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Chapter 1

Number systems
1.0.1 Exercises NS-1

**Topic of this homework:** Introduction to MATLAB/OCTAVE (see the Matlab or Octave tutorial for help).

Deliverable: Report with charts and answers to questions. Hint: Use \LaTeX\ \(^1\)

**Plotting complex quantities in Matlab** Plot real, imaginary, magnitude and phase quantities.

1. Consider the functions \( f(s) = s^2 + 6s + 25 \) and \( g(s) = s^2 + 6s + 5 \).
   
   (a) Find the zeros of functions \( f(s) \) and \( g(s) \) using the command `roots`.
   
   (b) On a single plot, show the roots of \( f(s) \) as red circles, and the roots of \( g(s) \) as blue plus signs. The x-axis should display the real part of each root, and the y-axis should display the imaginary part. Use `hold on` and `grid on` when plotting the roots.
   
   (c) Give your figure the title ‘Complex Roots of \( f(s) \) and \( g(s) \)’ using the command `title`. Label the x-axis ‘Real Part’ and the y-axis ‘Imaginary Part’ using `xlabel` and `ylabel`. Type `ylim([-10 10])` and `xlim([-10 10])`, to expand the axes.

2. Consider the function \( h(t) = e^{j2\pi ft} \) for \( f = 5 \) and \( t = [0:0.01:2] \).
   
   (a) Use `subplot` to show the real and imaginary parts of \( h(t) \) as two graphs in one figure. Label the x-axes ‘Time (s)’ and the y-axes ‘Real Part’ and ‘Imaginary Part’.
   
   (b) Use `subplot` to plot the magnitude and phase parts of \( h(t) \). Use the command `angle` or `unwrap(angle())` to plot the phase. Label the x-axes ‘Time (s)’ and the y-axes ‘Magnitude’ and ‘Phase (radians)’.

1. Prime numbers in Matlab
   
   (a) Use the Matlab function `factor` to find the prime factors of 123, 248, 1767, and 999,999.
   
   (b) Use the Matlab function `isprime` to check if 2, 3 and 4 are prime numbers. What does the function `isprime` return when a number is prime, or not prime? Why?
   
   (c) Use the Matlab/Octave function `primes.m` to generate prime numbers between 1 and \( 10^6 \) and save them in a vector \( x \). Plot this result using the command `hist(x)`.
   
   (d) Now try \( [n,bin_centers] = hist(x) \). Use `length(n)` to find the number of bins.
   
   (e) Set the number of bins to 100 by using an extra input argument to the function `hist`. Show the resulting figure and give it a title and axes labels.

2. Inf, NaN and logarithms in Matlab

\(^1\)http://www.overleaf.com
(a) Try $1/0$ and $0/0$ in the command window. What are the results? What do these ‘numbers’ mean in Matlab?

(b) In Matlab, the natural logarithm $\ln(\cdot)$ is computed using the function $\log$ ($\log_{10}$ and $\log_{2}$ are computed using $\log 10$ and $\log 2$). Try $\log(0)$ in the command window.

(c) Try $\log(-1)$ in the command window. Do you get what you expect for $\ln(-1)$? Show how Matlab arrives at the answer by considering $-1 = e^{i \pi}$.

(d) (not graded) What is a decibel? Look up decibels on the internet.

3. Find the largest prime number that can be stored on an Intel 64 bit computer, which we call $\pi_{\text{max}}$. Hint: As explained in the Matlab/Octave command help flintmax, the largest positive integer is $2^{53}$, however the largest integer that can be factored is $2^{32} = \sqrt{2^{64}}$. Explain the logic of your answer. Hint: help isprime().

```matlab
%Matlab code to find the largest prime in IEEE-floating point
clear variables; close all
clc
N=2^32; %flintmax says this is the largest integer
disp(sprintf('N %g',N));
for n=1:20
p=isprime(N-n);
if p
F=factor(N-n)
disp(sprintf('n= %g, N=%g; Factor: %d',n,N,factor(N-n)))
end
end
```

4. Suppose you are interested in primes that are greater than $\pi_{\text{max}}$. How can you find them on an Intel computer (i.e., one using IEEE-floating point)?

(a) Hint 1: Since every prime number greater than 2 is odd, there is no reason to check the even numbers. Thus consider a sieve containing only odd numbers, starting from 3 (not 2). Thus odd integers $n_{\text{odd}} \in \mathbb{N}/2$ contain all the primes other than 2.

5. The following identity is interesting:

\[
\begin{align*}
1 &= 1^2 \\
1 + 3 &= 2^2 \\
1 + 3 + 5 &= 3^2 \\
1 + 3 + 5 + 7 &= 4^2 \\
1 + 3 + 5 + 7 + 9 &= 5^2 \\
\vdots \\
\sum_{n=0}^{N-1} 2n + 1 &= N^2.
\end{align*}
\]

Can you find a proof?²

²This problem came from an exam problem for Math 213, Fall 2016.
1.0.2 Exercises NS-2

**Topic of this homework:** Prime numbers, greatest common divisors, the continued fraction algorithm

Deliverable: Answers to questions.

**Prime Numbers**

1. According to the fundamental theorem of arithmetic, every integer may be written as a product of primes.

   (a) Put the numbers 1,000,000, 1,000,004 and 999,999 in the form \( N = \prod_k \pi_k^{\beta_k} \) (Hint: Use Matlab/Octave to find the prime factors).

   (b) Give a generalized formula for the natural logarithm of a number, \( \ln(N) \), in terms of its primes \( \pi_k \) and their multiplicities \( \beta_k \). Express your answer as a sum of terms.

2. Explain why the following brief Matlab/Octave program returns the prime numbers \( \pi_k \) between 1 and 100.

   ```matlab
   n=2:100;
k = isprime(n);
n(k)
   ```

3. How many primes are there between 2 and 100?

4. Prime numbers may be identified using ‘sieves.’

   (a) By hand, perform the sieve of Eratosthenes for \( n = 1 \ldots 49 \). Circle each prime \( p \), then cross out each number which is a multiple of \( p \).

   (b) In part (a), what is the largest number you need to consider before only the primes remain?

   (c) Generalize: for \( n = 1 \ldots N \), what is the highest number you need to consider before only the primes remain?

   (d) Write each of these numbers as a product of primes:

      \[
      \begin{align*}
      22 &= \rule{0cm}{0cm} \\
      30 &= \rule{0cm}{0cm} \\
      34 &= \rule{0cm}{0cm} \\
      43 &= \rule{0cm}{0cm} \\
      44 &= \rule{0cm}{0cm} \\
      48 &= \rule{0cm}{0cm} \\
      49 &= \rule{0cm}{0cm}
      \end{align*}
      \]
5. Find the largest prime $\pi_k \leq 100$? Hint: Do not use Matlab/Octave other than to check your answer. Hint: Write out the numbers starting with 100 and counting backwards: 100, 99, 98, 97, · · · . Cross off the even numbers, leaving 99, 97, 95, · · · . Pull out a factor (only 1 is necessary to show that it is not prime).

6. Find the largest prime $\pi_k \leq 1000$? Hint: Do not use Matlab/Octave other than to check your answer.

7. Explain why $\pi_k^{-s} = e^{-s\ln \pi_k}$.

**GCD-1**

Consider the Euclidean algorithm to find the greatest common divisor (GCD; the largest common prime factor) of two numbers. Note this algorithm may be performed using one of two methods:

<table>
<thead>
<tr>
<th>Method</th>
<th>Division</th>
<th>Subtraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>On each iteration...</td>
<td>$a_{i+1} = b_i$</td>
<td>$a_{i+1} = \max(a_i, b_i) - \min(a_i, b_i)$</td>
</tr>
<tr>
<td></td>
<td>$b_{i+1} = a_i - b_i \cdot \floor(a_i/b_i)$</td>
<td>$b_{i+1} = \min(a_i, b_i)$</td>
</tr>
<tr>
<td>Terminates when...</td>
<td>$b = 0 \text{ (gcd} = a)$</td>
<td>$b = 0 \text{ (gcd} = a)$</td>
</tr>
</tbody>
</table>

The division method (Eq. 2.1, Sect. 2.1.2, Lec 5, Ch. 2) is preferred because the subtraction method is much slower.

1. Understand the Euclidean (GCD) algorithm
   
   (a) Use the Matlab command `factor` to find the prime factors of $a = 85$ and $b = 15$. What is the greatest common prime factor of these two numbers?
   
   (b) By hand, perform the Euclidean algorithm for $a = 85$ and $b = 15$.
   
   (c) By hand, perform the Euclidean algorithm for $a = 75$ and $b = 25$. Is the result a prime number?
   
   (d) Consider the first step of the GCD division algorithm when $a < b$ (e.g. $a = 25$ and $b = 75$). What happens to $a$ and $b$ in the first step? Does it matter if you begin the algorithm with $a < b$ vs. $b < a$?
   
   (e) Describe in your own words how the GCD algorithm works. Try the algorithm using numbers which have already been separated into factors (e.g. $a = 5 \cdot 3$ and $b = 7 \cdot 3$).

2. Coprimes
   
   (a) Define the term *coprime*.
   
   (b) How can the Euclidean algorithm be used to identify coprimes?
   
   (c) Give at least one application of the Euclidean algorithm.

3. Write a Matlab function, `function x = my_gcd(a,b),` which uses the Euclidean algorithm to find the GCD of any two inputs `a` and `b`. Test your function on the `(a,b)` combinations from parts (a) and (b). Include a printout (or handwrite) your algorithm to turn in.

   **Hints and advice:**
CHAPTER 1. NUMBER SYSTEMS

- Don’t give your variables the same names as Matlab functions! Since \( \text{gcd} \) is an existing Matlab/Octave function, if you use it as a variable or function name, you won’t be able to use \( \text{gcd} \) to check your \( \text{gcd()} \) function. Try \text{clear all} to recover from this problem.
- Try using a ‘while’ loop for this exercise (see Matlab documentation for help).
- You may need to make some temporary variables for \( a \) and \( b \) in order to perform the algorithm.

GCD-2

In this problem we are looking for integer solutions \((m, n) \in \mathbb{Z}\) to the equations \(ma + nb = \gcd(a, b)\) and \(ma + nb = 0\) given positive integers \((a, b) \in \mathbb{Z}^+\). Note that this requires that either \(m\) or \(n\) be negative. These solutions may be found using the Euclidean algorithm only if \((a, b)\) are coprime \((a \perp b)\). Note that integer (whole number) polynomial relations such as these are known as ‘Diophantine equations.’ The above equations are linear Diophantine equations, possibly the simplest form of such relations.

**Example: \( \text{gcd}(2, 3) = 1 \):** For \((a, b) = (2, 3)\), the result is as follows:

\[
\begin{bmatrix}
1 \\
0 \\
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
1 & -2 \\
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & -1 \\
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
2 \\
3 \\
\end{bmatrix}
\]

Thus from the above equation we find the solution \((m, n)\) to the integer equation

\[
2m + 3n = \gcd(2, 3) = 1,
\]

namely \((m, n) = (-1, 1)\) (i.e., \(-2 + 3 = 1\)). There is also a second solution \((3, -2)\) (i.e., \(3 \cdot 2 - 2 \cdot 3 = 0\)), which represents the terminating condition. Thus these two solutions are a pair and the solution only exists if \((a, b)\) are coprime \((a \perp b)\).

**Subtraction method:** This method is more complicated than the division algorithm, because at each stage we must check if \(a < b\). Define

\[
\begin{bmatrix}
a_0 \\
\end{bmatrix} = \frac{a}{b}
Q = \begin{bmatrix}
1 & -1 \\
0 & 1 \\
\end{bmatrix}
S = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\]

where \(Q\) sets \(a_{i+1} = a_i - b_i\) and \(b_{i+1} = b_i\) assuming \(a_i > b_i\), and \(S\) is a ‘swap-matrix’ which swaps \(a_i\) and \(b_i\) if \(a_i < b_i\). Using these matrices, the algorithm is implemented by assigning

\[
\begin{bmatrix}
a_{i+1} \\
b_{i+1} \\
\end{bmatrix} = Q \begin{bmatrix}
a_i \\
b_i \\
\end{bmatrix} \text{ for } a_i > b_i,
\begin{bmatrix}
a_{i+1} \\
b_{i+1} \\
\end{bmatrix} = QS \begin{bmatrix}
a_i \\
b_i \\
\end{bmatrix} \text{ for } a_i < b_i.
\]

The result of this method is a cascade of \(Q\) and \(S\) matrices. For \((a, b) = (2, 3)\), the result is as follows:

\[
\begin{bmatrix}
1 \\
1 \\
\end{bmatrix} =
\begin{bmatrix}
1 & -1 \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
2 \\
2 \\
\end{bmatrix}
\]

Thus we find two solutions \((m, n)\) to the integer equation \(2m + 3n = \gcd(2, 3) = 1\).

1. By inspection, find at least one integer pair \((m, n)\) that satisfies \(12m + 15n = 3\).
2. Using matrix methods for the Euclidean algorithm, find integer pairs \((m, n)\) that satisfy \(12m + 15n = 3\) and \(12m + 15n = 0\). Show your work!!!
3. Does the equation \(12m + 15n = 1\) have integer solutions for \(n\) and \(m\)? Why, or why not?
Matrix approach: It can be difficult to keep track of the a’s and b’s when the algorithm has many steps. Here is an alternative way to run the Euclidean algorithm, using matrix algebra. Matrix methods provide a more transparent approach to the operations on \((a, b)\). Thus the Euclidean algorithm can be classified in terms of standard matrix operations (discussed at the end of Lec. 5). Division method:

Define

\[
\begin{bmatrix}
a_0 \\
b_0
\end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \begin{bmatrix} a_{i+1} \\
b_{i+1}
\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor a_i/b_i \rfloor \end{bmatrix} \begin{bmatrix} a_i \\
b_i
\end{bmatrix}
\]

Continued fractions

Here we explore the continued fraction algorithm (CFA), as discussed in Lec. 6 (Chapters 1 and 2). In its simplest form the CFA starts with a real number, which we denote as \(\alpha \in \mathbb{R}\). Let us work with an irrational real number, \(\pi \in \mathbb{I}\), as an example, because its CFA representation will be infinitely long. We can represent the CFA coefficients \(\alpha\) as a vector of integers \(n_k, k = 1, 2 \cdots \infty\)

\[
\alpha = [n_1; n_2, n_3, n_4, \cdots]
\]

\[
= n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \frac{1}{n_4 + \cdots}}}
\]

As discussed in Section ?? (p. ??), the CFA is recursive (Graham et al., 1994), with three steps per iteration:

For \(\alpha_1 = \pi, n_1 = 3, r_1 = \pi - 3\) and \(\alpha_2 \equiv 1/r_1\).

\[
\alpha_2 = \frac{1}{0.1416} = 7.0625 \cdots
\]

\[
\alpha_1 = n_1 + \frac{1}{\alpha_2} = n_1 + \frac{1}{n_2 + \frac{1}{\alpha_3}} = \cdots
\]

In Matlab/Octave script

```matlab
alpha0 = pi;
K=10;
n=zeros(1,K); alpha=zeros(1,K);
alpha(1)=alpha0;

for k=2:K %k=1 to K
 n(k)=round(alpha(k-1));
 %n(k)=fix(alpha(k-1));
 alpha(k)= 1/(alpha(k-1)-n(k));
 %disp([fix(k), round(n(k)), alpha(k)]); pause(1)
end
disp([n; alpha]);
%Now compair this to matlab’s rat() function
rat(alpha0,1e-20)
```

1. By hand (you may use Matlab/Octave as a calculator), find the first 3 values of \(n_k\) for \(\alpha = e^\pi\).
2. For part (1), what is the error (remainder) when you truncate the continued fraction after $n_1, \ldots, n_3$? Give the absolute value of the error, and the percentage error relative to the original $\alpha$.

3. Use the Matlab/Octave program provided to find the first 10 values of $n_k$ for $\alpha = e^\pi$, and verify your result using the Matlab/Octave command `rat()`.

4. Discuss the similarities and differences between the Euclidean algorithm (EA) and CFA.
1.0.3 Exercises NS-3

**Topic of this homework:** Pythagorean triples, Pell’s equation, Fibonacci sequence

Deliverable: Answers to problems

Euclid’s formula for the Pythagorean triples $a, b, c$ is: $a = p^2 - q^2$, $b = 2pq$, and $c = p^2 + q^2$.

1. What condition(s) must hold for $p$ and $q$ such that $a$, $b$, and $c$ are always positive and nonzero?

2. Solve for $p$ and $q$ in terms of $a$, $b$ and $c$.

3. The ancient Babylonians (c2000BEC) cryptically recorded $(a, c)$ pairs of numbers on a clay tablet, archeologically denoted *Plimpton-322*.

To Do: Find $p$ and $q$ for the first five pairs of $a$ and $c$ from the tablet entries:

*Table 1: First five $(a, c)$ pairs of Plimpton-322.*

<table>
<thead>
<tr>
<th>$a$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>119</td>
<td>169</td>
</tr>
<tr>
<td>3367</td>
<td>4825</td>
</tr>
<tr>
<td>4601</td>
<td>6649</td>
</tr>
<tr>
<td>12709</td>
<td>18541</td>
</tr>
<tr>
<td>65</td>
<td>97</td>
</tr>
</tbody>
</table>

4. Based on Euclid’s formula, show that $c > (a, b)$.

5. What happens when $c = a$?

6. Is $b + c$ a perfect square? Discuss.

**Pell’s equation:**

Pell’s equation is one of the most historic (i.e., important) equations of Greek number theory because it was used to show that $\sqrt{2} \in \mathbb{Q}$. We seek integer solutions

$$x^2 - Ny^2 = 1.$$  

As shown in Lec 8 of the lecture notes, the solutions $x_n, y_n$ for the case of $N = 2$ are given by the 2x2 matrix recursion of the form

$$
\begin{bmatrix}
x_{n+1} \\
y_{n+1}
\end{bmatrix} = 1 \cdot 
\begin{bmatrix}
1 & 2 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
x_n \\
y_n
\end{bmatrix} = j^n 
\begin{bmatrix}
1 & 2 \\
1 & 1
\end{bmatrix}^n 
\begin{bmatrix}
x_0 \\
y_0
\end{bmatrix}.
$$
Diagonalization of a matrix ("eigenvalue/eigenvector decomposition"): As derived in Appendix C of the lecture notes, the most efficient way to compute \( A^n \) is to diagonalize the matrix \( A \), by finding its eigenvalues and eigenvectors.

The eigenvalues \( \lambda_k \) and eigenvectors \( v_k \) of a square matrix \( A \) are related by

\[
A v_k = \lambda_k v_k,
\]

such that multiplying an eigenvector \( v_k \) of \( A \) by the matrix \( A \) is the same as multiplying by a scalar, \( \lambda_k \in \mathbb{C} \) (the corresponding eigenvalue). The complete eigenvalue problem may be written as

\[
A E = E \Lambda.
\]

If \( A \) is a \( 2 \times 2 \) matrix,\(^3\) the matrices \( E \) and \( \Lambda \) (of eigenvectors and eigenvalues, respectively) are

\[
E = \begin{bmatrix} e_1 & e_2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.
\]

Thus, the matrix equation \( A E = \begin{bmatrix} A e_1 & A e_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 e_1 & \lambda_2 e_2 \end{bmatrix} = E \Lambda \) contains Eq. M.1 for each eigenvalue-eigenvector pair.

The diagonalization of the matrix \( A \) refers to the fact that the matrix of eigenvalues, \( \Lambda \), has non-zero elements only on the diagonal. The key result is found by post-multiplication of the eigenvalue matrix by \( E^{-1} \), giving

\[
A E E^{-1} = A = E \Lambda E^{-1}.
\]

If we now take powers of \( A \), the \( n \)th power of \( A \) is

\[
A^n = (E \Lambda E^{-1})^n = E \Lambda E^{-1} E \Lambda E^{-1} \cdots E \Lambda E^{-1} = E \Lambda^n E^{-1}.
\]

This is a very powerful result, because the \( n \)th power of a diagonal matrix is extremely easy to calculate:

\[
\Lambda^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}.
\]

Thus, from Eq. M.3 we can calculate \( A^n \) using only two matrix multiplications

\[
A^n = E \Lambda^n E^{-1}.
\]

Finding the eigenvalues: The eigenvalues \( \lambda_k \) are determined by Eq. M.1, by factoring out \( e_k \)

\[
A e_k = \lambda_k e_k
\]

Matrix \( I = [1, 0; 0, 1]^T \) is the identity matrix, having the dimensions of \( A \), with elements \( \delta_{ij} \) (i.e., diagonal elements \( \delta_{11,22} = 1 \) and off-diagonal elements \( \delta_{12,21} = 0 \)).

The vector \( e_k \) is not zero, yet when operated on by \( A - \lambda_k I \), the result must be zero. The only way this can happen is if the operator is degenerate (has no solution), that is

\[
\det(A - \lambda I) = \det \begin{bmatrix} (a_{11} - \lambda) & a_{12} \\ a_{21} & (a_{22} - \lambda) \end{bmatrix} = 0.
\]

\(^3\)These concepts may be easily extended to higher dimensions.
This means that the two equations have the same slope (the equation is degenerate).
This determinant equation results in a second degree polynomial in \( \lambda \)
\[
(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0,
\]
the roots of which are the eigenvalues of the matrix \( A \).

**Finding the eigenvectors:** An eigenvector \( e_k \) can be found for each eigenvalue \( \lambda_k \) from Eq. M.1,
\[
(A - \lambda_k I)e_k = 0.
\]
The left side of the above equation becomes a column vector, where each element is an equation in the elements of \( e_k \), set equal to 0 on the right side. These equations are always degenerate, since the determinant is zero. Thus the two equations have the same slope.

Solving for the eigenvectors is often confusing, because they have arbitrary magnitudes, \( ||e_k|| = \sqrt{e_k \cdot e_k} = \sqrt{e_{k,1}^2 + e_{k,2}^2} = d \). From Eq. M.1, you can only determine the relative magnitudes and signs of the elements of \( e_k \), so you will have to choose a magnitude \( d \). It is common practice to normalize each eigenvector to have unit magnitude (\( d = 1 \)).

**To do:** *Hint: Use Matlab’s function \([E, Lambda] = eig(A)\) to check your results!*

1. Solutions to Pell’s equation were used by the Pythagoreans to explore the value of \( \sqrt{2} \). Explain why Pell’s equation is relevant to \( \sqrt{2} \).
2. Find the first 3 values of \([x_n, y_n]^T\) by hand and show that they satisfy Pell’s equation for \( N=2 \).
3. By hand, find the eigenvalues \( \lambda_{\pm} \) of the 2 \( \times \) 2 Pell’s equation matrix
\[
A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}
\]
4. By hand, show that the matrix of eigenvectors, \( E \), is
\[
E = \begin{bmatrix} e_+ & e_- \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}
\]
5. Using the eigenvalues and eigenvectors you found for \( A \), verify that
\[
E^{-1}AE = \Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}
\]
6. Now that you have diagonalized \( A \) (Equation M.3), use your results for \( E \) and \( \Lambda \) to solve for the \( n = 10 \) solution \([x_{10}, y_{10}]^T\) to Pell’s equation with \( N = 2 \).

The Fibonacci sequence is famous in mathematics, and has been observed to play a role in the mathematics of genetics. Let \( x_n \) represent the Fibonacci sequence,
\[
x_n = x_{n-1} + x_{n-2},
\]
where the current output sample, \( x_n \), is equal to the sum of the previous two inputs. This is a ‘discrete time’ recurrence relation. To solve for \( x_n \), we require some initial conditions.
In this exercise, let us define $x_0 = 1$ and $x_{n<0} = 0$. This leads to the Fibonacci sequence \{1, 1, 2, 3, 5, 8, 13, \ldots\} for $n = 0, 1, 2, 3, \ldots$.

Here we seek the general formula for $x_n$. Like the Pell’s equation, Eq. M.5 has a recursive, eigen-decomposition solution. To find it we must recast $x_n$ as a $2\times2$ matrix relation, and then proceed as we did for the Pell case.

1. By example, show that the Fibonacci sequence $x_n$ as described above may be generated by

$$
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
$$

What is the relationship between $y_n$ and $x_n$?

2. Write a Matlab/Octave program to compute $x_n$ using the matrix equation above. Test your code using the first few values of the sequence. Using your program, what is $x_{40}$? Note: Consider using the eigen-decomposition of $A$, described by Eq. M.3 (p. 336).

3. Using the eigen-decomposition of the matrix $A$ (and a lot of algebra), it is possible to obtain the general formula for the Fibonacci sequence,

$$
x_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right].
$$

(1.6)

What are the eigenvalues $\lambda_{\pm}$ of the matrix $A$? How is the formula for $x_n$ related to these eigenvalues?

4. Consider Eq. M.6 in the limit as $n \to \infty$...

(a) What happens to each of the two terms $[(1 \pm \sqrt{5})/2]^{n+1}$?

(b) What happens to the ratio $x_{n+1}/x_n$?

5. Prove that

$$
\sum_{1}^{N} f_n^2 = f_N f_{N+1}.
$$

6. Replace the Fibonacci sequence with

$$
x_n = \frac{x_{n-1} + x_{n-2}}{2},
$$

such that the value $x_n$ is the average of the previous two values in the sequence.

(a) What matrix $A$ is used to calculate this sequence?

(b) Modify your computer program to calculate the new sequence $x_n$. What happens as $n \to \infty$?

(c) What are the eigenvalues of your new $A$? How do they relate to the behavior of $x_n$ as $n \to \infty$? Hint: you can expect the closed-form expression for $x_n$ to be similar to Eq. M.6.

---

4I found this problem on a worksheet for Math 213 midterm (213practice.pdf).
7. Now consider
\[ x_n = \frac{x_{n-1} + 1.01x_{n-2}}{2}. \]

(a) What matrix \( A \) is used to calculate this sequence?

(b) Modify your computer program to calculate the new sequence \( x_n \). What happens as \( n \to \infty \)?

(c) What are the eigenvalues of your new \( A \)? How do they relate to the behavior of \( x_n \) as \( n \to \infty \)? *Hint: you can expect the closed-form expression for \( x_n \) to be similar to Eq. M.6.*
Chapter 2

Algebraic Equations
Topic of this homework:  Fundamental theorem of algebra, polynomials, analytic functions, convolution, analytic geometry, composition, intersection.

Deliverable: Answers to problems

Note: The term ‘analytic’ is used in two different ways. (1) An analytic function is a function that may be expressed as a locally convergent power series; (2) analytic geometry refers to geometry using a coordinate system.

Polynomials and the fundamental theorem of algebra (FTA)

A polynomial of degree \( N \) is defined as

\[ P_N(x) = a_0 + a_1x + a_2x^2 \cdots a_Nx^N \]

1. How many coefficients \( a_n \) does a polynomial of degree \( N \) have?

2. How many roots does \( P_N(x) \) have?

3. The fundamental theorem of algebra (FTA)

   (a) State the FTA.

   (b) Using the FTA, prove your answer to question (2) above.

4. Consider the polynomial function \( P_2(x) = 1 + x^2 \) of degree \( N = 2 \), and the related function \( F(x) = 1/P_2(x) \).

   (a) What are the roots (e.g. ‘zeros’) \( x_\pm \) of \( P_2(x) \)?

   (b) \( F(x) \) may be expressed as \( (A, B, x_\pm \in \mathbb{C}) \)

\[ F(x) = \frac{A}{x - x_+} + \frac{B}{x - x_-}, \quad (2.1) \]

where \( x_\pm \) are the roots (zeros) of \( P_2(x) \), which become the poles of \( F(x) \), and \( A, B \) are the residues. The expression for \( F(x) \) is sometimes called a ‘partial fraction expansion’ or ‘residue expansion,’ and it appears in many engineering applications.

   i. Find \( A, B \in \mathbb{C} \) in terms of the roots \( x_\pm \) of \( P_2(x) \).

   ii. Verify your answers for \( A, B \) by showing that this expression for \( F(x) \) is indeed equal to \( 1/P_2(x) \).

(c) The poles of a function \( G(x) \) are defined as values \( x_p \) where \( G(x_p) \to \infty \); the zeros are defined as values \( x_z \) where \( G(x_z) = 0 \).

   Hint: Do not forget to consider \( f(x) \) as \( x \to \pm \infty \)

   i. Give the values of the poles and zeros of \( P_2(x) \).

   ii. Give the values of the poles and zeros of \( F(x) = 1/P_2(x) \).
Analytic functions

Analytic functions are defined by infinite (power) series. The function \( f(x) \) is analytic at any value of \( x = x_0 \) where there exists a convergent power series

\[
P(x) = \sum_{n=0}^{\infty} a_n x^n
\]

such that \( P(x_0) = f(x_0) \). The local power series for \( f(x) \) near \( x = x_0 \) is often obtained by finding the Taylor series:

\[
f(x) \approx f(x_0) + \frac{df}{dx}\bigg|_{x=x_0} (x-x_0) + \frac{1}{2!} \frac{d^2 f}{dx^2}\bigg|_{x=x_0} (x-x_0)^2 + \ldots
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n f}{dx^n}\bigg|_{x=x_0} (x-x_0)^n.
\]

The point \( x = x_0 \) is called the series expansion point.

When the expansion point is at \( x_0 = 0 \), the series is denoted a MacLaurin series. Two classic examples are the geometric series where \( a_n = 1 \)

\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n, \tag{2.1}
\]

and the exponential function where \( a_n = 1/n! \)

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{2.2}
\]

The coefficients for both series may be derived from the Taylor formula (or MacLaurin formula, when the expansion point is zero).

1. The geometric series

(a) What is the region of convergence (RoC) for the power series of \( 1/(1-x) \) given above (e.g. where does the power series \( P(x) \) converge to the function value \( f(x) \))?

State your answer as a condition on \( x \). Hint: What happens to the power series when \( x > 1 \)?

(b) In terms of the pole, what is the RoC for the geometric series (Eq. N.1)?

(c) How does the RoC relate to the location of the pole of \( 1/(1-x) \)?

(d) Where are the zeros, if any, in Eq. N.1?

(e) Assuming \( x \) is in the RoC, prove that the geometric series correctly represents \( 1/(1-x) \) by multiplying both sides of Eq. N.1 by \( (1-x) \).

(f) Use the geometric series to study the degree \( N \) polynomial (It is very important to note that all the coefficients of this polynomial are 1)

\[
P_N(x) = 1 + x + x^2 + \ldots + x^N = \sum_{n=0}^{N} x^n. \tag{2.3}
\]

\footnote{The geometric series is not defined as the function \( 1/(1-x) \), it is defined as the series \( 1 + x + x^2 + x^3 + \ldots \), such that the ratio of consecutive terms is \( x \).}
\[ P_N(x) = \frac{1 - x^{N+1}}{1 - x} \]  \hspace{1cm} (2.4)

i. Prove that

ii. What is the RoC for Eq. N.2?

iii. What is the RoC for Eq. N.3?

iv. What is the RoC for Eq. N.4?

v. Evaluate \( P_N(x) \) at \( x = 0 \) and \( x = .9 \) for the case of \( N = 100 \), and compare the result to that from Matlab.

vi. How many poles does \( P_N(x) \) have? Where are they?

vii. How many zeros does \( P_N(x) \) have? State where are they in the complex plane?

viii. Does the above expression have both poles and zeros? Explain.

ix. Explain why Eq. N.3 and N.4 have different numbers of poles and zeros.

(g) Is the function \( 1/(1 - x) \) analytic outside of the RoC stated in part (a)? \textit{Hint: Can it be represented by a different power series outside this RoC?}

2. The exponential series

(a) What is the region of convergence (RoC) for the exponential series given above (e.g. where does the power series \( P(x) \) converge to the function value \( f(x) \))?

(b) Let \( x = j \) in Eq. N.2, and write out the series expansion of \( e^x \) in terms of its real and imaginary parts.

(c) Let \( x = j\theta \) in Eq. N.2, and write out the series expansion of \( e^x \) in terms of its real and imaginary parts. How does your result relate to Euler’s identity \( e^{j\theta} = \cos(\theta) + j\sin(\theta) \)?

Inverse analytic functions and composition

It may be surprising, but every analytic function has an inverse function. Starting from the function \( (x, y \in \mathbb{C}) \)

\[ y(x) = \frac{1}{1 - x} \]

the inverse is

\[ x = \frac{y - 1}{y} = 1 - \frac{1}{y}. \]

1. Considering the inverse function described above

(a) Where are the poles and zeros of \( x(y) \)?

(b) Where (for what condition on \( y \)) is \( x(y) \) analytic?
2. Considering the exponential function \( z(x) = e^x \) (\( x, z \in \mathbb{C} \))
   (a) Find the inverse \( x(z) \).
   (b) Where are the poles and zeros of \( x(z) \)?

3. Compose these two functions \( (y \circ z)(x) \)
   (a) Give the expression for \( (y \circ z)(x) = y(z(x)) \).
   (b) Where are the poles and zeros of \( (y \circ z)(x) \)?
   (c) Where (for what condition on \( x \)) is \( (y \circ z)(x) \) analytic?

**Convolution**

Multiplying two polynomials, when they are short or simple, is not demanding. However if they have many terms, it can become tedious. For example, multiplying two \( 10^{th} \) degree polynomials is not something one would want to do every day.

An alternative is a method called convolution, as described in Lecture 13.

1. Convolution of sequences
   Practice convolution (by hand!!) using a few simple examples. Show you work!!! You may check your solution using Matlab.
   (a) Convolve the sequence [0 1 1 1 1] with itself.
   (b) Convolve [1 1] with itself, then convolve the result with [1 1] again (e.g., calculate \([1, 1] \ast [1, 1] \ast [1, 1])

2. Multiplication of polynomials
   In class, it was shown that multiplying two polynomials is the same as convolving their coefficients. Consider
   \[
   f(x) = x^3 + 3x^2 + 3x + 1 \\
   g(x) = x^3 + 2x^2 + x + 2
   \]
   In Matlab, compute \( h(x) = f(x) \cdot g(x) \) two ways using (a) the commands \texttt{roots} and \texttt{poly}, and (b) the convolution command \texttt{conv}. Confirm that both methods give the same result. That is, compute the convolution \([1, 3, 3, 1] \ast [1, 2, 1, 2])
   What is \( h(x) \)?

**Intersection and analytic geometry**

To find Euclid’s formula, it was necessary to study the intersection of a circle and a secant line. Consider the unit circle of radius 1, centered at \((x, y) = (0, 0)\)
\[
\begin{align*}
  x^2 + y^2 &= 1 \\
  y &= t(x + 1)
\end{align*}
\]
having slope \( t \) and intercept \( x = -1 \). If the slope \( 0 < t < 1 \), the line intersects the circle at a second point \((a, b)\) in the positive \( x,y \) quadrant. The goal is to find \( a,b \in \mathbb{N} \) and then show that \( c^2 = a^2 + b^2 \). Since the construction gives a right triangle with short sides \( a, b \in \mathbb{N} \), then it follows that \( c \in \mathbb{N} \).
Newton’s root-finding method

Newton used the iteration\(^2\)

\[ x_{n+1} = x_n - \frac{P_N(x_n)}{P_N'(x_n)} \]

to find roots of the polynomial \(P_N(x_n)\). Here \(P_N(x) = dP_N(x)/dx\). This relation may be explored as a graph, which puts Newton’s method in the realm of analytic geometry. The function \(P_N'(x)\) is the slope of the polynomial \(P_N(x)\) at \(x_n\). The value of \(x_n\) is the estimate of the root after \(n\) iterations. \(x_0\) is the initial guess.

Example: When the polynomial is \(P_2 = 1 - x^2\), so \(P_2'(x) = -2x\). Newton’s iteration becomes

\[ x_{n+1} = x_n + \frac{1 - x_n^2}{2x_n}. \]

To start the iteration \((n = 0)\) we need an initial guess for \(x_0\), which is a “best guess” of where the root will be. If we let \(x_0 = 1/2\), then

\[ x_1 = x_0 - \frac{1 - x_0^2}{2x_0} = x_0 + \frac{1}{2} (x_0 - 1/x_0). \]

1. Let \(P_2(x) = 1 - x^2\), and \(x_0 = 1/2\). Draw a graph describing the first step of the iteration.
2. Calculate \(x_1\) and \(x_2\). What number is the algorithm approaching? Is it a root of \(P_2\)?
3. Write a Matlab script to check your answer for part (a).
   (a) For \(n = 4\), what is the absolute difference between the root and the estimate, \(|x_r - x_4|\)?
   (b) What happens if \(x_0 = -1/2\)?
4. Does Newton’s method work for \(P_2(x) = 1 + x^2\)? Why?\(^3\)

\(^2\)https://en.wikipedia.org/wiki/Newton’s_method

\(^3\)https://en.wikipedia.org/wiki/Newton’s_method#Complex_functions
this case?

5. What if you let $x_0 = (1 + j)/2$ for the case of $P_2(x) = 1 + x^2$?

**Riemann zeta function $\zeta(s)$**

The zeta function $\zeta(s)$ is defined by the complex analytic power series

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots. \quad (2.5)$$

This series converges, and thus is valid, only in the region of convergence (ROC) given by $\Re(s) = \sigma > 1$ since there $|n^{-\sigma}| < 1$. To determine its formula in other regions of the $s$ plane one must extend the series via analytic continuation.

**Euler product formula:** As was first published by Euler in 1737, one may recursively factor out the leading prime term, resulting in Euler’s product formula.\(^4\) Multiplying $\zeta(s)$ by the factor $1/2^s$, and subtracting from $\zeta(s)$, removes all the terms $1/(2n)^s$ (e.g., $1/2^s + 1/4^s + 1/6^s + 1/8^s + \cdots$)

$$\left(1 - \frac{1}{2^s}\right) \zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} \cdots - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \cdots\right), \quad (2.6)$$

which results in

$$\left(1 - \frac{1}{2^s}\right) \zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \cdots. \quad (2.7)$$

1. Repeat this with a lead factor $1/3^s$ applied to Eq. N.7.

2. Repeat this process, with all prime scale factors (i.e., $1/5^s, 1/7^s, \cdots, 1/\pi_k^s, \cdots$), and show that

$$\zeta(s) = \prod_{\pi_k \in \mathbb{P}} \frac{1}{1 - \pi_k^{-s}} = \prod_{\pi_k \in \mathbb{P}} \zeta_k(s) \quad (2.8)$$

where $\pi_p$ represents the $p^{th}$ prime.

3. Given the product formula we may identify the poles of $\zeta_p(s)$ ($p \in \mathbb{Z}$), which is important for defining the ROC of each factor. For example, the $p^{th}$ factor of Eq. N.8, expressed as an exponential, is

$$\zeta_p(s) \equiv \frac{1}{1 - \pi_p^{-s}} = \frac{1}{1 - e^{-sT_p}}, \quad (2.9)$$

where $T_p \equiv \ln \pi_p$.

Plot $\zeta_p(s)$ using $zviz$ for $p = 1$. Describe what you see.

---

\(^4\)This is known as Euler’s sieve, as distinguish from the Eratosthenes sieve.
2.0.2 Exercises AE-2

**Topic of this homework:** Linear systems of equations; Gaussian elimination; Matrix permutations; Overspecified systems of equations; Analytic geometry; Ohm’s law; Two-port networks

Deliverable: Answers to problems

**Nonlinear (quadratic) to linear equations**

In the following problems we deal with algebraic equations in more than one variable, that are not linear equations. For example, the circle \( x^2 + y^2 = 1 \) is just such an equation. It may be solved for \( y(x) = \pm \sqrt{1 - x^2} \).

If we let \( z_+ = x + yj = x + j\sqrt{1 - x^2} = e^{i\theta} \), we obtain the equation for half a circle \( (y > 0) \). The entire circle is described by the magnitude of \( z \), as \( |z|^2 = (x + yj)(x - yj) = 1 \).

1. Given the curve defined by the equation:
   \[ x^2 + xy + y^2 = 1 \]  \hspace{1cm} (2.10)
   
   (a) Find the function \( y(x) \).
   
   (b) Using Matlab/Octave, plot \( y(x) \), and describe the graph.
   
   (c) What is the name of this curve?

2. Find the solution (in \( x \), \( p \), and \( q \)) to the following equations:
   \[
   \begin{align*}
   x + y &= p \\
   xy &= q
   \end{align*}
   \]

3. Find an equation that is linear in \( y \) starting from equations that are quadratic (2nd degree) in the two unknowns\(^5\) \( x, y \):
   \[
   \begin{align*}
   x^2 + xy + y^2 &= 1 \hspace{1cm} (2.11) \\
   4x^2 + 3xy + 2y^2 &= 3 \hspace{1cm} (2.12)
   \end{align*}
   \]

**Gaussian elimination**

1. Find the inverse of
   \[
   A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.
   \]

2. Verify that \( A^{-1}A = AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \).

\(^5\)This problem is taken from Stillwell, Exercise 6.2.1 (p. 91).
3. Find the solution to the following 3x3 matrix equation $Ax = b$ by Gaussian elimination. Show your intermediate steps. You can check your work at each step using Matlab.

$$
\begin{bmatrix}
1 & 1 & -1 \\
3 & 1 & 1 \\
1 & -1 & 4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
9 \\
8
\end{bmatrix}.
$$

(a) Show (i.e., verify) that the first GE matrix $G_1$, which zeros out all entries in the first column, is given by

$$
G_1 = 
\begin{bmatrix}
1 & 0 & 0 \\
-3 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix}
$$

Identify the elementary row operations that this matrix performs.

(b) Find a second GE matrix, $G_2$, to put $G_1A$ in upper triangular form. Identify the elementary row operations that this matrix performs.

(c) Find a third GE matrix, $G_3$, which scales each row so that its leading term is 1. Identify the elementary row operations that this matrix performs.

(d) Finally, find the last GE matrix, $G_4$, that subtracts a scaled version of row 3 from row 2, and scaled versions of rows 2 and 3 from row 1, such that you are left with the identity matrix ($G_4G_3G_2G_1A = I$).

(e) Solve for $[x_1, x_2, x_3]^T$ using the augmented matrix format $G_4G_3G_2G_1[A|b]$ (where $[A|b]$ is the augmented matrix). Note that if you’ve performed the preceding steps correctly, $x = G_4G_3G_2G_1b$.

**Permutations and Pivots**

(a) Find the pivot matrix $G$ that rescales the third row of the augmented matrix $A|b$ by 1/3.

**Two linear equations**

In this exercise we transition from a general pair of equations

$$
\begin{align*}
f(x, y) &= 0 \\
g(x, y) &= 0
\end{align*}
$$

to the important case of two linear equations

$$
\begin{align*}
y &= ax + b \\
y &= \alpha x + \beta.
\end{align*}
$$

Note that, to help keep track of the variables, Roman coefficients ($a, b$) are used for the first equation and Greek ($\alpha, \beta$) for the second.

1. What does it mean, graphically, if these two linear equations have

(a) a unique solution, 

(b) a non-unique solution, or
CHAPTER 2. ALGEBRAIC EQUATIONS

(c) no solution?

2. Assuming the two equations have a unique solution, find the solution for \( x \) and \( y \).

3. When will this solution fail to exist (for what conditions on \( a, b, \alpha, \) and \( \beta \))?

4. Write the equations as a 2x2 matrix equation of the form \( Ax = b \), where \( x = [x, y]^T \).

5. Finding the inverse of the 2x2 matrix, and solve the matrix equation for \( x \) and \( y \).

6. Discuss the properties of the determinant of the matrix (\( \Delta \)) in terms of the slopes of the two equations (\( a \) and \( \alpha \)).

7. An application of linear functional relationships between two variables:

2x2 matrices are used to describe 2-port networks, as will be discussed in Lec 16. Transmission lines are a great example, where both voltage and current must be tracked as they travel along the line. Figure N.2 shows an example segment of a transmission line.

![Figure 2.2: This figure shows a cell from an LC transmission line. The index 1 is at the input on the left and 2 represents the output, on the right.](image)

Suppose you are given the following pair of linear relationships between the input (source) variables \( V_1 \) and \( I_1 \), and the output (load) variables \( V_2 \) and \( I_2 \) of the transmission line.

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} =
\begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
V_2 \\
I_2
\end{bmatrix}.
\]

(a) Let the output (the load) be \( V_2 = 1 \) and \( I_2 = 2 \) (i.e., \( V_2/I_2 = 1/2 \ [\Omega] \)). Find the input voltage and current, \( V_1 \) and \( I_1 \).

(b) Let the input (source) be \( V_1 = 1 \) and \( I_1 = 2 \). Find the output voltage and current \( V_2 \) and \( I_2 \).

Linear equations with three unknowns

This problem is similar to the previous problem, except we consider 3 dimensions. Consider two linear equations in unknowns \( x, y, z \), representing planes:

\[
a_1x + b_1y + z = c_1 \\
a_2x + b_2y + z = c_2
\]

(2.13)  
(2.14)

1. In terms of the geometry (i.e., think graphically), under what conditions do these two linear equations have (a) a unique solution, (b) a non-unique solution, or (c) no solution?

2. Given 2 equations in 3 unknowns, the closest we can come to a ‘unique’ solution is an equation in \( (x, y) \), \( (y, z) \), or \( (x, z) \). Find a solution in terms of \( x \) and \( y \) by substituting one equation into the other.
3. Now consider the intersection of the planes at some arbitrary constant height, $z = z_0$. Write the modified plane equations as a 2x2 matrix equation in the form $Ax = b$ where $x = [x, y]^T$, and find the unique solution in $x$ and $y$ using matrix operations.

4. When will this solution fail to exist (for what conditions on $a_1, a_2, b_1, b_2$, etc.)?

5. Now, write the system of equations as a 3x3 matrix equation in $x, y, z$ given the additional equation $z = z_0$ (e.g. put it in the form $Ax = b$ where $x = [x, y, z]^T$).

6. The determinant of a 3x3 matrix is given by

$$
\begin{vmatrix}
 a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33}
\end{vmatrix}
= a_{11} \begin{vmatrix}
 a_{22} & a_{23} \\
 a_{32} & a_{33}
\end{vmatrix} - a_{12} \begin{vmatrix}
 a_{21} & a_{23} \\
 a_{31} & a_{33}
\end{vmatrix} + a_{13} \begin{vmatrix}
 a_{21} & a_{22} \\
 a_{31} & a_{32}
\end{vmatrix}
$$

For the 3x3 matrix equation you wrote in the previous part, find the determinant. How is the determinant related to the 2x2 case? Why?

7. Put the following systems of equations in matrix form, and use Matlab to find (i) the determinant of the matrix, (ii) the matrix inverse, and (iii) the solution $(x, y, z)$. If it is not possible to complete (i-iii), state why.

(a)

\begin{align*}
x + 3y + 2z &= 1 \\
x + 4y + z &= 1 \\
x + y &= 1
\end{align*}

(b)

\begin{align*}
x + 3y + 2z &= 1 \\
2x + 6y + 4z &= 1 \\
x + y &= 1
\end{align*}

**Integer equations: applications and solutions (20 pts)**

Any equation for which we seek only integer solutions is called a *Diophantine* equation.

**A practical example of using a Diophantine equation:** “A merchant had a 40-pound weight that broke into 4 pieces. When the pieces were weighed, it was found that each piece was a whole number of pounds and that the four pieces could be used to weigh every integral weight between 1 and 40 pounds. What were the weights of the pieces?” - *Bachet de Bèziriac (1623 CE)*

Here, weighing is performed using a balance scale having two pans, with weights being put on either pan. Thus, given weights of 1 and 3 pounds, one can weigh a 2-pound weight by putting the 1-pound weight in the same pan with the 2-pound weight, and the 3-pound weight in the other pan. Then, the scale will be balanced. A solution to the four weights for Bachet’s problem is $1 + 3 + 9 + 27 = 40$ pounds.

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6Taken from: Joseph Rotman, “A first course in abstract algebra,” *Chapter 1, Number Theory* p. 50
Problem: Show how the combination of 1, 3, 9, & 27 pound weights may be used to weigh 1, 2, 3, ..., 8, 28, and 40 pounds of milk (or something else, such as flour). Assuming that the milk is in the left pan, provide the position of the weights using a negative sign '-' to indicate the left pan and a positive sign '+' to indicate the right pan. For example, if the left pan has 1 pound of milk, then 1 pound of milk in the right pan, '+' will balance the scales.

Hint: It is helpful to write the answer in matrix form. Set the vector of values to be weighed equal to a matrix indicating the pan assignments, multiplied by a vector of the weights $[1, 3, 9, 27]^T$. The pan assignments matrix should only contain the values -1 (left pan), +1 (right pan), and 0 (leave out). You can indicate these using ‘-’, ‘+’, and blank spaces.

Ohm’s Law

In general, impedance is defined as the ratio of a force over a flow. For electrical circuits, the voltage is the ‘force’ and the current is the ‘flow.’ Ohm’s law states that the voltage across and the current through a circuit element are related by the impedance of that element (which may be a function of frequency). For resistors, the voltage over the current is called the resistance, and is a constant (e.g. the simplest case, $V/I = R$). For inductors and capacitors, the voltage over the current is a frequency-dependent impedance (e.g. $V/I = Z(s)$, where $s$ is the complex frequency $s \in \mathbb{C}$).

The impedance concept also holds in mechanics and acoustics. In mechanics, the ‘force’ is equal to the mechanical force on an element (e.g. a mass, dashpot, or spring), and the ‘flow’ is the velocity. In acoustics, the ‘force’ is pressure, and the ‘flow’ is the volume velocity or particle velocity of air molecules.

<table>
<thead>
<tr>
<th>Case</th>
<th>Force</th>
<th>Flow</th>
<th>Impedance</th>
<th>units</th>
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</thead>
<tbody>
<tr>
<td>Electrical</td>
<td>voltage (V)</td>
<td>current (I)</td>
<td>$Z$</td>
<td>Ohms [Ω]</td>
</tr>
<tr>
<td>Mechanics</td>
<td>force (F)</td>
<td>velocity (V)</td>
<td>$Z$</td>
<td>Mechanical Ohms [Ω]</td>
</tr>
<tr>
<td>Acoustics</td>
<td>pressure (P)</td>
<td>particle velocity (U)</td>
<td>$Z$</td>
<td>Acoustic Ohms [Ω]</td>
</tr>
<tr>
<td>Thermal</td>
<td>temperature (T)</td>
<td>heat-flux (J)</td>
<td>$Z$</td>
<td>Thermal Ohms [Ω]</td>
</tr>
</tbody>
</table>

1. The resistance of an incandescent (filament) lightbulb, measured cold, is about 100 ohms. As it lights up, the resistance of the metal filament increases. Ohm’s law says that the current

$$\frac{V}{T} = R(T),$$

where $T$ is the temperature. In the United States, the voltage is 120 volts (RMS) at 60 [Hz]. Find the current when the light is first switched on.

2. The power, in Watts [W], is the product of the force and the flow. What is the power of the light bulb of this example?

3. State the impedance $Z(s)$ of each of the following circuit elements:
(a) A resistor with resistance $R$
(b) An inductor with inductance $L$
(c) A capacitor with capacitance $C$

2-port network analysis

Perform a simple analysis of electrical 2-port networks, shown in Fig. N.1. This can be a mechanical system if the capacitors are taken to be springs, and inductors taken as mass, as in the suspension of the wheels of a car. In an acoustical circuit, the low-pass filter could be a car muffler. While the physical representations will be different, the equations and the analysis are exactly the same.

\[ V_1 \quad R_1 \quad R_2 \quad C \quad I_1 \quad V_2 \quad I_2 \]

\[ P_1 \quad U_1 \quad L = 1 \quad C = 3 \quad Z_{in} \quad C = 2 \quad U_2 \quad P_2 \]

Figure 2.1: Left: A low-pass RC electrical filter. The circuit elements $R_1$, $R_2$, and $C$ are defined in the problems below. Right: A band-pass acoustic filter. Here, the pressure $P$ is analogous to voltage, and the velocity $U$ is analogous to current. The circuit elements are labeled with their $L$ and $C$ values as integers, to make the algebra simple.

The definition of the ABCD transmission matrix $(T)$ is

\[
\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \quad (2.15)
\]

The impedance matrix, where the determinant $\Delta_T = AD - BC$, is given by

\[
\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{C} \begin{bmatrix} A & \Delta_T \\ 1 & D \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad (2.16)
\]

1. Derive the formula for the impedance matrix (Eq. N.16) given the transmission matrix definition (Eq. N.15). Show your work.

2. Consider a single circuit element with impedance $Z(s)$
   (a) What is the ABCD matrix for this element if it is in ‘series’?
   (b) What is the ABCD matrix for this element if it is ‘shunt’?

3. Find the ABCD matrix for each of the circuits of Figure N.1. For each circuit, (i) show the cascade of transmission matrices in terms of the complex frequency $s \in \mathbb{C}$, then (ii) substitute $s = 1j$ and calculate the total transmission matrix at this single frequency.
   (a) Left circuit (let $R_1 = R_2 = 10 \text{ k}\Omega$ ‘kilo-ohms’, and $C = 10 \text{ nF} \ ‘\text{nano-farads}’$)
   (b) Right circuit (use $L$ and $C$ values given in the figure), where the pressure $P$ is analogous to the voltage $V$, and the velocity $U$ is analogous to the current $I$.

4. Convert both transmission (ABCD) matrices to impedance matrices using Equation N.16. Do this for the specific frequency $s = 1j$, as in the previous part (feel free to use Matlab for your computation).
2.0.3 Exercises AE-3

**Topic of this homework:** Visualizing complex functions; Bilinear/Möbius transform; Riemann sphere.

**Algebra with complex variables**

1. One can always say that 3 < 4, namely that real numbers have order. One way to view this is to take the difference, and compare to zero, as in $4 - 3 > 0$. Here, we will explore how complex variables may be ordered. Define the complex variable $z = x + iy \in \mathbb{C}$.

   (a) Explain the meaning of $|z_1| > |z_2|$.

   (b) If $x_1, x_2 \in \mathbb{R}$ (are real numbers), define the meaning of $x_1 > x_2$. *Hint: Take the difference.*

   (c) Explain the meaning of $z_1 > z_2$.

   (d) (*not graded*) If time were complex how might the world be different?

2. It is sometimes necessary to consider a function $w(z) = u + iv$ in terms of the real functions, $u(x, y)$ and $v(x, y)$ (e.g. separate the real and imaginary parts). Similarly, we can consider the inverse $z(w) = x + iy$ where $x(u, v)$ and $y(u, v)$ are real functions.

   (a) Find $u(x, y)$ and $v(x, y)$ for $w(z) = 1/z$.

   (b) Find $u(x, y)$ and $v(x, y)$ for $w(z) = c^z$ with complex constant $c \in \mathbb{C}$ for the following cases

   i. $c = e$

   ii. $c = 1$ (recall that $1 = e^{i2\pi k}$ for $k = 0, 1, 2, \ldots$)

   (c) $c = j$. *Hint: $j = e^{\pi i/2}$.*

   (d) Find $u(x, y)$ for $w(z) = \sqrt{z}$. *Hint: Begin with the inverse function $z = w^2$.*

**Fundamental theorem of algebra (FTA)**

1. State the fundamental theorem of algebra (FTA).

**Möbius transforms and infinity**

The bilinear transform: The *bilinear z transform* (a specific case of the Möbius transformation) is used in signal processing to design a digital (discrete-time) filter $H(z)$ given an analog (continuous time) filter $H(s)$. The goal of the transform is to take a function of analog frequency $\omega_a$, where $\omega_a \in (-\infty, \infty)$, and map it to a finite digital frequency range, $\omega_d \in [-\pi, \pi]$. You will learn more about this if you take ECE 310.
The bilinear z transform is expressed in terms of the complex Laplace frequency $s \equiv \sigma_a + j\omega_a$, where $\omega_a$ is the analog frequency in radians/second, and $z \equiv \rho e^{j\omega_d}$, where $\rho = |z|$ and $\omega_d$ is the digital frequency (it is an angle, in radians). The bilinear transform is given by

$$s = \alpha \frac{1 - z^{-1}}{1 + z^{-1}},$$

(2.17)

where $\alpha$ is a real constant.

1. Suppose you are given the analog low-pass filter $h(t) = e^{-t}u(t)$, which has a frequency response given by

$$H(s) = \frac{1}{s + 1} = \int_0^\infty h(t)e^{-st}dt,$$

where $s = \sigma + j\omega$.

Use the bilinear z transform (Eq. N.17) to find the discrete time filter $H(z)$. Hint: Look at matlab/Octave command help bilinear. Your answer should be a composition of $H(s)$ and Eq. N.17.

2. Substitute $s = j\omega_a$ and $z = e^{j\omega_d}$ ($\sigma_a, \sigma_d = 0$) into the Eq. N.17 to determine the relationship between $\omega_a, \omega_d$. Express your final result using a tangent function. Hint: Try to form sine and cosine terms! Recall that $\sin(\omega) = (e^{j\omega} - e^{-j\omega})/2j$ and $\cos(\omega) = (e^{j\omega} + e^{-j\omega})/2$.

3. By hand, draw a graph of the relationship you found the previous part, $\omega_a = f(\omega_d)$. Make sure to specify the behavior of $\omega_a$ at $\omega_d = 0, \pm \pi/2, \pm \pi$.

4. Explain how this relationship maps the analog frequency $\omega_a \rightarrow \pm \infty$ to the digital frequency $\omega_d$.

5. Draw the $s$ and $z$ planes, showing the real parts on the horizontal axes and the imaginary parts on the vertical axes. Mark (e.g. using thick lines) which sets of values are considered when $\sigma_a, \sigma_d = 0$.

6. Geometrically, what is the effect of this Möbius transform? Consider your drawing in the previous part.
Chapter 3

Differential equations
3.0.1 Exercises DE-1

**Topic of this homework:** Complex numbers and functions (ordering and algebra); Complex power series; Fundamental theorem of calculus (real and complex); Cauchy-Riemann conditions; Multivalued functions (branch cuts and Riemann sheets)

**Two fundamental theorems of calculus**

**Fundamental Theorem of Calculus (Leibniz):** According to the Fundamental Theorem of (Real) Calculus (FTC)

\[ F(x) = F(a) + \int_a^x f(\xi) d\xi, \]

where \( x, a, \xi, F \in \mathbb{R} \). This is an *indefinite integral* (since the upper limit is unspecified). It follows that

\[ \frac{dF(x)}{dx} = \frac{d}{dx} \int_a^x f(x) dx = f(x). \]

This justifies also calling the indefinite integral the *anti-derivative*.

For a closed interval \([a, b]\), the FTC is

\[ \int_a^b f(x) dx = F(b) - F(a), \]

thus the integral is independent of the path from \( x = a \) to \( x = b \).

**Fundamental Theorem of Complex Calculus:** According to the Fundamental Theorem of Complex Calculus (FTCC)

\[ f(z) = f(z_0) + \int_{z_0}^z F(\zeta) d\zeta, \]

where \( z_0, z, \zeta, F \in \mathbb{C} \). It follows that

\[ \frac{df(z)}{dz} = \frac{d}{dz} \int_{z_0}^z F(\zeta) d\zeta = F(z). \]

**To do:**

1. Consider Equation O.1. What is the condition on \( f(x) \) for which this formula is true?
2. Consider Equation O.7. What is the condition on \( f(z) \) for which this formula is true?
3. Perform the following integrals \((z = x + iy \in \mathbb{C})\):
   (a) \( I = \int_0^{1+y} z dz \)
   (b) \( I = \int_0^{1+y} z dz \), but this time make the path explicit: from 0 to 1, with \( y=0 \), and then to \( y=1 \), with \( x=1 \).
   (c) Do your results agree with Equation O.9?
4. Perform the following integrals on the closed path C, which we define to be the unit circle. You should substitute \( z = e^{i\theta} \) and \( dz = i e^{i\theta} d\theta \), and integrate from \([-\pi, \pi]\) to go once around the unit circle.

(a) \( \int_C z\,dz \)
(b) \( \int_C \frac{1}{z}\,dz \)
(c) Do your results agree with Equation O.9? If not, do you know why not?

**Cauchy-Riemann Equations**

For the following problem: \( i = \sqrt{-1} \), \( s = \sigma + i\omega \), and \( F(s) = u(\sigma, \omega) + iv(\sigma, \omega) \).

In class I showed that the integration of a complex analytic function is independent of the path, formally known as the *Fundamental theorem of complex calculus*. The derivative of \( F(s) \) is defined as

\[
\frac{df}{ds} = \frac{d}{ds} [u(\sigma, \omega) + iv(\sigma, \omega)].
\]

(3.4)

If the integral is independent of the path, then the derivative must also be independent of direction

\[
\frac{df}{ds} = \frac{\partial f}{\partial \sigma} = \frac{\partial f}{\partial \omega}.
\]

(3.5)

1. The Cauchy-Riemann (CR) conditions

\[
\frac{\partial u(\sigma, \omega)}{\partial \sigma} = \frac{\partial v(\sigma, \omega)}{\partial \omega} \quad \text{and} \quad \frac{\partial u(\sigma, \omega)}{\partial \omega} = -\frac{\partial v(\sigma, \omega)}{\partial \sigma}
\]

may be used to show where Equation O.5 holds.

**To do:**

(a) Assuming Equation O.5 is true, use it to derive the CR equations.

(b) Merge the CR equations to show that \( u \) and \( v \) obey Laplace’s equation \( \nabla^2 u(\sigma, \omega) = 0 \) and \( \nabla^2 v(\sigma, \omega) = 0 \). One may conclude that the real and imaginary parts of complex analytic functions must obey these conditions.

2. Apply the CR equations to the following functions. State for which values of \( s = \sigma + i\omega \) the CR conditions do or do not hold (e.g. where the function \( F(s) \) is or is not analytic).

*Hint: Recall your answers to problem 1.2 of this assignment.*

(a) \( F(s) = e^s \)
(b) \( F(s) = 1/s \)

**Complex Power Series**

1. In each case derive (e.g. using Taylor’s formula) the power series of \( w(s) \) about \( s = 0 \) and state the ROC of your series. If the power series doesn’t exist, state why! *Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at \( s = 0 \).*
CHAPTER 3. DIFFERENTIAL EQUATIONS

To do:

(a) $1/(1-s)$
(b) $1/(1-s^2)$
(c) $1/(1-s)^2$
(d) $1/(1+s^2)$. \textit{Hint: This series will be very ugly to derive if you try to take the derivatives $\frac{d^n}{dn}[1/(1+s^2)]$. Using the results of our previous homework, you should represent this function as $w(s) = -0.5i/(s-i) + 0.5i/(s+i)$.
(e) $1/s$
(f) $1/(1-|s|^2)$

2. Consider the function $w(s) = 1/s$

(a) Expand this function as a power series about $s = 1$. \textit{Hint: Let $1/s = 1/(1-1+s) = 1/(1-(1-s))$.}
(b) What is the ROC?
(c) Expand $w(s) = 1/s$ as a power series in $s^{-1} = 1/s$ about $s^{-1} = 1$.
(d) What is the ROC?
(e) What is the residue of the pole?

3. Consider the function $w(s) = 1/(2-s)$

(a) Expand $w(s)$ as a power series in $s^{-1} = 1/s$. State the ROC as a condition on $|s^{-1}|$.
\textit{Hint: Multiply top and bottom by $s^{-1}$.}
(b) Find the inverse function $s(w)$. Where are the poles and zeros of $s(w)$, and where is it analytic?

4. If $a = 0.1$ what is the value of

$$x = 1 + a + a^2 + a^3 \cdots ?$$

Show your work.

5. If $a = 10$ what is the value of

$$x = 1 + a + a^2 + a^3 \cdots ?$$

Branch cuts and Riemann sheets

1. Consider the function $[w(z)]^2 = z$. This function can also be written as $w(z) = \sqrt{z}$. Define $z = re^{\theta j}$ and $w(z) = \rho e^{\theta j} = \sqrt{re^{\theta j}/2}$.

(a) How many Riemann sheets do you need in the domain ($z$) and the range ($w$) to fully represent this function as single valued?

Indicate (e.g. using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.

(b) Use \texttt{zviz.m} to plot the positive and negative square roots $+\sqrt{z}$ and $-\sqrt{z}$. Describe what you see.
(c) Where does \texttt{zviz.m} place the branch cut for this function?
(d) Must it necessarily be in this location?

2. Consider the function $w(z) = \log(z)$. As before define $z = re^{\theta j}$ and $w(z) = \rho e^{\theta j}$.

(a) Describe with a sketch, and then discuss the branch cut for $f(z)$.
(b) What is the inverse of this function, $z(f)$? Does this function have a branch cut (if so, where is it)?

3. Using \texttt{zviz.m}, show that

$$
\tan^{-1}(z) = -\frac{j}{2} \log \frac{j - z}{j + z}.
$$

(3.6)

That is, plot both these function and verify they are the same function.

4. Algebraically justify Eq. ??\footnote{\textbf{Hint:} Let $w(z) = \tan^{-1}(z)$, $z(w) = \tan w = \sin w / \cos w$, then solve for $e^{wj}$.}
3.0.2 Exercises DE-2

**Topic of this homework:** Cauchy-Riemann conditions; Integration of complex functions; Cauchy’s theorem, integral formula, residue theorem; power series; Riemann sheets and branch cuts; inverse Laplace transforms

**FTCC and integration in the complex plane**

Recall that, according to the Fundamental Theorem of Complex Calculus (FTCC),

\[ f(z) = f(z_0) + \int_{z_0}^{z} F(\zeta) d\zeta, \]  

(3.7)

where \( z_0, z, \zeta, F \in \mathbb{C} \). It follows that

\[ f(z) = \frac{d}{dz} F(z). \]  

(3.8)

Thus Eq. O.7 is also known as the anti-derivative of \( f(z) \).

**To do:**

1. For a closed interval \([a, b]\), the FTCC can be stated as

\[ \int_{a}^{b} F(z) dz = f(b) - f(a), \]  

(3.9)

meaning that the result of the integral is independent of the path from \( x = a \) to \( x = b \). What condition(s) on the integrand \( f(z) \) is (are) sufficient to assure that Eq. O.9 holds?

2. For the function \( f(z) = c^z \), where \( c \in \mathbb{C} \) is an arbitrary complex constant, use the Cauchy-Riemann (CR) equations to show that \( f(z) \) is analytic for all \( z \in \mathbb{C} \).

3. In the following problems, solve the integral

\[ I = \int_{\mathcal{C}} F(z) dz \]

for a given path \( \mathcal{C} \). In some cases this might be the definite integral (Eq. O.9).

Let the function \( F(z) = e^z \), where \( c \in \mathbb{C} \) is given for each problem below. **Hint: Can you apply the FTCC?**

(a) Find the anti-derivative of \( F(z) \).
(b) \( c = 1/e = 1/2.7183 \ldots \) where \( \mathcal{C} \) is \( \zeta = 0 \to i \to z \)
(c) \( c = 2 \) where \( \mathcal{C} \) is \( \zeta = 0 \to (1 + i) \to z \)
(d) \( c = i \) where the path \( \mathcal{C} \) is an inward spiral described by \( z(t) = 0.99^t e^{it\pi t} \) for \( t = 0 \to t_0 \to \infty \)
(e) \( c = e^{i\tau_0} \) where \( \tau_0 > 0 \) is a real number, and \( \mathcal{C} \) is \( z = (1 - i\infty) \to (1 + i\infty) \). **Hint: Do you recognize this integral? If you do not recognize the integral, please do not spend a lot of time trying to solve it via the ‘brute force’ method.**
Cauchy’s theorems for integration in the complex plane

There are three basic definitions related to Cauchy’s integral formula. They are all related, and can greatly simplify integration in the complex plane. When a function depends on a complex variable we shall use uppercase notation, consistent with the engineering literature for the Laplace transform.

1. **Cauchy’s (Integral) Theorem** (Stillwell, p. 319; Boas, p. 45)

\[ \oint_C F(z) \, dz = 0, \]

if and only if \( F(z) \) is complex-analytic inside of \( C \).

This is related to the Fundamental Theorem of Complex Calculus (FTCC)

\[ f(z) = f(a) + \int_a^z F(z) \, dz, \]

where \( f(z) \) is the anti-derivative of \( F(z) \), namely \( F(z) = df/dz \). The FTCC requires \( F(z) \) to be complex-analytic for all \( z \in \mathbb{C} \). By closing the path (contour \( C \)), Cauchy’s theorem (and the following theorems) allows us to integrate functions that may not be complex-analytic for all \( z \in \mathbb{C} \).

2. **Cauchy’s Integral Formula** (Boas, p. 51; Stillwell, p. 220)

\[ \frac{1}{2\pi j} \oint_C \frac{F(z)}{z - z_0} \, dz = \begin{cases} F(z_0), & z_0 \in C \text{ (inside)} \\ 0, & z_0 \notin C \text{ (outside)} \end{cases} \]

Here \( F(z) \) is required to be analytic everywhere within (and on) the contour \( C \). \( F(z_0) \) is called the residue of the pole.

3. **(Cauchy’s) Residue Theorem** (Boas, p. 72)

\[ \oint_C F(z) \, dz = 2\pi j \sum_{k=1}^K \text{Res}_k, \]

where \( \text{Res}_k \) are the residues of all poles of \( F(z) \) enclosed by the contour \( C \).

**How to calculate the residues:** The residues can be rigorously defined as

\[ \text{Res}_k = \lim_{z \to z_k} [(z - z_k)f(z)]. \]

This can be related to Cauchy’s integral formula: Consider the function \( F(z) = w(z)/(z - z_k) \), where we have factored \( F(z) \) to isolate the first-order pole at \( z = z_k \). If the remaining factor \( w(z) \) is analytic at \( z_k \), then the residue of the pole at \( z = z_k \) is \( w(z_k) \).

**To do:**

1. In one or two brief sentences, describe the relationships between the three theorems:
   
   (a) (1) and (2)
   
   (b) (1) and (3)
2. Consider the function with poles at \( z = \pm j \)

\[
F(z) = \frac{1}{1 + z^2} = \frac{1}{(z - j)(z + j)}
\]

Apply Cauchy’s theorems to solve the following integrals. **State which theorem(s) you used**, and **show your work**.

(a) \( \oint_C F(z) \, dz \) where \( C \) is a circle centered at \( z = 0 \) with a radius of \( \frac{1}{2} \).

(b) \( \oint_C F(z) \, dz \) where \( C \) is a circle centered at \( z = j \) with a radius of 1.

(c) \( \oint_C F(z) \, dz \) where \( C \) is a circle centered at \( z = 0 \) with a radius of 2.

**Integration in the complex plane**

In the following questions, you’ll be asked to integrate \( F(s) = u(\sigma, \omega) + iv(\sigma, \omega) \) around the contour \( C \) for complex \( s = \sigma + i\omega \),

\[
\oint_C F(s) \, ds.
\]

Follow the directions carefully for each question. When asked to state where the function is and is not analytic, you are not required to use the Cauchy-Riemann equations (but you should if you can’t answer the question ‘by inspection’).

**To do:**

1. \( F(s) = \frac{1}{s} \)
   
   (a) State where the function is and is not analytic.
   
   (b) Explicitly evaluate the integral when \( C \) is the unit circle, defined as \( s = e^{i\theta}, 0 \leq \theta \leq 2\pi \).
   
   (c) Evaluate the same integral using Cauchy’s theorem and/or the residue theorem.

2. \( F(s) = \frac{1}{s^2} \)
   
   (a) State where the function is and is not analytic.
   
   (b) Explicitly evaluate the integral when \( C \) is the unit circle, defined as \( s = e^{i\theta}, 0 \leq \theta \leq 2\pi \).
   
   (c) What does your result imply about the residue of the 2\textsuperscript{nd} order pole at \( s = 0 \)?

3. \( F(s) = e^{st} \)
   
   (a) State where the function is and is not analytic.
   
   (b) Explicitly evaluate the integral when \( C \) is the square \((\sigma, \omega) = (1, 1) \rightarrow (-1, 1) \rightarrow (-1, -1) \rightarrow (1, -1) \rightarrow (1, 1)\).
   
   (c) Evaluate the same integral using Cauchy’s theorem and/or the residue theorem.

4. \( F(s) = \frac{1}{s+2} \)
   
   (a) State where the function is and is not analytic.
(b) Let \( C \) be the unit circle, defined as \( s = e^{i\theta}, \) \( 0 \leq \theta \leq 2\pi \). Evaluate the integral using Cauchy’s theorem and/or the residue theorem.

(c) Let \( C \) be a circle of radius 3, defined as \( s = 3e^{i\theta}, \) \( 0 \leq \theta \leq 2\pi \). Evaluate the integral using Cauchy’s theorem and/or the residue theorem.

5. \( F(s) = \frac{1}{2\pi i} \left[ \frac{e^{st}}{s+2} \right] \)

(a) State where the function is and is not analytic.

(b) Let \( C \) be a circle of radius 3, defined as \( s = 3e^{i\theta}, \) \( 0 \leq \theta \leq 2\pi \). Evaluate the integral using Cauchy’s theorem and/or the residue theorem.

(c) Let \( C \) contain the entire left-half \( s \)-plane. Evaluate the integral using Cauchy’s theorem and/or the residue theorem. Do you recognize this integral?

6. \( F(s) = \pm \frac{1}{\sqrt{s}} \) (e.g. \( F^2 = \frac{1}{s} \))

(a) State where the function is and is not analytic.

(b) This function is multivalued. How many Riemann sheets do you need in the domain \((s)\) and the range \((f)\) to fully represent this function? Indicate (e.g. using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.

(c) Explicitly evaluate the integral
\[
\int_C \frac{1}{\sqrt{z}} \, dz
\]
when \( C \) is the unit circle, defined as \( s = e^{i\theta}, \) \( 0 \leq \theta \leq 2\pi \). Is this contour ‘closed’? State why or why not.

(d) Explicitly evaluate the integral
\[
\int_C \frac{1}{\sqrt{z}} \, dz
\]
when \( C \) is twice around the unit circle, defined as \( s = e^{i\theta}, \) \( 0 \leq \theta \leq 4\pi \). Is this contour ‘closed’? State why or why not. Hint: Note that \( \sqrt{e^{i(\theta+2\pi)}} = \sqrt{e^{i2\pi}e^{i\theta}} = e^{i\theta} \sqrt{e^{i\theta}} = -1 \sqrt{e^{i\theta}} \)

(e) What does your result imply about the residue of the (twice-around \( \frac{1}{2} \) order) pole at \( s = 0? \)

(f) Show that the residue is zero (apply the definition of the residue).

A two-port network application for the Laplace transform

The Laplace transform (LT) and inverse Laplace transform (ILT): Recall that the Laplace transform (LT) \( f(t) \leftrightarrow F(s) \)\(^1\) of a causal function \( f(t) \) is
\[
F(s) = \int_0^\infty f(t)e^{-st} \, dt,
\]

\(^1\)Many loosely adhere to the convention that the frequency domain uses upper-case [e.g. \( F(s) \)] while the time domain uses lower case \( [f(t)] \)
where \( s = \sigma + j\omega \) is complex frequency\(^2\) in [radians] and \( t \) is time in [seconds]. Causal functions and the Laplace transform are particularly useful for describing systems, which have no response until a signal enters the system (e.g. at \( t = 0 \)).

To define the inverse Laplace transform (ILT) we need to understand integration in the complex plane

\[
f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s)e^{st}ds = \frac{1}{2\pi j} \oint_C F(s)e^{st}ds.
\]

The Laplace contour \( C \) actually includes two pieces

\[
\oint_C = \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} + \int_{\subset \infty},
\]

where the path represented by ‘\( \subset \infty \)’ is a semicircle of infinite radius with \( \sigma \to -\infty \). It is somewhat tricky to do, but it may be proved that the integral over the contour \( \subset \infty \) goes to zero. For a causal, ‘stable’ (e.g. doesn’t blow up over time) signal, all of the poles of \( F(s) \) must be inside of the Laplace contour, in the left-half \( s \)-plane.

**Transfer functions** Linear, time-invariant systems are described by an ordinary differential equations. For example, consider the first-order linear differential equation

\[
a_1 \frac{d}{dt} y(t) = b_1 \frac{d}{dt} x(t) + b_0 x(t).
\]

This equation describes the relationship between the input \( x(t) \) and output \( y(t) \) of the system. If we define Laplace transforms \( y(t) \leftrightarrow Y(s) \) and \( x(t) \leftrightarrow X(s) \), then this equation may be written in the frequency domain as

\[
a_1 s Y(s) = b_1 s X(s) + b_0 X(s).
\]

The transfer function for this system is defined as

\[
H(s) = \frac{Y(s)}{X(s)} = \frac{b_1 s + b_0}{a_1 s} = \frac{b_1}{a_1} + \frac{b_0}{a_1 s}.
\]

In this problem, we will look at the transfer function of a simple two-port network, shown in Figure O.1. This network is an example of a RC low-pass filter, which acts as a leaky integrator.

**To do:**

1. Use the ABCD method to find the matrix representation of Fig. O.1.

2. Assuming that \( I_2 = 0 \), find the transfer function \( H(s) \equiv V_2/V_1 \). From the results of the ABCD matrix you determined above, show that

\[
H(s) = \frac{1}{1 + R_1 C s}.
\]

\(^2\)While radians are useful units for calculations, when providing physical insight in discussions of problem solutions, it is easier to work with Hertz, since frequency in [Hz] and time in [s] are mentally more more natural units than radians. The same is true of degrees vs. radians. Boas (p. 10) recommends the use degrees over radians. He gives the example of \( 3\pi/5 \) [radians], which is more easily visualize as 108°.
Figure 3.1: This three-element electrical circuit is a system that acts to low-pass filter the signal voltage $V_1(\omega)$, to produce signal $V_2(\omega)$.

3. The transfer function $H(s)$ has one pole. Where is the pole? Find the residue of this pole.

4. Find $h(t)$, the inverse Laplace transform of $H(s)$.

5. Assuming that $V_2 = 0$ find $Y_{12}(s) \equiv I_2/V_1$.

6. Find the input impedance to the right-hand side of the system, $Z_{22}(s) \equiv V_2/I_2$ for two cases:
   (a) $I_1 = 0$
   (b) $V_1 = 0$

7. Compute the determinant of the ABCD matrix. Hint: It is always 1.

8. Compute the derivative of $H(s) = \left. \frac{V_2}{V_1} \right|_{I_2=0}$.

**With the help of a computer**

In the following problems, we will look at some of the concepts from this homework using Matlab/Octave. We are using the `syms` function which requires Matlab’s/Octave’s symbolic math toolbox. Or you may use the EWS lab’s Matlab. Alternative symbolic-math tool, such as Wolfram Alpha.³

1. To find the Taylor series expansion about $s = 0$ of
   \[ F(s) = -\log(1 - s), \]
   first consider the derivative and its Taylor series (about $s = 0$)
   \[ F'(s) = \frac{1}{1 - s} = \sum_{n=0}^{\infty} s^n. \]
   Then, integrate this series term by term
   \[ F(s) = -\log(1 - s) = \int_0^s F'(s)ds = \sum_{n=0}^{\infty} \frac{s^n}{n}. \]
   Alternatively you may use Matlab/Octave commands:
   ```matlab
   syms s
   taylor(-log(1-s),'order',7)
   ```

³https://www.wolframalpha.com/
CHAPTER 3. DIFFERENTIAL EQUATIONS

To do:

(a) Try the above Matlab/Octave commands. Give the first 7 terms of the Taylor series (confirm that Matlab/Octave agrees with the formula derived above).

(b) What is the inverse Laplace transform of this series? Consider the series term by term.

2. The function $1/\sqrt{z}$ has a branch point at $z = 0$, thus it is singular there.

(a) Can you apply Cauchy’s integral theorem when integrating around the unit circle?

(b) Below is a Matlab/Octave code that computes $\int_0^{2\pi} \frac{dz}{\sqrt{z}}$ using Matlab’s/Octave’s symbolic analysis package:

```matlab
syms z
I=int(1/sqrt(z))
J = int(1/sqrt(z),exp(-j*pi),exp(j*pi))
eval(J)
```

To do: Run this script. What answers do you get for $I$ and $J$?

(c) Modify this code to integrate $f(z) = 1/z^2$ once around the unit circle. What answers do you get for $I$ and $J$?

3. Bessel functions can describe waves in a cylindrical geometry (e.g. vibrations of a drum head). The Bessel function has a Laplace transform with a branch cut

$$J_0(t)u(t) \leftrightarrow \frac{1}{\sqrt{1+s^2}}.$$ 

Draw a hand sketch showing the nature of the branch cut. Hint: Use zviz.

To do:

(a) Try the following Matlab/Octave commands, and then comment on your findings.

```matlab
%Take the inverse LT of 1/sqrt(1+s^2)
syms s
I=ilaplace(1/(sqrt((1+s^2))));
disp(I)

%Find the Taylor series of the LT
T = taylor(1/sqrt(1+s^2),10);
disp(T);

%Verify this
syms t
J=laplace(besselj(0,t));
disp(J);

%plot the Bessel function
t=0:0.1:10*pi;
b=besselj(0,t);
plot(t/pi,b);
grid on;
```
(b) When did Friedrich Bessel live? What did he use Bessel functions for?

4. Using \texttt{zviz}, for each of the following functions

i. Describe the plot generated by \texttt{zviz S=Z}.

ii. Are the functions defined below legal Brune impedances? \( \text{i.e., Do they function obey } \Re Z(\sigma > 0) \geq 0 \)? \textit{Hint: Consider the phase (color). Plot \texttt{zviz Z} for a reminder of the colormap.}

(a) \texttt{zviz 1./sqrt(1+S.^2)}
(b) \texttt{zviz 1./sqrt(1-S.^2)}
(c) \texttt{zviz 1./(1+sqrt(S))}

**Inverse Laplace transform of the zeta function \( \zeta(s) \)**

Take the inverse Laplace transform of \( \zeta_p(s) \leftrightarrow z_p(t) \) \eqref{eq:N.9} and describe the result in words. \textit{Hint: Consider the geometric series representation}

\[
\zeta_p(s) = \frac{1}{1 - e^{-sT_p}} = \sum_{k=0}^{\infty} e^{-skT_p}, \tag{3.11}
\]

for which you can easily look up (or may have memorized) the inverse Laplace transform of each term.

**Inverse transform of Product of factors:** The time domain version of Eq. N.8 \eqref{eq:N.8} may be written as the convolution of all the \( z_k(t) \) factors

\[
z(t) \equiv z_2 \ast z_3(t) \ast z_5(t) \ast z_7(t) \cdots \ast z_p(t) \cdots, \tag{3.12}
\]

where \( \ast \) represents time convolution.

![Feedback network diagram](fig:imped:DE2)

**Figure 3.2:** This feedback network is described by a time-domain difference equation with delay \( T_p \), has an all-pole transfer function \( \zeta_p(s) \equiv Q(s)/I(s) \) given by Eq. O.13, which physically corresponds to a stub of a transmission line, with the input at one end and the output at the other. To describe the \( \zeta(s) \) function we must take \( \alpha = -1 \). A transfer function \( Y(s) = V(s)/I(s) \) that has the same poles as \( \zeta_p(s) \), but with zeros as given by Eq. O.14, is the input admittance \( Y(s) = I(s)/V(s) \) of the transmission line, defined at the ratio of the Laplace transform of the current \( i(t) \leftrightarrow I(s) \) over the voltage \( v(t) \leftrightarrow V(s) \). \!(Same as Fig. ?? p. ??).

Explain what this means in physical terms. Start with two terms (e.g., \( z_1(t) \ast z_2 \)).
**Physical interpretation:** Such functions may be generated in the time domain as shown in Fig. O.2 (p. 369), using a feedback delay of $T_p$ seconds described by the two equations in the figure with a unity feedback gain $\alpha = -1$. Taking the Laplace transform of the system equation we see that the transfer function between the state variable $q(t)$ and the input $x(t)$ is given by $\zeta_p(s)$, which is an all-pole function, since

$$Q(s) = e^{-sT_p}Q(s) + V(s), \text{ or } \zeta_p(s) \equiv \frac{Q(s)}{V(s)} = \frac{1}{1 - e^{-sT_p}}. \quad (3.13)$$

Closing the feed-forward path gives a second transfer function $Y(s) = I(s)/V(s)$, namely

$$Y(s) \equiv \frac{I(s)}{V(s)} = \frac{1 - e^{-sT_p}}{1 + e^{-sT_p}}. \quad (3.14)$$

If we take $i(t)$ as the current and $v(t)$ as the voltage at the input to the transmission line, then $y_p(t) \leftrightarrow \zeta_p(s)$ represents the input impedance at the input to the line. The poles and zeros of the impedance interleave along the $j\omega$ axis. By a slight modification $\zeta_p(s)$ may alternatively be written as

$$Y_p(s) = \frac{e^{sT_p/2} + e^{-sT_p/2}}{e^{sT_p/2} - e^{-sT_p/2}} = j \tan(sT_p/2). \quad (3.15)$$

Every impedance $Z(s)$ has a corresponding *reflectance* function given by a Möbius transformation, which may be read off of Eq. O.14 as

$$\Gamma(s) \equiv \frac{1 + Z(s)}{1 - Z(s)} = e^{-sT_p} \quad (3.16)$$

since impedance is also related to the round-trip delay $T_p$ on the line. The inverse Laplace transform of $\Gamma(s)$ is the round trip delay $T_p$ on the line

$$\gamma(t) = \delta(t - T_p) \leftrightarrow e^{-sT_p} \quad (3.17)$$

In terms of the physics, these transmission line equations are telling us that $\zeta(s)$ may be decomposed into an infinite cascade of transmission lines (Eq. O.12), each having a delay given by $T_p = \ln \pi_p$. The input admittance of this cascade may be interpreted as an analytic continuation of $\zeta(s)$ which defines the eigen-modes of that cascaded impedance function.

Working in the time domain provides a key insight, as it allows us to parse out the best analytic continuation of the infinity of possible continuations, that are not obvious in the frequency domain. Transforming to the time domain is a form of analytic continuation of $\zeta(s)$, that depends on the assumption that $z(t)$ is one-sided in time (causal).
3.0.3 Exercises DE-3

Laplace transforms

Given a Laplace transform (LT) pair \( f(t) \leftrightarrow F(s) \), the frequency domain will always be upper-case [e.g. \( F(s) \)] and the time domain lower case [\( f(t) \)] and causal (i.e., \( f(t < 0) = 0 \)). The definition of the forward transform (\( f(t) \rightarrow F(s) \)) is

\[
F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt,
\]

where \( s = \sigma + j\omega \) is the complex Laplace frequency in [radians] and \( t \) is time in [seconds].

The inverse Laplace transform ((LT\(^{-1}\)), \( F(s) \rightarrow f(t) \)) is defined as

\[
f(t) = \frac{1}{2\pi j} \int_{\sigma_0-j\infty}^{\sigma_0+j\infty} F(s)e^{st}ds = \frac{1}{2\pi j} \oint_C F(s)e^{st}ds
\]

with \( \sigma_0 > 0 \in \mathbb{R} \) is a positive constant.

As discussed in the lecture notes (Section 1.4.7, p. 72) we may use the Cauchy Residue Theorem (CRT), to evaluate the \( \mathcal{L}T^{-1} \), by requiring closure of the contour \( C \) at \( \omega j \rightarrow \pm j\infty \)

\[
\oint_C = \int_{\sigma_0-j\infty}^{\sigma_0+j\infty} + \int_{\subset\infty},
\]

where the path represented by ‘\( \subset\infty \)’ is a semicircle of infinite radius. For a causal, ‘stable’ (e.g. doesn’t “blow up” in time) signal, all of the poles of \( F(s) \) must be inside of the Laplace contour, in the full (closed) left-half \( s \)-plane (\( \sigma \leq 0 \)).

Figure 3.3: Three-element mechanical resonant circuit consisting of a spring, mass and dash-pot (e.g., viscous fluid).

Hooke’s law for a spring states that the force \( f(t) \) is proportional to the displacement \( x(t) \), i.e., \( f(t) = Kx(t) \). The formula for a dash-pot is \( f(t) = Rv(t) \), and Newton’s famous formula for mass is \( f(t) = d[Mv(t)]/dt \), which for constant \( M \) is \( f(t) = Mdv/dt \).

The equation of motion for the mechanical oscillator in Fig. O.3 is given by Newton’s second law; the sum of the forces must balance to zero

\[
M \frac{d^2}{dt^2} x(t) + R \frac{d}{dt} x(t) + Kx(t) = f(t).
\]

These three constants – the mass \( M \), resistance \( R \) and stiffness \( K \) – are all real and positive. The dynamical variables are the driving force \( f(t) \leftrightarrow F(s) \), the position of the mass \( x(t) \leftrightarrow X(s) \) and its velocity \( v(t) \leftrightarrow V(s) \), with \( v(t) = dx(t)/dt \leftrightarrow V(s) = sX(s) \).
Brune Impedance

A Brune impedance is defined as the ratio of the force $F(s)$ over the flow $V(s)$, and may be expressed in residue form as

$$Z(s) = c_0 + \sum_{k=1}^{K} \frac{c_k}{s - s_k} = \frac{N(s)}{D(s)}$$

with

$$D(s) = \prod_{k=1}^{K} (s - s_k)$$

and

$$c_k = \lim_{s \to s_k} (s - s_k) D(s) = \prod_{n' = 1}^{K-1} (s - s_n).$$

The prime on index $n'$ means that $n = k$ is not included in the product.

To do:

1. Find the Laplace transform ($\mathcal{L}T$) of the three force relations in terms of the force $F(s)$ and the velocity $V(s)$, along with the electrical equivalent impedance:

   (a) Hooke’s Law $f(t) = K x(t)$.
   (b) Dash-pot resistance $f(t) = R v(t)$.
   (c) Newton’s Law for Mass $f(t) = M \frac{dv}{dt}$.

2. Take the Laplace transform ($\mathcal{L}T$) of Eq. O.18, and find the total impedance $Z(s)$ of the mechanical circuit.

3. What are $N(s)$ and $D(s)$ (e.g. Eq. O.19)?

4. Assume that $M = R = K = 1$, find the residue form of the admittance $Y(s) = 1/Z(s)$ (e.g. Eq. O.19) in terms of the roots $s_{\pm}$. You may check your answer with the Matlab’s residue command.

5. By applying the CRT, find the inverse Laplace transform ($\mathcal{L}T^{-1}$). Use the residue form of the expression that you derived in the previous exercise.

![Figure 3.4: Depiction of a train consisting of cars, treated as a mass $M$ and linkages, treated as springs of stiffness $K$ or compliance $C = 1/K$. Below it is the electrical equivalent circuit, for comparison. The mass is modeled as an inductor and the springs as capacitors to ground. The velocity is analogous to a current and the force $f_n(t)$ to the voltage $v_n(t)$. The train may be accurately modeled as a transmission line (TL), since the equivalent electrical circuit is a lumped model of a TL. This problem is further considered in Lecture O.0.3 (p. ??).](image-url)
**Transfer functions**

In this problem, we will look at the transfer function of a two-port network, shown in Fig. O.4. We wish to model the dynamics of a freight-train having \( N \) such cars. The model of the train consists of masses connected by springs.

The velocity *transfer function* for this system is defined as the ratio of the output to the input velocity. Consider the engine on the left pulling the train at velocity \( V_1 \) and each car responding with a velocity of \( V_n \). Then

\[
H(s) = \frac{V_N(s)}{V_1(s)}
\]

is the frequency domain ratio of the last car having velocity \( V_N \) to \( V_1 \), the velocity of the engine, at the left most spring (i.e., coupler).

**To do:** Use the ABCD method to find the matrix representation of Fig. O.4. Consistent with the figure, break the model into cells each consisting of three elements: a series inductor representing half the mass \( (L = M/2) \), a shunt capacitor representing the spring \( (C = 1/K) \), and another series inductor representing half the mass \( (L = M/2) \). Each cell is symmetric, making the model a cascade of identical cells (P7, p. ??).

At each node define the force \( f_n(t) \leftrightarrow F_n(\omega) \) and the velocity \( v_n(t) \leftrightarrow V_n(\omega) \) at junction \( n \).

1. Write the ABCD matrix \( T \) for a single cell, composed of series mass \( M/2 \), shunt compliance \( C \) and series mass \( M/2 \), that relates the input node 1 to node 2, where

\[
\begin{bmatrix}
F_1 \\
V_1
\end{bmatrix} = T
\begin{bmatrix}
F_2(\omega) \\
-V_2(\omega)
\end{bmatrix}.
\]

Note that here the mechanical force \( F \) is analogous to electrical voltage, and the mechanical velocity \( V \) is analogous to electrical current.

Defining the *wave velocity*

\[
c_o = \frac{1}{\sqrt{MC}} < \omega_c/2 < \omega_c\sqrt{2}/2,
\]

and the wave delay \( T_o \), the wavelength \( \lambda = c_o/f \), and the the distance between cars as \( \Delta_o = c_oT_o \), we may define the Nyquist sampling rate as approximation Eq. ?? follows when \( \omega < \omega_c \).

From Appendix ?? this matrix has eigenvalues

\[
\lambda_\pm = 1 \mp 2s/s_c \approx e^{\pm 2s/s_c} = e^{\mp 2sT_c}.
\]

From this we can interpret the eigenvalues as the cell delay \( T_c = 2/s_c \).

The eigenvectors are

\[
E_\pm = \begin{bmatrix}
\mp \sqrt{M/C} \\
1
\end{bmatrix},
\]

with the characteristic impedance defined as \( r_o = \sqrt{M/C} \).
2. Assuming that \( N = 2 \) and that \( F_2 = 0 \) (two mass problem), find the transfer function \( H(s) \equiv V_2/V_1 \). From the results of the \( T \) matrix you determined above, find

\[
H_{21}(s) = \left. \frac{V_2}{V_1} \right|_{F_2=0}
\]

3. Find \( h_{21}(t) \), the inverse Laplace transform of \( H_{21}(s) \).

4. What is the input impedance \( Z_2 = F_2/V_2 \) if \( F_3 = -r_0 V_3 \)?

5. Simplify the expression for \( Z_2 \) with \( N \to \infty \) by defining the characteristic impedance as \( r_0 = \sqrt{M/C} \) and again assuming that: 1) \( F_3 = -r_0 V_3 \) (i.e., \( -V_3 \) cancels), 2) \( |s/s_c| \to 0 \) (Nyquist approximation).

6. State the ABCD matrix relationship between the first and \( N \)th node, in terms of the cell matrix.

7. Given a \( T \) (ABCD) transmission matrix, the eigenvalues are and vectors are given in Appendix ??, p. ??, repeated here.

**Eigenvalues:**

\[
\begin{bmatrix}
\lambda_+ \\
\lambda_-
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
(A + D) - \sqrt{(A - D)^2 + 4BC} \\
(A + D) + \sqrt{(A - D)^2 + 4BC}
\end{bmatrix}
\]

Due to symmetry, \( A = D \), this simplifies to \( \lambda_\pm = A \mp \sqrt{BC} \) so that the eigenmatrix is

\[
\Lambda = \begin{bmatrix}
A - \sqrt{BC} & 0 \\
0 & A + \sqrt{BC}
\end{bmatrix}
\]

**Eigenvectors:** The eigenvectors simplifying even more

\[
\begin{bmatrix}
E_+ \\
E_-
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2\sqrt{C}} \left[(A - D) \mp \sqrt{(A - D)^2 + 4BC}\right] \\
1
\end{bmatrix} = \begin{bmatrix}
\mp \frac{1}{\sqrt{C}} \\
1
\end{bmatrix}
\]

**Eigenmatrix:**

\[
E = \begin{bmatrix}
-\sqrt{\frac{B}{C}} & +\sqrt{\frac{B}{C}} \\
1 & 1
\end{bmatrix}, \quad E^{-1} = \frac{1}{2} \begin{bmatrix}
-\sqrt{\frac{C}{B}} & 1 \\
+\sqrt{\frac{C}{B}} & 1
\end{bmatrix}
\]

**To do:** What is the velocity transfer function \( H_{N1} = \frac{V_N}{V_1} \)? Hint: Use an eigenmatrix diagonalization, as we did for the Pell equation (Appendix C).
Chapter 4

Vector differential equations
4.0.1 Exercises VC-1

**Topic of this homework:** Vector algebra and fields in \( \mathbb{R}^3 \); Gradient and scalar Laplacian operator; Definitions of Divergence and Curl; Gauss’s (divergence) & Stokes’ (Curl) Law; Schwarz inequality; Quadratic forms; System postulates

**Vector algebra in \( \mathbb{R}^3 \).**

Definitions of the dot, cross and triple product of vectors \( \mathbf{A} \cdot \mathbf{B}, \mathbf{A} \times \mathbf{B} \) and \( \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \) may be found in the class notes in Appendix A.2 (Vectors in \( \mathbb{R}^3 \)). Note: \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \).

\[
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}.
\]

**To Do:**

1. **Scalar product** \( \mathbf{A} \cdot \mathbf{B} \)
   
   (a) If \( \mathbf{A} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z} \) and \( \mathbf{B} = b_x \hat{x} + b_y \hat{y} + b_z \hat{z} \), write out the definition of \( \mathbf{A} \cdot \mathbf{B} \).
   
   (b) The dot product is often defined as \( ||\mathbf{A}|| \, ||\mathbf{B}|| \cos(\theta) \), where \( ||\mathbf{A}|| = \sqrt{\mathbf{A} \cdot \mathbf{A}} \) and \( \theta \) is the angle between \( \mathbf{A}, \mathbf{B} \). If \( ||\mathbf{A}|| = 1 \), describe how the dot product relates to the vector \( \mathbf{B} \).

2. **Vector (cross) product** \( \mathbf{A} \times \mathbf{B} \)
   
   (a) If \( \mathbf{A} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z} \) and \( \mathbf{B} = b_x \hat{x} + b_y \hat{y} + b_z \hat{z} \), write out the definition of \( \mathbf{A} \times \mathbf{B} \).
   
   (b) Show that the cross product is equal to the area of the parallelogram formed by \( \mathbf{A}, \mathbf{B} \), namely \( ||\mathbf{A}|| \, ||\mathbf{B}|| \sin(\theta) \), where \( ||\mathbf{A}|| = \sqrt{\mathbf{A} \cdot \mathbf{A}} \) and \( \theta \) is the angle between \( \mathbf{A} \) and \( \mathbf{B} \).
3. **Triple product** $A \cdot (B \times C)$

Let $A = [a_1, a_2, a_3]^T$, $B = [b_1, b_2, b_3]^T$, $C = [c_1, c_2, c_3]^T$ be three vectors in $\mathbb{R}^3$.

(a) Starting from the definition of the dot and cross product, explain using a diagram and/or words, how one shows that: $A \cdot (B \times C) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

(b) Describe why $|A \cdot (B \times C)|$ is the volume of parallelepiped generated by $A, B$ and $C$.

(c) Explain why three vectors $A, B, C$ are in one plane if and only if the triple product $A \cdot (B \times C) = 0$.

4. Given two vectors $A, \hat{B}$ in the $\hat{x}, \hat{y}$ plane (see Fig. 1), with $B = \hat{y}$ (i.e., $||\hat{B}|| = 1$). Show that $A$ may be split into two orthogonal parts, one in the direction of $B$ and the other perpendicular ($\perp$) to $B$.

$$A = (A \cdot \hat{B}) \hat{B} + \hat{B} \times (A \times \hat{B}) = A_\parallel + A_\perp.$$

**Scalar fields and the $\nabla$ operator**

**To Do:**

1. Let $T(x, y) = x^2 + y$ be an analytic scalar temperature field in 2 dimensions (single-valued $\in \mathbb{R}^2$).

   (a) Find the gradient of $T(x)$ and make a sketch of $T$ and the gradient.

   (b) Compute $\nabla^2 T(x)$, to determine if $T(x)$ satisfies Laplace’s equation.

   (c) Sketch the iso-temperature contours at $T = -10, 0, 10$ degrees.

   (d) The heat flux$^1$ is defined as $J(x, y) = -\kappa(x, y)\nabla T$ where $\kappa(x, y)$ is a constant denoting thermal conductivity at the point $(x, y)$. Assuming $\kappa = 1$ everywhere (the medium is homogenous), plot the vector $J(x, y) = -\nabla T$ at $x = 2, y = 1$. Be clear about the origin, direction and length of your result.

   (e) Find the vector $\perp$ to $\nabla T(x, y)$, namely tangent to the iso-temperature contours. Hint: Sketch it for one $(x, y)$ point (e.g., 2,1) and then generalize.

   (f) The thermal resistance $R_T$ is defined as the potential drop $\Delta T$ over the magnitude of the heat flux $|J|$. At a single point the thermal resistance is

   $$R_T(x, y) = -\nabla T / |J|.$$

   How is $R_T(x, y)$ related to the thermal conductivity $\kappa(x, y)$?

2. **Acoustic wave equation**: Note: In the following problem, we will work in the frequency domain.

---

$^1$The heat flux is proportional to the change in temperature times the thermal conductivity $\kappa$ of the medium, which can be derived from the Heat Equation. See [https://en.wikipedia.org/wiki/Heat_equation#Derivation_in_one_dimension](https://en.wikipedia.org/wiki/Heat_equation#Derivation_in_one_dimension).
The basic equations of acoustics in 1 dimension are

\[-\frac{\partial}{\partial x} P = \rho_0 s V\quad \text{and}\quad -\frac{\partial}{\partial x} V = \frac{s}{\eta_0 P_0} P.\]

Here \(P(x, \omega)\) is the pressure (in the frequency domain), \(V(x, \omega)\) is the volume velocity (integral of the velocity over the wave-front having area \(A\)), \(s = \sigma + \omega f\), \(\rho_0 = 1.2\) is the specific density of air, \(\eta_0 = 1.4\) and \(P_0\) is the atmospheric pressure (i.e., \(10^5\) [Pa]) (see the handout Appendix F.2 for details). Note that the pressure field \(P\) is a scalar (pressure does not have direction), while the volume velocity field \(V\) is a vector (velocity has direction).

We can generalize these equations to 3 dimensions using the \(\nabla\) operator

\[-\nabla P = \rho_0 s V\quad \text{and}\quad -\nabla \cdot V = \frac{s}{\eta_0 P_0} P.\]

(a) Starting from these two basic equations, derive the scalar wave equation in terms of the pressure \(P\),

\[\nabla^2 P = \frac{s^2}{c_0^2} P\]

where \(c_0\) is a constant representing the speed of sound.
(b) What is \(c_0\) in terms of \(\eta_0, \rho_0,\) and \(P_0\)?

(c) Rewrite the pressure wave equation in the time domain, using the time derivative property of the Laplace transform (e.g. \(dx/dt \leftrightarrow sX(s)\)). For your notation, define the time-domain signal using a lowercase letter, \(p(x, y, z, t) \leftrightarrow P\).

Vector fields and the \(\nabla\) operator

To Do:

Vector Algebra

1. Let \(R(x, y, z) \equiv x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}\):

   (a) If \(a, b, c\) are constants, what is \(R(x, y, z) \cdot R(a, b, c)\)?

   (b) If \(a, b, c\) are constants, what is \(\frac{d}{dt}[R(x, y, z) \cdot R(a, b, c)]\)?

2. Find the divergence and curl of the following vector fields:

   (a) \(v = \hat{x} + \hat{y} + 2\hat{z}\)

   (b) \(v(x, y, z) = x\hat{x} + xy\hat{y} + z^2\hat{z}\)

   (c) \(v(x, y, z) = x\hat{x} + xy\hat{y} + \log(z)\hat{z}\)

   (d) \(v(x, y, z) = \nabla(1/x + 1/y + 1/z)\)
Vector & scalar field identities

1. Find the divergence and curl of the following vector fields:
   
   (a) \( \mathbf{v} = \nabla \phi \), where \( \phi(x, y) = xe^y \)
   
   (b) \( \mathbf{v} = \nabla \times \mathbf{A} \), where \( \mathbf{A} = x \mathbf{\hat{x}} + y \mathbf{\hat{y}} + z \mathbf{\hat{z}} \)
   
   (c) \( \mathbf{v} = \nabla \times \mathbf{A} \), where \( \mathbf{A} = y \mathbf{\hat{x}} + x^2 \mathbf{\hat{y}} + z \mathbf{\hat{z}} \)

2. For any differentiable vector field \( \mathbf{V} \), write down two vector-calculus identities that are equal to zero.

3. What is the most general form of a vector field may be expressed in, in terms of scalar \( \Phi \) and vector \( \mathbf{A} \) potentials?

4. Perform the following calculations. If you can state the answer without doing the calculation, explain why.
   
   (a) Let \( \mathbf{v} = \sin(x) \mathbf{\hat{x}} + y \mathbf{\hat{y}} + z \mathbf{\hat{z}} \). Find \( \nabla \cdot (\nabla \times \mathbf{v}) \) \( \text{Hint: Look at Lec 41 on page 83 of the notes, Eq. 1.58, 59.} \)
   
   (b) Let \( \mathbf{v} = \sin(x) \mathbf{\hat{x}} + y \mathbf{\hat{y}} + z \mathbf{\hat{z}} \). Find \( \nabla \times (\nabla \sqrt{\mathbf{v} \cdot \mathbf{v}}) \)
   
   (c) Let \( \mathbf{v}(x, y, z) = \nabla [x + y^2 + \sin(\log(z))] \). Find \( \nabla \times \mathbf{v}(x, y, z) \).

Integral theorems

1. In a few words, identify the law, define what it means, and explain the following formula:
   
   \[
   \int_S \mathbf{n} \cdot \mathbf{v} \, dA = \int_V \nabla \cdot \mathbf{v} \, dV.
   \]

2. What is the name of this formula?
   
   \[
   \int_S (\nabla \times \mathbf{V}) \cdot dS = \oint_C \mathbf{V} \cdot d\mathbf{R}
   \]
   
   Give one important application.

3. Describe a key application of the vector identity
   
   \[
   \nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}.
   \]

Schwarz inequality

Below is a picture of these three vectors for an arbitrary value of \( a \) and a specific \( a = a^* \).
CHAPTER 4. VECTOR DIFFERENTIAL EQUATIONS

1. Find the value of \( a^* \in \mathbb{R} \) such that the length (norm) of \( E \) (i.e., \( ||E|| \geq 0 \)) is minimum? Hint minimize
\[
||E||^2 = E \cdot E = (V + aV) \cdot (V + aU) \geq 0
\] (4.1)
with respect to \( a \).

2. Find the formula for \( ||E(a^*)||^2 \geq 0 \). Hint: Substitute \( a^* \) into Eq. P.1, and show that this results in the \textit{Schwarz inequality}
\[
|U \cdot V| \leq ||U|| ||V||.
\]

3. What is the geometrical meaning of the dot product of two vectors?

4. Give the formula for the dot product between two vectors. Explain the meaning based on Fig. P.0.1.

5. Write the formula for the “dot product” between two vectors: \( U \cdot V \) in \( \mathbb{R}^n \) in polar form (e.g., assume the angle between the vectors is equal to \( \theta \)).

6. How is this related to the Pythagorean theorem?

7. Starting from \( ||U + V|| \) derive the \textit{triangle inequality}
\[
||U + V|| \leq ||U|| + ||V||.
\]

8. The \textit{triangular inequality} \( ||U + V|| \leq ||U|| + ||V|| \) is true for 2 and 3 dimensions: Does it hold for 5 dimensional vectors?

\textbf{Quadratic forms}

A matrix that has positive eigenvalues is said to be \textit{positive-definite}. The eigenvalues are real if the matrix is symmetric, so this is a necessary condition for the matrix to be positive-definite. This condition is related to conservation of energy since the power is the voltage times the current. Given an impedance matrix
\[
V = ZI,
\]
the power \( \mathcal{P} \) is
\[
\mathcal{P} = I \cdot V = I \cdot ZI,
\]
which must be positive definite for the system to obey conservation of energy. For the following problems, consider the \( 2 \times 2 \) \( Z \) matrix
\[
\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}.
\]

1. Solve for the power \( \mathcal{P}(i_1, i_2) \) by multiplying out the matrix equation below (which is in \textit{quadratic form}) \( (I \equiv \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}^T) \)
\[
\mathcal{P}(i_1, i_2) = I^T \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} I.
\]

2. Is the impedance matrix \textit{positive definite}? Show your work by finding the eigenvalues of the matrix \( Z \).

3. Should an impedance matrix always be positive definite? Explain.
System Classification

Provide a one-sentence definition of the following properties:

L/NL : linear(L)/nonlinear(NL):
TI/TV : time-invariant(TI)/time varying(TV):
P/A : passive(P)/active(A):
C/NC : causal(C)/non-causal(NC):
Re/Clx : real(Re)/complex(Clx):

1. Along the rows of the table, classify the following systems: In terms of a table having 5 columns, labeled with the abbreviations: L/NL, TI/TV, P/A, C/NC, Re/Clx:

<table>
<thead>
<tr>
<th>#</th>
<th>Case</th>
<th>Definition</th>
<th>L/NL</th>
<th>TI/TV</th>
<th>P/A</th>
<th>C/NC</th>
<th>Re/Clx</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Resistor</td>
<td>$v(t) = r_0i(t)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Inductor</td>
<td>$v(t) = L \frac{di(t)}{dt}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Switch</td>
<td>$v(t) = \begin{cases} 0 &amp; t \leq 0 \ V_0 &amp; t &gt; 0 \end{cases}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Transistor</td>
<td>$i_{out} = g_m(V_{in})$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>“Resistor”</td>
<td>$v(t) = r_0i(t + 3)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>modulator</td>
<td>$f(t) = e^{2\pi t}g(t)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Using the same classification scheme, characterize the following equations:

<table>
<thead>
<tr>
<th>#</th>
<th>Case:</th>
<th>L/NL</th>
<th>TI/TV</th>
<th>P/A</th>
<th>C/NC</th>
<th>Re/Clx</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A(x)\frac{d^2y(t)}{dx^2} + D(t)y(x, t) = 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\frac{dy(t)}{dt} + \sqrt{t}y(t) = \sin(t)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$y^2(t) + y(t) = \sin(t)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$\frac{\partial y}{\partial x} + xy(t + 1) + x^2y = 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$\frac{dy(t)}{dt} + (t - 1)y^2(t) = ie^t$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
4.0.2 Exercises VC-2

**Topic of this homework:** Maxwell’s equations (ME) and variables (E, D; B, H); Compressible and rotational properties of vector fields; fundamental theorem of vector calculus (Helmholtz’ Theorem); Riemann zeta function; Wave equation.

**Notation:** The following notation is used in this assignment:

1. \( s = \sigma + j\omega \) is the *Laplace frequency*, as used in the Laplace transform.
2. A Laplace transform pair are indicated by the symbol \( \leftrightarrow \): e.g., \( f(t) \leftrightarrow F(s) \).
3. \( \pi_k \) is the \( k^{th} \) prime (i.e., \( \pi_k \in \mathbb{P} \), e.g., \( \pi_k = [2, 3, 5, 7, 11, 13 \ldots] \) for \( k = 1 \ldots 6 \)).

**Partial differential equations (PDEs): Wave equation**

1. Show that d’Alembert’s solution, \( \rho(x, t) = f(t - x/c) + g(t + x/c) \), is a solution to the acoustic pressure wave equation, in 1-dimension:

\[
\frac{\partial^2 \rho(x, t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \rho(x, t)}{\partial t^2},
\]

where \( f(\xi) \) and \( g(\xi) \) are arbitrary functions.

2. Solution to the wave equation in spherical coordinates (i.e, 3-dimensions):

(a) Write out the wave equation in spherical coordinates \( \rho(r, \theta, \phi, t) \). Only consider the radial term \( r \) (i.e., dependence on angles \( \theta, \phi \) is assumed to be zero). *Hint: The form of the Laplacian as a function of the number of dimensions is given in the last appendix on Transmission lines and Acoustic Horns. Alternatively, look it up on the internet or in a calculus book.*

(b) Show that the following is true:

\[
\nabla_r^2 \rho(r) \equiv \frac{1}{r^2 \partial_r} r^2 \frac{\partial}{\partial r} \rho(r) = \frac{1}{r} \frac{\partial^2}{\partial r^2} r \rho(r).
\]

*Hint: Expand both sides of the equation.*

(c) Use the results from Eq. P.2 to show that the solution to the spherical wave equation is

\[
\nabla_r^2 \rho(r, t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \rho(r, t)
\]

\[\rho(r, t) = \frac{f(t - r/c)}{r} + \frac{g(t + r/c)}{r}.\]  \hspace{1cm} (4.4)

(d) With \( f(\xi) = \sin(\xi) \) and \( g(\xi) = e^\xi u(\xi) \) [\( u(\xi) \) is the step function] (Eq. P.4) write down the solutions to the spherical wave equation.
(e) Sketch this last case for several times (e.g., 0, 1, 2 seconds), and describe the behavior of the pressure \( p(r, t) \) as a function of time \( t \) and radius \( r \).

(f) What happens when the inbound wave reaches the center at \( r = 0 \)?

**Helmholtz formula**

Every differentiable vector field may be written as the sum of a *scalar potential* \( \phi \) and *vector potential* \( \mathbf{w} \). This relationship is best known as the *fundamental theorem of vector calculus* (Helmholtz’ formula).

\[
\mathbf{v} = -\nabla \phi + \nabla \times \mathbf{w},
\]

(4.5)

where \( \phi \) is the *scalar potential* and \( \mathbf{w} \) is the *vector potential*. This formula seems a natural extension of the algebraic \( \mathbf{A} \cdot \mathbf{B} - \mathbf{A} \times \mathbf{B} \), since \( \mathbf{A} \cdot \mathbf{B} \propto \|\mathbf{A}\| \|\mathbf{B}\| \cos(\theta) \) and \( \mathbf{A} \times \mathbf{B} \propto \|\mathbf{A}\| \|\mathbf{B}\| \sin(\theta) \) as developed in the notes (Fig. A.1). Thus these orthogonal components have magnitude 1 when we take the norm, due to Euler’s identity \( \cos^2(\theta) + \sin^2(\theta) = 1 \).

**Table 4.1:** Definitions of irrotational, rotational, incompressible and compressible. A *solenoidal* field is an alternative name for an incompressible field, and a conservative field is irrotational.

<table>
<thead>
<tr>
<th>Field type</th>
<th>Definition (most common)</th>
<th>Generator (form of potential)</th>
<th>Test (on ( \mathbf{v} ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Irrotational</td>
<td>( \nabla \times \mathbf{v} = 0 )</td>
<td>( \mathbf{v} = -\nabla \phi )</td>
<td>( \nabla \times \mathbf{v} = 0 )</td>
</tr>
<tr>
<td>Rotational</td>
<td>( \nabla \times \mathbf{v} \neq 0 )</td>
<td>( \mathbf{v} = -\nabla \phi + \nabla \times \mathbf{w} )</td>
<td>( \nabla \times \mathbf{v} \neq 0 )</td>
</tr>
<tr>
<td>Incompressible</td>
<td>( \nabla \cdot \mathbf{v} = 0 )</td>
<td>( \mathbf{v} = \nabla \times \mathbf{w} )</td>
<td>( \nabla \cdot \mathbf{v} = 0 )</td>
</tr>
<tr>
<td>Compressible</td>
<td>( \nabla \cdot \mathbf{v} \neq 0 )</td>
<td>( \mathbf{v} = -\nabla \phi + \nabla \times \mathbf{w} )</td>
<td>( \nabla \cdot \mathbf{v} \neq 0 )</td>
</tr>
<tr>
<td>Conservative</td>
<td>( \mathbf{v} = -\nabla \phi )</td>
<td>( \mathbf{v} = -\nabla \phi )</td>
<td>( \nabla \times \mathbf{v} = 0 )</td>
</tr>
<tr>
<td>Solenoidal</td>
<td>( \nabla \cdot \mathbf{v} = 0 )</td>
<td>( \mathbf{v} = \nabla \times \mathbf{w} )</td>
<td>( \nabla \cdot \mathbf{v} = 0 )</td>
</tr>
</tbody>
</table>

Helmholtz’ formula separates a vector field (i.e., \( \mathbf{v}(\mathbf{x}) \)) into *compressible* and *rotational* parts:

1. The *rotational* (e.g., angular) part is defined by the vector potential \( \mathbf{w} \), requiring \( \nabla \times \nabla \times \mathbf{w} \neq 0 \). A field is *irrotational* (conservative) when \( \nabla \times \mathbf{v} = 0 \), meaning that the field \( \mathbf{v} \) can be generated using only 2 a scalar potential, \( \mathbf{v} = -\nabla \phi \) (note this is how a conservative field is usually defined, by saying there exists some \( \phi \) such that \( \mathbf{v} = -\nabla \phi \)).

2. The *compressible* (e.g., radial) part of a field is defined by the scalar potential \( \phi \), requiring \( \nabla \cdot \nabla \phi = \nabla^2 \phi \neq 0 \). A field is *incompressible* (solenoidal) when \( \nabla \cdot \mathbf{v} = 0 \), meaning that the field \( \mathbf{v} \) can be generated using only a vector potential, \( \mathbf{v} = \nabla \times \mathbf{w} \).

---

2 A note about the relationship between the generating function and the test: You might imagine special cases where \( \nabla \times \mathbf{w} \neq 0 \) but \( \nabla \times \nabla \times \mathbf{w} = 0 \) (or \( \nabla \phi \neq 0 \) but \( \nabla^2 \phi = 0 \)). In these cases, the vector (or scalar) potential can be recast as a scalar (or vector) potential.

Example: Consider a field \( \mathbf{v} = \nabla \phi_0 + \mathbf{b} \) where \( \mathbf{b} = x \hat{x} + y \hat{y} + z \hat{z} \). Note that \( \mathbf{b} \) can actually be generated by either a scalar potential (\( \phi_1 = \frac{1}{2} [x^2 + y^2 + z^2] \), such that \( \nabla \phi_1 = \mathbf{b} \)) or a vector potential (\( \mathbf{w}_0 = \frac{1}{2} [2x \hat{x} + 2y \hat{y} + 2z \hat{z}] \), such that \( \nabla \times \mathbf{w}_0 = \mathbf{b} \)). We find that \( \nabla \times \mathbf{v} = 0 \), therefore \( \mathbf{v} \) must be irrotational. Therefore, we say this irrotational field is generated by \( \nabla \phi = \nabla (\phi_0 + \phi_1) \).
The definitions and generating potential functions of irrotational (conservative) and incompressible (solenoidal) fields naturally follow from two key vector identities:

1. \( \nabla \cdot (\nabla \times \mathbf{w}) = 0 \)
2. \( \nabla \times (\nabla \phi) = 0 \)

**Exercises:**

1. Define the following:
   (a) A conservative vector field
   (b) A irrotational vector field
   (c) An incompressible vector field
   (d) A solenoidal vector field

2. When is a conservative field irrotational?

3. When is a incompressible field irrotational?

4. For each of the following, (i) compute \( \nabla \cdot \mathbf{v} \), (ii) compute \( \nabla \times \mathbf{v} \), (iii) classify the vector field (e.g., conservative, irrotational, incompressible, etc.):
   (a) \( \mathbf{v}(x, y, z) = -\nabla[3yx^3 + y \log(xy)] \)
   (b) \( \mathbf{v}(x, y, z) = xy\mathbf{\hat{x}} - z\mathbf{\hat{y}} + f(z)\mathbf{\hat{z}} \)
   (c) \( \mathbf{v}(x, y, z) = \nabla \times [x\mathbf{\hat{x}} - z\mathbf{\hat{y}}] \)

**Maxwell’s Equations**

The variables have the following names and defining equations:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Equation</th>
<th>Name</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{E} )</td>
<td>( \nabla \times \mathbf{E} = -\mathbf{B} )</td>
<td>Electric Field strength</td>
<td>[Volts/m]</td>
</tr>
<tr>
<td>( \mathbf{D} = \varepsilon_0 \mathbf{E} )</td>
<td>( \nabla \cdot \mathbf{D} = \rho )</td>
<td>Electric Displacement (flux density)</td>
<td>[Col/m²]</td>
</tr>
<tr>
<td>( \mathbf{H} )</td>
<td>( \nabla \times \mathbf{H} = \mathbf{J} + \dot{\mathbf{D}} )</td>
<td>Magnetic Field strength</td>
<td>[Amps/m]</td>
</tr>
<tr>
<td>( \mathbf{B} = \mu_0 \mathbf{H} )</td>
<td>( \nabla \cdot \mathbf{B} = 0 )</td>
<td>Magnetic Induction (flux density)</td>
<td>[Webers/m²]</td>
</tr>
</tbody>
</table>

Note that \( \mathbf{J} = \sigma \mathbf{E} \) is the current density (which has units of [Amps/m²]). Furthermore the speed of light in vacuo is \( c_o = 3 \times 10^8 \) \( = 1/\sqrt{\mu_0 \varepsilon_0} \) [m/s], and the characteristic resistance of light \( r_0 = 377 = \sqrt{\mu_0/\varepsilon_0} \) [\( \Omega \) (i.e., ohms)].

**Speed of light**

The speed of light in vacuo is \( c_o = 1/\sqrt{\mu_0 \varepsilon_0} \approx 3 \times 10^8 \) [m/s]. The characteristic resistance in in-vacuo is \( r_o = \sqrt{\mu_0/\varepsilon_o} \approx 377 \) [\( \Omega \)]. Find a formula for the in-vacuo permittivity \( \varepsilon_o \) and permeability in terms of \( \varepsilon_o \) and \( r_o \). Based on your formula, what are the numeric values values of \( \varepsilon_o \) and \( \mu_o \)?
**Application of ME**

\( \mathbf{E} \) is the *electric field strength* and \( \dot{\mathbf{B}} \) is the time rate of change of the *magnetic induction field*, or simply the magnetic flux density. Consider this equation integrated over a two-dimensional surface \( S \), where \( \hat{n} \) is a unit vector normal to the surface (you may also find it useful to define the closed path \( C \) around the surface):

\[
\iint_S [\nabla \times \mathbf{E}] \cdot \hat{n} dS = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \hat{n} dS
\]

1. Apply Stokes’ theorem to the left-hand side of the equation.

2. Consider the right-hand side of the equation. How is it related to the magnetic flux \( \Psi \) through the surface \( S \)?

3. Assume the right-hand side of the equation is zero. Can you relate your answer to part (a) to one of Kirchhoff’s laws?

The magnetic Maxwell equation is \( \nabla \times \mathbf{H} = \mathbf{C} \equiv \mathbf{J} + \dot{\mathbf{D}} \), where \( \mathbf{H} \) is the *magnetic field strength*, \( \mathbf{J} = \sigma \mathbf{E} \) is the conductive (resistive) current density and the *displacement current* \( \dot{\mathbf{D}} \) is the time rate of change of the *electric flux density* \( \mathbf{D} \). Here we defined a new variable \( \mathbf{C} \) as the total current density.

1. First consider the equation over a two dimensional surface \( S \),

\[
\iint_S [\nabla \times \mathbf{H}] \cdot \hat{n} dS = \iint_S [\mathbf{J} + \dot{\mathbf{D}}] \cdot \hat{n} dS = \iint_S \mathbf{C} \cdot \hat{n} dS
\]

Apply Stokes’ theorem to the left-hand side of this equation. In a sentence or two, explain the meaning of the resulting equation. *Hint: What is the right-hand side of the equation?*

2. Now consider this equation in three dimensions. Take the divergence of both sides, and integrate over a volume \( V \) (closed surface \( S \)).

\[
\iiint_V \nabla \cdot [\nabla \times \mathbf{H}] dV = \iiint_V \nabla \cdot \mathbf{C} dV
\]

(a) What happens to the left-hand side of this equation? *Hint: Can you apply a vector identity?*

(b) Apply the divergence theorem (sometimes known as Gauss’s theorem) to the right-hand side of the equation, and interpret your result. *Hint: Can you relate your result to one of Kirchhoff’s laws?*

**Second-order differentials**

1. If \( \mathbf{v}(x, y, z) = \nabla \phi(x, y, z) \), then what is \( \nabla \cdot \mathbf{v}(x, y, z) \)?

2. Evaluate \( \nabla^2 \phi \) and \( \nabla \times \nabla \phi \) for \( \phi(x, y) = xe^y \).

3. Evaluate \( \nabla \cdot (\nabla \times \mathbf{v}) \) and \( \nabla \times (\nabla \times \mathbf{v}) \) for \( \mathbf{v} = x\mathbf{x} + y\mathbf{y} + z\mathbf{z} \).

4. When \( \mathbf{V}(x, y, z) = \nabla(1/x + 1/y + 1/z) \) what is \( \nabla \times \mathbf{V}(x, y, z) \)?

5. When was Maxwell born (and die)? How long did he live (within \( \pm 10 \) years)?
CHAPTER 4. VECTOR DIFFERENTIAL EQUATIONS

Capacitor analysis

1. Find the solution to the Laplace equation between two infinite \(^3\) parallel plates, separated by a distance of \(d\). Assume that the left plate, at \(x = 0\), is at a voltage of \(V(0) = 0\), and the right plate, at \(x = d\), is at a voltage of \(V_d \equiv V(d)\).

2. Write down Laplace’s equation in one dimension for \(V(x)\).

3. Write down the general solution to your differential equation for \(V(x)\).

4. Apply the boundary conditions \(V(0) = 0\) and \(V(d) = V_d\) to solve for the constants in your equation from the previous part.

5. Find the charge density per unit area (\(\sigma = Q/A\), where \(Q\) is charge and \(A\) is area) on the surface of each plate. **Hint:** \(E = -\nabla V\), and Gauss’s Law states that \(\int\int_S \mathbf{D} \cdot \hat{n} dS = Q_{\text{enclosed}}\).

6. Determine the per-unit-area capacitance \(C\) of the system.

Webster Horn Equation

Horns provide an important generalization of the solution of the 1D wave equation, in regions where the properties (i.e., area of the tube) vary along the axis of wave propagation. Classic applications of horns are vocal tract acoustics, loudspeaker design, cochlear mechanics, any case having wave propagation.

**To do:** Write out the formula for the Webster horn equation, and explain the variables.

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\(^3\)We study plates that are infinite because this means the electric field lines will be perpendicular to the plates, running directly from one plate to the other. However, we will solve for per-unit-area characteristics of the capacitor.
Bibliography