0.1 Problems NS-1

Topic of this homework:

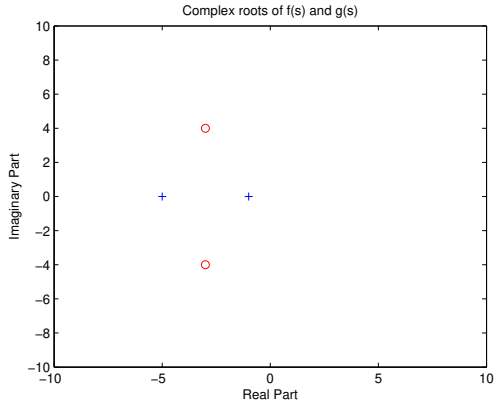
Introduction to Matlab/Octave (see the Matlab or Octave tutorial for help)
Deliverables: Report with charts and answers to questions.

Plotting complex quantities in Octave/Matlab

Problem # *1: Consider the functions* $f(s) = s^2 + 6s + 25$ *and* $g(s) = s^2 + 6s + 5$.

– 1.1: Find the zeros of functions $f(s)$ *and* $g(s)$ *using the command* `roots()`.
Sol: The roots of $f(s)$ are $-3 \pm 4i$ (in Matlab: `roots([1 6 25])`). The roots of $g(s)$ are -1 and -5 (in Matlab: `roots([1 6 5])`). You will find the program that generates all these figures at `https://jontalle.web.engr.illinois.edu/uploads/298.17/NS1.m`

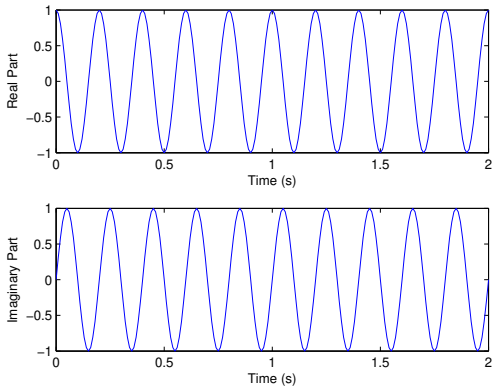
– 1.2: Show the roots of $f(s)$ *as red circles and of* $g(s)$ *as blue plus signs.*
The x-axis should display the real part of each root, and the y-axis should display the imaginary part. Use `hold on` and `grid on` when plotting the roots.
Sol:



– 1.3 Give your figure the title "Complex Roots of f(s) and g(s)." Label the x- and y-axes "Real Part" and "Imaginary Part." Hint: Use `xlabel`, `ylabel`, `ylim([-10 10])`, and `xlim([-10 10])` to expand the axes.

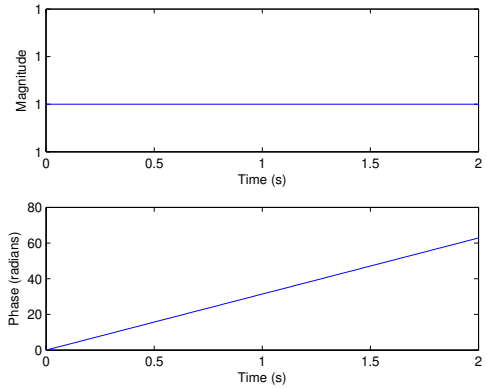
Problem # *2: Consider the function* $h(t) = e^{\jmath 2\pi f t}$ *for* $f = 5$ *and* `t=[0:0.01:2]`.

– 2.1: Use `subplot` *to show the real and imaginary parts of* $h(t)$.
Make two graphs in one figure. Label the x-axes "Time (s)" and the y-axes "Real Part" and "Imaginary Part."
Sol: Breaking $h(t)$ into real and imaginary parts gives $e^{\jmath 2\pi 5t} = \cos(10\pi t) + \jmath \sin(10\pi t)$.



– 2.2: Use `subplot` *to plot the magnitude and phase parts of* $h(t)$.
Use the command `angle` or `unwrap(angle())` to plot the phase. Label the x-axes "Time (s)" and the y-axes ' 'Magnitude" and "Phase (radians)."

Sol:



Prime numbers, infinity

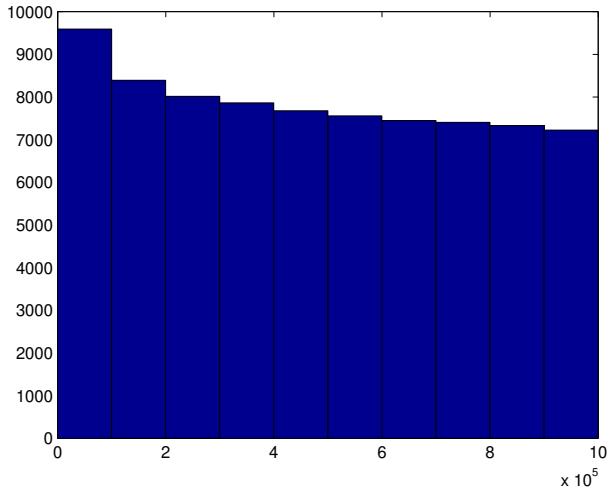
Problem # *3: Prime numbers*

– 3.1: Show that the prime number 5 may be factored.
Hint: There are two Gaussian prime factors. consider $(m + nj)(m - nj)$. Find m, n. **Sol:** We may factor 5 as follows $5 = (2 + i)(2 - i) = 4 + 1$. Interingly, this example seems to violate all the modern rules of matheatics, yet is is undeniably correct.

– 3.2: Use the Matlab/Octave function `factor` *to find the prime factors of 123, 248, 1767, and 999,999.*
Sol: Factors: 123 (3, 41), 248 (2,2,2,31), 1767 (3,19,31), 999999 (3,3,3,7,11,13,37)

– 3.3: Use the Matlab/Octave function `isprime` *to determine whether 2, 3, and 4 are prime numbers. What does the function* `isprime` *return when a number is prime or not prime? Why?*
Sol: Function `isprime(2)` returns 1, `isprime(3)` returns 1, and `isprime(4)` returns 0. 1 means 'yes' and 0 means 'no'

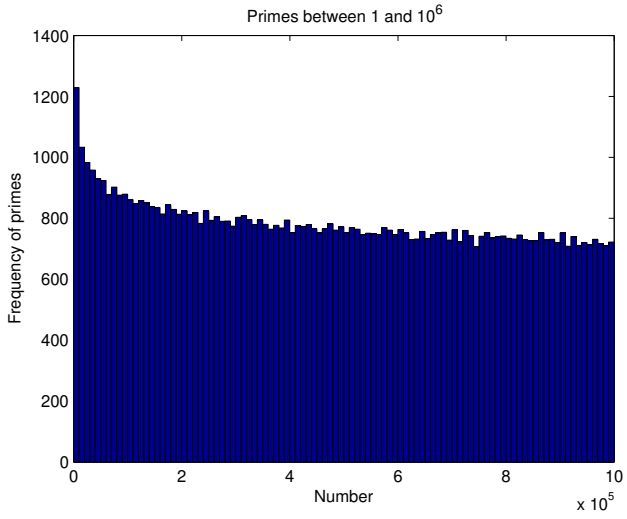
– 3.4: Use the Matlab/Octave function `primes` *to generate prime numbers between 1 and* 10^6. *Save them in a vector* `x`. *Plot this result using the command* `hist(x)`.
Sol:



– 3.5: Now try `[n,bincenters] = hist(x)`*. Use* `length(n)` *to find the number of bins.*

Sol: `length(n)` is 10

– 3.6: Set the number of bins to 100 by using an extra input argument to the function `hist`*. Show the resulting figure, give it a title, and label the axes.* Hint: `help hist` and `doc hist`.
Sol:



– 3.7: How does your Figure differ from the book's Fig. 2.2 ?
Sol: Fig 1.1 has log coordinates which reveals an analytic smaller ratio of primes whereas the larger ratios has random nature.

Problem # *4: Rounding numbers*

– 4.1: A special notation called floor $\lfloor \cdot \rfloor$ *and ceiling* $\lceil \cdot \rceil$ *are used for rounding numbers to the nearest integer*
For example $\lfloor \pi \rfloor = 3$ and $\lceil \pi \rceil = 4$. A simple combination results in the remainder part of π

$$\pi - \lfloor \pi \rfloor = \pi - 3 = 0.141592653589793$$

Use this operator to test for rational and irrational numbers. Are π and $1/\pi$ rational or irrational? **Sol:** Give the definition and examples of rational and irrational numbers.

Problem # *5: Very large primes on Intel computers.*

– 5.1: Find the largest prime number $\pi_{\max}$ that can be stored on an Intel 64-bit computer, Hint: As explained in the Matlab/Octave command `help flintmax`.

The largest positive integer is 2^{53}; however, the largest unsigned integer that can be factored is at least 2^{54}. Note that this number is even, so at least 2 is a factor. Thus taking out 2 as a factor is $2 = 2 \cdot 2^{53}$, or twice the largest number. Explain the logic of your answer. Hint: `help isprime()`.

Sol: It is obvious that it is already factored by 54 factors of 2. By trial and error, I discovered that the largest integer that Octave can factor is $2^{54} - 6 = 2 \cdot 2203 \cdot 5741 \cdot 712176643]$.

Problem # *6: We are interested in primes that are greater than $\pi_{\max}$.*

– 6.1: How can you find them on an Intel computer (i.e., one using IEEE floating point)? Hint: Consider a sieve that contains only odd numbers, starting from 3 (not 2). Since every prime number greater than 2 is odd, there is no reason to check the even numbers. $n_{\text{odd}} \in \mathbb{N}/2$ contain all the primes other than 2.

Sol: At this time, I don't see any way to do this, due to the Matlab/Octave limitation that it cannot factor numbers larger than $2^{54} - 6$.

Special functions in Octave/Matlab and infinity (∞)

Problem # *7: `Inf`, `NaN`, and logarithms in Octave/Matlab.*

– 7.1: Try `1/0` and `0/0` in the Octave/Matlab command window.
What are the results? What do these "numbers" mean in Octave/Matlab? **Sol:** `1/0` returns `Inf` (infinity) and `0/0` returns `NaN` ('not a number').

– 7.2: Try `log(0)`, `log10(0)`, and `log2(0)` in the command window.
In Matlab/Octave, the natural logarithm $\ln(\cdot)$ is computed using the function `log`. Functions $\log_{10}$ and $\log_2$ are computed using `log10` and `log2`. **Sol:** `log(0)` is `-Inf`. Working in any base results in a scale factor, so the vlaue does not change in these different bases. For example if $10^x = 2^y$ then $x = \log_{10} 2^y = y \log_{10} 2 = 0.30103y$.

– 7.3: Try `log(1)` in the command window. What do you expect for `log10(1)` and `log2(1)`?
Sol: As with `log(0)`, changing base of `log(1)`=0 gives the same result, because scaling 0 always gives 0.

– 7.4: Try `log(-1)` in the command window. What do you expect for `log10(-1)` and `log2(-1)`?
Sol: From Matlab/Octave `log(-1)`=$i\pi$. For the answer to the two other questions, see the next problems.

– 7.5: Explain how Matlab/Octave arrives at the answer in problem 6.4. Hint: $-1 = e^{i\pi}$.
Sol: `log(-1)` is $0 + i\pi$, because $\ln(-1) = \ln(e^{i\pi}) = i\pi \ln(e) = i\pi$. For base 10 let $e^x = 10^y$ and $y = \log_{10} e^x$. Thus $\log(-1) = \log_{10}(e^{\pi i}) = \pi i \log_{10} e = 1.364i$. Likewise $\log_2(-1) = \pi i \log_2(e) = 4.532i$.

– 7.6: Try `log(exp(j*sqrt(pi)))` (i.e., $\log e^{\jmath\sqrt{\pi}}$) in the command window. What do you expect?
Sol: $\log e^{\jmath\sqrt{\pi}} = \jmath\sqrt{\pi} = 1.7725\jmath$. because $\ln(\cdot) =$ is the inverse of $e^{(\cdot)}$.

– 7.7: What does "inverse" mean in this context? What is the inverse of $\ln f(x)$?
Sol: $\ln f(x) = e^{\ln f(x)}$? Conclusion: e^G and $\ln G$ are mutual inverses: that is: $\ln()$ of $e^{()}$ and $e^{()}$ of $\ln()$. Or said another way: $G = e^{\ln(G)}$, $G = \ln e^G$.

– 7.8: What is a decibel? (Look up decibels *on the internet.)*
Sol: The *decibel* is very important in engineering (and unused in mathematics). It is defined as the log of a *power ratio*. If a power ratio is 2, the dB value is 6 [dB]. A ratio of 10 is 20 [dB]. Thus the formula for the dB-ref is $10 \log_{10} \frac{P}{P_{\text{ref}}}$. Thus the decibel is defined

on the log (i.e., ratio) scale. Engineers quickly learn to "think" in dB units, because its so easy (once they learn to think in terms of ratios).

While the definition is in terms of power, it practice, is almost always used in terms of voltage, pressure, current, velocity, (force and flux), etc. Of course power is the product of force and flux, and the log of the power is the sum of the log of the force plus the log of the flux.

Problem # *8: The following identity is interesting. Can you find a proof?*

$$\begin{aligned} 1 &= 1^2 \\ 1+3 &= 2^2 \\ 1+3+5 &= 3^2 \\ 1+3+5+7 &= 4^2 \\ 1+3+5+7+9 &= 5^2 \\ &\vdots \\ \sum_{n=0}^{N-1} 2n+1 &= N^2. \end{aligned}$$

– 8.1: Can you find a proof? Hint: Add $(9+1)+(7+3)+5=25=5^2$.

Sol: Subtracting any line from the line following it, gives:

$$\begin{aligned} (1-1)+3 &= 2^2-1^2 \\ 5 &= 3^2-2^2 \\ 7 &= 4^2-3^2 \\ 9 &= 5^2-4^2 \\ &\dots \\ \sum_{n=0}^{N-1} 2n+1 - \sum_{n=0}^{N-2} 2n+1 &= N^2. - (N-1)^2 \\ 2N-1 &= N^2-(N^2-2N+1) \\ 2N-1 &= 2N-1. \end{aligned}$$

Thus the two sides are equal, as suggested by the above formula.

Can you find a simpler more constructive "proof?" Hint: assuming you know what *integration by parts* is, can you devise a concept called *Summation by parts*?

0.2 Problems NS-2

Topic of this homework:

Greatest common divisors (GCD), Continued fraction algorithm (CFA), Prime numbers (Primes),

Greatest common divisors

Consider using the *Euclidean algorithm* to find the *greatest common divisor* (i.e., GCD; the largest common prime factor) of two numbers (Allen 2020, p. 42). This algorithm may be performed using one of two methods:

Method	Division	Subtraction
On each iteration...	$a_{i+1} = b_i$ $b_{i+1} = a_i - b_i \cdot \text{floor}(a_i/b_i)$	$a_{i+1} = \max(a_i, b_i) - \min(a_i, b_i)$ $b_{i+1} = \min(a_i, b_i)$
Terminate when	$b = 0$ (GCD $= a$)	$b = 0$ (GCD $= a$)

The division method (Matlab's floor function) (Eq. 2.1, Sec. 2.1.2, Ch. 2) is preferred because the subtraction method may require a huge number of itterations steps.

Problem # *1: Understanding the* Euclidean algorithm *(GCD)*

– 1.1: Find the prime factors of $a = 85$ *and* $b = 15$.
Sol: From Octave's `factor()` we find $85 = 17 \cdot 5$, $15 = 3 \cdot 5$.

– 1.2: What is the greatest common prime factor of $a = 85$ *and* $b = 15$*?*
Sol: The largest common factor `gcd(85,15)` is 5.

– 1.3: By hand, perform the Euclidean algorithm for $a = 85$ *and* $b = 15$.
Sol: Division method:

$$a_1 = 15 \qquad b_1 = 85 - 15\lfloor \cancel{\tfrac{85}{15}}^{75} \rfloor = 10$$

$$a_2 = 10 \qquad b_2 = 15 - 10\lfloor \cancel{\tfrac{15}{10}}^{2} \rfloor = 5$$

$$a_3 = 5 \qquad b_3 = 10 - 5\lfloor \cancel{\tfrac{10}{5}}^{2} \rfloor = 0$$

$\therefore$ gcd$= 5$
Subtraction method:

$$\begin{aligned} a_1 &= 85 - 15 = 70 & b_1 &= 15 \\ a_2 &= 70 - 15 = 55 & b_2 &= 15 \\ a_3 &= 55 - 15 = 40 & b_3 &= 15 \\ a_4 &= 40 - 15 = 25 & b_4 &= 15 \\ a_5 &= 25 - 15 = 10 & b_5 &= 15 \\ & \text{swap} & & \\ a_6 &= 15 - 10 = 5 & b_6 &= 10 \\ a_7 &= 10 - 5 = 5 & b_7 &= 5 \\ & \text{terminate} & & \end{aligned}$$

$\therefore$ gcd$= 5$

– 1.4: By hand, perform the Euclidean algorithm for $a = 75$ *and* $b = 25$*. Is the result a prime number?*
Sol: Division method:

$$a_1 = 25 \qquad b_1 = 75 - 25\lfloor \cancel{\tfrac{75}{25}}^{3} \rfloor = 0$$

Subtraction method:

$$a_1 = 75 - 25 = 50 \qquad b_1 = 25$$
$$a_2 = 50 - 25 = 25 \qquad b_2 = 25$$

$\therefore$ gcd$= 25$
The result is $25 = 5^2$, the *square* of a prime number.

– 1.5: Consider the first step of the GCD division algorithm when $a < b$ (e.g., $a = 25$ and $b = 75$). What happens to a and b in the first step? Does it matter if you begin the algorithm with $a < b$ rather than $b < a$?
Sol: If $a < b$, the first step of the division algorithm swaps the terms ($a \to b$ and $b \to a$).

– 1.6: Describe in your own words how the GCD algorithm works. Try the algorithm using numbers that have already been divided into factors (e.g., $a = 5 \cdot 3$ and $b = 7 \cdot 3$).
Sol: Division method:

$$a_1 = 5 \cdot 3 \qquad b_1 = 7 \cdot 3 - 5 \cdot 3 \lfloor \frac{7 \cdot 3}{5 \cdot 3} \rfloor = 2 \cdot 3$$
$$a_2 = 2 \cdot 3 \qquad b_2 = 5 \cdot 3 - 2 \cdot 3 \lfloor \frac{5 \cdot 3}{2 \cdot 3} \rfloor = 1 \cdot 3$$
$$a_3 = 1 \cdot 3 \qquad b_3 = 2 \cdot 3 - 1 \cdot 3 \lfloor \frac{2 \cdot 3}{1 \cdot 3} \rfloor = 0$$

Subtraction method:

$$a_1 = 7 \cdot 3 - 5 \cdot 3 = 2 \cdot 3 \qquad b_1 = 5 \cdot 3$$
$$a_2 = 5 \cdot 3 - 2 \cdot 3 = 3 \cdot 3 \qquad b_2 = 2 \cdot 3$$
$$a_3 = 3 \cdot 3 - 2 \cdot 3 = 1 \cdot 3 \qquad b_3 = 2 \cdot 3$$
$$a_4 = 2 \cdot 3 - 1 \cdot 3 = 1 \cdot 3 \qquad b_4 = 1 \cdot 3$$

The algorithm iteratively converges on the GCD by subtracting out multiples of the GCD until only the GCD is left.

– 1.7: Find the GCD of $2 \cdot \pi_{25}$ and $3 \cdot \pi_{25}$.
Sol: π_{25}.

Problem # *2: Coprimes*

– 2.1: Define the term coprime.
Sol: when two integers have no common factors the are said to be *coprime*

– 2.2: How can the Euclidean algorithm be used to identify coprimes?
Sol: If `gcd(a,b)=1` they only have 1 as a common factor, thus they are coprime.

– 2.3: Give an important application of the Euclidean algorithm.
Sol: The obvious application is in network security using two-key encryption methods. Given two integers $n, d \in \mathbb{Z}$, if we wish to reduce the fraction n/d, we must cancel the common factors. Example: If $n = 9, d = 6$ then $9/6 = (3 \cdot 3)/(2 \cdot 3) = 3/2$, where the GCD, 3, may be identified using the Euclidean algorithm. While this fraction may be easily simplified via inspection, the GCD algorithm could be very helpful for larger numbers n, d.

– 2.4: Write a Matlab function, `function x = my_gcd(a,b)`, that uses the Euclidean algorithm to find the GCD of any two inputs `a` and `b`. Test your function on the (a, b) combinations from the previous problem. Include a printout (or hand-write) your algorithm to turn in.
Hints and advice:

- Don't give your variables the same names as Matlab functions! Since `gcd` is an existing Matlab/Octave function, if you use it as a variable or function name, you won't be able to use `gcd` to check your `gcd()` function. Try `clear all` to recover from this problem.
- Try using a "while" loop for this exercise (see Matlab documentation for help).
- You may need to use some temporary variables for `a` and `b` in order to perform the algorithm.

Sol: Division method:

```
function x = my_gcd(a,b)
while b>0
atmp= a; btmp = b;
a = btmp; b = atmp-btmp*floor(atmp/btmp);
end
```

Subtraction method:

```
function x = my_gcd(a,b)
while a~=b
atmp= a; btmp = b;
a = max(atmp,btmp) - min(atmp,btmp); b = min(atmp,btmp);
end
```

Algebraic generalization of the GCD (Euclidean) algorithm

Problem # *3: In this problem we are looking for integer solutions* $(m,n) \in \mathbb{Z}$ *to the equations* $ma+nb = gcd(a,b)$ *and* $ma+nb = 0$ *given positive integers* $(a,b) \in \mathbb{Z}^+$.
Note that this requires that either m or n be negative. These solutions may be found using the Euclidean algorithm.

Example: `gcd(2,3)= 1:` For $(a,b) = (2,3)$, the result is

$$\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & -2\end{bmatrix}\begin{bmatrix}0 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = \underbrace{\begin{bmatrix}-1 & 1\\3 & -2\end{bmatrix}}_{m\quad n}\begin{bmatrix}2\\3\end{bmatrix}.$$

From the above equation we find the solution (m,n) to the integer equation

$$2m+3n = \texttt{gcd}(2,3) = 1;$$

namely, $(m,n) = (-1,1)$ (i.e., $-2+3=1$). There is also a second solution $(3,-2)$ (i.e., $3\cdot 2 - 2\cdot 3 = 0$). Thus these two solutions are a pair, and the solution exists only if (a,b) are coprime ($a \perp b$).

– 3.1: By inspection, find at least one integer pair (m,n) *that satisfies* $12m+15n = 3$.
Sol: By inspection, $(m,n) = (-1,1)$ is one solution.

– 3.2: Using matrix methods for the Euclidean algorithm, find integer pairs (m,n) *that satisfy* $12m+15n = 3$ *and also for* $12m+15n = 0$. *Show your work!!!*
Sol: If we look for solutions in terms of (m,n) the both cases have solutions. For the first case where RHS = 3, $m = -1$ and $n = 1$, and $-12+15 = 3$. For the case of the RHS = 0, $m = 5$ and $n = -4$, thus $12*5 - 15*4 = 0$.

– 3.3: Does the equation $12m+15n = 1$ *have integer solutions for* n *and* m*? Why or why not?*
Sol: No, because $\text{gcd}(12,15) = gcd(3*4, 3*5) = 3$, not 1. Thus there are no Diophantine solutions to this equation.

Problem # *4: Matrix approach:*
It can be difficult to keep track of the a's and b's when the algorithm has many steps. We need an alternative way to run the Euclidean algorithm using matrix algebra. Matrix methods provide a more transparent approach to the operations on (a,b). Thus the Euclidean algorithm can be classified in terms of standard matrix operations. Write out the indirect matrix approach discussed at the end of Sec. **??** (Eq. **??**).

Sol: Division method:
Define

$$\begin{bmatrix}a\\b\end{bmatrix}_0 = \begin{bmatrix}a_0\\b_0\end{bmatrix}, \qquad \begin{bmatrix}a\\b\end{bmatrix}_{i+1} = \begin{bmatrix}0 & 1\\1 & -\lfloor a/b\rfloor\end{bmatrix}_i \begin{bmatrix}a\\b\end{bmatrix}_i$$

Continued fractions

Problem # *5: Here we explore the continued fraction algorithm (CFA)*

In its simplest form, the CFA starts with a real number, which we denote as $\alpha \in \mathbb{R}$. Let us work with an irrational real number, $\pi \in \mathbb{I}$, as an example because its CFA representation will be infinitely long. We can represent the CFA coefficients α as a vector[1] of integers n_k, $k = 1, 2, \ldots, \infty$:

$$\begin{aligned} \alpha &= [n_1; n_2, n_3, n_4, \ldots] \\ &= n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \frac{1}{n_4 + \cdots}}}. \end{aligned}$$

The CFA is recursive, with three repeated steps per iteration. For example $\alpha_1 = \pi \approx 3.14159\ldots$, $n_1 = 3$, $r_1 = \pi - 3$, and $\alpha_2 \equiv 1/r_1$.

$$\begin{aligned} \alpha_2 &= 1/0.1416 = 7.0625\ldots \\ \alpha_1 = n_1 + \frac{1}{\alpha_2} &= n_1 + \frac{1}{n_2 + \frac{1}{\alpha_3}} = \cdots . \end{aligned}$$

In terms of a Matlab/Octave script,

```
alpha0 = pi;
K=10;
n=zeros(1,K); alpha=zeros(1,K);
alpha(1)=alpha0;

for k=2:K  %k=1 to K
n(k)=round(alpha(k-1));
%n(k)=fix(alpha(k-1));
alpha(k)= 1/(alpha(k-1)-n(k));
%disp([fix(k), round(n(k)), alpha(k)]); pause(1)
end
disp([n; alpha]);
%Now compare this to matlab's rat() function
rat(alpha0,1e-20)
```

– 5.1: By hand find the first few values of n_k for $\alpha = e^\pi \approx 23.1407$.

Sol: The CFA for this is: $e^\pi = 23.1407\cdots = [23; 7, 9, 4, \cdots]$.

– 5.2: For the proceeding question, what is the error (remainder) when you truncate the continued fraction after $n_1, \ldots, n_3$? Give the absolute value of the error and the percentage error relative to the original α.

Sol: The remainder is $e^\pi - (23 + 1/(7 + (1/9)))$ which gives an error of $\epsilon = |e^\pi - (23 + 1/(7 + (1/9)))|/e^\pi = 2.92 \cdot 10^{-6} = 0.0003\%$

– 5.3: Use the Matlab/Octave program provided to find the first 10 values of n_k for $\alpha = e^\pi$, and verify your result using the Matlab/Octave command `rat()`.

Sol: $e^\pi = 23.1407\cdots = [23; 7, 9, 4, -2, -591, -2, -10, 3, -2, \cdots]$.

– 5.4: Discuss the similarities and differences between the Euclidean algorithm and the CFA.

Sol:

1. Both are recursive, meaning that the steps are repeated one after another.
2. The EA starts from two numbers (a,b). The output of the gca(a,b) is the GCD. The CFA starts with a single number and the output is a sequence of integers. If the sequence terminates the number was rational. If the sequence does not terminate, the number is irrational.

[1] This notation is widely used in many Number Theory text books.

3. The EA works with the difference between the minimum and maximum of the two numbers whereas the CFA works with the rounding function and the reciprocal of the error.

4. It would seem that the goals of the two algorithms, the starting point, and the results are totally different. Both are very useful and powerful. Both generalize to more difficult situations than working with simple numbers.

Prime numbers

Problem # *6: Every integer may be written as a product of primes.*

– 6.1: Write the numbers $1,000,000$, $1,000,004$, *and* $999,999$ *in the form* $N = \prod_k \pi_k^{\beta_k}$. Hint: Use Matlab/Octave to find the prime factors.

Sol: $1,000,000 = 2^6 \cdot 5^6$
$1,000,004 = 2^2 \cdot 53^2 \cdot 89$
$999,999 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$

– 6.2: Give a generalized formula for the natural logarithm of a number $\ln(N)$ *in terms of its primes* π_k *and their multiplicities* β_k. *Express your answer as a* sum *of terms.*

Sol: $\ln N = \sum_k \beta_k \ln(\pi_k)$

Problem # *7:CFA of ratios of large primes*

– 7.1: Expand 23/7 as a continued fraction. Express your answer in bracket notation (e.g., $\pi = [3., 7, 16, \cdots]$). Show your work. **Sol:** $23/7 = (21+2)/7 = 3 + 2/7 = 3 + 1/(6+1)/2 = 3 + 1/(6 + 1/2)$. In bracket notation $23/7 = [3., 6, 2]$. Matlab gives `rat(23/7)` $= 3 + 1/(4 + 1/(-2))$, or [1., 4, -2] because rounding $7/2$ can be taken as either 3+1/2 or 4-1/2.

– 7.2:Starting from the primes below 10^6, *form the CFA of* π_j/π_k *with* $j = 78498$ *and* $k < j$.

Sol: First generate 10^6 primes with the matlab command `π=primes(11+1e6)`.
The length of π is $j = 78499$, $\pi(j) = 1,000,003$, $\pi(j-1) = 999,983$ and $\pi(j-2) = 999,979$.
Let the target fraction be

$$T = \frac{\pi(end-1)}{\pi(end-2)} = \frac{999983}{999979} = 1.000004000084002.$$

Finding the CFA of T gives

$$rat(T) = 1 + 1/249995 = [1; 249995].$$

Factoring this integer gives `factor(249995)=5*49999`.

– 7.3: Look at other ratios of prime numbers and look for a pattern in the CFA of the ratios of large primes. What is the most obvious conclusion? **Sol:** *The CFA terminates in only one term, as in the above example.*

– 7.4: Try the Matlab/Octave functions `rats(23/7)`, `rats(3.2857)`, *and* `rats(3.2856)`. *What an you conclude?*

Sol: This function is similar to the CFA but uses rounding rather than truncation arithmatic. `rats(3.2857)`=32857/10000 but `rats(23/7)`=`23/7` because it rounds to 23/7, whereas `rats(3.2856)` = 4107/1250 because it does not.

– 7.5: Can $\sqrt{2}$ *be represented as a finite continued fraction? Why or why not?*

Sol: No, because it is irrational.

– *7.6: What is the CFA for* $\sqrt{2}-1$*?*

$$\text{Hint:} \qquad \sqrt{2}+1 = \frac{1}{\sqrt{2}-1} = [2;2,2,2,\cdots].$$

Sol: $1+\sqrt{2} = 2+1/(2+1/(2+\cdots))$ or $[2.,2,2,2,\cdots]$, thus

$$\sqrt{2}-1 = [2.,2,2,2,\cdots]-2 = 0+1/(2+1/(2+1/(2+\cdots))).$$

Repeat this for $2+\sqrt{5}$, $2+\sqrt{3}$

– *7.7: Show that*

$$\frac{1}{1-\sqrt{a}} = a^{\frac{11}{2}}+a^{\frac{9}{2}}+a^{\frac{7}{2}}+a^{\frac{5}{2}}+a^{\frac{3}{2}}+\sqrt{a}+a^5+a^4+a^3+a^2+a+1 = 1-a^6$$

```
syms a,b
b= taylor(1/( 1-sqrt(a) ))
simplify((1-sqrt(a))*b) = 1-a^6
```

Use symbolic analysis to show this, then explain. **Sol:** This seems like a very unlikely relationship. Unexpectedly the coefficents of this expansion are all 1, leading to a is a sixth degree polynomial. It is obvioiusly related to the six complex roots of unity. Thus we may find the companion matrix, followed by an eigen solution. This seems to be a Taylor expansion of six roots of unity, expressed in terms of removable singularities. See Cotes Theorem (1716) (**?**, p. 289).

0.3 Problems NS-3

Topic of this homework: Pythagorean triplets, Pell's equation, Fibonacci sequence

Pythagorean triplets

Problem # *1: Euclid's formula for the Pythagorean triplets a, b, c is $a = p^2 - q^2$, $b = 2pq$, and $c = p^2 + q^2$.*

– 1.1: What condition(s) must hold for p and q such that a, b, and c are always positive and nonzero?

Sol: $p > q > 0$ (strictly greater than)

– 1.2: Solve for p and q in terms of a, b, and c.

Sol:

Method 1: Given a, c, one may find p, q via matrix operations by solving the *nonlinear system of equations* for p, q.
First solve linear system of equations for p^2, q^2:

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p^2 \\ q^2 \end{bmatrix}$$

Inverting this 2x2 matrix gives (the determinant $\Delta = 2$)

$$\begin{bmatrix} p^2 \\ q^2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix}.$$

Thus $p = \pm\sqrt{(a+c)/2}$, $q = \pm\sqrt{(c-a)/2}$.

Method 2: The algebraic approach is:

$$a + c = (p^2 - q^2) + (p^2 + q^2) = 2p^2$$

$$-a + c = -(p^2 - q^2) + (p^2 + q^2) = 2q^2,$$

Thus $p = \sqrt{(a+c)/2}$, $q = \sqrt{(c-a)/2}$, where $p, q \in \mathbb{N}$.
Method 1 seems more "transparent" than Method 2.

Problem # *2: The ancient Babylonians (ca. 2000 BCE) cryptically recorded (a, c) pairs of numbers on a clay tablet, archeologically denoted* Plimpton-322 *(Fig.* **??***).*

– 2.1: Find p and q for the first five pairs of a and c shown here from Plimpton-322.

a	c
119	169
3367	4825
4601	6649
12709	18541
65	97

Find a formula for a in terms of p and q.
Sol:

$$(a,c) = (119, 169) \qquad (p,q) = \pm(12,5)$$
$$(a,c) = (3367, 4825) \qquad (p,q) = \pm(64,27)$$
$$(a,c) = (4601, 6649) \qquad (p,q) = \pm(75,32)$$
$$(a,c) = (12709, 18541) \qquad (p,q) = \pm(125,54)$$
$$(a,c) = (65, 97) \qquad (p,q) = \pm(9,4)$$

– 2.2: Based on Euclid's formula, show that $c > (a, b)$.

Sol: $c - a = (p^2 + q^2) - (p^2 - q^2) = 2q^2$
Because $2q^2$ is always positive, $c > a$
$c - b = (p^2 + q^2) - 2pq = (p - q)^2 > 0$
Note that by the definition of $p, q \in \mathbb{N}$, $p > q$.

– 2.3: What happens when $c = a$*?*

Sol: Then its not a triangle since $b = 0$. The triangle is degenerate.

– 2.4: Is $b + c$ *a perfect square? Discuss.*

Sol: $b + c = p^2 + 2pq + q^2 = (p + q)^2$. Since p and q are integers, $b + c$ will always be a perfect square ($\sqrt{b + c}$ will always be an integer).

Pell's equation:

Problem # *3: Pell's equation is one of the most historic (i.e., important) equations of Greek number theory because it was used to show that* $\sqrt{2} \in \mathbb{I}$*. We seek integer solutions of*

$$x^2 - Ny^2 = 1.$$

As shown in Sec. **??**, the solutions x_n, y_n for the case of $N = 2$ are given by the linear 2×2 matrix recursion

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = 1\jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

with $[x_0, y_0]^T = [1, 0]^T$ and $1\jmath = \sqrt{-1} = e^{j\pi/2}$. It follows that the general solution to Pell's equation for $N = 2$ is

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = (e^{\jmath\pi/2})^n \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

To calculate solutions to Pell's equation using the matrix equation above, we must calculate

$$A^n = e^{\jmath\pi n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n = e^{\jmath\pi n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix},$$

which becomes tedious for $n > 2$.

– 3.1: Find the companion matrix and thus the matrix A *that has the same eigenvalues as Pell's equation.*
Hint: Use Matlab's function `[E,Lambda] = eig(A)` to check your results!
Sol: The companion matrix is

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

– 3.2: Solutions to Pell's equation were used by the Pythagoreans to explore the value of $\sqrt{2}$. Explain why Pell's equation is relevant to $\sqrt{2}$.

Sol: As discussed in Sec. 2.5.2, as the iteration n increases, the ratio of the x_n/y_n approaches $\sqrt{2}$.

– 3.3: Find the first three values of $(x_n, y_n)^T$ by hand and show that they satisfy Pell's equation for $N = 2$.

Sol: *See class notes (slide 9.4.2) for this calculation. By hand, find the eigenvalues $\lambda_\pm$ of the 2×2 Pell's equation matrix*

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

Sol: The eigenvalues are given by the roots of the equation $(1 - \lambda_\pm)^2 = 2$. Thus $\lambda_\pm = 1 \pm \sqrt{2} = \{2.1412, -.4142\}$

– 3.4: By hand, show that the matrix of eigenvectors, E, is

$$E = \begin{bmatrix} \vec{e}_+ & \vec{e}_- \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}.$$

Sol: The eigenvectors $\vec{e}_\pm$ may be found by solving

$$A \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \lambda_\pm \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \rightarrow (A - \lambda_\pm I) \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0$$

For λ_+, this gives

$$0 = \begin{bmatrix} 1 - (1 + \sqrt{2}) & 2 \\ 1 & 1 - (1 + \sqrt{2}) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

which gives the relation between the elements of $\vec{e}_+$, e_1, e_2, as $e_1 = \sqrt{2} e_2$.

The eigenvectors are defined to be unit length and orthogonal, namely

1. $||\vec{e}_k||^2 = \vec{e}_k \cdot \vec{e}_k = 1$
2. $\vec{e}_+ \cdot \vec{e}_- = 0$.

Once we normalize $\vec{e}_+$ to have unit length, we obtain the first eigenvector

$$\vec{e}_+ = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$$

Repeating this for λ_- gives

$$\vec{e}_- = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$$

Thus, the matrix of eigenvalues is

$$E = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}$$

– 3.5: Using the eigenvalues and eigenvectors you found for A, verify that

$$E^{-1} A E = \Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$$

Sol: Using the formula for a matrix inverse, we find

$$E^{-1} = \frac{1}{\det(E)} \begin{bmatrix} e_{22} & -e_{12} \\ -e_{21} & e_{11} \end{bmatrix} = \frac{3}{-2\sqrt{2}} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} = \frac{-\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix}$$

Thus

$$E^{-1}AE = \frac{-\sqrt{3}}{2\sqrt{2}}\begin{bmatrix}1 & -\sqrt{2}\\ -1 & -\sqrt{2}\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix}-\sqrt{2} & \sqrt{2}\\ 1 & 1\end{bmatrix}$$
$$= \frac{-1}{2\sqrt{2}}\begin{bmatrix}1 & -\sqrt{2}\\ -1 & -\sqrt{2}\end{bmatrix}\begin{bmatrix}(-\sqrt{2}+2) & (\sqrt{2}+2)\\ (-\sqrt{2}+1) & (\sqrt{2}+1)\end{bmatrix}$$
$$= \begin{bmatrix}1-\sqrt{2} & 0\\ 0 & 1+\sqrt{2}\end{bmatrix} = \Lambda$$

– 3.6: Once you have diagonalized A, use your results for E and Λ to solve for the $n = 10$ solution $(x_{10}, y_{10})^T$ to Pell's equation with $N = 2$.

Sol: $x_{10} = -3363$ and $y_{10} = -2378$. Note this formulation gives the negative solution, but since the values for $n = 10$ are real, when they are squared in Pell's equation, it makes no difference whether they are negative or positive.

The Fibonacci sequence

The Fibonacci sequence is famous in mathematics and has been observed to play a role in the mathematics of genetics. Let x_n represent the Fibonacci sequence,

$$x_{n+1} = x_n + x_{n-1}, \tag{NS-3.1}$$

where the current input sample x_n is equal to the sum of the previous two inputs. This is a "discrete time" recurrence relationship. To solve for x_n, we require some initial conditions. In this exercise, let us define $x_0 = 1$ and $x_{n<0} = 0$. This leads to the Fibonacci sequence $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$ for $n = 0, 1, 2, 3, \ldots$.

Equation **??** is equivalent to the 2×2 matrix equations

$$\begin{bmatrix}x_n\\ y_n\end{bmatrix} = A\begin{bmatrix}x_{n-1}\\ y_{n-1}\end{bmatrix}, \qquad A = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}. \tag{NS-3.2}$$

Problem # *4: Here we seek the general formula for x_n. Like Pell's equation, the Fibonacci equation has a recursive eigenanalysis solution. To find it we must recast x_n as a 2×2 matrix relationship and then proceed, as we did for the Pell case.*

– 4.1: Show that the Fibonacci sequence $x_n = x_{n-1} + x_{n-2}$ may be generated by

$$\begin{bmatrix}x_n\\ y_n\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}^n\begin{bmatrix}x_0\\ y_0\end{bmatrix}, \qquad \begin{bmatrix}x_0\\ y_0\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix}. \tag{NS-3.3}$$

Sol: Using the Matrix Eigen-equation, powers of the eigen equation $\mathbf{A}^n = \mathbf{E}\Lambda^n\mathbf{E}^{-1}$. The final solution is

$$\begin{bmatrix}x_n\\ y_n\end{bmatrix} = \boldsymbol{e}\begin{bmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{bmatrix}^n\boldsymbol{e}^{-1}\begin{bmatrix}x_0\\ y_0\end{bmatrix}. \tag{NS-3.4}$$

– 4.2: What is the relationship between y_n and x_n?

Sol: This equation says that $x_n = x_{n-1} + y_{n-1}$ and $y_n = x_{n-1}$. The latter equation may be rewritten as $y_{n-1} = x_{n-2}$. Thus

$$x_n = x_{n-1} + x_{n-2}$$

as requested.

– 4.3: Write a Matlab/Octave program to compute x_n using the matrix equation above. Test your code using the first few values of the sequence. Using your program, what is x_{40}? Note: Consider using the eigenanalysis of A, described by Eq. **??** of the text.

Sol: You can try something like:

```
function xn = fib(n)
```

```
A = [1 1; 1 0]; [E,D] = eig(A); xy = E*D^n*inv(E)*[1; 0];
xn = xy(1);
```

For this initial condition, $x_{40} = 165,580,141 = \frac{1}{\sqrt{5}}\left(\frac{(1+\sqrt{5}}{2}\right)^{41}$.

– 4.4: Using the eigenanalysis of the matrix A (and a lot of algebra), show that it is possible to obtain the general formula for the Fibonacci sequence

$$x_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]. \tag{NS-3.5}$$

– 4.5: What are the eigenvalues $\lambda_\pm$ of the matrix A?

Sol: The eigenvalues of the Fibonacci matrix are given by

$$det\begin{bmatrix}1-\lambda & 1\\ 1 & -\lambda\end{bmatrix} = \lambda^2 - \lambda - 1 = (\lambda - 1/2)^2 - (1/2)^2 - 1 = (\lambda - 1/2)^2 - 5/4 = 0,$$

thus $\lambda_\pm = \frac{1\pm\sqrt{5}}{2} = [1.618, -0.618]$.

– 4.6: How is the formula for x_n related to these eigenvalues? Hint: Find the eigenvectors.

Sol: The eigenvectors (determined from the equation $(A - \lambda_\pm I)\vec{e}_\pm = \vec{0}$, and normalized to 1) are given by

$$\vec{e}_+ = \begin{bmatrix}\frac{\lambda_+}{\sqrt{\lambda_+^2+1}}\\ \frac{1}{\sqrt{\lambda_+^2+1}}\end{bmatrix} \qquad \vec{e}_- = \begin{bmatrix}\frac{\lambda_-}{\sqrt{\lambda_-^2+1}}\\ \frac{1}{\sqrt{\lambda_-^2+1}}\end{bmatrix} \qquad E = \begin{bmatrix}\vec{e}_+ & \vec{e}_-\end{bmatrix}$$

From the eigenanalysis, we find that

$$\begin{bmatrix}x_n\\ y_n\end{bmatrix} = E\begin{bmatrix}\lambda_+^n & 0\\ 0 & \lambda_-^n\end{bmatrix}E^{-1}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}e_{11} & e_{12}\\ e_{21} & e_{22}\end{bmatrix}\begin{bmatrix}\lambda_+^n & 0\\ 0 & \lambda_-^n\end{bmatrix}\frac{1}{(e_{11}e_{22}-e_{12}e_{21})}\begin{bmatrix}e_{22} & -e_{12}\\ -e_{21} & e_{11}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}.$$

Solving for x_n we find that

$$\begin{aligned} x_n &= \frac{1}{(e_{11}e_{22}-e_{12}e_{21})}\left(\lambda_+^n e_{11}e22 - \lambda_-^n e_{12}e_{21}\right)\\ &= \frac{1}{\frac{\sqrt{5}}{\sqrt{(\lambda_+^2+1)(\lambda_-^n+1)}}}\left[\lambda_+^n\left(\frac{\lambda_+^n}{\sqrt{(\lambda_+^2+1)(\lambda_-^n+1)}}\right) - \lambda_-^n\left(\frac{\lambda_-^n}{\sqrt{(\lambda_+^2+1)(\lambda_-^n+1)}}\right)\right]\\ &= \frac{1}{\sqrt{5}}\left[\lambda_+^{n+1} - \lambda_-^{n+1}\right] \end{aligned}$$

– 4.7: What happens to each of the two terms

$$\left[\frac{1\pm\sqrt{5}}{2}\right]^{n+1}$$

as $n \to \infty$?

Sol: $[\frac{1+\sqrt{5}}{2}]^{n+1} \to 0$ and $[\frac{1+\sqrt{5}}{2}]^{n+1} \to \infty$

– 4.8: What happens to the ratio x_{n+1}/x_n as $n \to \infty$ for

$$x_n \approx \left[\frac{(1+\sqrt{5})}{2}\right]^{n+1} ? \tag{NS-3.6}$$

Sol: $x_{n+1}/x_n \to (1+\sqrt{5})/2$, because $\left((1-\sqrt{5})/2\right)^n \to 0$

Problem # *5: Replace the Fibonacci sequence with the average of the previous two values in the sequence.*

$$x_n = \frac{x_{n-1} + x_{n-2}}{2},$$

– 5.1: What matrix A is used to calculate this sequence?

Sol:

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$$

– 5.2: Modify your computer program to calculate the new sequence x_n. What happens as $n \to \infty$?

Sol: As $n \to \infty$, $x_n \to 2/3$

– 5.3: What are the eigenvalues of your new A? How do they relate to the behavior of x_n as $n \to \infty$? Hint: You can expect the closed-form expression for x_n to be similar to Eq. **??**.

Sol: The eigenvalues are $\lambda_+ = 1$ and $\lambda_- = -0.5$. From Eq. **??**, the expression for A^n is

$$A^n = (E\Lambda E^{-1})^n = E\Lambda^n E^{-1} = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}^n = \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix}.$$

The solution is the sum of two sequences, one a constant and the other an oscillation that quickly fades. As $n \to \infty$, $\lambda_+^n = 1^n \to 1$ and $\lambda_-^n = (-1/2)^n \to 0$. The solution becomes

$$x_n = \frac{2}{3}\left[\lambda_+^n - \lambda_-^n\right] = \frac{2}{3}\left[1^n - (-1)^n\right] \to \frac{2}{3}.$$

Problem # *6: Consider the expression*

$$\sum_1^N f_n^2 = f_N f_{N+1}.$$

– 6.1: Find a formula for f_n that satisfies this relationship. Hint: It holds for only the Fibonacci recursion formula.

Sol: Write this out for N and $N-1$:

$$\begin{aligned} f_1^2 + f_2^2 + \cdots + f_{N-1}^2 + f_N^2 &= f_N f_{N+1} \\ f_1^2 + f_2^2 + \cdots + f_{N-1}^2 &= f_{N-1} f_N \end{aligned}$$

Subtracting gives

$$\begin{aligned} f_N^2 &= \cancel{f_N} f_{N+1} - f_{N-1}\cancel{f_N} = \cancel{f_N}(f_{N+1} - f_{N-1}) \\ f_N &= f_{N+1} - f_{N-1} \end{aligned}$$

Thus the relation only holds for the Fibonacci recursion formula.

CFA as a matrix recursion

Problem # *7: The CFA may be writen as a matrix recursion. For this we adopt a special notation, unlike other matrix notations,*[2] *with $k \in \mathbb{N}$:*

$$\begin{bmatrix} n \\ x \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & \lfloor x_k \rfloor \\ 0 & \frac{1}{x_k - \lfloor x_k \rfloor} \end{bmatrix} \begin{bmatrix} n \\ x \end{bmatrix}_k.$$

This equation says that $n_{k+1} = \lfloor x_k \rfloor$ and $x_{k+1} = 1/(x_k - \lfloor x_k \rfloor)$. It does *not* mean that $n_{k+1} = \lfloor x_k \rfloor x_k$, as would be implied by standard matrix notation. The lower equation says that $r_k = x_k - \lfloor x_k \rfloor$ is the *remainder*—namely, $x_k = \lfloor x - k \rfloor + r_k$ (Octave/Matlab's `rem(x,floor(x))` function), also known as `mod(x,y)`.

[2] This notation is highly nonstandard due to the nonlinear operations. The matrix elements are *derived* from the vector rather than multiplying them. These calculation may be done with the help of Matlab/Octave.

– 7.1: Start with $n_0 = 0 \in \mathbb{N}$, $x_0 \in \mathbb{I}$, $n_1 = \lfloor x_0 \rfloor \in \mathbb{N}$, $r_1 = x - \lfloor x \rfloor \in \mathbb{I}$, and $x_1 = 1/r_1 \in \mathbb{I}$, $r_n \neq 0$. For $k = 1$ this generates on the left the next CFA parameter $n_2 = \lfloor x_1 \rfloor$ and $x_2 = 1/r_2 = 1/(x_0 - \lfloor x_0 \rfloor)$ from n_0 and x_0. Find $[n, x]_{k+1}^T$ for $k = 2, 3, 4, 5$.

Sol: If $x_0 = \pi$, then $n_1 = \lfloor \pi \rfloor = 3$, $r_1 = \pi - n_1 = 0.14159\cdots$, and $x_1 = 1/r_1 \approx 7.06$:

$$\begin{bmatrix} 3 \\ 7.06251 \end{bmatrix}_1 = \begin{bmatrix} 0 & \lfloor \pi \rfloor \\ 0 & \frac{1}{\pi - \lfloor \pi \rfloor} \end{bmatrix} \begin{bmatrix} 0 \\ \pi \end{bmatrix}_0$$

and for $n = 2$

$$\begin{bmatrix} 7 \\ 15.99659 \end{bmatrix}_2 = \begin{bmatrix} 7 \\ \frac{1}{0.06251} \end{bmatrix}_2 = \begin{bmatrix} 0 & 7 \\ 0 & \frac{1}{7.0625-7} \end{bmatrix} \begin{bmatrix} 3 \\ 7.06251 \end{bmatrix}_1$$

For $n = 3$, $\pi_3 = [n_1; n_2, n_3] = [3; 7, 15]$. Continuing $n_4 = \lfloor 1.003418 \rfloor = 1$ and $n_5 = 292$.

0.4 Problems AE-1

Topics of this homework: Fundamental theorem of algebra, polynomials, analytic functions and their inverse, convolution, Newton's root finding method, Riemann zeta function. Deliverables: Answers to problems

Note: The term analytic is used in two different ways. (1) An analytic function is a function that may be expressed as a locally convergent power series; (2) analytic geometry refers to geometry using a coordinate system.

Polynomials and the fundamental theorem of algebra (FTA)

Problem # *1: A polynomial of degree N is defined as*

$$P_N(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N.$$

– 1.1: How many coefficients a_n does a polynomial of degree N have?

Sol: $N + 1$

– 1.2: How many roots does $P_N(x)$ have?

Sol: N

Problem # *2: The* fundamental theorem of algebra *(FTA)*

– 2.1: State and then explain the FTA.

Sol: The FTA says that every polynomial has at least one root $x = x_r$.

– 2.2: Using the FTA, prove *your answer to question 1.2.* Hint: Apply the FTA to *prove* how many roots a polynomial $P_N(x)$ of order N has.

Sol: When a root is determined, it may be factored out, leaving a new polynomial of degree one less than the first. Specifically,

$$P_{N-1}(x) = \frac{P_N(x)}{x - x_r}.$$

Thus it follows that by a recursive application of this theorem, a polynomial has a number of roots equal to its degree. All the roots must be counted, including repeated and complex roots and roots at ∞.

Problem # *3: Consider the polynomial function $P_2(x) = 1 + x^2$ of degree $N = 2$ and the related function $F(x) = 1/P_2(x)$. What are the roots (e.g., zeros) $x_\pm$ of $P_2(x)$?* Hint: Complete the square on the polynomial $P_2(x) = 1 + x^2$ of degree 2, and find the roots.

Sol: Solving for the roots by setting $P_2(x) = 0$ gives $x_\pm^2 = -1$, leading to $x_\pm = \pm 1j$.

Problem # *4: $F(x)$ may be expressed as ($A, B, x_\pm \in \mathbb{C}$)*

$$F(x) = \frac{A}{x - x_+} + \frac{B}{x - x_-}, \tag{AE-1.1}$$

where $x_\pm$ are the roots (zeros) of $P_2(x)$, which become the *poles* of $F(x)$; A and B are the *residues*. The expression for $F(x)$ is sometimes called a *partial fraction expansion* or *residue expansion*, and it appears in many engineering applications.

– 4.1:Find $A, B \in \mathbb{C}$ in terms of the roots $x_\pm$ of $P_2(x)$.
Sol: The fastest (i.e., easiest) way to find the constants A, B is to cross-multiply

$$\frac{1}{1+x^2} = \frac{A(x - x_-) + B(x - x_+)}{(x - x_+)(x - x_-)} = \frac{(A+B)x - (Ax_- + Bx_+)}{(x - x_+)(x - x_-)}$$

Since the numerator must equal 1, $B = -A$ and $A = 1/(x_+ - x_1)$.
In summary , in terms of the roots of Eq. **??**

$$A = -B = \frac{1}{(x_+ - x_-)}, \quad \text{thus} \quad F(x) = \frac{1}{1+x^2} = \frac{1}{2\jmath}\left(\frac{1}{x - 1\jmath} - \frac{1}{x + 1\jmath}\right).$$

– 4.2: Verify your answers for A and B by showing that this expression for $F(x)$ is indeed equal to $1/P_2(x)$.
Sol: This is easily verified by cross-multiplying and simplifying. In the numerator the x terms cancel and Eq. **??** is recovered.

– 4.3: Give the values of the poles and zeros of $P_2(x)$.
Sol: The zeros are at $x_z = \pm\jmath$, and the poles are at $x_p = \pm\infty$

– 4.4: Give the values of the poles and zeros of $F(x) = 1/P_2(x)$.
Sol: The poles are at $x_p = \pm\jmath$, and the zeros are at $x_z = \pm\infty$

0.4.1 Analytic functions

Overview: Analytic functions are defined by infinite (power) series. The function $f(x)$ is said to be *analytic* at any value of constant $x = x_o$, where there exists a convergent power series

$$P(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n$$

such that $P(x_o) = f(x_o)$. The point $x = x_o$ is called the *expansion point*. The region around x_o such that $|x - x_o| < 1$ is called the *radius of convergence*, or region of convergence (RoC). The local power series for $f(x)$ about $x = x_o$ is defined by the Taylor series:

$$\begin{aligned} f(x) &\approx f(x_o) + \frac{df}{dx}\Big|_{x=x_o}(x - x_o) + \frac{1}{2!}\frac{d^2 f}{dx^2}\Big|_{x=x_o}(x - x_o)^2 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n}{dx^n} f(x)\Big|_{x=x_o} (x - x_o)^n. \end{aligned}$$

Two classic examples are the geometric series[3] where $a_n = 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \tag{AE-1.2}$$

and the exponential function where $a_n = 1/n!$, Eq. **??** (p. **??**). The coefficients for both series may be derived from the Taylor formula.

[3]The geometric series is *not* defined as the function $1/(1-x)$, it is defined as the series $1 + x + x^2 + x^3 + \cdots$, such that the ratio of consecutive terms is x.

Problem # *5: The geometric series*

– 5.1: What is the region of convergence *(RoC) for the power series Eq.* **??** *of* $1/(1-x)$ *given above—for example, where does the power series* $P(x)$ *converge to the function value* $f(x)$*? State your answer as a condition on* x. Hint: What happens to the power series when $x > 1$?
Sol: $|x| < 1$ because for $|x| \geq 1$, the power series diverges to infinity.

– 5.2: In terms of the pole, what is the RoC for the geometric series in Eq. **??***?*
Sol: The nearest pole relative to the expansion point, at $x = 0$ is at at the nearest pole $x_p = 1$ to the expansion point at $x = 0$. Namely the RoC is 1 re 0.

– 5.3: How does the RoC relate to the location of the pole of $1/(1-x)$*?*
Sol: The pole is at $x = 1$, on the border of the RoC. The nearest pole relative to the expansion point, at $x = 0$ is at $x = 1$. Thus the RoC is 1.

– 5.4: Where are the zeros, if any, in Eq. **??***?*

Sol: There is a single zero at $x = \infty$.

– 5.5: Assuming x *is in the RoC, prove that the geometric series correctly represents* $1/(1-x)$ *by multiplying both sides of Eq.* **??** *by* $(1-x)$.
Sol:

$$
\begin{aligned}
1 &= \frac{1-x}{1-x} && \text{for all } x \neq 1 \\
&= (1-x)(1+x+x^2+x^2\cdots), && |x| < 1 \\
&= (1+x+x^2+x^2\cdots) - x(1+x+x^2\cdots) \\
&= 1 + \cancel{(x+x^2+x^3\cdots)} - \cancel{(x+x^2+x^3\cdots)} \\
&= 1 && \text{for all } x.
\end{aligned}
$$

The introduction of the pole introduces an added zero since $P_N(x)|_{x=1} = N$.

If one lets $z = 1/x$ the relation becomes

$$1 = \frac{1-z}{1-z},$$

which is valid for $z \neq 1$, which when expanded the RoC is $|z| < 1$, or $x > 1$. Once the removable pole and zero at $x = 1$ are cancelled, the solution is valid for all x.

Problem # *6: Use the geometric series to study the degree* N *polynomial. It is very important to note that all the coefficients* c_n *of this polynomial are* 1.

$$P_N(x) = 1 + x + x^2 + \cdots + x^N = \sum_{n=0}^{N} x^n. \tag{AE-1.3}$$

– 6.1: Prove that

$$P_N(x) = \frac{1-x^{N+1}}{1-x}. \tag{AE-1.4}$$

Sol:

$$\begin{aligned} P_N(x) &= 1 + x + x^2 \cdots x^N \\ &= \sum_{n=0}^{\infty} x^n - \sum_{n=N+1}^{\infty} x^n \\ &= \sum_{n=0}^{\infty} x^n - x^{N+1} \sum_{n=0}^{\infty} x^n \\ &= (1 - x^{N+1}) \sum_{n=0}^{\infty} x^n \\ &= \frac{1 - x^{N+1}}{1 - x} \end{aligned}$$

– 6.2: What is the RoC for Eq. **??***?*

Sol: There is no pole; thus the RoC is ∞. This polynomial has N zeros.

– 6.3: What is the RoC for Eq. **??***?*

Sol: A polynomial has no RoC.

– 6.4: How many poles does $P_N(x)$ (Eq. **??***) have? Where are they?*

Sol: Since $P_N(x)$ is defined by Eq. **??**, there is no poles at $x = 1$. However it still has a pole of order N at $x = \infty$. To show this, define $z = 1/x$ and study the zeros.

– 6.5: How many zeros does $P_N(x)$ (Eq. **??***) have? State where are they in the complex plane.*

Sol: $P_N(x)$ only has N zeros, at $s_z = \sqrt[N]{-1} = e^{j2\pi n/(N+1))}$ where $n = 1, 2, \ldots, N$. The zero at $s_z = 1$ ($n = 0$) of Eq. **??** exactly cancels with the pole at $s_p = 1$. This this zero-pole pair are referred to as a *removable singularity*.

– 6.6: Explain why Eqs. **??** *and* **??** *have different numbers of poles and zeros.*

Sol: The answer is very interesting. For Eq. **??**, $P_N(s_r) = 0$ has N roots and we are not sure where they are. The numerator of Eq. **??** has $N + 1$ roots at $s_r = e^{j2\pi n/(N+1)}$ for $n = 0, 1, 2, \ldots N$. However for $n = 0$, $s_r = e^{j0/N} = 1$ is not a root, since $P_N(1) = N$. This root and the pole exactly cancel. All the roots $N + 1$ of Eq. **??** are known as *the roots of unity*, but the root at $n = 0$ is special because it cancels with the pole at $s = 1$. Given the roots of Eq. **??**, we can see that the N roots of Eq. **??** are at $s_z = \sqrt[N]{-1} = e^{j2\pi n/(N+1))}$, with $n = 1, \ldots, N$ ($n \neq 0$). Perhaps even a bit clever.

– 6.7: Is the function $1/(1 - x)$ analytic outside of the RoC?

Sol: Yes, because it is analytice everwhere other than at the pole $x = 1$.

– 6.8: Extra credit. Evaluate $P_N(x)$ at $x = 0$ and $x = 0.9$ for the case of $N = 100$, and compare the result to that from Matlab.

```
%sum the geometric series and P_100(0.9)
clear all;close all;format long
N=100; x=0.9; S=0;
for n=0:N
S=S+x^n
end
P100=(1-x^(N+1))/(1-x);
disp(sprintf('S= %g, P100= %g, error= %g',S,P100, S-P100))
```

Sol: $P_N(0) = 1$ and $P_N(0.9) = \frac{1-.9^{N+1}}{1-0.9} = 9.999760947410010$. According to Matlab $P_{100}(0) = 1$ and $P_{100}(0.9) = 9.999760947410014$, with a difference of -3.55271×10^{-15} (i.e., -16×eps).

Problem # *7: The exponential series*

– 7.1: What is the RoC for the exponential series Eq. **??***?*

Sol: The exponential is convergent everywhere on the open real line.

*– 7.2: Let $x = \jmath$ in Eq. **??**, and write out the series expansion of e^x in terms of its real and imaginary parts.*
Sol:

$$\begin{aligned} e^{\jmath} &= \sum_0^\infty \frac{\jmath^n}{n!} \\ &= 1 + \jmath - \frac{1}{2!} - \jmath\frac{1}{3!} + \frac{1}{4!} + \jmath\frac{1}{5!} - \frac{1}{6!} + \cdots \\ &= \left(1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots\right) + \jmath\left(1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots\right) \\ &= \sum_{n=0,2,\ldots}^{\text{n even}} \frac{(-1)^n}{n!} + \jmath \sum_{n=1,3,\ldots}^{\text{n odd}} \frac{(-1)^n}{n!}. \end{aligned}$$

*– 7.3: Let $x = \jmath\theta$ in Eq. **??**, and write out the series expansion of e^x in terms of its real and imaginary parts. How does your result relate to Euler's identity ($e^{\jmath\theta} = \cos(\theta) + \jmath\sin(\theta)$)?*
Sol:

$$\begin{aligned} e^{\jmath\theta} &= \sum_0^\infty \frac{\jmath^n\theta^n}{n!} \\ &= 1 + \jmath\theta - \frac{\theta^2}{2!} - \jmath\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \jmath\frac{\theta^5}{5!} - \frac{\theta^6}{6!} + \cdots \\ &= \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)}_{\cos\theta} + \jmath\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)}_{\sin\theta} \\ &= \cos(\theta) + \jmath\sin(\theta). \end{aligned}$$

Inverse analytic functions and composition

Overview: It may be surprising, but every analytic function has an inverse function. Note that the inverse is not to be confused with the *reciprocal*. The function ($x, y \in \mathbb{C}$)

$$y(x) = \frac{1}{1-x},$$

has a pole at $x = 1$ and a zero at $x = \infty$. Its inverse

$$x(y) = \frac{y-1}{y} = 1 - \frac{1}{y}$$

has a zero at $y = 1$ and a pole at $y = 0$, with $x(\infty) = 1$. The reciprocal of $y(x)$ is $1/y(x) = (1-x)$, which is very different from $x(y)$.

Problem # *8 Consider the exponential function $z(x) = e^x$ ($x, z \in \mathbb{C}$).*

– 8.1: Find the inverse $x(z)$.
Sol: Taking the natural log (ln) of both sides gives $x(z) = \ln(z)$. Thus the natural log is the inverse of the exponential.

– 8.2: Where are the poles and zeros of $x(z)$?
Sol: Their is a branch cut between $z = 0, -\infty$, and the zero is at $z = 1$. There seems to be a pole at $z = 0$, where the branch cut terminates. I don't seem to fully understand this singular point.

Problem # *9: Composition.*

– 9.1: If $y(s) = 1/(1-s)$ and $z(s) = e^s$, compose these two functions to obtain $(y \circ z)(s)$.
Give the expression for $(y \circ z)(s) = y(z(s))$. **Sol:**

$$(y \circ z)(s) = \frac{1}{1 - e^s}$$

– 9.2: Where are the poles and zeros of $(y \circ z)(s)$?
Sol: It is best to analyize this function using `zviz 1./(1-exp(5.*s))`. There are an ∞ number of poles at $s_n = \jmath 2\pi n, n \in \mathbb{Z}$ (namely when $e^s_n = 1$). There is a single zero at $\Re s = \sigma \to \infty$, and $y(s)$ goes to 1 for $\Re s = \sigma \to -\infty$.

– 9.3: Where (for what condition on s) is $(y \circ z)(s)$ analytic?
Sol: It is analytic everywhere except at the poles $s_n = \jmath 2\pi n, n = \mathbb{Z}$.

Eigen-analysis

Problem # *10: The vectorized eigen-equation for a matrix $\mathcal{A}$ is*

$$\boldsymbol{A}\boldsymbol{e} = \boldsymbol{e}\boldsymbol{\Lambda}. \tag{AE-1.5}$$

– 10.1: Provide a formula for $\boldsymbol{A}^3$ in terms of the eigenvector $\mathbf{e}$ and eigenvalue $\boldsymbol{\Lambda}$ matricies.
Sol: To find powers of a matrix modify Eq. **??** by post multication by $\boldsymbol{e}$

$$\boldsymbol{A} = \boldsymbol{e}\boldsymbol{\Lambda}\boldsymbol{e}^{-1}.$$

Then

$$\boldsymbol{A}^3 = \boldsymbol{e}\boldsymbol{\Lambda}\boldsymbol{e}^{-1}\boldsymbol{e}\boldsymbol{\Lambda}\boldsymbol{e}^{-1}\boldsymbol{e}\boldsymbol{\Lambda}\boldsymbol{e}^{-1} = \boldsymbol{e}\boldsymbol{\Lambda}^3\boldsymbol{e}^{-1}.$$

– 10.2: Find the eigenvalues of the matrix, and find the roots, by completing the square, where $a, b, c, d \in \mathbb{C}$, and

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Sol: The definition of the eigenvalues is

$$\det \left| \boldsymbol{A} - \lambda \boldsymbol{I}_2 \right| = 0$$

which is

$$\det \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda - bc.$$

Completing the square

$$\left(\lambda - \frac{a+d}{2}\right)^2 - \left(\frac{a+2}{2}\right)^2 - bc = 0.$$

Thus

$$\lambda_\pm = \frac{a+d}{2} \pm \sqrt{bc + \left(\frac{a-d}{2}\right)^2}.$$

The eigenvalues are typically the damped resonant frequencies $\lambda_\pm = \sigma_o \pm \jmath\omega_o$ of a mechanical or electrical circuit. In these cases the radical is $\jmath\omega_0$ is the resonance radian frequency and $j\sigma_o \leq 0$ is the resonant damping. This requires that the constants $\{a, b, c, d\} \geq 0$ and $\in \mathbb{R}$.

0.4.2 Convolution

Multiplying two short or simple polynomials is not demanding. However, if the polynomials have many terms, it can become tedious. For example, multiplying two 10th-degree polynomials is not something one would like to do every day.

An alternative is a method called *convolution*. The inverse of convolution is called *deconvolution*, which is equivalent to long-division of polynomials, also known as factoring polynomials (Sec. 3.4.1, p. 109-111). Newton's method is a reliable and accurately algorithm to extract roots from polynomials using term by term deconvolution. When the roots are well approximated by fractional numbers, the method is accurate to within computational accuracy. For example, if the root is $\pi \approx \hat{\pi}_{19} \equiv 817696623/260280919 \in \mathbb{F}$, as given by `rats(pi,19)`. $\hat{\pi}_{19}$ is the 64 bit machine's internal representation of π since $\pi - \hat{\pi}_{19} = 0$ (See text Fig. 2.6, p. 48).

Problem # *11: Convolution of sequences. Practice convolution (by hand!!) using a few simple examples. Manually evaluate the following convolutions. Show your work!*

– 11.1: Multiplying two polynomials is the same as convolving their coefficients.
Given

$$f(x) = x^3 + 3x^2 + 3x + 1 \leftrightarrow [1; 3, 3, 1]$$
$$g(x) = x^3 + 2x^2 + x + 2 \leftrightarrow [1; 2, 1, 2].$$

show that

$$f(x)g(x) = x^6 + 5x^5 + 10x^4 + 12x^3 + 11x^2 + 7x + 2 \leftrightarrow [1; 3, 3, 1] \star [1; 2, 1, 2].$$

Sol: Do the convolution $[1; 3, 3, 1] \star [1; 2, 1, 2]$. Reverse the first vector and run it across the second. This produces $[1, [3, 1] \cdot [1, 2], [1, 3, 3] \cdot [1, 2, 1] \cdots$ = $[1; 5, 10, 12, 11, 7, 2]$.

Here is how you put [] inside the latex's
`begin{question}[: [bracketed text] ]`

```
\begin{question}
[:
  \( \char91 1;-1 \char93 \star \char91 0; 1, 2, 4,  7, 0\char93 \)
]
\end{question}
```

as used in the following question:

– 11.2: $[1; -1] \star [0; 1, 2, 4, 7, 0]$
Sol: $[1; -1] \star [0; 1, 2, 4, 7, 0]$ = $[0; 1, 2, 4, 3, -7, 0, \ldots]$. = $[0, 1, 1, 2, 3, -7, 0, \ldots]$.

– 11.3: $[1; 2, 1] \star [1; -1]$
Sol: $[1; 1, -1, -1]$

Newton's root-finding method

In 2024 I published a journal paper on Newton's method[4], and carefully explained Newton's chaotic convergence problem (**?**).

Problem # *12: Use Newton's iteration to find the roots of the polynomial*

$$P_3(x) = 1 - x^3.$$

– 12.1: Draw a graph describing the first step of the iteration starting with $x_0 = (1/2, 0)$.
Sol: Start with an (x, y) coordinate system and put points at and the vertex of $P_3(x)$.

– 12.2: Calculate x_1 *and* x_2*. What number is the algorithm approaching?*
Sol: First we must find $P_3'(x) = -3x^2$. Thus the equation we must iterate is:

$$x_{n+1} = x_n + \frac{1 - x_n^3}{3x_n^2}.$$

[4] https://jontalle.web.engr.illinois.edu/Public/Allen-Newton-Apr9.24.pdf

Given a first guess for the root x_0, the next are $x_1 = x_0 + \frac{1-x_0^3}{3x_0^2}$ and $x_2 = x_1 + \frac{1-x_1^3}{3x_1^2}$. Note that if $x+0$ is the root, then $x_1 = x_0$ and we are done. However, if $x_0 = 0$, then $x_1 = \infty$, since $x_0 = 0$ is a root of $P_3'(x)$. Thus we must not start at the roots of $P_n'(x_0) = 0$.

– 12.3: Here is an Octave/Matlab script for the $P_2(x)$ case. Modify it to find $P_3(x)$:

```
x(1)=1/2; %x(1)=0.9; %x(1)=-10
y(1)=x(1);
 for n=2:10
x(n) = x(n-1) + (1-x(n-1)^2)/(2*x(n-1));
y(n) = (1+y(n-1)^2)/(2*x(n-1));
 end
semilogy(abs(x)-1); hold on
semilogy(abs(7)-1,'or'); hold off
```

Sol:

```
x=1/2;
for n = 1:3
x = x+(1-x*x*x)/(3*x*x);
end
```

– 12.4: For $n = 4$, what is the absolute difference between the root and the estimate, $|x_r - x_4|$?
Sol: 4.6E-8 (very small!)

– 12.5: Does Newton's method work for $P_2(x) = 1+x^2$? If so, why? Hint: The poles and zeros are exactly known!
Sol: Here $P_2'(x) = 2x$. Thus the iteration gives

$$x_{n+1} = x_n - \frac{1+x_n^2}{2x_n}.$$

In this case the roots are $x_\pm = \pm 1\jmath$—namely, purely imaginary. The solution will converge for complex roots as long as the starting point is complex. If we start with a real number for x_0, and use real arithmetic, Newton's method fails because there is no way for the answer to become complex. Real in = Real out.

– 12.6: What if we let $x_0 = (1+\jmath)/2$ for the case of $P_2(x) = 1 + x^2$?
Sol: By starting with a complex initial value, we fix the Real in = Real out problem.

Problem # *13: In this problem we consider the case of fractional roots, and take advantage of this fact during the itteration. Given that the roots are integers, composed of primes, we may uniquely identify the primes by factoring the numerator and denominator of the rational approximation of the root.*
The method is:

1. Start the Newton itteration
$$s_{n+1} = s_n - \frac{M(s_n)}{M'(s_n)}$$
2. Apply the CFA to the next output `rats`(s_{n+1})
3. Factor the Num and Dem of the CFA
4. Terminate when the factors converge

Using this method, show that we can find either the best possible fractional approximation to the roots (or even the exact roots, when the answer is within machine accuracy).

– 13.1: Find the roots of a Monic having coefficents $m_k \in \mathbb{F}$.

Let

$$M_3(x) = (x - 254/17)(x - 2047/13)(x - 17/13)$$

In this case the root vector R becomes

$$R = [14.9412, 157.4615, 1.3077].$$

Verify that `rats(M)` returns the rational set of roots. **Sol:** In double precision this returns M_3. (Not sure what happens in single precision.)

0.5 Problems AE-2

Topics of this homework:

Linear vs nonlinear systems of equations, Euclid's formula, Gaussian elimination, matrix permutations, Ohm's law, two-port networks,

Deliverables: Answers to problems

Gaussian elimination Book Ver 2

Problem # *1: Gaussian elimination*

– 1.1: Find the inverse of

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Sol:

$$A^{-1} = \frac{1}{3-8}\begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}.$$

– 1.2: Verify that $A^{-1}A = AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Sol: Multiply them to show this.

Problem # *2: Find the solution to the following* 3×3 *matrix equation* $Ax = b$ *by GE. Show your intermediate steps. You can check your work at each step using Octave/Matlab.*

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 8 \end{bmatrix}.$$

– 2.1: Show (i.e., verify) that the first GE matrix G_1*, which zeros out all entries in the first column is given by*

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Identify the elementary row operations that this matrix performs. **Sol:** Operate with GE matrix on A

$$G_1[A|b] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ -3 & 1 & 0 & \\ -1 & 0 & 1 & \end{array}\right] \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 3 & 1 & 1 & 9 \\ 1 & -1 & 4 & 8 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 4 & 6 \\ 0 & -2 & 5 & 7 \end{array}\right]$$

It scales the first row by -3 and adds it to the second row, and scales the first row by -1 and adds it to the third row.

– 2.2 Find a second GE matrix, G_2, to put G_1A in upper triangular form. Identify the elementary row operations that this matrix performs.

Sol:

$$G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

which scales the second row by -1 and adds it to the third row. Thus we have

$$G_2G_1[A|b] = \left[\begin{array}{ccc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 3 & 1 & 1 & 9 \\ 1 & -1 & 4 & 8 \end{array}\right]$$
$$= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 4 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

– 2.3 Find a third GE matrix G_3 that scales each row so that its leading term is 1. Identify the elementary row operations that this matrix performs.

Sol:

$$G_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which scales the second row by -1/2. Thus we have

$$G_3G_2G_1[A|b] = \left[\begin{array}{ccc|c} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 3 & 1 & 1 & 9 \\ 1 & -1 & 4 & 8 \end{array}\right]$$
$$= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

– 2.4: Find the last GE matrix, G_4, which subtracts a scaled version of row 3 from row 2, and scaled versions of rows 2 and 3 from row 1, such that you are left with the identity matrix ($G_4G_3G_2G_1A = I$).

Sol:

$$G_4 = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus we find $G_4G_3G_2G_1[A|b]$ is

$$= \left[\begin{array}{ccc|c} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 3 & 1 & 1 & 9 \\ 1 & -1 & 4 & 8 \end{array}\right]$$
$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

– 2.5: Solve for $\{x_1, x_2, x_3\}^T$ using the augmented matrix format $G_4G_3G_2G_1\{A|b\}$ (where $\{A|b\}$ is the augmented matrix). Note that if you've performed the preceding steps correctly, $x = G_4G_3G_2G_1b$.

Sol: From the preceding problems, we see that $[x_1, x_2, x_3]^T = [3, -1, 1]^T$

– 2.6: Find the pivot matrix G that rescales the second row of the augmented matrix $A|b$ by 1/3.

Sol:

$$G_1 = \begin{bmatrix} 1 & & \\ & 1/3 & \\ & & 1 \end{bmatrix}.$$

Proceeding

$$G_1 A = \begin{bmatrix} 1 & & \\ & 1/3 & \\ & & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 3 & 1 & 1 & 9 \\ 1 & -1 & 4 & 8 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & 1/3 & 1/3 & 3 \\ 1 & -1 & 4 & 8 \end{array}\right]$$

Two linear equations

Problem # *3 In this problem we transition from a general pair of equations*

$$f(x, y) = 0$$
$$g(x, y) = 0$$

to the important case of two linear equations

$$y = ax + b$$
$$y = \alpha x + \beta.$$

Note that to help keep track of the variables, roman coefficients (a, b) are used for the first equation and Greek (α, β) for the second.

– 3.1: What does it mean, graphically, if these two linear equations have (1) a unique solution, (2) a nonunique solution, or (3) no solution?

Sol: There are three possibilities:

1. When they have different slopes, they meet at one (x,y) point, which is the solution.
2. If the two lines are identical, any point on the line is a solution.
3. If they have the same slope but different intercepts (are parallel to each other) there is no solution.

– 3.2: Assuming the two equations have a unique solution, find the solution for x and y.

Sol: Since there must be one point where the two are equal, we may solve for that by setting the y values equal to each other:

$$ax + b = \alpha x + \beta$$

Thus

$$x = \frac{\beta - b}{a - \alpha}$$
$$y = a\frac{\beta - b}{a - \alpha} + b$$

– 3.3: When will this solution fail to exist (for what conditions on a, b, α, and β)?

Sol: As stated above, if they have the same slope $\alpha = a$ but different intercepts $\beta \neq b$, there is no solution. When $\beta = b$ and $\alpha = a$ every point on the line is a solution.

– 3.4: Write the equations as a 2×2 matrix equation of the form $A\vec{x} = \vec{b}$, where $\vec{x} = \{x, y\}^T$.

Sol:

$$\begin{bmatrix} 1 & -a \\ 1 & -\alpha \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} b \\ \beta \end{bmatrix}$$

– 3.5: Find the inverse of the 2×2 matrix, and solve the matrix equation for x and y.

Sol:

$$\begin{bmatrix} y \\ x \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} -\alpha & a \\ -1 & 1 \end{bmatrix} \begin{bmatrix} b \\ \beta \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} -\alpha b + a\beta \\ -b + \beta \end{bmatrix}$$

where the determinant is $\Delta \equiv a - \alpha$.

– 3.6: Discuss the properties of the determinant of the matrix (Δ) in terms of the slopes of the two equations (a and α).

Sol: When the slopes are the same there is no solution and $\Delta = 0$. Thus the matrix solution is consistent with the geometry. This is our first result in analytic geometry.

Problem # *4: The application of* linear functional relationships *between* two variables

We use 2×2 matrices to describe two-port networks, as discussed in Sec. **??** (p. **??**). Transmission lines are a great example: Both voltage and current must be tracked as they travel along the line. The Figure below shows an example segment of a transmission line.

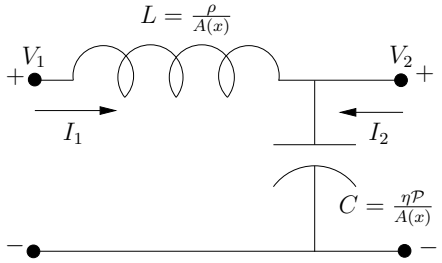


Fig. AE-2.1 *This figure shows a cell from an LC transmission line. The index* 1 *is at the input on the left and 2 represents the output, on the right.*

Suppose you are given the following pair of linear relationships between the input (source) variables V_1 and I_1 and the output (load) variables V_2 and I_2 of the transmission line:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \jmath & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}.$$

– 4.1: Let the output (the load) be $V_2 = 1$ and $I_2 = 2$ (i.e., V_2/I_2 =1/2 $\{\Omega\}$). Find the input voltage and current, V_1 and I_1.

Sol: This case corresponds to

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \jmath & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1\jmath + 2 \\ 1 - 2 \end{bmatrix}$$

Thus $V_1 = 2 + 1\jmath$ and $I_1 = -1$.

– 4.2: Let the input (source) be $V_1 = 1$ and $I_1 = 2$. Find the output voltage and current, V_2 and I_2.

Sol: With the input specified the two equations are

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \jmath & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}.$$

To find the input we must invert the matrix ($\Delta = -j - 1$)

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \frac{1}{1 + j} \begin{bmatrix} 1 & 1 \\ 1 & -\jmath \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Thus $V_2 = 3/(1 + \jmath) = 3(1 - \jmath)/2$, $I_2 = (1 - 2\jmath)/(1 + 1\jmath) = -(1 + 3\jmath)/2$. The point of this exercise is that the two lines have a complex intersection point, not easily visualized.

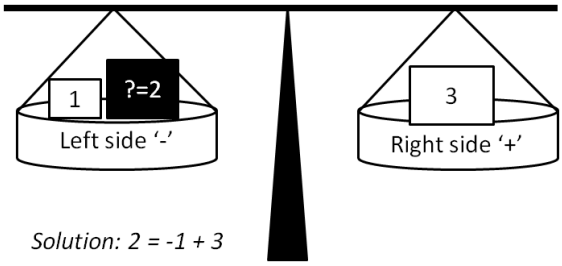
Integer equations: applications and solutions

Any equation for which we seek only integer solutions is called a *Diophantine* equation.

Problem # *5: A practical example of using a Diophantine (Number theory) equation:*

"A merchant had a 40-pound weight that broke into 4 pieces. When the pieces were weighed, it was found that each piece was a whole number of pounds and that the four pieces could be used to weigh every integral weight between 1 and 40 pounds. What were the weights of the pieces?" - *Bachet de Bèziriac (1623)*[5].

Here, weighing is performed using a balance scale that has two pans, with weights on either pan. Thus, given weights of 1 and 3 pounds, one can weigh a 2-pound weight by putting the 1-pound weight in the same pan with the 2-pound weight, and the 3-pound weight in the other pan. Then the scale will be balanced. A solution to the four weights for Bachet's problem is $1+3+9+27 = 40$ pounds.



– 5.1: Show how the combination of 1-, 3-, 9-, and 27-pound weights can be used to weigh $1, 2, 3, \ldots, 8, 28$*, and* 40 *pounds of milk. Assuming that the milk is in the* left *pan, provide the position of the weights using a negative sign* $-$ *to indicate the left pan and a positive sign* $+$ *to indicate the right pan. For example, if the left pan has 1 pound of milk, then 1 pound of milk in the right pan,* $+1$ *will balance the scales.*

Hint: It is helpful to write the answer in matrix form. Set the vector of values to be weighed equal to a matrix indicating the pan assignments, multiplied by a vector of the weights $[1, 3, 9, 27]^T$. The pan assignments matrix should contain only the values -1 (left pan), $+1$ (right pan), and 0 (leave weight out). We indicate these weights using $-$, $+$, and blanks.

Sol: Any integer between 1 and 40 may be expanded using the weights 1, 3, 9, 27 ($\in \mathbb{Z}$). Here is the problem stated in matrix form:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \ldots \\ 28 \\ \ldots \\ 40 \end{bmatrix} = \begin{bmatrix} + & & & \\ - & + & & \\ & + & & \\ + & + & & \\ - & - & + & \\ & - & + & \\ + & - & + & \\ - & & + & \\ \ldots & & & \\ + & & & + \\ \ldots & & & \\ + & + & + & + \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 9 \\ 27 \end{bmatrix}$$

The left column is the weight of the milk in integer units. The right-most column is the four weights. Note that these four weights span the integers from 1-40 with binary weights. Each weight may be computed recursively from twice the sum of the previous weights +1, that is

$$W_{n+1} = 2W_n + 1 = 2^{n+1} \quad \text{since} \quad W_n = 2^n.$$

For example, to get 26 we place weights 9+3+1 in the pan. The next weight is 27 = 2*(9+3+1)+1. Recursively, the weights are 3=2*1+1, 9=2*(3+1)+1, 27=2*(9+3+1)+1. To go above 40 a fifth weight (not shown) must be: 81=2*(27+9+3+1)+1 = 2*40+1.

0.5.1 Vector algebra in $\mathbb{R}^3$

Basic setup: Let $\boldsymbol{A} = [a_1, a_2, a_3]^T$, $\boldsymbol{B} = [b_1, b_2, b_3]^T$, $\boldsymbol{C} = [c_1, c_2, c_3]^T$ be three vectors in $\mathbb{R}^3$. Definitions of the scalar (also called a dot product) $\boldsymbol{A} \cdot \boldsymbol{B}$, cross $\boldsymbol{A} \times \boldsymbol{B}$ and triple product $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$, may be found in Appendix **??** (p. **??**), where $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ in $\mathbb{R}^3 \subset \mathbb{C}^3$), as shown in Fig. **??**, p. **??**.

[5] Taken from **?**, p. 50

A fourth "double-cross" ($\mathbf{x}$) vector product is:[6]

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \alpha_o \boldsymbol{B} - \beta_o \boldsymbol{C}.$$

where $\alpha_o = \boldsymbol{A} \cdot \boldsymbol{C}$ and $\beta_o = \boldsymbol{A} \cdot \boldsymbol{B}$ (Note: $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) \neq (\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C}$).

Problem # *6:* Scalar product $\mathbf{A} \cdot \mathbf{B}$

– 6.1: If $\boldsymbol{A} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}$ *and* $\boldsymbol{B} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}} + b_z\hat{\mathbf{z}}$*, write out the definition of* $\mathbf{A} \cdot \mathbf{B}$.
Sol: See the definition in the above figure. $\boldsymbol{A} \cdot \boldsymbol{B} = a_x b_x + a_y b_y + a_z b_z$. In general: $\boldsymbol{A} \cdot \boldsymbol{B} = \sum_k A_k B_k$.

– 6.2: The dot product is often defined as $||\boldsymbol{A}||\,||\boldsymbol{B}||\cos(\theta)$*, where* $||\boldsymbol{A}|| = \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}}$ *and* θ *is the angle between* $\boldsymbol{A}, \boldsymbol{B}$*. If* $||\boldsymbol{A}|| = 1$*, describe how the dot product relates to the vector* $\mathbf{B}$.
Sol: See the definition in the above figure. The vector product is the portion of $\mathbf{B}$ in the direction of $\mathbf{A}$.

Problem # *7:* Vector (cross) product $\mathbf{A} \times \mathbf{B}$

– 7.1: If $\boldsymbol{A} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}$ *and* $\boldsymbol{B} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}} + b_z\hat{\mathbf{z}}$*, write out the definition of* $\mathbf{A} \times \mathbf{B}$.
Sol:

$$\boldsymbol{A} \times \boldsymbol{B} \equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{\mathbf{x}} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{\mathbf{y}} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}.$$

– 7.2: Show that the cross product is equal to the area of the parallelogram formed by $\boldsymbol{A}, \boldsymbol{B}$*, namely* $||\boldsymbol{A}||\,||\boldsymbol{B}||\sin(\theta)$*, where* $||\boldsymbol{A}|| = \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}}$ *and* θ *is the angle between* $\boldsymbol{A}$ *and* $\boldsymbol{B}$.
Sol: A parallelogram's area is equal to its base times its height. Therefore, let's say the base is length $||A||$, and the height $||B||\sin(\theta)$, which is the portion of $\mathbf{B}$ that is perpendicular to $\mathbf{A}$.

Problem # *8:* *Triple product* $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$

Sol: ?

– 8.1: Starting from the definition of the dot and cross product, explain using a diagram and/or words, how one shows that: $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.
Sol: Using the determinate-definition of the cross product,

$$\boldsymbol{B} \times \boldsymbol{C} \equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \hat{\mathbf{x}} \begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} - \hat{\mathbf{y}} \begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix}.$$

Let $\boldsymbol{D} = \boldsymbol{B} \times \boldsymbol{C}$ and compute $\boldsymbol{A} \cdot \boldsymbol{D} = \boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$. Finally compute the requested right-hand side, and compare the two. It should be clear that they are the same, because the dot product transfers the elements of vector $\boldsymbol{A}$ to cross product and reduces the product to the scalar.

– 8.2: Describe why $|\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})|$ *is the volume of parallelepiped generated by* $\boldsymbol{A}$*,*$\boldsymbol{B}$*, and* $\boldsymbol{C}$.
Sol: Note that the norm of $\boldsymbol{B} \times \boldsymbol{C}$ is the area of the parallelogram generated by $\boldsymbol{C}$ and $\boldsymbol{B}$. Taking the dot product with $\mathbf{A}$ results in the volume of the corresponding parallelepiped (prism). So the absolute value of triple product is volume of parallelepiped.

– 8.3: Explain why three vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ *are in one plane if and only if the triple product* $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{0}$.
Sol: (triple product is zero) if and only if: (volume is zero), if and only if: (they are in the same plane)

[6] Greenberg p. 694, Eq. 8.

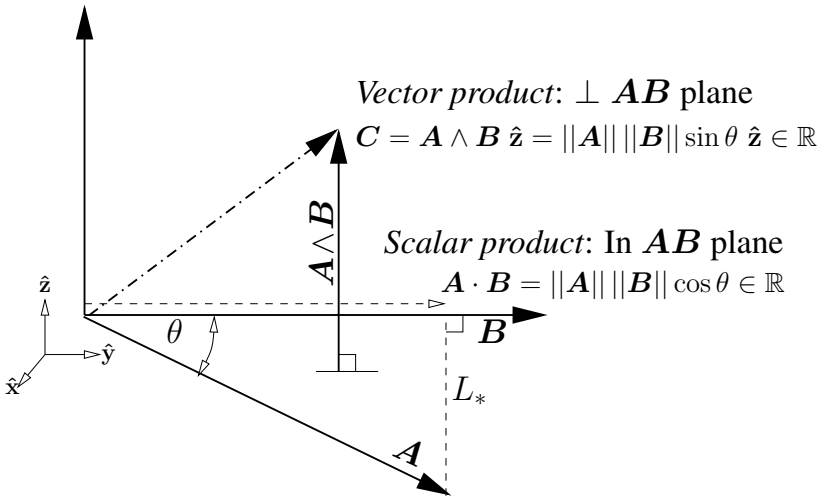


Figure 1: This figure is identical to Fig. **??** (p. **??**).

Problem # *9: Given two vectors* $\boldsymbol{A}, \boldsymbol{B}$ *in the* $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ *plane shown in Fig.* **??** *(same as* **??** *on page* **??***), with* $\boldsymbol{B} = \hat{\mathbf{y}}$ *(i.e.,* $||\boldsymbol{B}|| = 1$*).*

– 9.1: Show that $\boldsymbol{A}$ *may be split into two orthogonal parts, one in the direction of* $\boldsymbol{B}$ *and the other perpendicular (*$\perp$*) to* $\boldsymbol{B}$*. Hint: Express the vector products of* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *(dot and cross) in polar coordinates (***?***).*

$$\begin{aligned}\boldsymbol{A} &= (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{B} + \boldsymbol{B}\times(\boldsymbol{A}\times\boldsymbol{B}) \\ &= \boldsymbol{A}_{\parallel} + \boldsymbol{A}_{\perp}.\end{aligned}$$

Sol: Expressing the vector products in polar form makes this result transparent:

$$\boldsymbol{A}\cdot\boldsymbol{B} = ||\boldsymbol{A}||\cos(\theta) \quad \text{and} \quad \boldsymbol{A}\times\boldsymbol{B} = ||\boldsymbol{A}||\sin(\theta)$$

The first quantity is in the direction of $\boldsymbol{B}$, while the second is in the direction $\boldsymbol{A}\times\boldsymbol{B}$, which is $\perp$ to $\boldsymbol{B}$. Thus

$$\begin{aligned}\boldsymbol{A} &= ||\boldsymbol{A}||\,(\boldsymbol{B}\cos(\theta) + \boldsymbol{A}\times\boldsymbol{B}\sin(\theta)) \\ &= \boldsymbol{A}_{\parallel} \qquad\qquad\quad + \boldsymbol{A}_{\perp}.\end{aligned}$$

Ohm's Law Ver 2

In general, impedance is defined as the ratio of a force to a flow. For electrical circuits, the voltage is the force and the current is the flow. Ohm's law states that the voltage across and the current through a circuit element are related by the impedance of that element (which may be a function of frequency). For resistors, the voltage over the current is called the *resistance* and is a constant (e.g., the simplest case is $V/I = R$). For inductors and capacitors, the voltage over the current is a frequency-dependent impedance (e.g., $V/I = Z(s)$, where s is the complex frequency $s \in \mathbb{C}$).

As shown in Table **??** (p. **??**), the impedance concept also holds in mechanics and acoustics. In mechanics, the force is equal to the mechanical force on an element (e.g., a mass, dashpot, or spring) and the flow is the velocity. In acoustics, the force is pressure and the flow is the volume velocity or particle velocity of air molecules.

Case	Force	Flow	Impedance	units
Electrical	voltage (V)	current (I)	Z	Ohms [Ω]
Mechanics	force (F)	velocity (V)	Z	Mechanical Ohms [Ω]
Acoustics	pressure (P)	particle velocity (U)	Z	Acoustic Ohms [Ω]
Thermal	temperature (T)	heat-flux (J)	Z	Thermal Ohms [Ω]

Problem # *10: The resistance of an incandescent (filament) lightbulb, measured cold, is about 100 ohms. As the bulb lights up, the resistance of the metal filament increases.*
Ohm's law says that the current

$$\frac{V}{I} = R(T),$$

where T is the temperature. In the United States, the voltage is 120 volts (RMS) at 60 [Hz]. Find the current when the light is first switched on. **Sol:** Thus the current is

$$I = 120/R = 120/100 = 1.2. \qquad \text{[Amps]}$$

As the bulb heats up, the current rapidly drops, and the resistance increases. This typically takes less than a milliseconds [ms], which depends on the wattage of the light bulb. Such lightbulbs are *nonlinear.* These rules don't apply to LED bulbs.

Problem # *11: The power in watts is the product of the force and the flow. What is the power of the lightbulb of Problem* **??***?*
Sol: $P = V \cdot I = 120 \times 1.2 = 120 + 24 = 144$ [W].

Problem # *12: State the impedance $Z(s)$ of each of the following circuit elements: (1) a resistor with resistance R, (2) an inductor with inductance L, and (3) a capacitor with capacitance C.*

Sol: (1) For the resistor, $Z = R$.
(2) For the inductor, $Z = sL$ with $s = \sigma + \omega\jmath$. Note the flux $\psi(t) = Li(t)$. The voltage $v(t)$ is the time derivative of the flux

$$v(t) = \frac{d\psi(t)}{dt} = L\frac{di(t)}{dt}.$$

(3) For the capacitor, $Z = 1/sC$. Note the charge $q(t) = Cv(t)$, thus the current $i(t)$ is the time derivative of the charge

$$i(t) = \frac{d}{dt}q(t) = C\frac{dv(t)}{dt}.$$

Problem # *13: Consider what happens at the triple point of water. As water freezes or thaws, the temperature remains constant at 0 (C°). Once all the water is frozen and more heat is removed, the temperature drops below 0°. As heat is added, water thaws but the temperature remains at 0°.*

– 13.1: Once all the ice has melted, what is the temperature as more heat is added?
Model the triple point using a Zener diode, a resistor, and a capacitor. A Zener diode holds the voltage constant independent of current. For the case of water's triple point, the voltage represents the temperature of water at the triple point, clamped at 0 [C°]. The current represents the heat flux. The latent heat of water at the triple point is 32 Cal/gm. Thus as the temperature rises from below freezing, the water is clamped at 0° once the triple point is reached. At that point, adding more heat flux has no effect on the temperature until all the ice melts. Once the ice has melted, the temperature again begins to rise until it hits the boiling point, where it again stays at 100° until all the water has evaporated.

Sol: Need a figure here showing how to model the triple point of water. The Heat capacity may be modeled by a capacitor, which is fixed at 0°as the capacitor discharges. Once it is empty, the temperature again begins to rise as the heat Q from the sun is added

$$T^\circ = C_w Q,$$

where C_w is the heat capacity of the water mass. Thus the required circuit needs to emulate this temperature behavior due to the latent heat of melting ice and boiling water into steam.

Nonlinear (quadratic) to linear equations

In the following problems we deal with algebraic equations in more than one variable that are not linear equations. For example, the circle $x^2+y^2=1$ may be solved for $y(x)=\pm\sqrt{1-x^2}$. If we let $z_+ = x+yj = x+j\sqrt{1-x^2} = e^{\theta j}$, we obtain the equation for half a circle ($y>0$). The entire circle is described by the magnitude of z as $|z|^2=(x+yj)(x-yj)=1$.

Problem # *14: Give the curve defined by the equation:*

$$x^2+xy+y^2=1$$

– 14.1: Find the function $y(x)$.

Sol: Completing the square in y and solve for $y(x)$ to isolate x and y

$$\begin{aligned}(y+x/2)^2 - x^2/4 + x^2 &= 1\\ (y+x/2)^2 &= 1-\frac{3}{4}x^2\\ y+x/2 &= \pm\sqrt{\frac{4-3x^2}{4}}\\ y &= \frac{1}{2}\left(\pm\sqrt{4-3x^2}-x\right)\\ (2y)^2 &= \left(\pm\sqrt{4-3x^2}-x\right)^2\end{aligned}$$

Added Dec8, 2021: Expanding the RHS square gives

$$4y^2 = (4-3x^2) \mp 2x\sqrt{4-3x^2} + x^2$$

and setting $q=x^2$.

We would like to complete the square for $x=\sqrt{q}$, like this: $(\sqrt{x}\mp x)^2 = x \mp 2x\sqrt{x} + \cancel{(x^2-x^2)}$ where $x=(4-3q)/2$ gives (this need fixing).

$$4y^2 = (4-3q) \mp 2\sqrt{q}\sqrt{4-3q} + q,$$

gives

$$\mp 4y^2 = \mp q \mp (4-3q) + 2\sqrt{4q-3q^2}$$

$$\mp 4y^2 = \mp q + \left(\mp\sqrt{4q-3q^2} \mp (4-3q)/2\right)^2 - (4-3q)^2/4$$

and verifying:

$$\mp 4y^2 = \mp q + \left(4q-3q^2 \mp (4-3q)\sqrt{4q-3q^2} + (4-3q)^2/4\right) - (4-3q)^2/4$$

$$\mp 4y^2 = \mp q + 4q - 3q^2 \mp (4-3q)\sqrt{4q-3q^2} + \cancel{\left((4-3q)^2/4 - (4-3q)^2/4\right)}^{0}$$

$$4y^2 = \mp(4q-3q^2) + (4-3q)\sqrt{4q-3q^2} + q$$

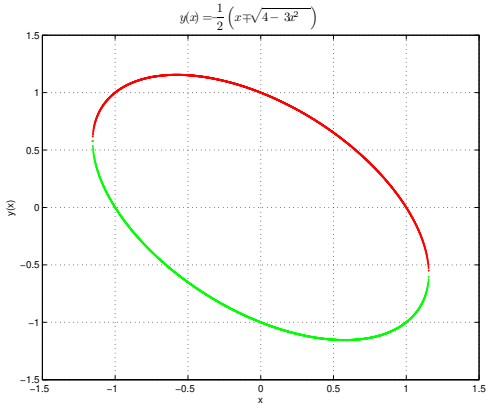
The verification does not check out. Need to try again using

$$(\sqrt{x}\mp x)^2 = x \mp 2x\sqrt{x} + \cancel{(x^2-x^2)}$$

This last line uses the identity $(\mp)^2=+$ twice. Was moving the $\mp$ necessary?

– 14.2: Using Matlab/Octave, plot $y(x)$ and describe the graph.

Sol:



Thus we find the equation is a rotated ellipse.

– 14.3: What is the name of this curve?

Sol: It is an ellipse, rotated by 45 degrees.

– 14.4: Find the solution (in x, p, and q) to these equations:

$$\begin{aligned} x + y &= p \\ xy &= q. \end{aligned}$$

Sol: Solve the first equation for y as $y = p - x$, and then substitute it into the second equation

$$x(p - x) = -x^2 + px = q.$$

Thus we find the quadratic

$$x^2 - px + q = 0$$

having roots given by completing the square

$$(x - p/2)^2 = (p/2)^2 - q.$$

resulting in $x = p/2 \pm \sqrt{(p/2)^2 - q}$, $y = p - x$.

Summary: Here we started with one linear and one quadratic (hyperbola). By the use of composition we found the roots.

– 14.5: Find an equation that is linear in y starting from equations that are quadratic (second-degree) in the two unknowns x and y:

$$x^2 + xy + y^2 = 1 \tag{AE-2.1}$$
$$4x^2 + 3xy + 2y^2 = 3. \tag{AE-2.2}$$

Sol: The goal is to obtain a linear equation in y.

Method 1: remove xy term: Scale the upper equation by 3 and subtract from the lower:

$$\begin{aligned} 4x^2 + 3xy + 2y^2 &= 3 \\ 3x^2 + 3xy + 3y^2 &= 3 \end{aligned}$$

giving $x^2 - y^2 = 0$, or $x = \pm y$.

This results in the two equations

$$\begin{aligned} x^2 \qquad -y^2 &= 0 \\ x^2 + xy + y^2 &= 1 \end{aligned}$$

Adding these gives $2x^2 \pm x^2 = 1$, which is $3x^2 = 1$ and $x^2 = 1$. Thus the final solutions are $x = \pm y = \pm 1/\sqrt{3}$ and $x = \pm y = \pm 1$.

– 14.6: Compose the following two quadratic equations and describe the results.

$$\begin{aligned} x^2 + xy + y^2 &= 1 \\ 2x^2 + xy \quad\ &= 1 \end{aligned}$$

Sol: By isolating y from one of the two equations, we may remove it from the other equation, giving us a single 4^{th} degree equation in x:

$$x^2 + (1 - 2x^2) + (1 - 2x^2)^2/x^2 = 1$$

or

$$x^4 + x^2 - 2x^4 + 1 - 4x^2 + 4x^4 - x^2 = 0$$

Collecting terms

$$3x^4 - 4x^2 + 1 = 0$$

This is a quartic, but quadratic in x^2. Thus it may be solved for x^2 by the completion of squares

$$\begin{aligned} x^4 - \frac{4}{3}x^2 &= -\frac{1}{3} \\ \left(x^2 - \frac{2}{3}\right)^2 &= \frac{1}{3}\left(\frac{4}{3} - 1\right) \\ x^2 &= \frac{2}{3} \pm \frac{1}{3} \\ x &= \pm\frac{\sqrt{2 \pm 1}}{\sqrt{3}} = \pm\frac{1}{\sqrt{3}} \text{ and } \pm\frac{2}{\sqrt{3}} \end{aligned}$$

resulting in four roots.

Nonlinear intersection in analytic geometry

Euclid's formula for Pythagorean triplets (Eq. **??**, p. **??**) can be derived by intersecting a circle and a secant line. Consider the nonlinear equation of a unit circle having radius 1, centered at $(x, y) = (0, 0)$,

$$x^2 + y^2 = 1,$$

and the secant line through $(-1, 0)$,

$$y = t(x + 1),$$

a linear equation having slope t and intercept $x = -1$. If the slope $0 < t < 1$, the line intersects the circle at a second point (a, b) in the positive x, y quadrant. The goal is to find $a, b \in \mathbb{N}$ and then show that $c^2 = a^2 + b^2$. Since the construction gives a right triangle with short sides $a, b \in \mathbb{N}$, then it follows that $c \in \mathbb{N}$.

Problem # *15: Derive Euclid's formula*

– 15.1: Draw the circle and the line, given a positive slope $0 < t < 1$.
Sol: Sol in given in Fig. **??**

Problem # *16: Substitute* $y = t(x + 1)$ *(the line equation) into the equation for the circle, and solve for* $x(t)$.
Hint: Because the line intersects the circle at two points, you will get two solutions for x. *One of these solutions is the trivial solution* $x = -1$. **Sol:** $x(t) = (1 - t^2)/(1 + t^2)$

– 16.1: Substitute the $x(t)$ *you found back into the line equation, and solve for* $y(t)$.
Sol: $y(t) = 2t/(1 + t^2)$

– 16.2: Let $t = q/p$ *be a rational number, where* p *and* q *are integers. Find* $x(p, q)$ *and* $y(p, q)$.
Sol: $x(p, q) = 2pq/(p^2 + q^2)$ and $y(p, q) = (p^2 - q^2)/(p^2 + q^2)$

– 16.3: Substitute $x(p,q)$ and $y(p,q)$ into the equation for the circle, and show how Euclid's formula for the Pythagorean triples is generated.
Sol: Multiplying out gives $(p^2+q^2) = (p^2-q^2)+2pq$

For full points you must show that you understand the argument. Explain the meaning of the comment "magic happens" when t^4 cancels.

0.6 Problems AE-3

Topics of this homework:

Visualizing complex functions, bilinear/Möbius transformation, Riemann sphere.
Deliverables: Answers to problems

Two-port network analysis

Problem # *1: Perform an analysis of electrical two-port networks, shown in Fig.* **??** *(page* **??***). This can be a mechanical system if the capacitors are taken to be springs and inductors taken as mass, as in the suspension of the wheels of a car. In an acoustical circuit, the low-pass filter could be a car muffler. While the physical representations will be different, the equations and the analysis are exactly the same.*
The definition of the ABCD *transmission matrix* ($\mathcal{T}$) is

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \tag{AE-3.1}$$

The *impedance matrix*, where the determinant $\Delta_\mathcal{T} = AD - BC$, is given by

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{\mathcal{C}} \begin{bmatrix} \mathcal{A} & \Delta_\mathcal{T} \\ 1 & \mathcal{D} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{AE-3.2}$$

– 1.1: Derive the formula for the impedance matrix (Eq. **??***) given the transmission matrix definition (Eq.* **??***). Show your work.*
Sol: The formula may be easily derived by re-arranging the equations from the matrix (Eq. **??**). Begin with

$$V_1 = \mathcal{A}V_2 - \mathcal{B}I_2$$
$$I_1 = \mathcal{C}V_2 - \mathcal{D}I_2$$

From the second equation, we get

$$V_2 = \frac{1}{\mathcal{C}}I_1 + \frac{\mathcal{D}}{\mathcal{C}}I_2$$

which gives (upon substitution)

$$V_1 = \frac{\mathcal{A}}{\mathcal{C}}I_1 + \frac{\mathcal{A}\mathcal{D}}{\mathcal{C}}I_2 - \mathcal{B}I_2 = \frac{\mathcal{A}}{\mathcal{C}}I_1 + \Big(\frac{\mathcal{A}\mathcal{D}}{\mathcal{C}} - \mathcal{B}\Big)I_2$$

which yields the matrix equation

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \mathcal{A}/\mathcal{C} & (\mathcal{A}\mathcal{D}/\mathcal{C} - \mathcal{B}) \\ 1/\mathcal{C} & \mathcal{D}/\mathcal{C} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{\mathcal{C}} \begin{bmatrix} \mathcal{A} & \Delta_\mathcal{T} \\ 1 & \mathcal{D} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{AE-3.3}$$

Problem # *2: Consider a single circuit element with impedance $Z(s)$.*

– 2.1: What is the ABCD matrix for this element if it is in series?

Sol:

$$\begin{bmatrix} 1 & Z(s) \\ 0 & 1 \end{bmatrix}$$

– 2.2: What is the ABCD matrix for this element if it is in shunt?

Sol:

$$\begin{bmatrix} 1 & 0 \\ 1/Z(s) & 1 \end{bmatrix}$$

Problem # *3: Find the ABCD matrix for each of the circuits of Fig.* **??**.

For each circuit, (i) show the cascade of transmission matrices in terms of the complex frequency $s \in \mathbb{C}$, then (ii) substitute $s = 1\jmath$ and calculate the total transmission matrix at this single frequency.

– 3.1: Left circuit (let $R_1 = R_2 = 10$ kilo-ohms and $C = 10$ nano-farads)

Sol: Write the system in chain matrix form:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & Z_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y_C & 1 \end{bmatrix} \begin{bmatrix} 1 & Z_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC & 1 \end{bmatrix} \begin{bmatrix} 1 & R_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

– 3.2: Right circuit (use L and C values given in the figure), where the pressure P is analogous to the voltage V, and the velocity U is analogous to the current I.

Sol: Write the system in chain matrix form:

$$\begin{bmatrix} P_1 \\ U_1 \end{bmatrix} = \begin{bmatrix} 1 & sL_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{sC_3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{sL_4} & 1 \end{bmatrix} \begin{bmatrix} P_2 \\ -U_2 \end{bmatrix}$$

Now we substitute the given values:

$$\begin{bmatrix} P_1 \\ U_1 \end{bmatrix} = \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2j & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{3j} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{4j} & 1 \end{bmatrix} \begin{bmatrix} P_2 \\ -U_2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{4}{3}j \\ \frac{19}{12}j & \frac{5}{3} \end{bmatrix} \begin{bmatrix} P_2 \\ -U_2 \end{bmatrix}$$

I used Matlab/Octave to evaluate this script:

```
a=[1 j;0 1];b=[1 0;2j 1];c=[1 1/3j; 0 1];d=[1 0;1/4j 1]; T=a*b*c*d.
```

Finally I found $T(2,1)$ to be $19/12$ using the Matlab/Octave command: `rats(1.5833,6)`

– 3.3: Convert both transmission (ABCD) matrices to impedance matrices using Eq. **??**. *Do this for the specific frequency $s = 1\jmath$ as in the previous part (feel free to use Matlab/Octave for your computation).*

Sol: Left circuit: Using the previous solution, and Matlab:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{j10^{-8}} \begin{bmatrix} 1 + j10^{-4} & 1 \\ 1 & 1 + j10^{-4} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

– 3.4: Right circuit: Repeat the analysis as in question 3.3.

Sol:

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{1}{1.5833j} \begin{bmatrix} -\frac{2}{3} & 1 \\ 1 & \frac{5}{3} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

Algebra

Problem # *4: Fundamental theorem of algebra (FTA).*

– 4.1: State the fundamental theorem of algebra (FTA).

Sol: There are multiple definitions of the FTA, which of course must be equivalent.
Here are three (equivalent) answers from Wikipedia

1. The fundamental theorem of algebra states that every non-constant single-variable polynomial with complex coefficients has at least one complex root. This may then be applied recursively till the degree is zero.
2. Every degree n polynomial with complex coefficients has, counted with multiplicity, exactly n roots. The equivalence of the two statements can be proven through the use of successive polynomial division.
3. The field of complex numbers is algebraically closed. Note: this one requires an understanding of the term *algebraically closed*.

Wikipedia warns:

> In spite of its name, there is no purely algebraic proof of the theorem, since any proof must use the completeness of the reals (or some other equivalent formulation of completeness), which is not an algebraic concept.

Algebra with complex variables

Problem # *5: Order and complex numbers:*

One can always say that $3 < 4$—namely, that real numbers have order. One way to view this is to take the difference and compare it to zero, as in $4 - 3 > 0$. Here we will explore how complex variables may be ordered. In the following define $\{x, y\} \in \mathbb{R}$ and complex variable $z = x + y\jmath \in \mathbb{C}$.

– 5.1: Explain the meaning of $|z_1| > |z_2|$.

Sol: $|z| = \sqrt{x^2 + y^2}$ is the length of z, so the above expression says that a disk of radius $|z_1|$ contains a second disk of radius $|z_2|$.

– 5.2: If $x_1, x_2 \in \mathbb{R}$ *(are* real *numbers), define the meaning of* $x_1 > x_2$.

Sol: This conditions is the same as $x_1 - x_2 > 0$. Order is meaningful on the real line, as a length.

– 5.3: Explain the meaning of $z_1 > z_2$.

Sol: It makes no sense to order complex numbers. A complex number has both a length and an angle (it is the same as a vector). The concept of an angle extends the sign of a real number, making order impossible. To show this, place two points on a plane, and ask which is larger than the other. The order of the x and y components, each have order. Thus order cannot be defined.

– 5.4: What is the meaning of $|z_1 + z_2| > 3$*?*

Sol: Define $z_3 = z_1 + z_2$. Then the problem becomes $|z_3| > 3$, which is a disk of radius $3 > 0$. Thus the solution is all values of $z_1 + z_2$ outside, but not including, a circle of radius 3.

– 5.5: If time were complex, how might the world be different?

Sol: As best we know, time is real, thus it has the *order* property: the is a past, present and future. If time were complex this would not be the case. Thus if time were complex, the past could be the future.

Problem # *6: It is sometimes necessary to consider a function* $w(z) = u + v\jmath$ *in terms of the real functions* $u(x, y)$ *and* $v(x, y)$ *(e.g. separate the real and imaginary parts). Similarly, we can consider the inverse* $z(w) = x + y\jmath$*, where* $x(u, v)$ *and* $y(u, v)$ *are real functions.*

– 6.1: Find $u(x, y)$ *and* $v(x, y)$ *for* $w(z) = 1/z$.

Sol: Multiply by the complex conjugate $x - y\jmath$

$$w = \frac{1}{x + y\jmath} = \frac{x - y\jmath}{x^2 + y^2}$$

Therefore $u(x, y) = \frac{x}{x^2+y^2}$ and $v(x, y) = \frac{-y}{x^2+y^2}$.

Problem # *7: Find $u(x,y)$ and $v(x,y)$ for $w(z)=c^z$ with complex constant $c\in\mathbb{C}$ for questions* **??**, **??**, *and* **??**:

– *7.1:* $c=e$

Sol: Since $u+iv=e^z=e^{x+yj}=e^x(\cos y+j\sin y)$,

$$u=e^x\cos y$$

and

$$v=e^x\sin y.$$

– *7.2:* $c=1$ *(recall that* $1=e^{\pm j2\pi k}$ *for* $k\in\mathbb{Z}$

Sol: From the general formula with $c=1$

$$1^z=e^{z\log 1}\,e^{j k2\pi z}=e^0e^{jk2\pi z}=e^{-yk2\pi}e^{-jxk2\pi}$$

where $k\in\mathbb{Z}$ is a signed counting integer. Thus $u=e^{-k2\pi y}\cos k2\pi x$ and $v=e^{-k2\pi y}\sin k2\pi x$.

– *7.3:* $c=j$. Hint: $j=e^{j\pi/2+j2\pi k},\quad k\in\mathbb{Z}$.

Sol: $j^j=\left(e^{j\pi/2+j2\pi m}\right)^j=e^{jj\pi/2+jj2\pi m}=e^{-\pi/2}\,e^{-2\pi m}=0.2079\,e^{-2\pi m}$.

Thus for $m=0$, $j^z=\left(e^{j\pi/2}\right)^z=e^{jz\pi/2}=e^{j(x+jy)\pi/2}=e^{(jx-y)\pi/2}=e^{-\pi y/2}(\cos(x\pi/2)+j\sin(x\pi/2))$.

– *7.4: What is* j^j*?*

Sol: Since $j=e^{\frac{\pi}{2}j}$, then $j^j=e^{\frac{\pi}{2}jj}=e^{-\pi/2}\approx 0.20788$.

Expanding this in a continued-fraction expansion using Matlab's `rat(exp(-pi/2))` function gives $[0;5,-5,-4,3,-3,3,\cdots]$.

Schwarz inequality

Problem # *8: The above figure shows three vectors for an arbitrary value of $\alpha\in\mathbb{R}$ and a specific value of $\alpha=\alpha^*$.*

– *8.1: Find the value of $\alpha\in\mathbb{R}$ such that the length (norm) of $\vec{E}$ (i.e., $||\vec{E}||\geq 0$) is minimum. Show your derivation, not the answer ($\alpha=\alpha^*$).*

Sol: In the above figure we see vectors V, U, and for reference, $V+\alpha^*U$. Also shown are scaled values of U, αU and α^*U. The setup for the derivation is

$$||\boldsymbol{E}(a)||^2=\boldsymbol{E}\cdot\boldsymbol{E}=(\vec{V}+\alpha\vec{U})\cdot(\vec{V}+\alpha\vec{U})\geq 0. \qquad \text{(AE-3.4)}$$

Minimize with respect to α.

When U is scaled by α^*, length $||E(\alpha^*)||$ is minimum, and $(V-\alpha^*U)\perp U$, namely vector $E(\alpha^*)$ is $\perp$ to vector U. This follows from $\frac{\partial}{\partial\alpha}||\vec{E}||^2=\frac{\partial}{\partial\alpha}((\vec{V}+\alpha\vec{U})\cdot(\vec{V}+\alpha\vec{U}))=2(\vec{V}+\alpha\vec{U})\cdot\vec{U}=0$. Thus

$$\alpha^*=-\frac{\vec{V}\cdot\vec{U}}{||\vec{U}||^2}$$

– *8.2: Find the formula for $||\boldsymbol{E}(\alpha^*)||^2\geq 0$.* Hint: Substitute α^* into Eq. **??** (p. **??**) and show that this results in the Schwarz inequality

$$|\vec{U}\cdot\vec{V}|\leq||\vec{U}||||\vec{V}||.$$

Sol: From Eq. **??**

$$||\boldsymbol{V}||^2+2\alpha^*\boldsymbol{V}\cdot\boldsymbol{U}+(\alpha^*)^2\,||\boldsymbol{U}||^2\geq 0$$

Substituting α^* gives

$$||\boldsymbol{V}||^2||\boldsymbol{U}||^2 - 2(\boldsymbol{V}\cdot\boldsymbol{U})^2 + |\boldsymbol{U}\cdot\boldsymbol{V}|^2 \geq 0.$$

Simplifying

$$||\boldsymbol{V}||^2||\boldsymbol{U}||^2 \geq |\boldsymbol{U}\cdot\boldsymbol{V}|^2$$

and taking the square root (and swap order), gives the *Schwarz inequality*

$$|\vec{U}\cdot\vec{V}| \leq ||\vec{U}||||\vec{V}||.$$

Problem # *9: Geometry and scaler products*

– 9.1: What is the geometrical meaning of the dot product of two vectors?

Sol: The dot product of two vectors is the length of the $\perp$ projection of one vector on the other. According to the Schwarz inequality, this project length must be less than the product of the lengths of the two vectors.

– 9.2: Give the formula for the dot product of two vectors. Explain the meaning based on Fig. **??** *(page* **??***).*

Sol: $\vec{V}\cdot\vec{U} = ||\vec{V}||||\vec{U}||\cos\theta_{\vec{V},\vec{U}}$. $\vec{V}\cdot\vec{U} = ||\vec{V}||||\vec{U}||\cos\theta_{\vec{V},\vec{U}}$. It represents the amount of one vector going in the direction of the other. In a drawing, it is a projection of the one on the other, found by dropping the $\perp$ from the tip of one, on the other.

– 9.3: Write the formula for the dot product of two vectors $\vec{U}\cdot\vec{V}$ in $\mathbb{R}^n$ in polar form (e.g., assume the angle between the vectors is θ).

Sol: $\vec{U}\cdot\vec{V} = \sum_{i=1}^{n} a_i b_i (= ||\vec{U}||\ ||\vec{V}||\cos(\theta))$. This last relationship defines the angle between two vectors.

– 9.4: How is the Schwarz inequality related to the Pythagorean theorem?

Sol: It says that for a right triangle, the case when $a = a^*$, the lengths of the two vectors must be greater than the projection of one on the other, unless they are co-linear (i.e., the angle between them is zero).

– 9.5: Starting from $||\boldsymbol{U}+\boldsymbol{V}||$, derive the triangle inequality

$$||\vec{U}+\vec{V}|| \leq ||\vec{U}|| + ||\vec{V}||.$$

Sol: $||\vec{U}+\vec{V}||^2 = (\vec{U}+\vec{V})\cdot(\vec{U}+\vec{V}) = ||\boldsymbol{U}||^2 + ||\boldsymbol{V}||^2 + 2\boldsymbol{U}\cdot\boldsymbol{V} \leq ||\boldsymbol{U}||^2 + ||\boldsymbol{V}||^2 + 2|\boldsymbol{U}\cdot\boldsymbol{V}|$ Using the Schwarz inequality we find $||\vec{U}+\vec{V}||^2 \leq ||\boldsymbol{U}||^2 + ||\boldsymbol{V}||^2 + 2||\boldsymbol{U}||\ ||\boldsymbol{V}||$. Completing the square on the right gives $||\vec{U}+\vec{V}||^2 \leq (||\boldsymbol{U}|| + ||\boldsymbol{V}||)^2$. Final taking the square root gives the *triangle inequality*.

– 9.6: The triangle inequality $||\vec{U}+\vec{V}|| \leq ||\vec{U}|| + ||\vec{V}||$ *is true for two and three dimensions: Does it hold for five-dimensional vectors?*

Sol: It is true in any number of dimensions.

– 9.7: Show that the wedge product $\vec{U}\wedge\vec{V} \perp \vec{U}\cdot\vec{V}$.

Sol: $\vec{V}\wedge\vec{U} = ||\vec{V}||||\vec{U}||\sin\theta_{\vec{V},\vec{U}}$ while $\vec{V}\cdot\vec{U} = ||\vec{V}||||\vec{U}||\cos\theta_{\vec{V},\vec{U}}$. Thus they are perpendicular. This is true in any number of dimensions. See the discussion in the text on the wedge product.

0.7 System Classification

Problem # *10: Complete this system classification problem about physical systems using the system postulates.*

– *10.1: Provide a brief definition of these classificatioms*L/NL : linear (L)/nonlinear (NL) **Sol:** Superposition and scaling hold

TI/TV : time-invariant (TI)/time-varying (TV) **Sol:** The measurement time is irrelevant

P/A : passive (P)/active (A) **Sol:** An active system has a power source, a passive system does not.

C/NC : causal (C)/noncausal (NC) **Sol:** Responds only when driven for $t \geq 0$. Does not anticipate for negative time.

Re/Clx : real (Re)/complex (Clx) **Sol:** The time function is real (or complex).

– *10.2: Along the rows of the table, classify each system using the abbreviations L/NL, TI/TV, P/A, C/NC, and Re/Clx:*

					Category		
#	Case	Definition	L/NL	TI/TV	P/A	C/NC	Re/Clx
1	Resistor	$v(t) = r_0\, i(t)$	**Sol:** L	**Sol:** TI	**Sol:** P	**Sol:** C	**Sol:** Re
2	Inductor	$v(t) = L\frac{di}{dt}$	**Sol:** L	**Sol:** TI	**Sol:** P	**Sol:** C	**Sol:** Re
3	Switch	$v(t) \equiv \begin{cases} 0 & t \leq 0 \\ V_0 & t > 0. \end{cases}$	**Sol:** L	**Sol:** TV	**Sol:** P	**Sol:** C	**Sol:** Re
5	Transistor	$I_{out} = g_m(V_{in})$	**Sol:** NL	**Sol:** TI	**Sol:** P	**Sol:** C	**Sol:** Re
7	Resistor	$v(t) = r_0\, i(t+3)$	**Sol:** L	**Sol:** TI	**Sol:** P	**Sol:** NC	**Sol:** Re
8	Modulator	$f(t) = e^{i2\pi t} g(t)$	**Sol:** L	**Sol:** TV	**Sol:** P	**Sol:** C	**Sol:** Clx

Sol: Notes:

1. is a nonlinear system and is active system only when it is connected to a battery, similar to a diode.
2. The current is non-causal since it has a 3 [s] *negative time delay*, specified in the time domain.
3. is 1 Hz complex-modulation, so it is both complex and time-varying (TV)

– *10.3: Classify each equation:*

#	Case:	L/NL	TI/TV	P/A	C/NC	Re/Clx
1	$A(x)\frac{d^2y(t)}{dt^2} + D(t)y(x,t) = 0$	**Sol:** L	**Sol:** TV	**Sol:** P	**Sol:** C	**Sol:** Re
2	$\frac{dy(t)}{dt} + \sqrt{t}\, y(t) = \sin(t)$	**Sol:** L	**Sol:** TV	**Sol:** ?	**Sol:** C	**Sol:** Re
3	$y^2(t) + y(t) = \sin(t)$	**Sol:** NL	**Sol:** TI	**Sol:** ?	**Sol:** C	**Sol:** Re
4	$\frac{\partial^2 y}{\partial t^2} + xy(t+1) + x^2 y = 0$	**Sol:** L	**Sol:** TI	**Sol:** P	**Sol:** NC	**Sol:** Re
5	$\frac{dy(t)}{dt} + (t-1)\, y^2(t) = ie^t$	**Sol:** NL	**Sol:** TV	**Sol:** A?	**Sol:** C	**Sol:** Clx

Probability

Problem # *11: Basic terminology of experiments*

– *11.1: What is the* mean *of a trial, and what is the average over all trials?*
Sol: The mean and average are the same. What is different in these two questions is the size of the set being averaged.

– *11.2: What is the* expected value *of a random variable X?*
Sol: This is a mathematical definition of how to compute the mean, as an inner product.

– *11.3: What is the* standard deviation *about the mean?*
Sol: This is the expected value of the second moment of the random variable.

– *11.4: What is the definition of* information *of a random variable?*
Sol: The *information* is $\mathcal{I} = 1/P(X_k)$.

– *11.5: How do you combine events? Hint: If the event is the flip of a biased coin, the events are $H = p$, $T = 1 - p$, so the event is $\{p, 1-p\}$. To solve the problem, you must find the probabilities of two independent events.*
Sol: If one event has probability p it may be captured by a vector $[p, 1-p]^T$. Two uncorrelated (independent) events then have probability $[p, 1-p] \star [p, 1-p] = [p^2, 2p(1-p), (1-p^2)]$. Here $\star$ represents convolution (Section **??**, p. **??**). Three events have four outcomes $[p, 1-p] \star [p, 1-p] \star [p, -1-p]$. Pascal's triangle is a related structure defined by recursive convolutions of $[1, 1]$, assuming $p = 1/2$.

– *11.6: What does the term* independent *mean in the context of question* **??***? Give an example.*
Sol: This term means that one event (flip of a coin) has no influence on the next (or any other flip) of that same coin. An example of non-independent events might be that upon flipping the coin, it bent. thus changing the probability for any following flips.

– *11.7: Define* odds.
Sol: The *odds* are the ratio of the two outcomes. Namely the odds are $p/(1-p)$, or equivalently $(1-p)/p = 1/p - 1$.

Ad

0.8 Problems DE-1

0.8.1 Topics of this homework:

Complex numbers and functions (ordering and algebra), complex power series, fundamental theorem of calculus (real and complex); Cauchy-Riemann conditions, multivalued functions (branch cuts and Riemann sheets)

0.8.2 Complex Power Series

Problem # *1: In each case derive (e.g., using Taylor's formula) the power series of $w(s)$ about $s = 0$ and give the RoC of your series. If the power series doesn't exist, state why!* Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at $s = 0$.

– *1.1:* $1/(1-s)$
Sol: $1/(1-s) = \sum_{n=0}^{\infty} s^n$, which converges for $|s| < 1$ (e.g., the RoC is $|s| < 1$)

– *1.2:* $1/(1-s^2)$
Sol: $1/(1-s^2) = \sum_{n=0}^{\infty} s^{2n}$, which converges for $|s^2| < 1$. (e.g., the RoC is $|s| < 1$). One can also factor the polynomial, thus write it as: $\frac{1}{(1-s)(1+s)}$. There are two poles, at $s = \pm 1$, and each has an RoC of 1.

– *1.3:* $1/(1+s^2)$.
Sol: The resulting series is $1/(1+s^2) = 0.5 \sum_{n=0}^{\infty} s^n((-i)^n + (i)^n)$. The RoC is $|s| < 1$. We can see this by considering the poles of the function at $s = \pm i$; both poles are 1 from $s = 0$, the point of expansion. An alternative is to write the function as $1/(1-(is)^2) = \sum (is)^n$.

– *1.4:* $1/s$
Sol: If you try to do a Taylor expansion at $s = 0$, the first term, $w(0) \to \infty$. Thus, the Taylor series expansion in s does not exist.

– *1.5:* $1/(1-|s|^2)$
Sol: The imaginary part is zero. Thus the derivative of the imaginary part is zero. Thus the CR conditions cannot be obeyed.

Problem # *2: Consider the function $w(s) = 1/s$*

– 2.1: Expand this function as a power series about $s = 1$. Hint: Let $1/s = 1/(1-1+s) = 1/(1-(1-s))$.
Sol: The power series is

$$w(s) = \sum_{n=0}^{\infty} (-1)^n (s-1)^n,$$

which converges for $|s-1| < 1$.

To convince you this is correct, use the Matlab/Octave command `syms s; taylor(1/s,s,'ExpansionPoint',1)`, which is equivalent to the shorthand `syms s; taylor(1/s,s,1)`. What is missing is the logic behind this expansion, given as follows: First move the pole to $z = -1$ via the Möbius "translation" $s = z + 1$, and expand using the Taylor series

$$\frac{1}{s} = \frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n.$$

Next back-substitute $z = s - 1$ giving

$$\frac{1}{s} = \sum (-1)^n (s-1)^n.$$

It follows that the RoC is $|z| = |s-1| < 1$, as provided by Matlab/Octave.

– 2.2: What is the RoC?
Sol: As stated in the solution of 2.1, $|s-1| < 1$.

– 2.3: Expand $w(s) = 1/s$ *as a power series in* $s^{-1} = 1/s$ *about* $s^{-1} = 1$.
Sol: Let $z = s^{-1}$ and expand about 1: The solution is $w(z) = z$, which has a zero at 0 thus a pole at ∞.

– 2.4: What is the RoC?
Sol: $|s| > 0$ or $|z| < \infty$.

– 2.5: What is the residue of the pole?
Sol: The pole is at ∞. Since $w(s) = 1/s$ and applying the definition for the residue $c_{-1} = \lim_{s\to\infty} s(1/s) = 1$. Thus residue is 1. Note that it is the amplitude of the pole, which is 1.

Problem # *3: Consider the function* $w(s) = 1/(2-s)$

– 3.1: Expand $w(s)$ *as a power series in* $s^{-1} = 1/s$. *State the RoC as a condition on* $|s^{-1}|$. Hint: Multiply top and bottom by s^{-1}.
Sol: $1/(2-s) = -s^{-1}/(1-2s^{-1}) = -s^{-1}\sum 2^n s^{-n}$. The RoC is $|2/s| < 1$, or $|s| > 2$.

– 3.2: Find the inverse function $s(w)$. *Where are the poles and zeros of* $s(w)$, *and where is it analytic?*
Sol: Solving for $s(w)$ we find $2 - s = 1/w$ and $s = 2 - 1/w = (2w-1)/w$. This has a pole at 0 and a zero at $w = 1/2$. The RoC is therefore from the expansion point out to, but not including $w = 0$.

Problem # *4:Summing the series*
The Taylor series of functions have more than one region of convergence.

– 4.1: Given some function $f(x)$, *if* $a = 0.1$, *what is the value of*

$$f(a) = 1 + a + a^2 + a^3 + \cdots ?$$

Show your work. **Sol:** To sum this series, we may use the fact that

$$f(a) - af(a) = (1 + a + a^2 + a^3 + \cdots) - a(1 + a + a^2) = 1 + a(1-1) + a^2(1-1) + \cdots$$

This gives $(1-a)f(a) = 1$, or $f(a) = 1/(1-a)$. Now since $a = .1$, the sum is $1/(1-0.1) = 1.11$.

– 4.2: Let $a = 10$. *What is the value of*

$$f(a) = 1 + a + a^2 + a^3 + \cdots ?$$

Sol: In this case the series clearly does not converge. To make it converge we need to write a formula for $y = 1/x$ rather than for x.

$$f(1/y) - f(1/y)/a = (1 + 1/a + 1/a^2 + 1/a^3 + \cdots) - 1/a(1 + 1/a + a1/^2) = 1 + (1-1)/a + (1-1)/a^2 + \cdots$$

This gives $f(1/a) = -a^{-1}/(1-a^{-1})$. Now since $a = 10$, the series sums to $f(10) = -0.1/(1-0.1) = -1/9$.

0.8.3 Cauchy-Riemann Equations

Problem # *5: For this problem $\jmath = \sqrt{-1}$, $s = \sigma + \omega\jmath$, and $F(s) = u(\sigma,\omega) + \jmath v(\sigma,\omega)$. According to the fundamental theorem of complex calculus (FTCC), the integration of a complex–analytic function is independent of the path. It follows that the derivative of $F(s)$ is defined as*

$$\frac{dF}{ds} = \frac{d}{ds}\left[u(\sigma,\omega) + \jmath v(\sigma,\omega)\right]. \tag{DE-1.1}$$

If the integral is independent of the path, then the derivative must also be independent of the direction:

$$\frac{dF}{ds} = \frac{\partial F}{\partial \sigma} = \frac{\partial F}{\partial \jmath\omega}. \tag{DE-1.2}$$

The Cauchy-Riemann (CR) conditions

$$\frac{\partial u(\sigma,\omega)}{\partial \sigma} = \frac{\partial v(\sigma,\omega)}{\partial \omega} \quad \text{and} \quad \frac{\partial u(\sigma,\omega)}{\partial \omega} = -\frac{\partial v(\sigma,\omega)}{\partial \sigma}$$

may be used to show where Equation **??** holds.

*– 5.1: Assuming Equation **??** is true, use it to derive the CR equations.*

Sol: First form the partial derivatives as indicated and then set the real and imaginary parts equal. This results in the two CR equations.

– 5.2: Merge the CR equations to show that u and v obey Laplace's equations.

$$\nabla^2 u(\sigma,\omega) = 0 \quad \text{and} \quad \nabla^2 v(\sigma,\omega) = 0.$$

Sol: Take partial derivatives with respect to σ and ω and solve for one equation in each of u and v.

– 5.3: What can you conclude?

Sol: We can conclude that the real and imaginary parts of complex–analytic functions must obey these conditions.

Problem # *6: Apply the CR equations to the following functions. State for which values of $s = \sigma + i\omega$ the CR conditions do or do not hold (e.g., where the function $F(s)$ is or is not analytic).* Hint: Review where CR-1 and CR-2 hold.

– 6.1: $F(s) = e^s$

Sol: CR conditions hold everywhere.

– 6.2: $F(s) = 1/s$

Sol: CR conditions are violated at $s = 0$. The function is analytic everywhere except $s = 0$.

0.8.4 Branch cuts and Riemann sheets

Problem # *7: Consider the function $w^2(z) = z$. This function can also be written as $w_\pm(z) = \sqrt{z_\pm}$. Assume $z = re^{\phi\jmath}$ and $w(z) = \rho e^{\theta\jmath} = \sqrt{r}e^{\phi\jmath/2}$.*

– 7.1: How many Riemann sheets do you need in the domain *(z) and the* range *(w) to fully represent this function as single-valued?*

Sol: There is one sheet for z and two sheet for $w = \pm\sqrt{z}$. When any point in the domain z (being mapped to $w(z)$) crosses the z branch cut, the codomain (range) $w_\pm(z)$ switches from the w_+ sheet to the w_- sheet. $w(z)$ remains analytic on the cut. Look at Fig. **??** in Chap. **??** (p. **??**) to see how this works.

– 7.2: Indicate (e.g., using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.
Sol: Above we show the mapping for the square root function $w(z) = \sqrt{z_\pm} = \sqrt{r}e^{\jmath\phi/2}$.

– 7.3: Use `zviz.m` *to plot the positive and negative square roots $+\sqrt{z}$ and $-\sqrt{z}$. Describe what you see.*
Sol: The sheet for the positive root is shown in Fig 3.2 (page 106 of the Oct 24 version of the class notes.) Two view the two sheets use Matlab command `zviz sqrt(W) -sqrt(W)`.

– 7.4: Where does `zviz.m` *place the branch cut for this function?*
Sol: Typically the cut is placed along the negative real z axis $\phi = \pm\pi$. This is Matlab's/Octave's default location. In the figure above, it has been placed along the positive real axis, $\phi = 0 = 2\pi$.

– 7.5: Must the branch cut necessarily be in this location?
Sol: No, it can be moved, at will. It must start from $z = 0$ and end at $|z| \to \infty$. The cut may be move when using `zviz.m` by multiplying z by $e^{\jmath\phi_o}$. For example, `zviz W = sqrt(j*Z)` rotates the cut by $\pi/2$. The colors of $w(z)$ (angle maps to color) always 'jump' at the branch cut, as you make the transition across the cut.

Problem # *8: Consider the function $w(z) = \log(z)$. As in Problem* **??**, *let $z = re^{\phi\jmath}$ and $w(z) = \rho e^{\theta\jmath}$.*

– 8.1: Describe with a sketch and then discuss the branch cut *for $f(z)$.*
Sol: From the plot of `zviz w(z) = log(z)` of Lecture 18, we see a branch cut going from $w = 0$ to $w = -\infty$. If we express z in polar coordinates ($z = re^{\jmath\phi}$), then

$$w(z) = \log(r) + \phi\jmath = u(x,y) + v(x,y)\jmath,$$

where $r(x,y) = |z| = \sqrt{x^2+y^2}$ and $\phi = \angle z = \phi(x,y)$. Thus a zero in $w(z)$ appears at $z = 1 + 0\jmath$, and only appears on the principle sheet of z (between $[-\pi < \angle z = \phi < \pi]$), because this is the only place where $\phi = 0$. As the angle ϕ increases, the imaginary part of $w = \angle z$, which increases without bound. Thus w is like a spiral stair case, or cork-screw. If $\rho = 1$ and $\phi \neq 0$, $w(r) = log(1) + \phi\jmath$ is not zero, since the angle is not zero.

– 8.2: What is the inverse of the function $z(f)$? Does this function have a branch cut? If so, where is it?
Sol: $z(w) = e^w$ is a single valued function, so a branch cut is not appropriate. Only multi-valued functions require a branch cut.

– 8.3: Using `zviz.m`*, show that*

$$\tan^{-1}(z) = -\frac{\jmath}{2}\log\frac{\jmath - z}{\jmath + z}. \tag{DE-1.3}$$

Sol: Use the Matlab commands `zviz atan(Z)` and `zviz -(j/2)*log((j+Z)./(j-Z))`.

– 8.4: Algebraically justify Eq. **??**. Hint: Let $w(z) = \tan^{-1}(z)$ and $z(w) = \tan w = \sin w/\cos w$; then solve for $e^{w\jmath}$.
Sol: Following the hint gives

$$z(w) = -\jmath\frac{e^{w\jmath} - e^{-w\jmath}}{e^{w\jmath} + e^{-w\jmath}} = -\jmath\frac{e^{2w\jmath} - 1}{e^{2w\jmath} + 1}.$$

Solving this bilinear equation for $e^{2w\jmath}$ gives

$$e^{2w\jmath} = \frac{1 + z\jmath}{1 - z\jmath} = \frac{\jmath - z}{\jmath + z}$$

Taking the log and using our definition of $w(z)$ we find

$$w(z) = \tan^{-1}(z) = -\frac{\jmath}{2}\log\frac{\jmath - z}{\jmath + z}.$$

0.8.5 A Cauer synthesis of any Brune impedance

Problem # *9: One may synthesize a transmission line (ladder network) from a positive real impedance $Z(s)$ by using the* continued fraction algorithm **??**. *To obtain the series and shunt impedance values, we can use a residue expansion* **?**. *Here we shall explore this method.*

– 9.1: Starting from the Brune impedance $Z(s) = \frac{1}{s+1}$, find the impedance network as a ladder network.
Sol: Taking the reciprocal we find the sum of two shunt admittances, and capacitor and resistor

$$Y(s) = s + 1.$$

The the impedance is $Z(s) = 1/(s+1)$.

– 9.2: Use a residue expansion in place of the CFA floor function (Sec. **??**, *p.* **??**) *for polynomial expansions. Find the residue expansion of $H(s) = s^2/(s+1)$ and express it as a ladder network.*
Sol: Verify that

$$Z(s) = s^2/(s+1) = s - 1 + 1/(s+1). \tag{DE-1.4}$$

Thus the Cauer synthesis is a series combination $s - 1$ (an inductor $L = 1$ and a resistor $R = -1$ ohms) and a shunt $1||s$ (i.e., $Y(s) = 1 + s$, a resistor of $R = 1$ in parallel with a capacitor $C = 1$.) It would appear that $Z(s)$ is not a positive real impedance.

– 9.3: Discuss how the series impedance $Z(s, x)$ and shunt admittance $Y(s, x)$ determine the wave velocity $\kappa(s, x)$ and the characteristic impedance $z_o(s, x)$ when (1) $Z(s)$ and $Y(s)$ are both independent of x; (2) $Y(s)$ is independent of x, $Z(s, x)$ depends on x; (3) $Z(s)$ is independent of x, $Y(s, x)$ depends on x; and (4) both $Y(s, x), Z(s, x)$ depend on x.
Sol: In the most general case

$$z_o(s, x) = \sqrt{Z(s, x)/Y(s, x)}$$

and

$$\kappa(s, x) = \sqrt{Z(s, x)Y(s, x)}.$$

The general equations for $z_o(s, s)$ and $\kappa(s, x)$ are given in **?**, and discussed in Appendix **??** (p. **??**). When z_o and κ depend on x, the area function $A(x)$ of the WHEN will depend on x. Thus the eigenfunction will critically depend on the characteristic impedance and the propagation function.

For example, $\kappa(s)$ can be independent of the area because it cancels out in the product. This is called the case of *constant k* because the speed of sound is independent of the area function. It follows that the area function only depends on $z_o(s, x)$.

This shows that a Cauer synthesis may be implemented with the residue expansion replacing the floor function in the CFA. This seems to solve Brune's network synthesis problem.

0.9 Problems DE-2

0.9.1 Topics of this homework:

Integration of complex functions, Cauchy's theorem, integral formula, residue theorem, power series, Riemann sheets and branch cuts, inverse Laplace transforms, Quadratic forms.

Two fundamental theorems of calculus

According to the fundamental theorem of (real) calculus (FTC),

$$f(x) = f(a) + \int_a^x F(\xi)d\xi, \tag{DE-2.1}$$

where $x, a, \xi, F, f \in \mathbb{R}$. This is an indefinite integral (since the upper limit is unspecified). It follows that

$$\frac{df(x)}{dx} = \frac{d}{dx}\int_a^x F(x)dx = F(x).$$

This justifies also calling the indefinite integral the *antiderivative*.

For a closed interval $[a, b]$, the FTC is

$$\int_a^b F(x)dx = f(b) - f(a), \tag{DE-2.2}$$

thus the integral is independent of the path from $x = a$ to $x = b$.

Fundamental Theorem of Complex Calculus: According to the fundamental theorem of complex calculus (FTCC),

$$f(z) = f(z_0) + \int_{z_0}^z F(\zeta)d\zeta, \tag{DE-2.3}$$

where $z_0, z, \zeta, f, F \in \mathbb{C}$. It follows that

$$\frac{df(z)}{dz} = \frac{d}{dz}\int_{z_0}^z F(\zeta)d\zeta = F(z). \tag{DE-2.4}$$

For a closed interval $[s, s_o]$, the FTCC is

$$\int_{s_o}^s F(\zeta)d\zeta = f(s) - f(s_o), \tag{DE-2.5}$$

ck eq number thus the integral is independent of the path from $x = a$ to $x = b$.

Problem # *1*

– 1.1: Consider Equation **??**. *What is the condition on* $F(x)$ *for which this formula is true?*

Sol: The sufficient condition is that the integrand $F(x)$ is be analytic, namely $F(x) = \sum_{x=0}^{\infty} a_n x^n$. This assures that $F(x)$ is single valued and may be integrated, since it may integrated term by term. It follows that as long as $x <$ ROC, this integral exists. Thus the integral equals $F(x) - F(a)$. Note that if the integrand has a Taylor series, all of its derivatives exist within the ROC, because the coefficients depend on derivatives of $F(x)$.

– 1.2: Consider Equation **??**. *What is the condition on* $F(z)$ *for which this formula is true?*

Sol: The sufficient condition is that the integrand $F(z)$ must be complex analytic, namely $F(z) = \sum_{z=0}^{\infty} c_n z^n$, with $c \in \mathbb{C}$.

– 1.3: Let $F(z) = \sum_{k=0}^{\infty} c_k z^k$.

Sol: Applying term by term integration gives

$$I = \int_{\mathcal{C}} \sum_{k=0}^{\infty} c_k z^k dz = \sum_{k=0}^{\infty} c_k \int_{\mathcal{C}} z^k dz$$
$$= \sum_{k=0}^{\infty} \frac{c_k}{k+1} z^{k+1}$$

– 1.4: Let

$$F(z) = \frac{\sum_{k=0}^{\infty} c_k z^k}{z - \jmath}.$$

Sol: Applying term by term integration, and using CT-3, gives

$$I = \int_{\mathcal{C}} \frac{\sum_{k=0}^{\infty} c_k z^k}{z-\jmath} dz = \sum_{k=0}^{\infty} c_k \int_{\mathcal{C}} \frac{z^k}{z-\jmath} dz$$
$$= \begin{cases} 0 & z = \jmath \notin \mathcal{C} \\ \frac{1}{2\pi\jmath} \sum_{k=0}^{\infty} c_k \jmath^k & z = \jmath \in \mathcal{C} \end{cases}$$

Note:

$$\int_{\mathcal{C}} \frac{z^k}{z-\jmath} dz = \sum_{m=1}^{k} \frac{(\jmath z)^m}{m} + \ln(z-\jmath),$$

but the residue is $\jmath^k$ which saves the day.

Problem # *2: In the following problems, solve the integral*

$$I = \int_{\mathcal{C}} F(z) dz$$

for a given path $\mathcal{C} \in \mathbb{C}$.

– 2.1: Perform the following integrals ($z = x + iy \in \mathbb{C}$):
$I = \int_0^{1+\jmath} z dz$

Sol: $I = \frac{1}{2} z^2 \big|_0^{1+\jmath} = \frac{1}{2}(1+\jmath)^2 = \frac{1}{2}(1-1+2\jmath) = \jmath$

– 2.2: $I = \int_0^{1+j} z dz$, but this time make the path explicit: from 0 to 1, with $y = 0$, and then to $y = 1$, with $x = 1$.

Sol: Some text to start the ball rolling

$$I = \int_{x=0}^{1} (x + 0\jmath)\, dx + \int_{y=0}^{1} (1 + y\jmath)\ dy\, \jmath$$
$$I = \frac{1}{2}\, x^2 \big|_0^1 + \int_{y=0}^{1} (\jmath - y)\, dy$$
$$= \frac{1}{2} + \left(y\jmath - \frac{1}{2} y^2 \right) \Big|_0^1$$
$$= \frac{1}{2} + j - \frac{1}{2}$$
$$= j$$

We conclude that the integration of z is independent of the path. This is true for any integrand z^n with $n \in \mathbb{Z}$.

– 2.3: Discuss whether your results agree with Eq. **??***?*

Sol: Yes the two integrals must agree, because the function is analytic, and the integral must be the same, independent of the path.

Problem # *3: Perform the following integrals on the closed path $\mathcal{C}$, which we define to be the unit circle. You should substitute $z = e^{i\theta}$ and $dz = ie^{i\theta} d\theta$, and integrate from $\{-\pi, \pi\}$ to go once around the unit circle.*

Discuss whether your results agree with Eq. **??***?*

– 3.1: $\int_{\mathcal{C}} z dz$

Sol: $\int_{\mathcal{C}} z dz = \int_{-\pi}^{\pi} e^{i\theta} de^{i\theta} = \int_{-\pi}^{\pi} e^{i2\theta} i d\theta = e^{i2\theta}|_{-\pi}^{\pi} = 0.$
This example obeys the FTCC because $f(z) = z$ is analytic everywhere;

– 3.2: $\int_{\mathcal{C}} \frac{1}{z} dz$

Sol: $\int_{-\pi}^{\pi} i d\theta = 2\pi i$.
This example does not obey FTCC because $f(z) = 1/z$ is not analytic at $z = 0$ (inside C), instead it satisfies CT-2;

– 3.3: $\int_{\mathcal{C}} \frac{1}{z^2} dz$

Sol: $\int_{\mathcal{C}} \frac{dz}{z^2} = \int_{-\pi}^{\pi} e^{-i2\theta} de^{i\theta} = \int_{-\pi}^{\pi} e^{-i\theta} i d\theta = -\frac{e^{-i\theta}}{\theta}|_{-\pi}^{\pi} = -\frac{e^{-i\pi}}{\pi} + \frac{e^{i\pi}}{\pi} = \frac{1-1}{\pi} = 0.$
This example obeys the FTCC because the residue is $-e^{-i\theta}/\theta$, and the loop is closed (starting and ending points are the same).

– 3.4: $I = \int_{\mathcal{C}} \frac{1}{(z+2\jmath)^2} dz$.

Recall that the path of integration is the unit circle, starting and ending at -1.

Sol: Let $\zeta = z + 2\jmath$, then the limits become $[-1 + 2\jmath, 1 + 2\jmath] = [2\jmath + e^{-\jmath\pi}, 2\jmath + e^{\jmath\pi + 2\jmath}]$.

$$I = \int_{\mathcal{C}} \frac{d\zeta}{\zeta^2} = \int_{-\pi}^{\pi} e^{-i2\theta} de^{i\theta} = \int_{-\pi}^{\pi} e^{-i\theta} i d\theta = -\frac{e^{-i\theta}}{\theta}|_{-\pi}^{\pi} = -\frac{e^{-i\pi}}{\pi} + \frac{e^{i\pi}}{\pi} = \frac{1-1}{\pi} = 0.$$

This example reduces to the case of (3), and therefore must have the same conclusion as (3). But in this case the reasoning is different because the second order pole (singular point) is outside the unit circle, thus the function is analytic inside $\mathcal{C}$, so CT-1 applies.

Problem # *4: FTCC and integration in the complex plane*
Let the function $F(z) = c^z$, where $c \in \mathbb{C}$ is given for each question. *Hint: Can you apply the FTCC?*

– 4.1: For the function $f(z) = c^z$, where $c \in \mathbb{C}$ is an arbitrary complex constant, use the Cauchy-Riemann (CR) equations to show that $f(z)$ is analytic for all $z \in \mathbb{C}$.

Sol: We may rewrite this function as $f(z) = e^{\ln(c)z}$, where $z = x + iy$ and $f = u + iv$. Thus

$$u(x, y) = e^{\ln(c)x} \cos(\ln(c)y),$$
$$v(x, y) = e^{\ln(c)x} \sin(\ln(c)y)$$
$$\frac{\partial u}{\partial x} = \ln(c)\, e^{\ln(c)x} \cos(\ln(c)y) = \frac{\partial v}{\partial y} = \ln(c)\, e^{\ln(c)x} \cos(\ln(c)y)$$

$$\frac{\partial u}{\partial y} = -\ln(c)\, e^{\ln(c)x} \sin(\ln(c)y) = -\frac{\partial v}{\partial x} = -\ln(c)\, e^{\ln(c)x} \sin(\ln(c)y)$$

Thus the CR conditions are satisfied everywhere and the function is analytic for all $z \in \mathbb{C}$.

– 4.2: Find the antiderivative of $F(z)$.

Sol: Since $c^z = e^{\ln(c)z}$, the indefinite integral (anti-derivative) is

$$I(z) = \frac{1}{\ln c} e^{\ln(c)z} \quad \text{since} \quad \frac{d}{dz} I(z) = \frac{d}{dz} \frac{1}{\ln c} e^{\ln(c)z} = e^{\ln(c)z} = F(z).$$

– 4.3: $c = 1/e = 1/2.7183, \ldots$ *where $\mathcal{C}$ is* $\zeta = 0 \to i \to z$

Sol: The integrand is $F(z) = e^{-z}$, which is *entire*. Thus the the integral is independent of the path (i.e., $\mathcal{C}$ is not relevant to the final answer).

$$I(z) = \int_0^i e^{-\zeta} d\zeta + \int_i^z e^{-\zeta} d\zeta = F(z) - F(i) + F(i) - F(0)$$
$$= \int_0^z e^{-\zeta} d\zeta = -e^z \Big|_0^z = -(e^{-z} - 1)$$

– 4.4: $c = 2$, where $\mathcal{C}$ is $\zeta = 0 \to (1+i) \to z$

Sol: The integrand is $F(z) = 2^z$, where $2 = e^{\ln 2}$. The path $\mathcal{C}$ is not relevant to the final answer.

$$I(z) = \int_0^z 2^\zeta d\zeta = \int_0^z e^{\zeta \ln 2} d\zeta = \frac{e^{\zeta \ln 2}}{\ln 2}\Big|_0^z = (e^{z \ln 2} - 1)/\ln 2 \approx 1.443(e^{0.693z} - 1)$$

– 4.5: $c = i$, where the path $\mathcal{C}$ is an inward spiral described by $z(t) = 0.99^t e^{i2\pi t}$ for $t = 0 \to t_0 \to \infty$

Sol: $i = e^{i\pi/2} e^{i2\pi n}$. We have already proved that the path doesn't matter for any $F(z) = c^z$, so we just need to evaluated $z(t)$ for $t = 0$ and $t \to \infty$. This gives $z(0) = 1$ and $z(t \to \infty) = 0$.

$$I = \int_{z(0)}^{z(t\to\infty)} i^z dz = \int_{z(0)}^{z(t\to\infty)} e^{i\pi z/2} dz = \frac{2e^{i\pi z/2}}{i\pi}\Big|_1^0 = \frac{2}{i\pi}(1 - e^{i\pi/2}) = \frac{-2(i+1)}{\pi}$$

– 4.6: $c = e^{t-\tau_0}$, where $\tau_0 > 0$ is a real number and $\mathcal{C}$ is $z = (1 - i\infty) \to (1 + i\infty)$. Hint: Do you recognize this integral? If you do not, please do not spend a lot of time trying to solve it via the "brute force" method.

Sol: This is the basically the inverse Laplace transform of $e^{-\tau_0 z}$, we are just missing the scale factor $\frac{1}{2\pi i}$.

$$I(t) = \int_{1-i\infty}^{1+i\infty} e^{(t-\tau_0)z} dz = \int_{1-i\infty}^{1+i\infty} e^{-\tau_0 z} e^{zt} dz = 2\pi i \delta(t - \tau_0)$$

0.9.2 Cauchy's theorems CT-1, CT-2, CT-3

There are three basic definitions related to Cauchy's integral formula. They are all related and can greatly simplify integration in the complex plane. When a function depends on a complex variable, we use uppercase notation, consistent with the engineering literature for the Laplace transform.

Problem # *5: Describe the relationships between the theorems:*

– 5.1: CT-1 and CT-2

Sol: When z_0 falls outside of $\mathcal{C}$, CT-2 reduces to CT-1.

– 5.2: CT-1 and CT-3

Sol: When there are no poles inside $\mathcal{C}$, all the residues are zero, and CT-3 reduces to CT-1.

– 5.3: CT-2 and CT-3

Sol: Case CT-2 has only one induced pole at $z = z_0$, having residue $F(z_0)$. Thus CT-3 is the same as CT-2 when $K = 1$, the pole at z_0 is within contour $\mathcal{C}$, and the single residue is $F(z_0)$.

– 5.4: Consider the function with poles at $z = \pm j$,

$$F(z) = \frac{1}{1+z^2} = \frac{1}{(z-j)(z+j)}.$$

Find the residue expansion.

Sol:

$$F(z) = \frac{j}{2}\Big(\frac{1}{z+j} - \frac{1}{z-j}\Big).$$

Problem # *6: Apply Cauchy's theorems to solve the following integrals. State which theorem(s) you used and show your work.*

– *6.1:* $\oint_{\mathcal{C}} F(z)dz$, *where* $\mathcal{C}$ *is a circle centered at* $z = 0$ *with a radius of* $\frac{1}{2}$

Sol: Because the contour $\mathcal{C}$ does not include the poles, $F(z)$ is analytic everywhere inside $\mathcal{C}$). Using *Cauchy's integral theorem*, the integral is 0.

– *6.2:* $\oint_{\mathcal{C}} F(z)dz$, *where* $\mathcal{C}$ *is a circle centered at* $z = j$ *with a radius of* 1

Sol: Since we only enclose the pole at $z = j$, use the *integral formula* with $F(z) = 1/(z + j)$:

$$\oint_{\mathcal{C}} \frac{F(z)}{z-j}dz = 2\pi j \text{Res}_j = 2\pi j\Big[\frac{1}{z+j}\Big]_{z=j} = 2\pi j\frac{1}{2j} = \pi$$

– *6.3:* $\oint_{\mathcal{C}} F(z)dz$, *where* $\mathcal{C}$ *is a circle centered at* $z = 0$ *with a radius of* 2

Sol: Since we enclose both poles, using the *residue theorem*:

$$\oint_{\mathcal{C}} F(z)dz = 2\pi j(\text{Res}_j + \text{Res}_{-j}) = 2\pi j\Big(\frac{1}{2j} - \frac{1}{2j}\Big) = 0$$

As a side note, the inverse Laplace transform for $F(z)$ is $\sin(t)$, which is zero for $t = 0$, consistent with this result.

Integration of analytic functions

Problem # *7: In the following questions, you'll be asked to integrate* $F(s) = u(\sigma, \omega) + iv(\sigma, \omega)$ *around the contour* $\mathcal{C}$ *for complex* $s = \sigma + i\omega$,

$$\oint_{\mathcal{C}} F(s)ds. \tag{DE-2.6}$$

Follow the directions carefully for each question. When asked to state where the function is and is not analytic, you are not required to use the Cauchy-Riemann equations

– *7.1:* $F(s) = \sin(s)$

Sol: Analytic everywhere. $-\cos(s) = \int_{\theta=0}^{2\pi} \sin(s)ds = 0$. This function is *entire* (i.e., has no poles) so the integral must be zero.

– *7.2: Given function* $F(s) = \frac{1}{s}$ *State where the function is and is not analytic.*

Sol: Analytic everywhere except at $s = 0$, where it has a pole.

– *7.3: Explicitly evaluate the integral when* $\mathcal{C}$ *is the unit circle, defined as* $s = e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

Sol:

$$\oint_{\mathcal{C}} F(s)ds = \int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta}d\theta = \int_0^{2\pi} id\theta = 2\pi i$$

– *7.4: Evaluate the same integral using Cauchy's theorem and/or the residue theorem.*

Sol: The residue is 1 so the integral is $2\pi i$.

– *7.5:* $F(s) = \frac{1}{s^2}$ *State where the function is and is not analytic.*

Sol: Analytic everywhere except at $s = 0$, where it has a 2^{nd} order pole.

– *7.6: Explicitly evaluate the integral when* $\mathcal{C}$ *is the unit circle, defined as* $s = e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

Sol:

$$\oint_{\mathcal{C}} F(s)ds = \int_0^{2\pi} \frac{1}{e^{i2\theta}} ie^{i\theta}d\theta = \int_0^{2\pi} ie^{-i\theta}d\theta = i\frac{1}{-i}e^{-i\theta}\Big|_0^{2\pi} = 1(e^{-i2\pi} - e^0) = 0$$

– 7.7: What does your result imply about the residue of the second-order pole at $s = 0$?
Sol: The residue is 0.

– 7.8: $F(s) = e^{st}$: State where the function is and is not analytic.
Sol: Analytic everywhere.

– 7.9: Explicitly evaluate the integral when $\mathcal{C}$ is the square $(\sigma, \omega) = (1, 1) \to (-1, 1) \to (-1, -1) \to (1, -1) \to (1, 1)$.
Sol: When you perform this integral piece-wise, you will find that all terms cancel out and the result is 0.

– 7.10: Evaluate the same integral using Cauchy's theorem and/or the residue theorem.
Sol: The function is analytic everywhere, so the integral is 0 by Cauchy's theorem.

– 7.11: $F(s) = \frac{1}{s+2}$: State where the function is and is not analytic.
Sol: Analytic everywhere except at $s = -2$, where it has a pole.

– 7.12: Let $\mathcal{C}$ be the unit circle, defined as $s = e^{i\theta}$, $0 \leq \theta \leq 2\pi$. Evaluate the integral using Cauchy's theorem and/or the residue theorem.

Sol: The function is analytic everywhere inside $\mathcal{C}$, so the integral is 0 by Cauchy's theorem.

– 7.13: Let $\mathcal{C}$ be a circle of radius 3, defined as $s = 3e^{i\theta}$, $0 \leq \theta \leq 2\pi$. Evaluate the integral using Cauchy's theorem and/or the residue theorem.

Sol: This contour contains the pole. The residue is 1, therefore the integral is equal to $2\pi i$.

– 7.14: $F(s) = \frac{1}{2\pi i}\frac{e^{st}}{(s+4)}$ State where the function is and is not analytic.
Sol: Analytic everywhere except at $s = -4$, where it has a pole.

– 7.15: Let $\mathcal{C}$ be a circle of radius 3, defined as $s = 3e^{i\theta}$, $0 \leq \theta \leq 2\pi$. Evaluate the integral using Cauchy's theorem and/or the residue theorem.

Sol: This contour contains the pole. The residue is $\frac{1}{2\pi i}e^{-2t}$, therefore the integral is equal to e^{-2t}.

– 7.16: Let $\mathcal{C}$ contain the entire left half s plane. Evaluate the integral using Cauchy's theorem and/or the residue theorem. Do you recognize this integral?
Sol: This contour contains the pole. The residue is $\frac{1}{2\pi i}e^{-2t}$, therefore the integral is equal to e^{-2t}. This contour is the inverse Laplace transform.

– 7.17: $F(s) = \pm\frac{1}{\sqrt{s}}$ (e.g., $F^2 = \frac{1}{s}$) State where the function is and is not analytic.
Sol: Analytic everywhere except $s = 0$, where there is a pole.

– 7.18: This function is multivalued. How many Riemann sheets do you need in the domain (s) and the range (f) to fully represent this function? Indicate (e.g., using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.

Sol: There are 2 sheets in the domain (for the $\pm$ square root) which map to 1 sheet in the range.

– 7.19: Explicitly evaluate the integral $\int_{\mathcal{C}} \frac{1}{\sqrt{z}} dz$ when $\mathcal{C}$ is the unit circle, defined as $s = e^{i\theta}$, $0 \leq \theta \leq 2\pi$. Is this contour closed? State why or why not.

Sol: The solution is

$$2\sqrt{z}\Big|_{\theta=0}^{2\pi} = 2e^{j\theta/2}\Big|_{0}^{2\pi} = 2(e^{j\pi} - e^{0}) = -4.$$

In polar coordinates

$$\begin{aligned}\int_0^{2\pi} \frac{ds}{\sqrt{s}} &= \int_0^{2\pi} \frac{de^{i\theta}}{e^{i\theta/2}} \\ &= i\int_0^{2\pi} \frac{e^{i\theta}}{e^{i\theta/2}} d\theta \\ &= i\int_0^{2\pi} e^{i\theta/2} d\theta \\ &= 2\, e^{i\theta/2}\Big|_0^{2\pi} \\ &= 2[e^{i\pi} - 1] = 2(-2) = -4.\end{aligned}$$

This contour is *not* closed. One way to determine this is to see if going once around the unit circle returns $F(s)$ to its original value.

$$F(e^{i0}) = 1 \neq F(e^{i2\pi}) = e^{-i\pi} = -1.$$

– 7.20: Explicitly evaluate the integral $\int_{\mathcal{C}} \frac{1}{\sqrt{z}} dz$ *when* $\mathcal{C}$ *is* twice *around the unit circle, defined as* $s = e^{i\theta}$, $0 \leq \theta \leq 4\pi$. *Is this contour closed? State why or why not. Hint: Note that* $\sqrt{e^{i(\theta+2\pi)}} = \sqrt{e^{i2\pi}e^{i\theta}} = e^{i\pi}\sqrt{e^{i\theta}} = -1\sqrt{e^{i\theta}}$

Sol:

$$\begin{aligned}\int_0^{4\pi} \frac{ds}{\sqrt{s}} &= \int_0^{4\pi} \frac{de^{i\theta}}{e^{i\theta/2}} \\ &= i\int_0^{4\pi} \frac{e^{i\theta}}{e^{i\theta/2}} d\theta \\ &= i\int_0^{4\pi} e^{i\theta/2} d\theta \\ &= 2\, e^{i\theta/2}\Big|_0^{4\pi} \\ &= 2[e^{i2\pi} - 1] = 2(0) = 0.\end{aligned}$$

This contour *is* closed. One way to determine this is to see if going twice around the unit circle returns $F(s)$ to its original value. $F(e^{i0}) = 1 = F(e^{i4\pi}) = e^{-i2\pi} = 1$.

– 7.21: What does your result imply about the residue of the (twice-around $\frac{1}{2}$ *order) pole at* $s = 0$*?*

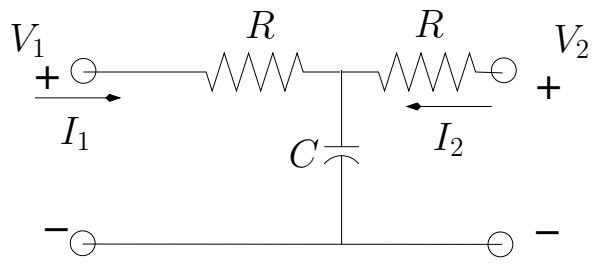
Sol: The residue is 0.

– 7.22: Show that the residue is zero. Hint: Apply the definition of the residue.

Sol: $c_{-1} = \lim_{z\to z_k} z/\sqrt{z} = \lim_{z\to z_k} \sqrt{z} = 0$.

0.9.3 Laplace transform applications

Problem # *8: A two-port network application for the Laplace transform*



This three-element electrical circuit is a system that acts to low-pass filter the signal voltage $V_1(\omega)$, to produce signal $V_2(\omega)$. It is convenient to define the dimensionless ratio $s/s_c = RCs$ in terms of a time constant $\tau = RC$ and cutoff frequency $s_c = 1/\tau$.

0.9.4 Computer exercises with Matlab/Octave

Problem # *9: With the help of a computer*

Now we look at a few important concepts using Matlab/Octave's `syms` commands or Wolfram Alpha's symbolic math toolbox.[7]

For example, to find the Taylor series expansion about $s = 0$ of

$$F(s) = -\log(1-s),$$

we first consider the derivative and its Taylor series (about $s = 0$)

$$F'(s) = \frac{1}{1-s} = \sum_{n=0}^{\infty} s^n.$$

Then, we integrate this series term by term:

$$F(s) = -\log(1-s) = \int^s F'(s)ds = \sum_{n=0}^{\infty} \frac{s^n}{n}.$$

Alternatively we can use Matlab/Octave commands:

```
syms s
taylor(-log(1-s),'order',7)
```

– 9.1: Use Octave's `taylor(-log(1-s))` to the seventh order, as in the example above. Try the above Matlab/Octave commands. Give the first seven terms of the Taylor series (confirm that Matlab/Octave agrees with the formula derived above).

Sol:

$$F(s) = \cdots + \frac{s^7}{7} + \frac{s^6}{6} + \frac{s^5}{5} + \frac{s^4}{4} + \frac{s^3}{3} + \frac{s^2}{2} + s$$

– 9.2: What is the inverse Laplace transform of this series? Consider the series term by term.

Sol: $f(t) = \sum \delta^{(n)}/n$

– 9.3: The function $1/\sqrt{z}$ has a branch point at $z = 0$; thus it is singular there. Can you apply Cauchy's integral theorem when integrating around the unit circle?

Sol: No, one cannot apply the Cauchy Theorem since it is not analytic at $z = 0$. But the integral may be evaluated.

– 9.4: This Matlab/Octave code computes $\int_0^{4\pi} \frac{dz}{\sqrt{z}}$ using Matlab's/Octave's symbolic analysis package. Run the following script:

```
syms  z
I=int(1/sqrt(z))
J = int(1/sqrt(z),exp(-j*pi),exp(j*pi))
eval(J)
```

What answers do you get for I and J?

Sol: This script returns the answers $I = 2 * \sqrt{z}$ and $J = 2.4493e-16$, which is numerically the same as zero.

[7] https://www.wolframalpha.com/

– 9.5: Modify this code to integrate $f(z) = 1/z^2$ once *around the unit circle. What answers do you get for* I *and* J*?*

Sol: This function has a 2d order pole at $s = 0$. Thus from the CIT, the integral evaluates to zero.

Proof:

$$I = \oint \frac{ds}{s^2} = -\left.\frac{1}{s}\right|_0^{2\pi} = -\left.e^{-i\theta}\right|_0^{2\pi} = -(1-1) = 0$$

More generally $I = \oint \frac{ds}{s^n} = 0$ for $n \neq 1$. As best I know, this holds for any $n \in \mathbb{Z}, \mathbb{Q}, \mathbb{F}, \mathbb{R}, \mathbb{C}$. For $n = 1$ it has a value of $2\pi j$.

– 9.6: Bessel functions can describe waves in a cylindrical geometry.

The Bessel function has a Laplace transform with a branch cut

$$J_0(t)u(t) \leftrightarrow \frac{1}{\sqrt{1+s^2}}.$$

Draw a hand sketch showing the nature of the branch cut. Hint: Use `zviz`. **Sol:** The roots are given by $s_\pm = \pm j$. The branch cut connects the two roots, or can go from each root to ∞. Either choice is valid.

Problem # *10: Matlab/Octave exercises*

– 10.1: Try the following Matlab/Octave commands, and then comment on your findings.

```
%Take the inverse LT of 1/sqrt(1+s^2)
syms s
I=ilaplace(1/(sqrt((1+s^2))));
disp(I)
```

Sol: $I = J_o(t)u(t)$.

```
%Find the Taylor series of the LT
T = taylor(1/sqrt(1+s^2),10); disp(T);
```

Sol:

$$T = \cdots + \frac{3s^4}{8} - \frac{s^2}{2} + 1$$

```
%Verify this
syms t
J=laplace(besselj(0,t));
disp(J);
```

Sol: $I = \frac{1}{\sqrt{1+s^2}}$.

```
%plot the Bessel function
t=0:0.1:10*pi;
b=besselj(0,t);
plot(t/pi,b);
grid on;
```

Sol: Plot of $J_o(t)u(t)$.

– 10.2: When did Friedrich Bessel live?

Sol: 1784–1846, in Königsberg, Germany.

– 10.3: What did he use Bessel functions for?

Sol: Solving the Bessel equation, which is the wave equation in 2D. Bessel functions were first introduced by the Daniel Bernoulli.

Problem # *11: Colorized plots of analytic functions. Use* `zviz` *for each of the following.*

– 11.1: Describe the plot generated by `zviz S=Z`.

Sol: It is a polar plot of the function, with intensity coding the magnitude and color coding the phase. Red is a positive real number while and blue is a negative real number.

– 11.2: Describe the plot generated by `zviz` $1./sqrt(1 + S.^2)$.

Sol: No. The RHP has blue near the branch cut, in the RHP.

– 11.3: Describe the plot generated by `zviz` $1./sqrt(1 - S.^2)$. *Is this function a Brune impedance (i.e., does this function obey*

Sol: NO, there is a branch cut in the RHP.

– 11.4: `zviz` $1./(1 + sqrt(S))$

Sol: Yes, it's red almost everywhere even though it has a branch cut from $[-\infty < \sigma \leq -10]$. Since $1/\sqrt{s}$ has an $\mathcal{LT}^{-1}$, this function must as well. Matlab found

$$\frac{1}{\sqrt{1+s}} \leftrightarrow \frac{e^{-t}}{\sqrt{\pi}\sqrt{t}}u(t),$$

however Octave failed to find the inverse transform, (but was able to find the forward transform).

0.9.5 Inverse of Riemann $\zeta(s)$ function

Problem # *12: Inverse zeta function (This problem is for extra credit).*

– 12.1: Find the $\mathcal{LT}^{-1}$ of one factor of the Riemann zeta function $\zeta_p(s)$, where $\zeta_p(s) \leftrightarrow z_p(t)$. Describe your results in words. Hint: Consider the geometric series representation

$$\zeta_p(s) = \frac{1}{1 - e^{-sT_p}} = \sum_{k=0}^{\infty} e^{-skT_p}, \tag{DE-2.7}$$

for which you can look up the $\mathcal{LT}^{-1}$ of each term.

Sol: Since each term in the series is a pure delay[8]

$$z_p(t) = \delta(t))_{T_p} \equiv \sum_{k=0}^{\infty} \delta(t - kT_p) \leftrightarrow \frac{1}{1 - e^{-sT_p}}. \tag{DE-2.8}$$

Problem # *13: Inverse transform of products:*
The time-domain version of Eq. **??** may be written as the convolution of all the $z_k(t)$ factors:

$$z(t) \equiv z_2(t) \star z_3(t) \star z_5(t) \star z_7(t) \star \cdots \star z_p(t) \star \cdots, \tag{DE-2.9}$$

where $\star$ represents time convolution.

Explain what this means in physical terms. Start with two terms (e.g., $z_1(t) \star z_2$). Hint: The input admittance of this cascade may be interpreted as the analytic continuation of $\zeta(s)$ by defining a cascade of eigenfunctions with eigenvalues derived from the primes. For a discussion of this idea see Sec. **??** and **??**.

Sol: In terms of the physics, these transmission line equations are telling us that $\zeta(s)$ may be decomposed into an infinite cascade of transmission lines, each having a delay given by $T_p = \ln \pi_p$.

Physical interpretation: Such functions may be generated in the time domain, as shown in Fig. **??**, using a feedback delay of T_p seconds, described by the two equations in the Fig. **??** with a unity feedback gain $\alpha = -1$. Taking the Laplace transform of the system equation, we see that the transfer function between the state variable $q(t)$ and the input $x(t)$ is given by $\zeta_p(s)$, which is an all-pole function, since

$$Q(s) = e^{-sT_n} Q(s) + V(s), \text{ or } \zeta_p(s) \equiv \frac{Q(s)}{V(s)} = \frac{1}{1 - e^{-sT_p}}. \tag{DE-2.10}$$

Closing the feed-forward path gives a second transfer function $Y(s) = I(s)/V(s)$—namely,

$$Y(s) \equiv \frac{I(s)}{V(s)} = \frac{1 - e^{-sT_p}}{1 + e^{-sT_p}}. \tag{DE-2.11}$$

If we take $i(t)$ as the current and $v(t)$ as the voltage at the input to the transmission line, then $y_p(t) \leftrightarrow \zeta_p(s)$ represents the input impedance at the input to the line. The poles and zeros of the impedance interleave along the $j\omega$ axis. By a slight modification, $\zeta_p(s)$ may alternatively be written as

$$Y_p(s) = \frac{e^{sT_p/2} + e^{-sT_p/2}}{e^{sT_p/2} - e^{-sT_p/2}} = j \tan(sT_p/2). \tag{DE-2.12}$$

Every impedance $Z(s)$ has a corresponding *reflectance* function given by a Möbius transformation, which may be read off of Eq. **??** as

$$\Gamma(s) \equiv \frac{1 + Z(s)}{1 - Z(s)} = e^{-sT_p}, \tag{DE-2.13}$$

since impedance is also related to the round-trip delay T_p on the line. The inverse Laplace transform of $\Gamma(s)$ is the round-trip delay T_p on the line

$$\gamma(t) = \delta(t - T_p) \leftrightarrow e^{-sT_p}. \tag{DE-2.14}$$

Working in the time domain provides a key insight, as it allows us to parse out the best analytic continuation of the infinity of possible continuations that are not obvious in the frequency domain (See Sec. **??**). Transforming to the time domain is a form of analytic continuation of $\zeta(s)$ that depends on the assumption that $Z^{eta}(t) \leftrightarrow \zeta(s)$ is one-sided in time (causal).

0.9.6 Quadratic forms

A matrix that has positive eigenvalues is said to be positive-definite. The eigenvalues are real if the matrix is symmetric, so this is a necessary condition for the matrix to be positive-definite. This condition is related to conservation of energy, since the power is the voltage times the current. Given an impedance matrix

$$\mathbf{V} = \mathcal{Z}\mathbf{I},$$

[8] Here we use a shorthand double-parentheses notation to define the infinite (one-sided) sum $f(t))_T \equiv \sum_{k=0}^{\infty} f(t - kT)$.

the power $\mathcal{P}$ is

$$\mathcal{P} = \mathbf{I} \cdot \mathbf{V} = \mathbf{I} \cdot \mathcal{Z}\mathbf{I},$$

which must be positive-definite for the system to obey conservation of energy.

Problem # *14: In this problem, consider the* 2×2 *impedance matrix*

$$\mathcal{Z} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}.$$

– 14.1: Solve for the power $\mathcal{P}(i_1, i_2)$ *by multiplying out this matrix equation (which is a* quadratic form*):*

$$\mathcal{P}(i_1, i_2) = \mathbf{I}^T \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \mathbf{I}.$$

Sol:

$$\mathcal{P}(i_1, i_2) = \begin{bmatrix} i_1 & i_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} i_1 & i_2 \end{bmatrix} \begin{bmatrix} 2i_1 + i_2 \\ i_1 + 4i_2 \end{bmatrix} = 2i_1^2 + 2i_1 i_2 + 4i_2^2.$$

– 14.2: Is the impedance matrix positive-definite? Show your work by finding the eigenvalues of the matrix **Z**.

Sol: Yes, as it is positive-definite if the eigenvalues are both positive. You need to show that the eigenvalues are positive (not zero or negative). They are, so it is.

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 3 \pm \sqrt{2} > 0$$

– 14.3: Should an impedance matrix always be positive-definite? Explain.

Sol: Yes.

0.10 Problems DE-3

0.10.1 Topics of this homework: Brune impedance

lattice transmission line analysis

0.10.2 Brune Impedance

Problem # *1: Residue form*

A Brune impedance is defined as the ratio of the force $F(s)$ to the flow $V(s)$ and may be expressed in residue form as

$$Z(s) = c_0 + \sum_{k=1}^{K} \frac{c_k}{s - s_k} = \frac{N(s)}{D(s)} \tag{DE-3.1}$$

with

$$D(s) = \prod_{k=1}^{K} (s - s_k) \qquad \text{and} \qquad c_k = \lim_{s \to s_k} (s - s_k) D(s) = \prod_{n'=1}^{K-1} (s - s_n).$$

The prime on the index n' means that $n = k$ is not included in the product.

– 1.1: Find the Laplace transform ($\mathcal{LT}$*) of a (1) spring, (2) dashpot, and (3) mass.*

Express these in terms of the force $F(s)$ and the velocity $V(s)$, along with the electrical equivalent impedance: (1) Hooke's law $f(t) = Kx(t)$, (2) dashpot resistance $f(t) = Rv(t)$, and (3) Newton's law for mass $f(t) = Mdv(t)/dt$. **Sol:**

1. Hooke's Law $f(t) = Kx(t)$. Taking the $\mathcal{LT}$ gives

$$F(s) = KX(s) = KV(s)/s \leftrightarrow f(t) = Ku(t) \star v(t) = K \int^t v(t),$$

since

$$v(t) = \frac{d}{dt}x(t) \leftrightarrow V(s) = sX(s).$$

Thus the impedance of the spring is

$$Z_s(s) = \frac{K}{s} \leftrightarrow z(t) = Ku(t),$$

which is analogous to the impedance of an electrical capacitor. The relationship may be made tighter by specifying the compliance of the spring as $C = 1/K$.

2. Dashpot resistance $f(t) = Rv(t)$. From the $\mathcal{LT}$ this becomes

$$F(s) = RV(s)$$

and the impedance of the dashpot is then

$$Z_r = R \leftrightarrow R\delta(t),$$

analogous to that of an electrical resistor.

3. Newton's law for mass $f(t) = Mdv(t)/dt$. Taking the $\mathcal{LT}$ gives

$$f(t) = M\frac{d}{dt}v(t) \leftrightarrow F(s) = M\,sV(s),$$

thus

$$Z_m(s) = sM \leftrightarrow M\frac{d}{dt},$$

analogous to an electrical inductor.

– 1.2: Take the Laplace transform ($\mathcal{LT}$) of Eq. **??** *and find the total impedance $Z(s)$ of the mechanical circuit.*

$$M\frac{d^2}{dt^2}x(t) + R\frac{d}{dt}x(t) + Kx(t) = f(t) \leftrightarrow (Ms^2 + Rs + K)X(s) = F(s). \tag{DE-3.2}$$

Sol: From the properties of the $\mathcal{LT}$ that $dx/dt \leftrightarrow sX(s)$, we find

$$f(t) \leftrightarrow F(s) = Ms^2X(s) + RsX(s) + KX(s).$$

In terms of velocity this is $(Ms + R + K/s)V(s) = F(s)$. Thus the circuit impedance is

$$z(t) \leftrightarrow Z(s) = \frac{F}{V} = \frac{K + Rs + Ms^2}{s}.$$

– 1.3: What are $N(s)$ and $D(s)$ (see Eq. **??***)?*

Sol: $D(s) = s$ and $N(s) = K + Rs + Ms^2$.

– 1.4: Assume that $M = R = K = 1$ and find the residue form of the admittance $Y(s) = 1/Z(s)$ (see Eq. **??***) in terms of the roots $s_\pm$. Hint: Check your answer with Octave's/Matlab's* `residue` *command.*

Sol: First find the roots of the numerator of $Z(s)$ (the denominator of $Y(s)$):

$$s_\pm^2 + s_\pm + 1 = (s_\pm + 1/2)^2 + 3/4 = 0,$$

which is

$$s_{\pm} = \frac{-1 \pm \jmath\sqrt{3}}{2}.$$

Second form a partial fraction expansion

$$\frac{s}{1+s+s^2} = c_0 + \frac{c_+}{s-s_+} + \frac{c_-}{s-s_-} = \frac{s(c_+ + c_-) - (c_+ s_- + c_- s_+)}{1+s+s^2}.$$

Comparing the two sides shows that $c_0 = 0$. We also have two equations for the residues $c_+ + c_- = 1$ and $c_+ s_- + c_- s_+ = 0$. The best way to solve this is to set up a matrix relation and take the inverse

$$\begin{bmatrix} 1 & 1 \\ s_- & s_+ \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{thus:} \quad \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \frac{1}{s_+ - s_-} \begin{bmatrix} s_+ & -1 \\ -s_- & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

which gives $c_{\pm} = \pm\frac{s_{\pm}}{s_+ - s_-}$ The denominator is $s_+ - s_- = j\sqrt{3}$ and the numerator is $\pm 1 + \jmath\sqrt{3}$. Thus

$$c_{\pm} = \pm\frac{s_{\pm}}{s_+ - s_-} = \frac{1}{2}\Big(1 \pm \frac{\jmath}{\sqrt{3}}\Big).$$

As always, finding the coefficients is always the most difficult part. Using 2x2 matrix algebra automates the process. Always check your final result as correct.

– 1.5: By applying Eq. **??** *(page* **??***), find the inverse Laplace transform ($\mathcal{LT}^{-1}$). Use the residue form of the expression that you derived in question* **??**.
Sol:

$$z(t) = \frac{1}{2\pi j} \oint_{\mathcal{C}} Z(s) e^{st} ds.$$

were $\mathcal{C}$ is the Laplace contour which encloses the entire left-half s plane. Applying the CRT

$$z(t) = c_+ e^{s_+ t} + c_- e^{s_- t}.$$

where $s_{\pm} = -1/2 \pm \jmath\sqrt{3}/2$ and $c_{\pm} = 1/2 \pm \jmath/(2\sqrt{3})$.

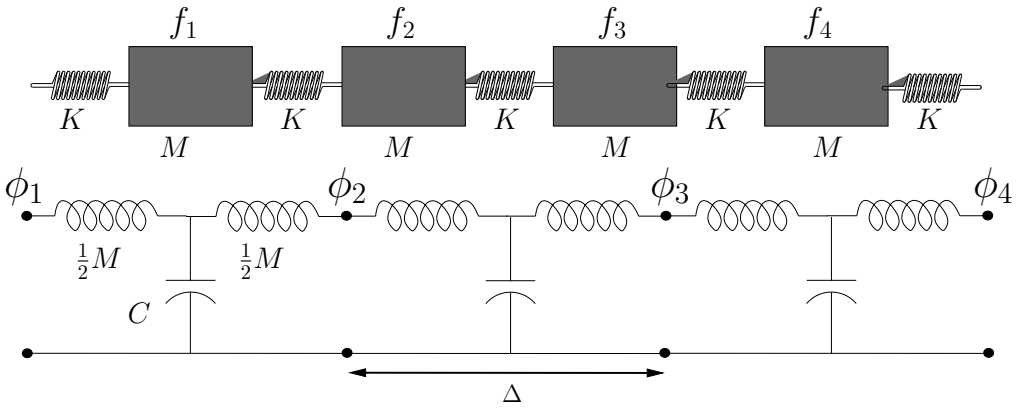


Figure 2: *Depiction of a train consisting of cars treated as masses M and linkages treated as springs of stiffness K or compliance $C = 1/K$. Below it is the electrical equivalent circuit for comparison. The masses are modeled as inductors and the springs as capacitors to ground. The velocity is analogous to a current and the force $f_n(t)$ to the voltage $\phi_n(t)$. The length of each cell is Δ [m]. The train may be accurately modeled as a transmission line (TL), since the equivalent electrical circuit is a lumped model of a TL. This method, called a* Cauer synthesis, *is based on the ABCD transmission line method of Sec.* **??** *(p.* **??***).*

0.10.3 Transmission-line analysis

Problem # *2:* **Train-mission-line** *We wish to model the dynamics of a freight train that has N such cars and study the* velocity transfer function *under various load conditions.*

As shown in Fig. **??**, the train model consists of masses connected by springs.

Problem # *3: Transfer functions*

Use the ABCD method (see the discussion pages 148-151) to find the matrix representation of the system of the above figure. Define the force on the nth train car $f_n(t) \leftrightarrow F_n(\omega)$ and the velocity $v_n(t) \leftrightarrow V_n(\omega)$.

Break the model into cells consisting of three elements: a series inductor representing half the mass ($M/2$), a shunt capacitor representing the spring ($C = 1/K$), and another series inductor representing half the mass ($L = M/2$), transforming the model into a cascade of symmetric ($\mathcal{A} = \mathcal{D}$) identical cell matrices $\mathcal{T}(s)$.

– 3.1: Find the elements of the ABCD matrix $\mathcal{T}$ for the single cell that relate the input node 1 to output node 2

$$\begin{bmatrix} F \\ V \end{bmatrix}_1 = \mathcal{T} \begin{bmatrix} F(\omega) \\ -V(\omega) \end{bmatrix}_2 . \tag{DE-3.3}$$

Sol:

$$\begin{aligned} \mathcal{T} &= \begin{bmatrix} 1 & sM/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC & 1 \end{bmatrix} \begin{bmatrix} 1 & sM/2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 + s^2MC/2 & (sM)(1 + s^2MC/4) \\ sC & 1 + s^2MC/2 \end{bmatrix} \end{aligned} \tag{DE-3.4a}$$

– 3.2: Express each element of $\mathcal{T}(s)$ in terms of the complex Nyquist ratio $s/s_c < 1$ ($s = 2\pi jf$, $s_c = 2\pi jf_c$). The Nyquist wavelength sampling condition is $\lambda_c > 2\Delta$. It says the critical wavelength $\lambda_c > 2\Delta$. condition is $\lambda_c > 2\Delta$.[9] *It says the critical wavelength $\lambda_c > 2\Delta$. Namely it is defined in terms the minimum number of cells 2Δ, per minimum wavelength λ_c.*

The Nyquist wavelength sampling theorem says that there are at least two cars per wavelength.

Proof: From the figure, the distance between cars $\Delta = c_oT_o$ [m], where

$$c_o = \frac{1}{\sqrt{MC}} \quad \text{[m/s]}.$$

[9] The history of this relation has been traced back to 1841, as discussed by (**?**, Chap. I,II, Eq. 4.7).

The cutoff frequency obeys $f_c\lambda_c = c_o$. The Nyquist critical wavelength is $\lambda_c = c_o/f_c > 2\Delta$. Therefore the Nyquist sampling condition is

$$f < f_c \equiv \frac{c_o}{\lambda_c} = \frac{c_o}{2\Delta} = \frac{1}{2\Delta\sqrt{MC}} \quad \text{[rad/sec]}. \tag{DE-3.5}$$

Finally, $s_c = \jmath 2\pi f_c$.

Sol: The solution is a repeat what is summarized above: the system in Fig. **??** represents a transmission line having a wave speed of $c_o = 1/\sqrt{MC}$ and characteristic impedance $r_o = \sqrt{M/C}$. Each cell, composed of 2 masses M connected by one spring K, has length Δ.

We wish to define the *Nyquist frequency* f_c such that the wavelength $\lambda > 2\Delta$, where Δ is the cell length. Using the formula for the wavelength in terms of the wave velocity and frequency we find

$$\lambda = c_o/f_c = 2\Delta,$$

thus we conclude that

$$f < f_c = \frac{c_o}{2\Delta} = \frac{1}{2\Delta\sqrt{MC}}. \tag{DE-3.6}$$

If we wish to have the system be accurate for a given frequency we may make the cell length Δ smaller, while keeping the velocity constant (MC is held constant). Thus the characteristic resistance [ohms/unit length] r_o must change as $f_c \to \infty$ and $\Delta \to 0$. We can either let $M \to \infty$ and $C \to 0$ (their product remains constant), or the other way around. In one case $r_o \to \infty$ and in the other case it goes to 0.

– 3.3: Use the property of the Nyquist sampling frequency $\omega < \omega_c$ *(Eq.* **??***) to remove higher order powers of frequency*

$$1 + \left(\frac{s}{s_c}\right)^{2} \approx 1 \tag{DE-3.7}$$

to determine a band-limited approximation of $\mathcal{T}(s)$.

Sol:

$$\mathcal{T} = \begin{bmatrix} 1 + 2(s/s_c)^2 & sM(1 + (s/s_c)^2) \\ sC & 1 + 2(s/s_c)^2 \end{bmatrix}$$
$$\approx \begin{bmatrix} 1 & sM \\ sC & 1 \end{bmatrix}$$

The approximation is highly accurate below the Nyquist cutoff frequency $s < s_c$. Given any desired frequency f, we can always make the cell size Δ smaller by decreasing M and C, while keeping $f < f_c$ and the cell velocity constant ($c_o = 1/\sqrt{MC}$). Thus the Nyquist condition represents a computational bound, not a physical limitation.

Problem # *4: Now consider the cascade of* N *such* $\mathcal{T}(s)$ *matrices and perform an eigenanalysis.*

– 4.1: Find the eigenvalues and eigenvectors of $\mathcal{T}(s)$ *as functions of* s/s_c.

Sol: Matrix $\mathcal{T}(s)$ has eigenvalues

$$\lambda_\pm = 1 \mp 2s/s_c \approx e^{\pm 2s/s_c} = e^{\mp sT_c}.$$

From this we can interpret the eigenvalues as the cell delay $T_c = 2/s_c$.

The corresponding unnormalized eigenvectors are

$$\boldsymbol{e}_\pm = \begin{bmatrix} \mp\sqrt{M/C} \\ 1 \end{bmatrix},$$

where the characteristic impedance defined is $r_o = \sqrt{M/C}$.

Problem # *5: Find the velocity transferfunction* $H_{12}(s) = V_2/V_1|_{F_2=0}$.

– 5.1: Assuming that $N = 2$ and $F_2 = 0$ (two half-mass problem), find the transfer function $H(s) \equiv V_2/V_1$. From the results of the $\mathcal{T}$ matrix, find

$$H_{21}(s) = \left.\frac{V_2}{V_1}\right|_{F_2=0}$$

Express H_{12} in terms of a residue expansion.

Sol: From Eq. **??**, $V_1 = sCF_2 - (s^2MC/2 + 1)V_2$. Since $F_2 = 0$

$$\frac{V_2}{V_1} = \frac{-1}{s^2MC/2 + 1} = \left(\frac{c_+}{s - s_+} + \frac{c_-}{s - s_-}\right)$$

having eigenfrequencies $s_\pm = \pm\jmath\sqrt{\frac{2}{2MC}} = \pm s_c$ and residues $c_\pm = \pm\jmath/\sqrt{2MC} = \pm s_c$.

– 5.2: Find $h_{21}(t) \leftrightarrow H_{21}(s)$.

Sol:

$$h(t) = \oint_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} \frac{e^{st}}{s^2MC/2 + 1}\,\frac{ds}{2\pi j} = c_+ e^{-s_+ t}u(t) + c_- e^{-s_- t}u(t).$$

The integral follows from the Cauchy Residue theorem (CRT).

– 5.3: What is the input impedance $Z_2 = F_2/V_2$, assuming $F_3 = -r_0V_3$?

Sol: Starting from Eq. **??** find Z_2

$$Z_2(s) = \frac{F_2}{V_2} = T\begin{bmatrix} F \\ -V \end{bmatrix}_2 = \frac{-(1 + s^2CM/2)r_0\cancel{V_2} - sM(1 + s^2CM/4)\cancel{V_2}}{-sCr_0\cancel{V_2} - (1 + s^2CM/2)\cancel{V_2}}$$

– 5.4: Simplify the expression for Z_2 as follows:

1. Assuming the *characteristic impedance* $r_0 = \sqrt{M/C}$,
2. terminate the system in r_0: $F_2 = -r_0V_2$ (i.e., $-V_2$ cancels).
3. Assume higher-order frequency terms are less than 1 ($|s/s_c| < 1$).
4. Let the number of cells $N \to \infty$. Thus $|s/s_c|^N = 0$.

When a transmission line is terminated in its characteristic impedance r_0, the input impedance $Z_1(s) = r_0$. Thus, when we simplify the expression for $\mathcal{T}(s)$, it should be equal to r_0. Show that this is true for this setup.

Sol: Applying the Nyquist approximation (i.e., ignore second order frequency terms $(s/s_c)^2 \approx 0$)

$$\begin{aligned}
Z_1(s) &= \frac{r_o(1 + \cancelto{0}{s^2CM/2}) + sM(1 + \cancelto{0}{s^2CM/4})}{r_o sC + \cancelto{0}{(1 + s^2CM/2)}} \\
&\approx \frac{r_o + sM}{1 + r_o sC} = \frac{MC}{MC}\cdot\frac{r_o + sM}{1 + r_o sC} = \frac{M}{C}\cdot\frac{r_oC + sMC}{M + r_o sMC} = r_o^2\,\frac{r_oC + s/s_c}{M + r_o s/s_c} \\
&\approx r_o^2\,\frac{r_oC + \cancelto{0}{s/s_c}}{M + \cancelto{0}{r_o s/s_c}} = r_o^3\,\frac{C}{M} \\
&= r_o.
\end{aligned}$$

We conclude that below the Nyquist cutoff frequency, as $N \to \infty$ the system equals a transmission line terminated by its characteristic impedance thus $Z_1(s) = r_o$.

– 5.5: State the ABCD matrix relationship between the first and Nth nodes in terms of the cell matrix. Write out the transfer function for one cell, H_{21}.

Sol:

$$\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$$

Now use the formulae for the eigenvalues and vectors to obtain $\mathcal{T}$ for $N = 1$:

$$\mathcal{T} = E\Lambda E^{-1} = E \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} E^{-1}.$$

– 5.6: What is the velocity transfer function $H_{N1} = \frac{V_N}{V_1}$?

Sol:

$$\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = \mathcal{T}^N \begin{bmatrix} F_N(\omega) \\ -V_N(\omega) \end{bmatrix}$$

along with the eigenvalue expansion

$$\mathcal{T}^N = E\Lambda^N E^{-1} = E \begin{bmatrix} \lambda_+^N & 0 \\ 0 & \lambda_-^N \end{bmatrix} E^{-1}.$$

where $\lambda_\pm^N = e^{\mp sNT_o}$. Recall that NT_o is the one way delay.

We conclude that as we add more cells, the delay linearly increases with N, since each eigenvalue represents the delay of one cell, and delay adds.

0.11 Problems VC-1

0.11.1 Topics of this homework:

Vector algebra and fields in $\mathbb{R}^3$, gradient and scalar Laplacian operators, definitions of divergence and curl, Gauss's (divergence) and Stokes's (curl) laws, system classification (postulates).

0.11.2 Scalar fields and the ∇ operator

Problem # *1: Let $T(x,y) = x^2 + y$ be an analytic scalar temperature field in two dimensions (single-valued $\in \mathbb{R}^2$).*

– 1.1: Find the gradient of $T(\boldsymbol{x})$ and make a sketch of T and the gradient.

Sol: $\nabla(x^2+y) = 2x\hat{\mathbf{x}} + \hat{\mathbf{y}}$. The temperature is quadratic in x and linear in y, which has the shape of a trough in x, linearly increasing in y. In the y ($\hat{\mathbf{y}}$) direction the gradient is constant, and in the $\hat{\mathbf{x}}$ direction, it is linear, and goes through zero at $x = 0$, with $T(0) = 0$. Skiing in the y direction would be a constant ride of slope 1. If the snow had no friction, you would accelerate, but the terminal velocity would be due to the friction of the snow on the skis. Along the x direction, you would accelerate, at first, coming down, and at $x = 0$ you would stop accelerating, and begin slow down. This would be a more interesting problem if you treated it in terms of the forces on the skis and included friction as well as gravity.

– 1.2: Compute $\nabla^2 T(\boldsymbol{x})$ to determine whether $T(\boldsymbol{x})$ satisfies Laplace's equation.

Sol: Forming this operation we find that

$$\frac{\partial^2}{\partial x^2}x^2 + \cancel{\frac{\partial^2}{\partial y^2}y} = 2.$$

So $T(\boldsymbol{x})$ does not satisfy laplace's equation, rather it satisfies the Poisson equation $\nabla^2 T(\boldsymbol{x}) = 2$.

– 1.3: Sketch the iso-temperature contours at $T = -10, 0, 10$ degrees.

Sol: The iso-potential contours are the concave parabolas $y = T_0 - x^2$.

– 1.4: The heat flux[10] *is defined as $\boldsymbol{J}(x,y) = -\kappa(x,y)\nabla T$, where $\kappa(x,y)$ is denotes the thermal conductivity at the point (x,y). Given that $\kappa = 1$ everywhere (the medium is homogeneous), plot the vector $\boldsymbol{J}(x,y) = -\nabla T$ at $x = 2$, $y = 1$. Be clear about the origin, direction, and length of your result.*

Sol: $\mathbf{J} = \nabla T = -2x\hat{\mathbf{x}} + -\hat{\mathbf{y}}$ thus $-\kappa\nabla T(2,1) = \mathbf{J} = -(4\hat{\mathbf{x}} + \hat{\mathbf{y}})$, which has a length of $\sqrt{17}$ and is pointed $1\sqrt{1}$ unit down and $4/\sqrt{17}$ units to the left.

– 1.5: Find the vector $\perp$ to $\nabla T(x,y)$—that is, tangent to the iso-temperature contours. Hint: Sketch it for one (x,y) point (e.g., $2,1$) and then generalize.

Sol: We may invoke the third dimension $\hat{\mathbf{z}}$ to generate this vector: $\pm\hat{\mathbf{z}} \times \nabla T = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & \pm 1 \\ 2x & 1 & 0 \end{bmatrix} = \mp(1\hat{\mathbf{x}} - 2x\hat{\mathbf{y}} + 0\hat{\mathbf{z}})$. Alternatively, rotate ∇T by $\pm\pi/2$ in the (x,y) plane.

[10] The heat flux is proportional to the change in temperature times the thermal conductivity κ of the medium.

– 1.6: The thermal resistance R_T is defined as the potential drop ΔT over the magnitude of the heat flux $|\boldsymbol{J}|$. At a single point the thermal resistance is

$$R_T(x,y) = -\nabla T/|\boldsymbol{J}|.$$

How is $R_T(x,y)$ related to the thermal conductivity $\kappa(x,y)$?

Sol: $R_T(x,y) = 1/\kappa(x,y)$. In general, resistance is the reciprocal of conductivity (conductance). This is true for electrical and acoustic systems as well.

Problem # *2: Acoustic wave equation*

Note: In this problem, we will work in the frequency domain.

– 2.1: The basic equations of acoustics in one dimension are

$$-\frac{\partial}{\partial x}\mathcal{P} = \rho_o s\,\mathcal{V} \quad \text{and} \quad -\frac{\partial}{\partial x}\mathcal{V} = \frac{s}{\eta_o P_o}\mathcal{P}.$$

Here $\mathcal{P}(x,\omega)$ is the pressure (in the frequency domain), $\mathcal{V}(x,\omega)$ is the volume velocity (the integral of the velocity over the wavefront with area A), $s = \sigma + \omega\jmath$, $\rho_o = 1.2$ is the specific density of air, $\eta_o = 1.4$, and P_o is the atmospheric pressure (i.e., 10^5 Pa). Note that the pressure field $\mathcal{P}$ is a scalar (pressure does not have direction), while the volume velocity field $\mathcal{V}$ is a vector (velocity has direction).

We can generalize these equations to three dimensions using the ∇ operator

$$-\nabla\mathcal{P} = \rho_o s\,\mathcal{V} \quad \text{and} \quad -\nabla\cdot\mathcal{V} = \frac{s}{\eta_o P_o}\mathcal{P}.$$

– 2.2: Starting from these two basic equations, derive the scalar wave equation in terms of the pressure $\mathcal{P}$,

$$\nabla^2\mathcal{P} = \frac{s^2}{c_0^2}\mathcal{P},$$

where c_0 is a constant representing the speed of sound.

Sol: We wish to remove $\mathcal{V}$ from the two equations, to obtain a single equation in pressure. If we take the partial wrt x of the pressure equation, and then substitute the velocity equation, to remove the velocity:

$$\nabla^2\mathcal{P} = -\rho_o s\nabla\cdot\mathcal{V} = \frac{s^2\rho_o}{\eta_o P_o}\mathcal{P} = \frac{s^2}{c_o^2}\mathcal{P}$$

– 2.3: What is c_0 in terms of η_0, ρ_0, and P_0?

Sol: Comparing the last two terms from the previous solution we see that

$$c_o = \sqrt{\eta_o P_o/\rho_o}.$$

– 2.4: Rewrite the pressure wave equation in the time domain using the time derivative property of the Laplace transform [e.g., $dx/dt \leftrightarrow sX(s)$]. For your notation, define the time-domain signal using a lowercase letter, $p(x,y,z,t) \leftrightarrow \mathcal{P}$.

Sol:

$$\nabla^2 p(x,y,z,t) = \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}p(x,y,z,t)$$

0.11.3 Vector fields and the ∇ operator

0.11.4 Vector algebra

Problem # *3: Let* $\boldsymbol{R}(x,y,z) \equiv x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$.

– *3.1: If* a, b, *and* c *are constants, what is* $\boldsymbol{R}(x,y,z)\cdot\boldsymbol{R}(a,b,c)$*?*
Sol: Using the formula for a scalar dot product:

$$\mathbf{R}(x,y,z)\cdot\mathbf{R}(a,b,c) \equiv [x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}]\cdot[a\hat{\mathbf{x}} + b\hat{\mathbf{y}} + c\hat{\mathbf{z}}]$$

$$= x(t)a + y(t)b + z(t)c.$$

– *3.2: If* a, b, *and* c *are constants, what is* $\frac{d}{dt}\left(\boldsymbol{R}(x,y,z)\cdot\boldsymbol{R}(a,b,c)\right)$*?*
Sol: $\left(a\frac{d}{dt}x(t) + b\frac{d}{dt}y(t) + c\frac{d}{dt}z(t)\right)$.

Problem # *4: Find the divergence and curl of the following vector fields:*

– *4.1:* $\boldsymbol{v} = \hat{\mathbf{x}} + \hat{\mathbf{y}} + 2\hat{\mathbf{z}}$
Sol: $\nabla\cdot\mathbf{v} = 0$, $\nabla\times\mathbf{v} = 0$

– *4.2:* $\boldsymbol{v}(x,y,z) = x\hat{\mathbf{x}} + xy\hat{\mathbf{y}} + z^2\hat{\mathbf{z}}$
Sol: $\nabla\cdot\mathbf{v} \equiv \partial_x x + \partial_y xy + \partial_z z^2 = 1 + x + 2z$ $\nabla\times\mathbf{v} \equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ x & xy & z^2 \end{vmatrix} = (0-0)\hat{\mathbf{x}} + (0-0)\hat{\mathbf{y}} + (y-0)\hat{\mathbf{z}} = y\hat{\mathbf{z}}$

– *4.3:* $\boldsymbol{v}(x,y,z) = x\hat{\mathbf{x}} + xy\hat{\mathbf{y}} + \log(z)\hat{\mathbf{z}}$
Sol: Divergence: $\partial_x x + \partial_y xy + \partial_z \log(z) = 1 + x + 1/z$, Curl: $\hat{\mathbf{x}}\left(\partial_y \log(z) - \partial_z xy\right) + \hat{\mathbf{y}}\left(\partial_z x - \partial_x \log(z)\right) + \hat{\mathbf{z}}\left(\partial_x xy - \partial_y x\right) = \hat{\mathbf{z}}y$

– *4.4:* $\boldsymbol{v}(x,y,z) = \nabla(1/x + 1/y + 1/z)$
Sol: First find $\mathbf{v} = -(\hat{\mathbf{x}}/x^2 + \hat{\mathbf{y}}/y^2 + \hat{\mathbf{z}}/z^2)$. Divergence of $\mathbf{v}$: $-(\partial_x 1/x^2 + \partial_y 1/y^2 + \partial_z 1/z^2) = 2(1/x^3 + 1/y^3 + 1/z^3)$, Curl of $\mathbf{v}$: 0, because the curl of the gradient is always zero.

0.11.5 Vector and scalar field identities

Problem # *5: Find the divergence and curl of the following vector fields:*

– *5.1:* $\boldsymbol{v} = \nabla\phi$*, where* $\phi(x,y) = xe^y$
Sol: $\nabla\times\nabla\phi = 0$, and $\nabla^2\phi = xe^y$

– *5.2:* $\boldsymbol{v} = \nabla\times\boldsymbol{A}$*, where* $\boldsymbol{A} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$
Sol: $\nabla\cdot(\nabla\times\mathbf{A}) = 0$, and $\nabla\times(\nabla\times\mathbf{A}) = 0$

– *5.3:* $\boldsymbol{v} = \nabla\times\boldsymbol{A}$*, where* $\boldsymbol{A} = y\hat{\mathbf{x}} + x^2\hat{\mathbf{y}} + z\hat{\mathbf{z}}$
Sol: $\nabla\cdot(\nabla\times\mathbf{A}) = 0$, and $\nabla\times(\nabla\times\mathbf{A}) = -2\hat{\mathbf{y}}$

– *5.4: For any differentiable vector field* $\mathbf{V}$*, write two vector calculus identities that are equal to zero.*
Sol: *Curl of the gradient* $\nabla\times\nabla\Phi(x,y,z) = 0$ and the *divergence of the curl* $\nabla\cdot\nabla\times\mathbf{V}(x,y,z) = 0$ are both zero. (Page 780, Stillwell)

– *5.5: What is the most general form a vector field may be expressed in, in terms of scalar* Φ *and vector* $\boldsymbol{A}$ *potentials?*
Sol: $\mathbf{V} = \nabla\Phi(x,y,z) + \nabla\times\mathbf{A}(x,y,z)$, where Φ is the scalar potential and $\mathbf{A}$ is the vector potential.

Problem # *6: Perform the following calculations. If you can state the answer without doing the calculation, explain why.*

– *6.1: Let* $\boldsymbol{v} = \sin(x)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. *Find* $\nabla \cdot (\nabla \times \boldsymbol{v})$.
Sol: 0

– *6.2: Let* $\boldsymbol{v} = \sin(x)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. *Find* $\nabla \times (\nabla\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}})$
Sol: 0

– *6.3: Let* $\boldsymbol{v}(x,y,z) = \nabla(x + y^2 + \sin(\log(z))$. *Find* $\nabla \times \boldsymbol{v}(x,y,z)$.
Sol: It is zero because $\nabla \times \nabla f(x,y,z)$ is *always* zero.

0.11.6 Integral theorems

Problem # *7: For each of the following problems, in a few words, identify either Gauss's or Stokes's law, define what it means, and explain the formula that follows the question.*

– *7.1: What is the name of this formula?*

$$\int_{\mathcal{S}} \hat{\mathbf{n}} \cdot \boldsymbol{v}\, dA = \int_{\mathcal{V}} \nabla \cdot \boldsymbol{v}\, dV.$$

Sol: This is the integral form of *Gauss' law*. The unit normal vector is $\perp$ to the surface $\mathcal{S}$ having area $A \equiv \int_{\mathcal{S}} dA$ The integral represents the total flow normal to the surface. The surface integral is equal to the integral of the divergence of the vector field $\nabla \cdot \mathbf{v}$ over the volume contained by the surface, and defined as $\mathcal{V}$.

– *7.2: What is the name of this formula?*

$$\int_S (\nabla \times \boldsymbol{V}) \cdot d\boldsymbol{S} = \oint_C \boldsymbol{V} \cdot d\boldsymbol{R}$$

Give one important application. **Sol:** Stokes Theorem, which relates the differential to the integral form of Maxwell's equations.

– *7.3: Describe a key application of the vector identity*

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \boldsymbol{\nabla^2 \mathbf{V}}.$$

Sol: When we wish to reduce Maxwell's two curl equations to the vector wave equation, we must use this identity.

0.12 Problems VC-2

0.12.1 Topics of this homework:

Partial differential equations; fundamental theorem of vector calculus (Helmholtz's theorem); wave equation; Maxwell's equations (ME) and variables ($\boldsymbol{E}$, $\boldsymbol{D}$; $\boldsymbol{B}$, $\boldsymbol{H}$); Second-order vector differentials; Webster horn equation.

Notation: The following notation is used in this homework:

1. $s = \sigma + j\omega$ is the Laplace frequency, as used in the Laplace transform.
2. A Laplace transform pair is indicated by the symbol $\leftrightarrow$: for example, $f(t) \leftrightarrow F(s)$.
3. Here π_k is the kth prime: $\pi_k \in \mathbb{P}$, $\pi_k = [2, 3, 5, 7, 11, 13, \ldots]$.

0.12.2 Partial differential equations (PDEs): Wave equation

Problem # *1: Solve the wave equation in one dimension by defining $\xi = t \mp x/c$.*

– 1.1: Show that d'Alembert's solution, $\varrho(x,t) = f(t - x/c) + g(t + x/c)$, is a solution to the acoustic pressure wave equation in one dimension:

$$\frac{\partial^2 \varrho(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \varrho(x,t)}{\partial t^2},$$

where $f(\xi)$ and $g(\xi)$ are arbitrary functions. **Sol:**

$$\frac{\partial}{\partial x}\varrho(x,t) = \frac{\partial}{\partial x}f(t - x/c) + \frac{\partial}{\partial x}g(t + x/c) = \frac{-1}{c}f'(t - x/c) + \frac{1}{c}g'(t + x/c) \tag{VC-2.1}$$

$$\frac{\partial^2}{\partial x^2}\varrho(x,t) = \frac{\partial^2}{\partial x^2}f(t - x/c) + \frac{\partial^2}{\partial x^2}g(t + x/c) = \frac{1}{c^2}f''(t - x/c) + \frac{1}{c^2}g''(t + x/c) \tag{VC-2.2}$$

$$\frac{\partial^2}{\partial t^2}\varrho(x,t) = \frac{\partial^2}{\partial t^2}f(t - x/c) + \frac{\partial^2}{\partial t^2}g(t + x/c) = f''(t - x/c) + g''(t + x/c) \tag{VC-2.3}$$

Problem # *2: Solving the wave equation in spherical coordinates (i.e., three dimensions)*

– 2.1: Write the wave equation in spherical coordinates $\varrho(r, \theta, \phi, t)$. Consider only the radial term r (i.e., dependence on angles θ and ϕ is assumed to be zero). Hint: The form of the Laplacian is a function of the number of dimensions is

Sol: Given the formula for the Laplacian in spherical coordinates, the wave equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\varrho(r,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varrho(r,t)$$

– 2.2: Show that this equation is true:

$$\nabla_r^2 \varrho(r) \equiv \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\varrho(r) = \frac{1}{r}\frac{\partial^2}{\partial r^2} r\varrho(r). \tag{VC-2.4}$$

Hint: Expand both sides of the equation. **Sol:** Both sides of the equation expand to

$$\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}$$

– 2.3: Use the results from Eq. **??** *to show that the solution to the spherical wave equation is*

$$\nabla_r^2\, \varrho(r,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\, \varrho(r,t) \tag{VC-2.5}$$

$$\varrho(r,t) = \frac{f(t-r/c)}{r} + \frac{g(t+r/c)}{r}. \tag{VC-2.6}$$

Sol: This proceed exactly as in the rectangular case (see above) except one must first recognize that the Laplacian in spherical coordinates may be written as

$$\frac{1}{r}\frac{\partial^2}{\partial r^2} r\varrho(r). \tag{VC-2.7}$$

One then may proceed to use the solution for the rectangular case, but for $r\varrho(r)$, and then divide that solution by r.

– 2.4: Using $f(\xi) = \sin(\xi)u(\xi)$ and $g(\xi) = e^{\xi}u(\xi)$, write the solutions to the spherical wave equation, where $u(\xi)$ is the Heaviside step function.
Sol: In each case replace $\xi = t - x/c$ to obtain the solution to the wave equation for 1 dimensional waves. Thus

$$\begin{aligned}\varrho(r,t) &= \frac{f(t-r/c)}{r} + \frac{g(t+r/c)}{r} \\ &= \frac{\sin(t-r/c)}{t-r/c}u(t-r/c) + \frac{e^{(t+r/c)}u(t+r/c)}{t+r/c}\end{aligned}$$

– 2.5: Sketch this $f(\xi)$ and $g(\xi)$ for several times (e.g., 0, 1, and 2 seconds), and describe the behavior of the pressure $\varrho(r,t)$ as a function of time t and radius r.
Sol: Plot the functions at several times (e.g., 0, 1 2 seconds), as a function of x. The first function becomes smaller as the radius grows. The second function becomes larger as the inbound waves approaches $r = 0$.

– 2.6: What happens when the inbound wave reaches the center at $r = 0$?
Sol: Stand back. It blows up. The equations fail when the solution becomes so large that the linearity assumption fails. I'm not sure what actually happens, in practice. This seems to be how they detonate nuclear weapons.

0.12.3 Helmholtz's formula

Every differentiable vector field may be written as the sum of a scalar potential ϕ and a vector potential $\boldsymbol{w}$. This relationship is best known as the fundamental theorem of vector calculus (also called Helmholtz's formula):

$$\boldsymbol{v} = -\nabla\phi + \nabla\times\boldsymbol{w}. \tag{VC-2.8}$$

This formula seems to be a natural extension of the algebraic products $\boldsymbol{a}\cdot\boldsymbol{b} \perp \boldsymbol{a}\times\boldsymbol{b}$, since $\boldsymbol{a}\cdot\boldsymbol{b} \propto \|\boldsymbol{a}\|\|\boldsymbol{b}\|\cos(\theta)$ and $\boldsymbol{a}\times\boldsymbol{b} \propto \|\boldsymbol{a}\|\|\boldsymbol{b}\|\sin(\theta)$, as developed in Appendix **??**, page **??**. Thus these orthogonal components have magnitude 1 when we take the norm, due to Euler's identity ($\cos^2(\theta) + \sin^2(\theta) = 1$).

As shown in Table **??** (p. **??**), Helmholtz's formula separates a vector field (i.e., $\mathbf{v}(\mathbf{x})$) into compressible and rotational parts:

1. The rotational (e.g., angular) part is defined by the vector potential $\boldsymbol{w}$, which requires that $\nabla \times \nabla \times \boldsymbol{w} \neq 0$. A field is irrotational (conservative) when $\nabla \times \boldsymbol{v} = 0$, meaning that the field $\boldsymbol{v}$ can be generated using only a scalar potential, $\boldsymbol{v} = \nabla\phi$ (note that this is how a conservative field is usually defined, by saying there exists some ϕ such that $\boldsymbol{v} = \nabla\phi$).[11]
2. The compressible (e.g., radial) part of a field is defined by the scalar potential ϕ, which requires that $\nabla \cdot \nabla\phi = \nabla^2\phi \neq 0$. A field is incompressible (solenoidal) when $\nabla \cdot \boldsymbol{v} = 0$, meaning that the field $\boldsymbol{v}$ can be generated using only a vector potential, $\boldsymbol{v} = \nabla \times \boldsymbol{w}$.

The definitions and generating potential functions of irrotational (conservative) and incompressible (solenoidal) fields naturally follow from two key vector identities: (1) $\nabla \cdot (\nabla \times \boldsymbol{w}) = 0$ and (2) $\nabla \times (\nabla\phi) = 0$.

Problem # *3: Define the following:*

– 3.1: A conservative vector field

Sol: A conservative vector field is defined as the gradient of a scalar potential $\mathbf{v} = \nabla\phi(x, y, z)$. Every conservative field is necessarily *irrotational* (the test for an irrotational field is $\nabla \times \mathbf{v} = 0$).

– 3.2: An irrotational vector field

Sol: The vector field $\mathbf{v}$ is rotational if there exists a vector potential $\mathbf{w}$ such that $\mathbf{v} = \nabla \times \mathbf{w}(x, y, z)$. The for *irrotational* is $\nabla \times \mathbf{v} = 0$. A purely rotational field is not conservative.

– 3.3: An incompressible vector field

Sol: A field $\mathbf{v}$ is incompressible if $\nabla \cdot \mathbf{v} = 0$.

– 3.4: A solenoidal vector field

Sol: A rotational field is one having a divergence of zero, i.e., $\nabla \cdot \mathbf{v} = 0$, or alternatively, $\mathbf{v} \equiv \nabla \times \mathbf{w}(x, y, z)$, since any field defined by a curl is rotational, since the divergence of the curl is always zero.

– 3.5: When is a conservative field irrotational?

Sol: Always!

– 3.6: When is an incompressible field irrotational?

Sol: A field is incompressible if $\nabla \cdot v = 0$ and irrotational if $\nabla \times v = 0$. So, almost never. The only case is the trivial solution $\mathbf{v} = 0$, or a constant field $\mathbf{v} = x_0\hat{\mathbf{x}} + y_0\hat{\mathbf{y}} + z_0\hat{\mathbf{z}}$.

Problem # *4: For each of the following, (i) compute $\nabla \cdot \boldsymbol{v}$, (ii) compute $\nabla \times \boldsymbol{v}$, and (iii) classify the vector field (e.g., conservative, irrotational, incompressible, etc.).*

– 4.1: $\boldsymbol{v}(x, y, z) = -\nabla(3yx^3 + y\log(xy))$

Sol: The field is conservative (or irrotational) because it is defined by a gradient. To test for irrotational, show that the curl is zero. But $\nabla \times \nabla\phi(x, y, z) = 0$ for any $\phi(x, y, z)$. Thus you do not need to do any computation, just state the answer.

– 4.2: $\boldsymbol{v}(x, y, z) = xy\hat{\mathbf{x}} - z\hat{\mathbf{y}} + f(z)\hat{\mathbf{z}}$

Sol: To test for a irrotational field, take the curl, to see if it is zero:

$$\nabla \times \mathbf{v} \equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ xy & -z & f(z) \end{vmatrix} = \hat{\mathbf{x}} - x\hat{\mathbf{z}}, \tag{VC-2.9}$$

which is not zero. We can also see by inspection that $\nabla \cdot \mathbf{v} \neq 0$. Thus the vector field is rotational and compressible.

– 4.3: $\boldsymbol{v}(x, y, z) = \nabla \times (x\hat{\mathbf{x}} - z\hat{\mathbf{y}})$

Sol: $\mathbf{v} = \hat{\mathbf{x}}$. Therefore, $\nabla \times \mathbf{v} = 0$, and $\nabla \cdot \mathbf{v} = 0$. This field is technically incompressible and irrotational, but it is also very boring, since it is a constant.

[11] A note about the relationship between the generating function and the test: You might imagine special cases where $\nabla \times \boldsymbol{w} \neq 0$ but $\nabla \times \nabla \times \boldsymbol{w} = 0$ (or $\nabla\phi \neq 0$ but $\nabla^2\phi = 0$). In these cases, the vector (or scalar) potential can be recast as a scalar (or vector) potential. For example, consider a field $\boldsymbol{v} = \nabla\phi_0 + \boldsymbol{b}$, where $\boldsymbol{b} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. Note that $\boldsymbol{b}$ can actually be generated by either a scalar potential ($\phi_1 = \frac{1}{2}[x^2 + y^2 + z^2]$, such that $\nabla\phi_1 = \boldsymbol{b}$) or a vector potential ($\boldsymbol{w}_0 = \frac{1}{2}[z^2\hat{\mathbf{x}} + x^2\hat{\mathbf{y}} + y^2\hat{\mathbf{z}}]$, such that $\nabla \times \boldsymbol{w}_0 = \boldsymbol{b}$). We find that $\nabla \times \boldsymbol{v} = 0$; therefore $\boldsymbol{v}$ must be irrotational. We say this irrotational field is generated by $\nabla\phi = \nabla(\phi_0 + \phi_1)$.

0.12.4 Maxwell's Equations

The four key definitions have the following names and definitions:

Symbol	Equation	Name	Units
$\boldsymbol{E}$	$\nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$	Electric field strength	[volts/m]
$\boldsymbol{D} = \epsilon_o \boldsymbol{E}$	$\nabla \cdot \boldsymbol{D} = \rho$	Electric displacement (flux density)	[coul/m^2]
$\boldsymbol{H}$	$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \dot{\boldsymbol{D}}$	Magnetic field strength	[amps/m]
$\boldsymbol{B} = \mu_o \boldsymbol{H}$	$\nabla \cdot \boldsymbol{B} = 0$	Magnetic induction (flux density)	[webers/m^2]

0.12.5 Speed of light

Problem # *5: The* speed of light *in vacuo is* $c_o = 1/\sqrt{\mu_o \epsilon_o} \approx 3 \times 10^8$ *[m/s]. The* characteristic resistance *in vacuo is* $r_o = \sqrt{\mu_o/\epsilon_o} \approx 377\,[\Omega]$.

– 5.1: Find a formula for the in-vacuo permittivity ϵ_o *and* permeability *in terms of* c_o *and* r_o. **Sol:** $\epsilon_o = 1/r_o c_o$ *and* $\mu_o = r_o/c_o$. *Approximately, what are the numeric values of* ϵ_o *and* μ_o*?*
Sol: $\epsilon_o \approx 10^{-8}/3 \cdot 377 = 8.84 \cdot 10^{-12}$ and $\mu_o \approx 377/3 \cdot 10^8 = 1.26 \cdot 10^{-6}$.

– 5.2: In a few words, identify the law given by this equation, define what it means, and explain the formula:

$$\int_{\mathcal{S}} \hat{\mathbf{n}} \cdot \boldsymbol{v}\, dA = \int_{\mathcal{V}} \nabla \cdot \boldsymbol{v}\, dV.$$

Sol: This is the integral form of *Gauss' law*. The unit normal vector is $\perp$ to the surface $\mathcal{S}$ having area $A \equiv \int_{\mathcal{S}} dA$ The integral represents the total flow normal to the surface. The surface integral is equal to the integral of the divergence of the vector field $\nabla \cdot \boldsymbol{v}$ over the volume contained by the surface, and defined as $\mathcal{V}$.

0.12.6 Application of Maxwell's equations

Problem # *6: The electric Maxwell equation is* $\nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$*, where* $\boldsymbol{E}$ *is the* electric field strength *and* $\dot{\boldsymbol{B}}$ *is the time rate of change of the* magnetic induction field*, or simply the magnetic flux density. Consider this equation integrated over a two-dimensional surface* S*, where* $\hat{\mathbf{n}}$ *is a unit vector normal to the surface (you may also find it useful to define the closed path* C *around the surface):*

$$\iint_S [\nabla \times \boldsymbol{E}] \cdot \hat{\mathbf{n}} dS = -\frac{\partial}{\partial t} \iint_S \boldsymbol{B} \cdot \hat{\mathbf{n}} dS.$$

– 6.1: Apply Stokes's theorem to the left-hand side of the equation.
Hint: view this relation in terms of the "integral forms" of the curl.
Sol: The surface S must be open, with its edge C defining the path for the line integral.

$$\text{emf} \equiv \iint_S \nabla \times \boldsymbol{E} \cdot \hat{\mathbf{n}}\, dS = \oint_C \boldsymbol{E} \cdot d\mathbf{R}. \tag{VC-2.10}$$

From Stokes' theorem: the *electromotive force* (emf) is the line integral of $\mathbf{E}$ around the rim of the open surface. Think of the flux change as the Thévenin source driving the voltage.

– 6.2: Consider the right-hand side of the equation. How is it related to the magnetic flux Ψ *through the surface* S*?*
Sol: It is equal to the time rate of change of the flux, $-\dot{\Psi}$. From Gauss' Law the total magnetic flux Ψ is the surface integral over the normal component of the magnetic flux density $\boldsymbol{B}$. After applying Gauss' Laws, the surface integral becomes

$$\Psi = -\iint_S \boldsymbol{B} \cdot \hat{\mathbf{n}} dS \tag{VC-2.11}$$

– 6.3: Assume the right-hand side of the equation is zero. Can you relate your answer in question **??** *to one of Kirchhoff's laws?*
Sol: This result is well know as Kirchhoff's first (voltage) law (KVL), emf $= \sum_k V_k = -\dot{\Psi}$. The sum of the voltages around the loop is zero. For a planar surface this is simply $\Phi = B_n A$, where A is the area and B_n the normal component of B ($\perp$ to the surface S).

0.13 Second-order differentials

Problem # *7: This problem is about second-order vector differentials.*

– 7.1: If $\boldsymbol{v}(x,y,z) = \nabla\phi(x,y,z)$*, then what is* $\nabla \cdot \boldsymbol{v}(x,y,z)$*?*
Sol: Since $\nabla \cdot \nabla = \nabla^2$ this is $\nabla^2\phi(x,y,z)$.

– 7.2: When $\boldsymbol{V}(x,y,z) = \nabla(1/x + 1/y + 1/z)$*, what is* $\nabla \times \boldsymbol{V}(x,y,z)$*?*
Sol: The CoG is always zero.

0.13.1 Webster horn equation

Problem # *8: Horns illustrate an important generalization of the solution of the one dimensional wave equation in regions where the properties (i.e., area of the tube) vary along the axis of wave propagation. Classic applications of horns are in vocal tract acoustics, loudspeaker design, cochlear mechanics, and any case that has wave propagation. Write the formula for the Webster horn equation, and explain the variables.*
Sol: The horn equation may be written as

$$\frac{1}{A(x)}\frac{\partial}{\partial x}\left(A(x)\frac{\partial \varrho}{\partial x}\right) = \frac{1}{c^2}\frac{\partial^2 \varrho}{\partial t^2}. \tag{VC-2.12}$$

where $A(x)$ is the area of the horn at x (range variable). $\varrho(x,t)$ is the pressure and c is the wave speed.