \([x_1, x_2, x_3, \ldots]\)^T (where \(^T\) indicates the transpose). In this text we will typically use column vectors. The vector may also be written out using unit vector notation to indicate direction. For example: \(x_1, x_2, x_3, \ldots = x_1 e_x + x_2 e_y + x_3 e_z\), where \(e_x, e_y, e_z\) are unit vectors in the \(x, y, z\) Cartesian directions (here the vector’s subscript \(3 \times 1\) indicates its dimensions). The type of notation used may depend on the engineering problem you are solving.

3. Matrix: \(A = [a_{i,j}]_{N \times M} = [a_{x,y}]_{N \times M}\) can be a non-square matrix if the number of elements in each of the vectors \((N)\) is not equal to the number of vectors \((M)\). When \(M = N\), the matrix is square. It may be inverted if its determinant \(|A| = \prod \lambda_i \neq 0\) (where \(\lambda_i\) are the eigenvalues). In this text we work only with \(2 \times 2\) and \(3 \times 3\) square matrices.

4. Linear system of equations: \(Ax = b\) where \(x\) and \(b\) are vectors and matrix \(A\) is a square.

   a. Inverse: The solution of this system of equations may be found by finding the inverse \(x = A^{-1}b\).

   b. Equivalence: If two systems of equations \(A_1x = b_1\) and \(A_2x = b_2\) have the same solution (i.e., \(x = A_2^{-1}b_2 - A_1^{-1}b_1\)), they are said to be equivalent.

   c. Augmented matrix: The first type of augmented matrix is defined by combining the matrix with the right-hand side for each equation in the system of equations of the form \(Ax = y\):

\[
\begin{bmatrix}
a & b & x_1 \\
c & d & y_1
\end{bmatrix}
\]

the augmented matrix is:

\[
[A|y] = \begin{bmatrix}
a & b & x_1 \\
c & d & y_1
\end{bmatrix}
\]

A second type of augmented matrix may be used for finding the inverse of a matrix (rather than solving a specific instance of linear equations \(Ax = b\)). In this case the augmented matrix is:

\[
[A|I] = \begin{bmatrix}
a & b & 1 \\
c & d & 0
\end{bmatrix}
\]

Performing Gaussian elimination on this matrix, until the left side becomes the identity matrix, yields \(A^{-1}\). This is because multiplying both sides by \(A^{-1}\) gives \(A^{-1}A[A|I] = [I|A^{-1}]\).

5. Permutation matrix \(P\): A matrix that is equivalent to the identity matrix, but with scrambled rows (or columns). Such a matrix has the properties \(det(P) = \pm 1\) and \(P^2 = I\). For the \(2 \times 2\) case, there is only one permutation matrix:

\[
P = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

A permutation matrix \(P\) swaps rows or columns of the matrix it operates on. For example, in the \(2 \times 2\) case, pre-multiplication swaps the rows:

\[
PA = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
a & b \\
\alpha & \beta
\end{bmatrix} = \begin{bmatrix}
\alpha & \beta \\
a & b
\end{bmatrix}
\]

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\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

Figure C.2: This feedback network is described by a time-domain difference equation with delay \(T = \ln r_1\). It has an all-pole transfer causal function given by Eq. C.1.11. Physically this delay corresponds to \(T_r\).

Such functions may be generated in the time domain as shown in Fig. C.2, using a feedback delay of \(T_r\) as described by the equation in the figure, with a unity feedback gain \(\alpha = 1\).

\[
Z^\alpha(t) = Z^\alpha(t - T_r) + \delta(t).
\]

Taking the Laplace transform of the system equation, we see that the transfer function between the state variable \(q(t)\) and the input \(x(t)\) is given by \(Z^\alpha(s)\). Taking the FT, we see that \(\zeta(s)\) is a causal all-pole function,

\[
\zeta(s) = e^{-sT_p}\zeta_p(s) + 1(t) \quad \zeta(s) = \frac{1}{1 - e^{-sT_p}}.
\]

This is a key expression in the theory of Black Body radiation where the Black Body radiation formula is (Allen, 2004)

\[
S(s) = \frac{-s^2}{1 - e^{-sT_p}}
\]

In terms of the physics, this transmission line equations are telling us that \(\zeta(s)\) may be decomposed into an infinite cascade of transmission lines (Eq. C.1.10), each having a unique delay given by \(T_k = \ln r_k\), \(r_k \in P\), the log of the primes. The input admittance of this cascade may be interpreted as an analytic continuation of \(\zeta(s)\) that defines the eigenmodes of that cascaded impedance function.

In this text we will typically use column vectors. The vector may also be written out using unit vector notation to indicate direction. For example: \(x_1, x_2, x_3, \ldots = x_1 e_x + x_2 e_y + x_3 e_z\), where \(e_x, e_y, e_z\) are unit vectors in the \(x, y, z\) Cartesian directions (here the vector’s subscript \(3 \times 1\) indicates its dimensions). The type of notation used may depend on the engineering problem you are solving.

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\[
\begin{bmatrix}
a & b & x_1 \\
c & d & y_1
\end{bmatrix}
\]

the augmented matrix is:

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c & d & y_1
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\[
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1 & 0
\end{bmatrix}
\]

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\[
PA = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
a & b \\
\alpha & \beta
\end{bmatrix} = \begin{bmatrix}
\alpha & \beta \\
a & b
\end{bmatrix}
\]
whereas post-multiplication swaps the columns.

For the $3 \times 2$ case there are $3 \cdot 2 = 6$ such matrices (swap a row with the other two, then swap the remaining two rows).

6. Gaussian elimination (GE) operations. There are three types of elementary row operations which may be performed on a system of equations (excluding row interchanges), each of which produces a new matrix:

- (1) perform addition/subtraction of two scaled rows. All such operations can be performed using matrices.
- (2) swap rows. We will categorize any matrix that performs only elementary row operations (but not column operations) as an GE matrix. Therefore, a cascade of GE matrices is also a GE matrix.
- (3) scale rows.

For lack of a better term, we'll describe these as 'Gaussian elimination' or 'GE' matrices.

We can now identify the poles of this function. It is clear that the RoC of $w(s)$ is $-\infty < \zeta < \infty$. These poles are required to determine the RoC of $Z(s)\eta(t)$, which are required to determine the RoC of $Z(s)\eta(t)$. Therefore, if we scale by a number greater than 1, we may be guaranteed that we keep all of the poles in the RoC.

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For lack of a better term, we'll describe these as 'Gaussian elimination' or 'GE' matrices.
Derivation of the inverse of a $2 \times 2$ matrix

1. Step 1: To derive (ii) starting from (i), normalize the first column to 1.

$$\begin{bmatrix} \frac{1}{a} & \frac{b}{a} \\ \frac{c}{a} & \frac{d}{a} \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

2. Step 2: Subtract row (1) from row (2): (2) ← (2)−(1)

$$\begin{bmatrix} 1 & \frac{b}{a} \\ \frac{c}{a} & \frac{d}{a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{a} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

3. Step 3: Multiply row (2) by $\Delta$ and express result in terms of the determinant $\Delta = ad - bc$.

$$\begin{bmatrix} 1 & \frac{b}{a} \\ \frac{c}{a} & \frac{d}{a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \frac{d}{a} & -\frac{b}{a} \\ -\frac{c}{a} & \frac{d}{a} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

4. Step 4: Solve row (2) for $x_2$: $x_2 = -\frac{d}{a}y_1 + \frac{c}{a}y_2.$

5. Step 5: Solve row (1) for $x_1$:

$$x_1 = -\frac{b}{a}y_1 - \frac{c}{a}y_2 = \left[ 1 + \frac{b}{a} \frac{c}{a} \right] y_1 - \frac{b}{a}\Delta y_2.$$

Rewriting in matrix format, in terms of $\Delta = ad - bc$, gives:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{d}{a} & -\frac{b}{a} \\ -\frac{c}{a} & \frac{d}{a} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

since $d = (\Delta + bc)/a.$

**Summary:** This is a lot of messy algebra, so it is essential that you memorize the final result: (1) swap the diagonal, (2) change the off-diagonal signs, and (3) normalize by the determinant $\Delta$.

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Finding the RoC of the product formula: It would be interesting to find the RoC for $\mathcal{L}(s)$, and rigor, this question demands further investigation. To find the RoC, we need to evaluate

$$|\tau_n| = |e^{-\sigma T_n}| = |e^{-\sigma T_1}| = \left( \frac{1}{\tau_1} \right)^n < 1 \quad \text{for} \quad \sigma > 0,$$

where $T_n = -n\tau_1$. For example,

$$\mathcal{L}(s) = \frac{1}{\left( 1 - \frac{s}{\tau_1} \right)^2} = 1 + \frac{1}{\tau_1} + \frac{1}{\tau_1^2} + \frac{1}{\tau_1^3} + \cdots, \quad \Re\{s\} = \sigma > 0.$$

The RoC for each root is $\sigma \geq 0$ since when $\sigma < 0$.

Since $1/\tau_1 < 1$ for all $k \in \mathbb{N}$, the Taylor series of $\zeta(s)$ is entire except at its poles. Note that the RoC of a Taylor series in powers of $\zeta(s)$ increases with $k$.

**Exercise #1**

Work out the RoC for $k = 2$.

**Solution:** The formula for the RoC is given above, which for $\tau_1 = 3$ is

$$|\tau_n| = \left( \frac{1}{\tau_1} \right)^n < 1 \quad \text{for} \quad \sigma > 0,$$

where $\sigma$ is the boundary of the RoC.

**Exercise #2**

Show how to construct $Z^{(n)}(t) \leftrightarrow \mathcal{L}(s)$ by working in the time domain.

**Solution:** The basic procedure for building a sieve is to sum the integers

$$S_k = \sum_{r=1}^{n} n \pi^{-1} = 1 + 2^1 + 2^2 + 3^1 + 3^2 + \cdots,$$

while the sieve for the $k$th prime $\tau_k$ is

$$S_k = \sum_{r=1}^{n} n \tau^{-1} = 1 + 2^1 \tau_k + 3^1 \tau_k + 2^2 \tau_k + 3^2 \tau_k + \cdots.$$

This sum may be written in terms of the convolution with the Heaviside step function $u_n$, since

$$u_n \ast u_k = n u_k = n u_k = 0 \ast 0 + 1 \ast u_1 + 2 u_2 + \cdots + k u_k + \cdots.$$

**Poles of $\zeta(s)$**

Riemann proposed that Euler’s zeta function $\zeta(s) \in \mathbb{C}$ has a complex argument [first explored by Chebyshev in 1850 (Bombieri, 2000)] that extends $\zeta(s)$ into the complex plane ($s \in \mathbb{C}$), thus making it a complex-analytic function. Thus we might presume that $\zeta(s)$ has an inverse Laplace transform. There seems to be very little written on this topic (Hill, 2007). We explore this question further here.
Appendix B

Eigenanalysis

Eigenanalysis is ubiquitous in engineering applications. It is useful in solving differential equations, data-science applications, numerical approximation and computing, and other areas. Thus the eigenvalues of the matrix are key to eigenvector analysis, and you need to memorize them. You will use them repeatedly throughout this text.

The sieve of Eratosthenes (Fig. 2.3, p. 45) was first published by Euler in 1737, we can recursively factor out the leading prime term, ensuring that the lead term, along with all of its multiplicative factors, is subtracted out, just like the cancellations with the sieve of Eratosthenes. It is instructive to compare each iteration with the cancellations that of the sieve (see Fig. 2.3, p. 45).

The Riemann's complex–analytic product formula (Eq. B.1.2) = 1 + 1/(sπ)3 · · · · · · corresponding to the poles of ζ(s) for prime π k defines the poles of ζ(s) for prime π k.

3 This is known as Euler's sieve, as distinguished from the Eratosthenes sieve, as referred to in the text.
If we restrict Eq. B.1.1 to a single eigenvector (one of $e_{\pm}$), along with the corresponding eigenvalue $\lambda_{\pm}$, we obtain the two matrix equations

$$a e_{\pm} - \lambda_{\pm} e_{\pm} = 0.$$ \hspace{1cm} (B.1.6)

Note the swap in the order of $e_{\pm}$ and $\lambda_{\pm}$. Since $\lambda_{\pm}$ is a scalar, this is legal (and critically important), since this allows us to factor out $e_{\pm}$

$$\left( a - \lambda_{\pm} I \right) e_{\pm} = 0.$$ \hspace{1cm} (B.1.6)

The matrix $a - \lambda_{\pm} I$ must be singular because when it operates on $e_{\pm}$, having nonzero norm, it must be zero. It follows that its determinant (i.e., $|a - \lambda_{\pm} I| = 0$) must be zero. This equation uniquely determines the eigenvalues $\lambda_{\pm}$.

### B.1.1 Calculating the eigenvalues $\lambda_{\pm}$

The eigenvalues $\lambda_{\pm}$ of $a$ may be determined from $|a - \lambda_{\pm} I| = 0$. As an example we let $a$ be Pell’s equation (Eq. 2.5.4, p. 62. In this case the eigenvalues may be found from

\[
\begin{vmatrix}
1 - \lambda_{\pm} & N \\
1 & 1 - \lambda_{\pm}
\end{vmatrix}
= (1 - \lambda_{\pm})^2 - N = 0,
\]

thus $\lambda_{\pm} = (1 \mp \sqrt{N}).$ \hspace{1cm} (C.1.1)

### B.1.2 Calculating the eigenvectors $e_{\pm}$

Once the eigenvalues have been determined, they are substituted into Eq. B.1.6, which determines the eigenvectors $e = [e_{\pm}, e_{\mp}]$, by solving

\[
\left( a - \lambda_{\pm} I \right) e_{\pm} = \left( \begin{array}{c}
1 - \lambda_{\pm} \\
1
\end{array} \right) N e_{\pm} = 0,
\]

where $1 - \lambda_{\pm} = 1 - (1 \mp \sqrt{N}) = \pm \sqrt{N}$, thus the Pell equation eigenvalues are

\[
\lambda_{\pm} = 1 \mp \sqrt{N}.
\]

Recall that Eq. B.1.6 is singular because we are using an eigenvalue, and each eigenvector is pointing in a unique direction (this is why it is singular). You might expect that this equation has no solution. In some sense you would be correct. When we solve for $e_{\pm}$, the two equations defined by Eq. B.1.6 are co-linear (the two equations describe parallel lines so their scalar product is one). This follows from the fact that there is only one eigenvector for each eigenvalue.

Since there is only one eigenvalue we are expecting trouble, yet we may proceed to solve for $e_{\pm} = [e_{\pm}, e_{\pm}]^T$ with eigenvalue $\pm \sqrt{N}$

$$\left( \begin{array}{cc}
\sqrt{N} & N \\
1 & \sqrt{N}
\end{array} \right) \left( \begin{array}{c}
e_{\pm} \\
e_{\mp}
\end{array} \right) = 0.$$

\hspace{1cm} (C.1.1)

It is a convention to order the eigenvalues from largest to smallest.

---

1. as described in his 1851 paper. But is his definition correct? To resolve this one needs to improve our understanding of the core definition of analytic continuation, as it relates to the $\zeta^{-1}$. In the case of the geometric series the analytic continuation is the closed form expression $\sum_{n=0}^{\infty} x^n = 1/(1-x)$ which is valid for all $x \neq 1$. This is not Riemann’s definition of analytic continuation.

2. Sanitize check: For example, let $n = 2$ and $x > 0$. Then $R = 2^{1-x}$, where $x = \lim_{n \to 0+}$. Taking the log gives $\log R = -\log 2 - \infty < 0$. Since $\log R < 0$, $R < 1$.

---

C.1. TABLES OF LAPLACE TRANSFORMS

as described in his 1851 paper. But is his definition correct? To resolve this one needs to improve our understanding of the core definition of analytic continuation, as it relates to the $\zeta^{-1}$. In the case of the geometric series the analytic continuation is the closed form expression $\sum_{n=0}^{\infty} x^n = 1/(1-x)$ which is valid for all $x \neq 1$. This is not Riemann’s definition of analytic continuation.

This section is a beginners review of $(s)$, building on the developments of analytic functions from Chapter 3, especially in Secs. 3.2.6, 5.4, and Fig. 5.3 p. 526. We will use the location of the poles of zeta, which depend on the prime numbers. Not so well understood are the remaining analytic properties over the entire plane, such as the zeros of $(s)$, namely the poles of $\zeta(s)$. A key function is $\ln(z)$.

Consider $\zeta^{-1}$ as $T$ is the sample period at which data are taken (every T seconds). For example, if $T = 22.676 = 10^3/44.100 [\mu s]$, then the data is sampled at 44.10 $[kHz]$. This is how modern digital audio works, for CD-quality music. The unit-time delay time operator $\zeta^{-1}$ is

$$\delta(t - T) \leftrightarrow e^{-\zeta T}.$$

When we deal with the Euler and Riemann zeta functions, the only sampling period that makes sense is $T = 1 [s]$ or $1 [Hz]$ (i.e., $n \in \mathbb{Z}$). In this case, the samples of interest are $\mod(n, n_i)$. Starting from the sieve of Eratosthenes, Euler showed that the counting numbers $n \in \mathbb{Z}$, presented at a rate of one per second [1-Hz], may be uniquely reduced to multiples of the primes. This is the basis for the fundamental theorem of arithmetic, the theorem of the concept of the prime number, which states that every integer may be uniquely factored into a product of prime numbers.

The zeta function

The poles of the zeta function depend explicitly on the primes, which makes it a very special function. In 1737 Euler proposed the real-valued function $\zeta(s) \in \mathbb{R}$, and $s \in \mathbb{R}$ to prove that the number of primes is infinite (Goldstein, 1973). Euler’s definition of $\zeta(s) \in \mathbb{R}$ is given by the analytic power series,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $s > 1 \in \mathbb{R}$. \hspace{1cm} (C.1.1)

This series converges for $s > 0$, since $R = n^{-s} < 1, n > 1 \in \mathbb{N}$. \hspace{1cm} (C.1.1)

In 1860 Riemann extended the zeta function into the complex plane, resulting in $\zeta(s)$, defined by the complex–analytic power series, identical to the Euler formula except

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\Re(s) = \sigma > 1$. \hspace{1cm} (C.1.2)

This formula converges for $\Re(s) \geq 1$ (Goldstein, 1973). To determine the formula in other regions of the s plane, we need to extend the series via analytic continuation. As it turns out, Euler’s formulation provided detailed information about the structure of primes, going far beyond his original goal.
### Table C.2: An extended table of Laplace transforms

<table>
<thead>
<tr>
<th>Name</th>
<th>Transform</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_0$, $J_1$</td>
<td>$\frac{\sin r\pi}{r\pi}$</td>
<td>Bessel functions of the first and second kind.</td>
</tr>
<tr>
<td>$K_0$, $K_1$</td>
<td>$\frac{1}{r\pi} \cdot \text{erfc}(\sqrt{s})$</td>
<td>Error function</td>
</tr>
<tr>
<td>$\sqrt{t}u(t)$</td>
<td>$\frac{1}{\sqrt{s}} K_{1/2}(s/2)/\sqrt{s}$</td>
<td>K-Bessel</td>
</tr>
<tr>
<td>$\zeta(t)$</td>
<td>$\zeta(s)$</td>
<td>Riemann zeta function</td>
</tr>
</tbody>
</table>

#### B.2. Pell equation solution example

If we divide the top row by $\sqrt{N}$, the two rows are identical, since the matrix must be singular. The eigenvectors are:

\[
\begin{bmatrix}
1 \\
1
\end{bmatrix}
\text{and}
\begin{bmatrix}
1 \\
-1
\end{bmatrix}
\]

Thus the eigenvector is:

\[
\begin{bmatrix}
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}}
\end{bmatrix}
\]

This results in the general solution given by:

\[
(I + N A)/s = \begin{bmatrix}
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}}
\end{bmatrix}
\]

where $c$ is a normalization constant.

The constant $\Lambda$ is equal to $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$. Thus the solution to Pell's equation (Eq. 2.5.2) is Eq. 2.5.4 (p. 62). Thus we need powers of the Pell matrix $A$, which gives an explicit expression for $c = 1/(\sqrt{2})$.

Once the matrix has been diagonalized, one may compute powers of that matrix as powers of the eigenvalues. This results in the general solution given by:

\[
\begin{bmatrix}
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}}
\end{bmatrix}
\]

where $\Lambda = 1.414$.

For $\sqrt{b}$, the Pell matrix is:

\[
\begin{bmatrix}
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}}
\end{bmatrix}
\]

Thus we need powers of the Pell matrix $A$, which gives an explicit expression for $c = 1/(\sqrt{2})$.
B.2.1 Pell equation eigenvalue-eigenvector analysis

Here we show how to compute the eigenvalues and eigenvectors for the $2 \times 2$ Pell matrix for $N = 2$

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

The Matlab/Octave command $[E, D] = eig(A)$ returns the eigenvector matrix $e$

$$e = [e_+, e_-] = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.8165 & -0.8165 \\ 0.5774 & 0.5774 \end{bmatrix}$$

and the eigenvector matrix $\Lambda$ (Matlab/Octave’s $D$)

$$\Lambda = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} = \begin{bmatrix} 1 + \sqrt{2} & 0 \\ 0 & 1 - \sqrt{2} \end{bmatrix} = \begin{bmatrix} 2.4142 & 0 \\ 0 & -0.4142 \end{bmatrix}$$

My student Kehan found the general formula for $\beta_0$. It seems likely that the powers of $\beta_0$ could be absorbed in the starting solution, and then be removed from the recursion.

### Pell’s Equation for $N = 3$

1. Case of $N = 3 & |(x, y)\|^2 = 3$:
   - $\beta_0 = \sqrt{3}$
   - $e_+ = e_-$

   Note: $x_+^2 - 3y_+^2 = 1$, $x_+/y_+ \to \sqrt{3}$

<table>
<thead>
<tr>
<th>$x_+$</th>
<th>$y_+$</th>
<th>$\beta_0$</th>
<th>$\frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} &amp; -\sqrt{2} \ 1 &amp; 1 \end{bmatrix}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_0$</td>
<td>1</td>
<td>$\frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} &amp; -\sqrt{2} \ 1 &amp; 1 \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>2</td>
<td>$\frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} &amp; -\sqrt{2} \ 1 &amp; 1 \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>3</td>
<td>$\frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} &amp; -\sqrt{2} \ 1 &amp; 1 \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>4</td>
<td>$\frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} &amp; -\sqrt{2} \ 1 &amp; 1 \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>5</td>
<td>$\frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} &amp; -\sqrt{2} \ 1 &amp; 1 \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>6</td>
<td>$\frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} &amp; -\sqrt{2} \ 1 &amp; 1 \end{bmatrix}$</td>
<td></td>
</tr>
</tbody>
</table>

Pell’s equation for $N = 3$

In Table B.1, Pell’s equation for $N = 3$ is given, with $\beta_0 = \sqrt{3}$. Perhaps try other trivial solutions such as $[-1, 0]^T$ and $[\pm \sqrt{3}, 0]^T$, to provide clues to the proper value of $\beta_0$ for cases where $N > 3$.

**Example:** I suggest that you verify $e\Lambda = \Lambda e$ and $ae = eA$ with Matlab/Octave. Here is the Matlab/Octave program which does this:

### Appendix C

#### Laplace transforms $\mathcal{L}$

Laplace transforms are discussed in Sec. 3.10 (p. 155), with the definition of the $\mathcal{L}$ in Eq. 3.10.1 (p. 156). Level-I (basic) $\mathcal{L}$s are listed in Table 3.6 (p. 156).

### C.1 Tables of Laplace transforms

The following tables of $\mathcal{L}$ and $\mathcal{L}^{-1}$ are a convenient summary of their properties and evaluations for many different functions. Table 3.8 gives basic function properties such as convolution and the properties of step functions and frequency scaling. Table C.1 provides the commands for doing symbolic (computer algebra and calculus) transformations, which includes some unusual $\mathcal{L}$’s and Taylor series of the $\Gamma$ function (Graham et al., 1994), a complex–analytic extension of the factorial. Table 3.9 gives the basic transforms typically used for more common calculations. Table C.2 provides extended less common transforms, such as the half-derivative and integration and Bessel functions.

These tables are available in most books on differential equations and remain a core technology for analytic methods for solving differential equations.

<table>
<thead>
<tr>
<th>$\text{sym}$</th>
<th>$\text{command}$</th>
<th>$\text{result}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>sym $t$</td>
<td>$\text{laplace}(\mathcal{L}^{-1})$</td>
<td>$\Gamma(p)x^p$</td>
</tr>
<tr>
<td>sym $s$</td>
<td>$\text{ilaplace}(\text{gamma}(s))$</td>
<td>$e^{-s}$</td>
</tr>
<tr>
<td>sym $s$, a</td>
<td>$\text{ilaplace}(\text{gamma}(s, a))$</td>
<td>$\text{Heaviside}(s-a)$</td>
</tr>
<tr>
<td>sym Gamma, t</td>
<td>$\text{taylor}(\text{Gamma}(s, a))$</td>
<td>$\frac{1}{2} - s + s \left( \frac{1}{2} + a \right) + \frac{1}{s} K_0 + \cdots$</td>
</tr>
</tbody>
</table>

#### C.1.1 $\mathcal{L}^{-1}$ of the Riemann zeta function

The analytic properties of the zeta function $\zeta(s)$ have been a holy grail for mathematicians, starting with Euler, all of whom have made their reputation on that function. For the neophyte, $\zeta(s)$ is important because it is an analytic extension of the sieve, which is the prime identification method. Analytic continuation of the $\zeta(s)$ function was first stated by Riemann.
B.3. SYMBOLIC ANALYSIS OF $T = \mathbf{E}\Lambda$

A = \[
\begin{bmatrix}
1 & 2 \\
1 & 1
\end{bmatrix}
\]; %define the matrix

\([E, D]\) = eig(A); %compute the eigenvector and eigenvalue matrices

A*E - E*D %This should be $\approx 0$, within numerical error.

E*D - D*E %This is not zero

Summary:
Thus far we have shown that for the case of Pell matrix with $N = 2$, the normalized eigenmatrix and its inverse is

\[
\begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{bmatrix}
\]

and the eigenmatrix is

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

Here we assume

B.3.1 The $2 \times 2$ transmission matrix

$V = \mathbf{E}_v \mathbf{E}_t$

\[
\begin{bmatrix}
\mathbf{v}^\top \mathbf{S} - 1 & 0 \\
0 & \mathbf{v}^\top + 1
\end{bmatrix} = \begin{bmatrix}
\mathbf{v} & 0 \\
0 & \mathbf{v}
\end{bmatrix} - \mathbf{V}
\]

and the coefficients in

\[
\begin{bmatrix}
\mathbf{v}^\top \mathbf{S} - 1 & 0 \\
0 & \mathbf{v}^\top + 1
\end{bmatrix} = \begin{bmatrix}
\mathbf{v}^\top & 0 \\
0 & \mathbf{v}^\top
\end{bmatrix} - \mathbf{V}
\]

and in inverse is

$\mathbf{v}$

and in matrix is

\[
\begin{bmatrix}
\mathbf{v}^\top \mathbf{S} - 1 & i \mathbf{v}^\top \\
i \mathbf{v}^\top + 1 & \mathbf{v}^\top
\end{bmatrix} - \mathbf{V}
\]

and in vector is

$\mathbf{v}$

Then we note that the case of Pell matrix with $\mathbf{E}_v = \mathbf{E}_t$.

Summary:
B.3.2 Matrices with symmetry

Reversible systems:

When $A = D$

$$e = \begin{bmatrix} -\sqrt{\frac{B}{2}} + \sqrt{\frac{C}{2}} \\ 1 \\ 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} A - \sqrt{BC} & 0 \\ 0 & A + \sqrt{BC} \end{bmatrix}$$

(B.3.5)

the transmission matrix is said to be reversible, and the properties greatly simplify.

In this case that $r_e = \sqrt{\frac{B}{2C}}$ is the characteristic impedance and $\nu(s) = \frac{1}{\sqrt{BC}}$ is the propagation function.

Reciprocal systems

For the case of the ABCD matrix the eigenvalues depend critically on reciprocity. A reciprocal matrix is when $\Delta_T = 1$ and is anti-reciprocal for $\Delta_T = -1$. It is helpful to display the eigenfunctions and values in terms of $\Delta_T$ so this distinction is clear.

When the matrix is symmetric ($B = C$), thus $r_e = 1$ and the corresponding system is said to be reciprocal. Most physical systems are reciprocal. The determinant of the transmission matrix of a reciprocal network $\Delta_T = AD - BC = 1$. For example, electrical networks composed of inductors, capacitors and resistors are always reciprocal. It follows that the complex impedance matrix is symmetric (Van Valkenburg, 1964a).

Magnetic systems such as dynamic loudspeakers are anti-reciprocal, and correspondingly $\Delta_T = -1$. The impedance matrix of these loudspeakers is skew symmetric (Kim and Allen, 2013). All impedance matrices are either symmetric or anti-symmetric, depending on whether they are reciprocal (LRC networks) or anti-reciprocal (magnetic networks). These systems have complex eigenvalues with negative real parts, corresponding to damped (lossy) causal systems. This follows from conservation of energy.

The impedance matrix cannot be Hermitian, or the losses would be zero, because Hermitian matrices have real eigenvalues. A physical system having power losses cannot be Hermitian because the eigenvalues must have a negative real part.

In summary, given a reciprocal system, the $T$ matrix has $\Delta_T = 1$, and the corresponding impedance matrix is symmetric (not Hermitian).

B.3.3 The impedance matrix

As previously discussed in Sec. 3.8 (p. 144), the impedance matrix $Z$ is

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = Z(s) \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{C} \begin{bmatrix} A & D \\ \frac{A}{C} & \frac{D}{C} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}.$$

Reciprocal systems have skew-symmetric impedance matrices, namely $z_{12} = -z_{21}$ (i.e., $\Delta_T = 1$). This condition is best understood using the $T$ form of the impedance matrix, as shown in Fig. 3.9 (p. 146). When the system is both reversible $A = D$ and reciprocal ($B = C$), the impedance matrix simplifies to

$$Z(s) = \frac{1}{C} \begin{bmatrix} A & 1 \\ 1 & A \end{bmatrix}.$$

For such systems there are only two degrees of freedom, $A$ and $C$. As discussed previously in Sec. 3.8 (p. 144), each of these has a physical meaning: $1/A$ is the Thévenin source voltage assuming a voltage drive, and $B/A$ is the Thévenin impedance (Sec. 3.8.3, p. 145).

Impedance is not Hermitian: By definition, when a system is Hermitian its matrix is conjugate symmetric

$$Z(s) = Z^*(s),$$

a stronger condition than reciprocal, but not the symmetry of the Brune impedance matrix. A reciprocal Brune impedance is symmetric (not Hermitian).