Riemann Sphere analytics

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Abstract

The Riemann sphere (RS), also known as the extended plane, was a breakthrough in complex analysis, introduced in B. Riemann’s Doctorial thesis (1851). His presentation was geometrical. We recall the formula for stereographic projection from the Riemann sphere to \( \mathbb{C} \), and we derive a formula for its inverse. This is a mapping from \( Z \) to \( P(x, y, z) \). We then discuss the physical interpretation of the inverse mapping when the complex variable denotes an impedance.¹

1 Introduction

Here we derive the mapping from a point on the finite plane \( Z \) to its “image” on the Riemann Sphere \( S \). We then interpret the meaning of this transformation when the plane defines an impedance \( Z(s) \) as a function of the complex frequency variable \( s = \sigma + i\omega \).

There are two sets of coordinates required to set up this problem. First there is any point in \( \mathbb{R}^3 \) denoted \( R \equiv [x, y, z] \). The North Pole is given by \([0, 0, 1]\) and the South Pole as \([0, 0, -1]\). Second the points \( Z = X + iY \) on the finite plane \((z = 0)\) are \( X = x \) and \( Y = y \). The points on the extended plane are a subset of \( R \), denoted \( P(x, y, z) \), such that \( \|P\| = 1 \).

The mapping from the sphere to the finite plane \( Z \), defined as \( Z = P^{-1}(x, y, z) \), may be expressed in either rectangular \((x, y, z)\) or in spherical \((\phi, \theta)\) coordinates as²

\[
Z(x, y, z) = \frac{x + iy}{1 - z} = \cot\left(\frac{\phi}{2}\right) e^{i\theta}.
\]

as shown in Fig. 1.³ We desire the mapping from \( Z \) to \([x, y, z] \) on the unit sphere (i.e., \( \alpha = P(A) \) of Fig. 1).

The spherical \( \cot(\phi/2) \) formula comes from the “law of cotangents” described in Appendix A.

The problem then is to determine \( P(Z) \) \(([x, y, z] \) given \( Z \), namely find the mapping from any point \( Z \) on the finite \( Z \) plane (indicated as \( A \) in Fig. 1), to the corresponding “puncture point” coordinates on \( S \) \( \alpha = P \). Formally we may define this mapping as \([x, y, z] = P(Z) \). In other words, given a point \( Z \) on the finite plane, determine the points \([x, y, z] \) on \( S \), such that \( \|[x, y, z]\| = 1 \).

¹Eventually we hope to discuss the Mobius transformation of the plane to the sphere.
²\( \text{wikipedia.org/wiki/Riemann_sphere} \)
³Jean-Christophe BENOIST \( \text{wikipedia.org/wiki/Riemann_sphere} \)
The solution: The final result is

\[ [x, y, z] = P(Z) = \frac{[2X, 2Y, |Z|^2 - 1]}{|Z|^2 + 1}, \]  

(2)

where \( X = \Re Z \) and \( Y = \Im Z \).

A more compact way of stating \( P(Z) \) is to express \( P \) in terms of a complex number \( \zeta \), proportional to

\[ \zeta = x + iy = \frac{2Z}{|Z|^2 + 1} \]  

(3)

along with the corresponding \( z \) coordinate

\[ z = \frac{|Z|^2 - 1}{|Z|^2 + 1}. \]  

(4)

Equations 1-4 “make sense” in terms of the construction of Fig. 1:

- Eq. 1 and Eq. 3: \( \theta = \angle Z(x, y) = \angle \zeta \). From Eq. 3 we see that \( |Z| / |\zeta| = (1 + |Z|^2) / 2 \). Thus when \( |Z| \geq 1, |Z| / |\zeta| \geq 1 \). From the construction this is easy to visualize, as \( |\zeta| \) is always inside the unit disk. Less obvious is what happens to \( |\zeta| \) for \( |Z| < 1 \).

- Eq. 2: This equation describes the coordinates for \( \alpha \) in terms of \( Z \), whereas Eq. 1 is the inverse relationship.

- Eq. 4 is the “height” of point \( \alpha(|Z|) \). When \( |Z| = 0, z = -1 \). When \( |Z| = 1, z = 0 \), and when \( |Z| \to \infty, z \to 1 \).

1.1 Mappings between the finite and extended planes

We are looking for the formula for the image point \( \alpha \) given any point \( Z = X + iY \) on the finite plane. The approach is to derive the formula for the mapping from the north pole of \( S \) to any point \( R \in \mathbb{R}^2 \).

\[ \text{http://www.encyclopediaofmath.org/index.php/Riemann_sphere} \]
A line $R(t) = p + t(q - p)$ is defined by two points $p, q \in \mathbb{R}^3$. When $t = 0$, $R(0) = p$ and when $t = 1$, $R(1) = q$. The line from the north pole $p = [0, 0, 1]$ to point $q = [x, y, z]$ (any point in $\mathbb{R}^3$) is thus given by

$$R(t) = [tx, ty, 1 + t(z - 1)].$$

**Line from the north pole to the finite plane $Z$:** Note $-1 \leq z \leq 1$ is limited to be between the two poles. We define our line $P(t)$ to go from the North pole to the $Z$ plane at $z = 0$. When $z = 0$, $R(t)$ becomes

$$P(t) = [tX, tY, 1 - t].$$

### 1.2 Restricting $[x, y, z]$ to the Riemann Sphere

To restrict the points $[x, y, z]$ to be on $S$ we require that

$$\|P(t)\|^2 = t^2(X^2 + Y^2) + t^2 - 2t + 1 = 1.$$ 

or in terms of $|Z|$

$$\|P(t)\|^2 = t^2(1 + |Z|^2) - 2t + 1 = 1.$$ 

Solving this equation for $t$ we have

$$t = \left\{ \frac{2}{1 + |Z|^2}, 0 \right\}.$$ 

The root 0 corresponds to the north pole. Thus

$$P(Z) = \frac{[2X, 2Y, |Z|^2 - 1]}{|Z|^2 + 1},$$ 

which is the desired Eq. 2.

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![Diagram](image-url)

Figure 2: Superimposed mappings. The point $z = 1$ is indicated on the $z$ axis (dark-blue) and $w = 1$ is indicated on the $w$ axis (light blue). The projections of these points are then reflected to the other’s axis. E.G., $w = 1$ is projected onto the $z$ axis as indicated by the solid dark-blue filled circle.
2 Examples of important mappings

Here we wish to discuss some important examples, mapping out $P(Z)$ for some classic case of impedance $Z(s)$ and reflectance $\Gamma(s)$.

We begin with the item in Fig. 2 which shows two variables, $z$ and $w$ which are rotated by 30° relative to each other.

Some ideas

- $Z = 1/\sqrt{(s)}$

- The map for various bilinear transformations.

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\[ A \quad \text{Law of cotangents} \]

For our case, $\phi$ is the polar angle and $a$ be the length of the chord from the North Pole ($N$) to the puncture point $\alpha$, then the triangle’s sides are $a, 1, 1$. The semi-perimeter $s$ is defined one-half the sum of the three sides (i.e., $s = 1 + a/2$), while the inradius (the radius of the inscribed circle)\(^5\) is

\[
r = \sqrt{\frac{(s-a)(s-1)(s-1)}{s}} = \frac{a}{2}\sqrt{\frac{a}{2+a}}.
\]

The law of cotangents is $\cot(\phi/2) = (s-a)/r$. From Fig. 1 $a$ is the chord form $N$ to $\alpha$.

\[^5\text{http://en.wikipedia.org/wiki/Law_of_cotangents}\]