Abstract

Group theory in physics classifies properties of physical systems. For example, conservation of energy is equivalent to time-invariance in the system equations. This follows from Noether’s Theorem. When Emma Noether died, Einstein spoke at her funeral, noting that the theory of relativity would not have been possible without her deep insights and theorems. These contributions reach down to Acoustic Metamaterials. In fact there are 1:1 relations between symmetry and system postulates [Kim et al., 2016]. Six of these postulates are the best known [Carlin and Giordano, 1964]: $P_1$: Causality, $P_2$: Linearity, $P_3$: Real time response, $P_4$: Passive/Active, $P_5$: Time-invariance, $P_6$: Reciprocity [Beranek and Mellow, 2012]. These relations will be defined and discussed in the presentation, along with others, the most interesting being Brune impedance [Brune, 1931], which explains $P_4$. As noted, $P_5$ implies conservation of energy, which is also related to $P_4$. Additional postulates include $P_7$: reversibility and $P_8$: transnational-invariance. We propose that all the postulates have a symmetry counter-part. For example, rotation and translation symmetries have corresponding momentum conservation laws. Since the mathematics of group-theoretic symmetry is a highly developed mathematical science, perhaps we might use such symmetry classification schemes to help us think about metamaterial properties.
What is a Metamaterial?

1. Traditional definition: *A material that does not appear in nature.*
   - This definition depends on the meaning of the word *natural.*
   - Is a semiconductor or PZT “natural?”
   - What is needed is a scientific taxonomy of materials.

2. Metamaterials are symmetric arrays of transducers

3. Network theory has defined such properties [Carlin and Giordano, 1964]
   - \( P_1 \) Causal (no response prior to stimulus)
   - \( P_2 \) Linear (super-position + no internal state)
   - \( P_3 \) Passive/active (no power/battery)
   - \( P_4 \) Real/complex “impulse response”
   - \( P_5 \) Time-invariant/time-varying: (response independent of time)
   - \( P_6 \) Reciprocal: \( \left. \frac{V_2}{I_1} \right|_{I_1=0} = \left. \frac{V_1}{I_2} \right|_{I_2=0} \)

4. Additional postulates [Kim et al., 2016]
   - \( P_7 \) Spatially Reversibility (invariant to swapping ports)
   - \( P_8 \) Spacial invariant (independent of location)
   - \( P_9 \) Quasi-static (\( \lambda > \Delta_x \), i.e., \( ka = a/\lambda < 1 \))
   - \( P_{10} \) Deterministic
Taxonomy of metamaterials

- Configurations of Metamaterials:
  - 1, 2, 3 dimensional arrays of interconnected cells of transducers

- Taxonomy of a cell: Assumed vs. alternative postulates
  - Assumed: $P_1$: Causal, $P_4$: Real, $P_9$: Quasi-static; $P_{10}$: Deterministic

- Examples:
  - $P_2$: Linear/nonlinear: filter-bank/Diode envelope detectors
  - $P_4$: Passive/Active: (transistor/amplifier)
  - $P_5$: modulation: (frequency shifting)
  - $P_6$: non-reciprocal/anti-reciprocal: (PZT/dynamic loudspeaker)
  - $P_7$: non-reversible (transformer, Horn)
  - $P_8$: Spacial invariance: (Plasma moving in a magnetic field)
  - $P_{10}$: Deterministic: ($F = ma$ vs. motions of universe)
Transmission lines: Webster Horn Equation

\[ \frac{d}{dx} \left[ \mathcal{P}(x, \omega) \right] = -\left[ \begin{array}{cc} 0 & s \frac{\rho}{A(x)} \\ s \frac{A(x)}{\eta \mathcal{P}} & 0 \end{array} \right] \left[ \begin{array}{c} \mathcal{P}(x, \omega) \\ \mathcal{V}(x, \omega) \end{array} \right], \]

Two key properties: Propagation constant; characteristic admittance

\[ \kappa(s, x) = s \sqrt{\frac{\rho}{A(x)}} \cdot \frac{A(x)}{\eta \mathcal{P}} = \rho \frac{s}{c} \quad \mathcal{Y}(x) = \sqrt{\frac{sA(x)}{\eta \mathcal{P}}} \cdot \frac{A(x)}{s \rho} = \frac{A(x)}{\rho c} \]
Modeling of a Metamaterial delay line cell

\[ L = \frac{\rho}{A(x)} \]

\[ V_1 - I_1 \rightarrow L \rightarrow V_2 + I_2 \]

\[ C = \frac{\eta P}{A(x)} \]

Cell model:

\[
\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & s \frac{\rho}{A(x)} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s \frac{\eta P}{A(x)} & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}
\]

Note that the change in the area \( A(x) \) is directly related to

\[ Z_{rad}(s, x) = \frac{V_1}{I_1} \]

Given \( Z_{rad}(s, x) \) one may determine \( A(x) \) [Youla, 1964]
Characterizing transmission lines

Figure: Depiction of a train: cars are mass’s $M$ and linkages are springs $C$.

\[
\begin{bmatrix}
F_1 \\
V_1
\end{bmatrix} = \mathbf{T}^N \begin{bmatrix}
F_N(\omega) \\
-V_N(\omega)
\end{bmatrix}
\]

\[
\mathbf{T} = \begin{bmatrix}
1+s^2/c^2 & sM(1 + \frac{1}{4}s^2/c^2) \\
sC & 1+s^2/c^2
\end{bmatrix} \approx \begin{bmatrix}
1 & sM \\
sC & 1
\end{bmatrix}
\]

Eigen expansion (uniform elements):

\[
\mathbf{T}^N = E \Lambda^N E^{-1} = E \begin{bmatrix}
\lambda_+^N & 0 \\
0 & \lambda_-^N
\end{bmatrix} E^{-1}.
\]
2x2 Eigenvalues

Defining:

\[ T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \]

Eigenvectors:

\[ \begin{bmatrix} e_\pm \end{bmatrix} = \frac{1}{2c} \begin{bmatrix} (A - D) \pm \sqrt{(A - D)^2 + 4BC} \\ 1 \end{bmatrix} \]

Eigenvalues:

\[ \lambda_\pm = \frac{1}{2} \left( \frac{(A + D) - \sqrt{(A - D)^2 + 4BC}}{(A + D) + \sqrt{(A - D)^2 + 4BC}} \right) \quad (1) \]

Interesting special cases:

- What if \( B = C \)? Reciprocal
- What if \( C = -B \)? Anti-reciprocal (Loudspeakers; Electron spin)
- What if \( A = D \)? Reversible.; \( A \neq D \)? reversible
- What if \( A = -D \)? Anti-reversible?? Never heard of it.
- What if \( \det T = 0 \)? Impossible: \( \det T = 1 \)
- What if \( C = 0 \)? Impossible:


