# An Invitation to Mathematical Physics
# *and its History*


*Jont B. Allen*

Help from Steve Levinson, John D'Angelo, Michael Stone

*University of Illinois at Urbana-Champaign*


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## Abstract

An understanding of physics requires knowledge of mathematics. The contrary is not true. By definition, pure mathematics contains no physics. Yet historically, mathematics has a rich history filled with physical applications. Mathematics was developed by people with intent of making things work. In my view, as an engineer, I see these creators of early mathematics, as budding engineers. This book is an attempt to tell this story, of the development of mathematical physics, as viewed by an engineer.

The book is broken down into three topics, called streams, presented as five chapters: 1) Introduction, 2) Number systems, 3) Algebra Equations, 4) Scalar Calculus, and 5) Vector Calculus. The material is delivered as 41 "Lectures" spread out over a semester of 15 weeks, three lectures per week, with time out for administrative duties. Problems are provided for each week's assignment. These problems are written out in LaTeX, with built in solutions, that may be expressed by uncommenting one line. Once the home-works are turned in, each student is given the solution. With regard to learning the material, the students rated these Assignments as the most important part of the course. There is a built in interplay between these assignments and the lectures. On many occasions I solved the homework in class, as motivation for coming to class. Four exams were given, one at the end of each of the three sections, and a final. Some of the exams were in class and some were evening exams, that ran over two hours. The final was two hours. Each of the exams, like the assignments, is provided as a LaTeX file, with solutions encoded with a one line software switch. The Exams are largely based on the Assignments. It is my philosophy that, in principle, the students could see the exam in advance of taking it.

## Author's Personal Statement

It has been said that an expert is someone who has made all the mistakes in a small field. That perfectly describes me. If you learn from your mistakes, and openly admit them, you will come away the better. For years I have said "I love making mistakes, because I learn so much from it." One might call that the "expert's corollary."

This book has been written out of both my love for the topic of mathematical physics, and a frustration for wanting to share many key concepts, and even new ideas on these basic concepts. Over the years I have developed a certain *physical sense* of math, along with a related mathematical sense of physics, that I wish to share. While doing my research,[1] I have come across what I feel are certain conceptual holes that need filling, and sense many deep relationships between math and physics, that remain unidentified. While what we presently teach is not wrong, it is missing these relationships. What is lacking is an intuition for how math "works." We need to start listening to the language of mathematics. We need to let mathematics guide us toward our engineering goals.

It is my strong suspicion that over the centuries many others before me have used a similar insights, and like me, have been unable to convey this slight misdirection. I hope these views can be useful to open young minds.

In my view it is the marriage of math and physics will help us to make progress in understanding the world around us.[2] I turn to mathematics and physics when trying to understand the universe. I have arrived in my views following life long attempt in understand human communication. This

[1] http://auditorymodels.org/index.php/Main/Publications

[2] Venn diagram here.

research arose from my 32 years at Bell Labs in the Acoustics Research Department. There such lifelong pursuits were not only possible, they were openly encouraged. The idea was that if you were successful at something, take it as far as you can. But on the other side, don't do something well that's not worth doing. People got fired for the latter. I should have left for University after a mere 20 years,[3] but the job was just too cushy.

In this text it is my goal to clarify some of the conceptual errors when telling the story about physics and mathematics, that young engineering minds can learn from. My views have been often inspired by classic works, as documented in the bibliography. This present book was inspired by my careful reading of Stillwell (2002), through Chapter 21 (Fig. 2). Somewhere in Chapter 22 I stopped reading and switched to the third edition (Stillwell, 2010), where I saw there was much more to master. At that point I saw that teaching this material to sophomores would allow me to absorb the more advanced material at a reasonable pace. This idea led to to the present book.

[3] I should have left when AT&T Labs was formed, c1997. I started around December 1970, fresh out of Graduate school, and retired in December 2002.

### Back Cover Summary

This is foremost a math book, but not the typical math book. First, this book is for the engineering minded, for those who need to understand math to do engineering, to learn how things work. In that sense it is more about physics and engineering. Math skill are critical to making progress in building things, be it pyramids or computers, as clearly shown by the many great civilizations of the Chinese, Egyptians, Arabs (people of Mesopotamia), Greeks and Romans.

Second, this is a book about the math that developed to explain physics, to allow people to engineer complex things. To sail around the world one needs to know how to navigate. This requires a model of the planets and stars. You can only know where you are on earth if you understand where earth is, relative to the heavens. The answer to such a deep questions will depend on who you ask. The utility and accuracy of that answer depends critically on the depth of understanding of how the worlds and heavens work. Who is qualified to answer such question? It is best answered by those who study mathematics applied to the physical world.

Halley (1656–1742), the English astronomer, asked Newton (1643–1727) for the equation that describes the orbit of the planets. Halley was obviously interested in comets. Newton immediately answered "an ellipse." It is said that Halley was stunned by the response (Stillwell, 2010, p. 176), as this was what had been experimentally observed by Kepler (c1619), and thus he knew Newton must have some deeper insight (Stillwell, 2010, p. 176).

When Halley asked Newton to explain how he knew this correct answer, Newton said he calculated it. But when challenged to show the calculation, Newton was unable to reproduce it. This open challenge eventually led to Newton's grand treatise, *Philosophiae Naturalis Principia Mathematica* (July 5, 1687). It had a humble beginning, more as a letter to Halley, explaining how to calculate the orbits of the planets. To do this Newton needed mathematics, a tool he had mastered. It is widely accepted that Isaac Newton and Gottfried Leibniz invented calculus. But the early record shows that perhaps Bhāskara II (1114–1185 AD) had mastered this art well before Newton.[4]

Third, the main goal of this book is to teach engineering mathematics, in a way that it can be understood, remembered, and mastered, by anyone motivated to learn this topic. How can this near impossible goal be achieved? The answered is to fill in the gaps with "who did what, and when." There is an historical story that may be told and mastered, by anyone serious about the science of making things.

One cannot be an expert in a field if they do not know the history of that field. This includes who the people were, what they did, and the credibility of their story. Do you believe the Pope or Galileo, on the topic of the relative position of the sun and the earth? The observables provided by science are clearly on Galileo's side. Who were those first engineers? They are names we all know: Archimedes, Pythagoras, Leonardo da Vinci, Galileo, Newton, .... All of these individuals had mastered mathematics. This book teaches the tools taught to every engineer. Do not memorize complex formulas, rather make the equations "obvious" by teaching the simplicity of the underlying concept.

Units are SI; Angles in degrees unless otherwise noted. Ex: $\sin(\pi), e^{\jmath 90^\circ} e^{\jmath\pi/2}$.

[4] http://www-history.mcs.st-and.ac.uk/Projects/Pearce/Chapters/Ch8_5.html

### Credits

Besides my parents, I would like to credit John Stillwell for his constructive, historical summary of mathematics. My close friend and colleague Steve Levinson somehow drew me into this project, without my even knowing it. My brilliant graduate student Sarah Robinson was constantly at my side, grading home-works and exams, and tutoring the students. Without her, I would not have survived the first semester the material was taught. Her proof-reading skills are amazing. Thank you Sarah for your infinite help. Finally I would like to thank John D'Angelo for putting up with my many silly questions. When it comes to the heavy hitting, John was always there to provide a brilliant explanation that I could easily understand.

To write this book I had to master the language of mathematics (John's language). I had already mastered the language of engineering, and a good part of physics.[5] But we are all talking about the same thing. Via the physics and engineering, I already had a decent understanding of the mathematics, but I did not know that language. Hopefully, now I can get by.

Finally I would like to thank my wife (Sheau Feng Jeng aka Patricia Allen) for her unbelievable support and love. She delivered constant piece of mind, without which this project could never have been started, much less finish.

There are many others who played important roles, but they must remain anonymous, out of my fear of offending someone I forgot to mention.

–Jont Allen, Mahomet IL, Dec. 24, 2015

[5]Each genre (i.e, group ) speaks their own dialect. One of my secondary goals is to bring down this scientific Tower of Babble.

# Contents

# Preface

It is widely acknowledged that interdisciplinary science is the backbone of modern scientific investigation. This is embodied in the STEM (Science, Technology, Engineering, and Mathematics) programs. Contemporary research is about connecting different areas of knowledge, thus it requires an understanding of cross-disciplines. However, while STEM is being taught, interdisciplinary science is not, due to its inherent complexity and breadth. Furthermore there are few people to teach it. Mathematics, Engineering and Physics (MEP) are at the core of such studies.[6]

## STEM vs. MEP

Mathematics is based on the application rigor. Mathematicians specifically attend to the definitions of increasingly general concepts. Thus mathematics advances slowly, as these complex definitions must be collectively agreed upon. Mathematics shuns controversy, and embraces rigor, the opposite of uncertainty. Physics explores the fringes of uncertainty. Physicists love controversy. Engineering addresses the advancement the technology. Engineers, much like mathematicians, are uncomfortable with uncertainty, but are trained to deal with it.

To create such an interdisciplinary STEM program, a unified MEP curriculum is needed. In my view this unification could (should) take place based on a core mathematical training, from a historical perspective, starting with Euclid or before (i.e., Chinese mathematics), up to modern information theory and logic. As a bare minimum, the *fundamental theorems of mathematics* (arithmetic, algebra, calculus, vector calculus, etc.) need to be appreciated by every MEP student. The core of this curriculum is outlined in Table 1.1.

If, in the sophomore semester, students are taught a common MEP methodology and vocabulary, presented in terms of the history of mathematics, they will be equipped to

1. Exercise interdisciplinary science (STEM)
2. Communicate with other MEP trained (STEM) students and professors.

Students with such a training would end up being our top students, as defined in terms of long-term outcomes, because they would have the necessary comprehensive understanding of the fundamental concepts needed to do engineering.

The key tool is methodology. The traditional approach is a five to six course sequence: Calc I, II, III, DiffEq IV, Linear Algebra V and Complex Variables VI, over a time frame of three years (six semesters). This was the way I learned math. Following such a formal training regime, I felt I had not fully mastered the material, so I started over. I now consider myself to be self-taught. We need a more effective teaching method. I am not suggesting we replace the standard 6 semester math curriculum, rather I am suggesting replacing Calc I, II with this mathematical physics course, based on the historical thread, for those students who have demonstrated advanced ability. One needs more than a high school education to succeed in college engineering courses.

By teaching mathematics in the context of history, the student can fully appreciate the underlying principles. Including the mathematical history provides a uniform terminology for understanding the

[6] I prefer MEP over STEM, as being better focused on the people that do the work, organized around their scientific point of view.

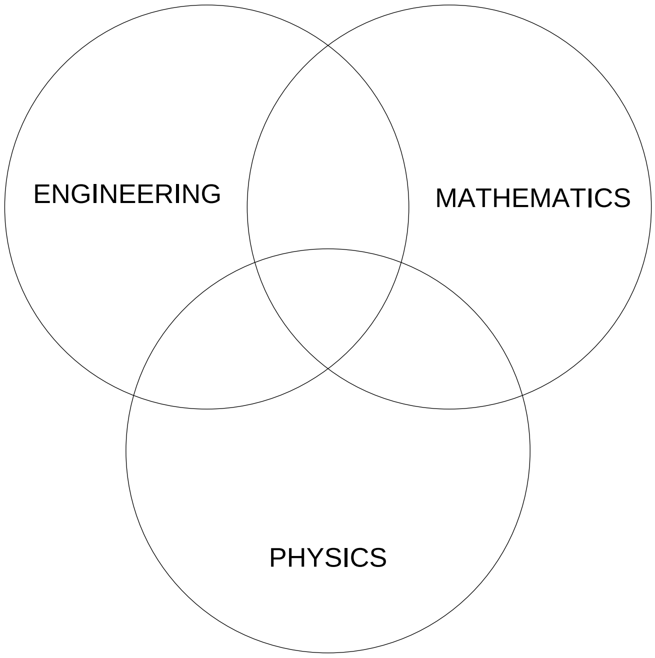


Figure 1: *There is a natural symbiotic relationship between Physics, Mathematics and Engineering, as depicted by this Venn diagram. Physics explores the boundaries. Mathematics provides the method and rigor. engineering transforms the method into technology. While these three disciplines work well together, there is poor communication due to a different vocabulary.*

fundamentals of mathematics. The present teaching method, using abstract proofs, with no (or few) figures or physical principles, by design removes intuition and the motivation that was available to the creators of these early theories. This present six semester approach does not function for many students, leaving them with a poor intuition.

## Mathematics and its History (Stillwell, 2002)

Contents



*Figure 2: Table of contents of Stillwell (2002)*

# Chapter 1

# Introduction

Much of early mathematics centered around the love of art and music, due to our sensations of light and sound. Exploring our physiological senses required a scientific understanding of vision and hearing, as first explored by Newton (1687) and Helmholtz (1863a) (Stillwell, 2010, p. 261).[1] Our sense of color and musical pitch are determined by the frequencies of light and sound. The Chinese and Pythagoreans are well known for their early contributions to music theory. Pythagoras strongly believed that "all is integer," meaning that every number, and every concept, could be explained by integral relationships. It may be that this belief was based on Chinese mathematics from thousands of years earlier. It is also known that his ideas about the importance of integers were based on what was known about music theory in those days. For example it was known that the relationships between the musical notes (pitches) obey natural integral relationships.

Other important modern applications of number theory are present with

- Public-private key encryption: which requires the computationally intensive factoring of large integers
- IEEE Floating point[2]

As acknowledged by Stillwell (2010, p. 16), the Pythagorean view is relevant today

> *With the digital computer, digital audio, and digital video coding everything, at least approximately into sequences of whole numbers, we are closer than ever to a world in which "all is number."*

Mersenne (1588-1647) contributed to our understanding of the relationship between the wavelength and the length of musical instruments. These results were extended by Galileo's father, and then Galileo himself (1564-1642). Many of these musical contributions resulted in new mathematics, such as the discovery of the wave equation by Newton (c1687), followed by its one-dimensional general solution by d'Alembert (c1747).

Galileo famously conceptualized an experiment in 1589 where he suggested dropping two different weights from the Leaning Tower of Pisa, and showed that they must take the same time to hit the ground. Conceptually this is an important experiment, driven by a mathematical argument in which he considered the two weights to be connected by an elastic cord. This resulted in the concept of conservation of energy, one of the cornerstones of modern physical theory.

By that time there was a basic understanding that sound and light traveled at very different speeds.

> Ole Rõmer first demonstrated in 1676 that light travels at a finite speed (as opposed to instantaneously) by studying the apparent motion of Jupiter's moon Io. In 1865, James Clerk Maxwell proposed that light was an electromagnetic wave, and therefore traveled at the speed c appearing in his theory of electromagnetism.[3]

[1] https://en.wikipedia.org/wiki/Acoustics
[2] https://en.wikipedia.org/wiki/IEEE_floating_point\#Formats
[3] https://en.wikipedia.org/wiki/Speed_of_light

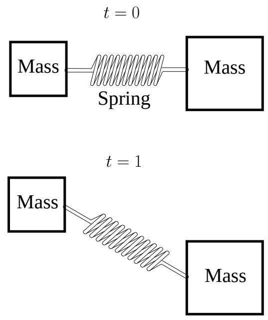


Figure 1.1: *Depiction of the argument of Galileo (unpublished book of 1638) as to why weights of different masses (size) must fall with identical velocity. By joining them with an elastic cord they become one. Thus if the velocity depended on the mass, the joined masses would fall even faster. This results in a logical fallacy. This may have been the first time that the principle of* conservation of energy *was clearly stated.*

While Newton may be best known for his studies on light, he was the first to predict the speed of sound. However his theory was in error by[4] $\sqrt{c_p/c_v} = \sqrt{1.4} = 1.183$. This famous error would not be resolved for over two hundred years, awaiting the formulation of thermodynamics by Laplace, Maxwell and Boltzmann, and others. What was needed was the concept of constant-heat, or *adiabatic process*. For audio frequencies (0.02-20 [kHz]), the small temperature gradients cannot diffuse the distance of a wavelength in one cycle (Pierce, 1981; Boyer and Merzbach, 2011), "trapping" the heat energy in the wave. There were several other physical enigmas, such as the observation that sound disappears in a vacuum and that a vacuum cannot draw water up a column by more than 34 feet.

There are other outstanding examples where physiology impacted mathematics. Leonardo da Vinci is well known for his studies of the human body. Helmholtz's theories of music and the perception of sound are excellent examples of under-appreciated fundamental mathematical contributions (Helmholtz, 1863a). Lord Kelvin (aka William Thompson),[5] was one of the first true engineer-scientists, equally acknowledged as a mathematical physicist and well known for his interdisciplinary research, knighted by Queen Victoria in 1866. Lord Kelvin coined the term *thermodynamics*, a science more fully developed by Maxwell (the same Maxwell of electrodynamics). Thus the interdisciplinary nature of science has played many key roles in the development of thermodynamics.[6] Lord Rayleigh's book on the theory of sound (Rayleigh, 1896) is a classic text, read even today by anyone who studies acoustics.

It seems that we have detracted from this venerable interdisciplinary view of science by splitting the disciplines into into smaller parts whenever we perceived a natural educational boundary. Reforging these natural connections at some point in the curriculum is essential for the proper training of students, both scientists and engineers.

[7]

## WEEK 1: 1.1.0

L1 Intro+timeline;
History of early mathematics: Diophantus, Archimedes, Euclid, ...
The Pythagorean theorem & its three streams:
1) Number Systems, 2) Geometry, 3) Infinity (sets)
Mathematical terminology and definitions

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[4] The square root of the ratio of the specific heat capacity at constant pressure to that at constant volume

[5] Lord Kelvin was one of half a dozen interdisciplinary mathematical physicists, all working about the same time, that made a fundamental change in our scientific understanding. Others include Helmholtz, Stokes, Green, Heaviside, Rayleigh and Maxwell.

[6] Thermodynamics is another example of a course that needs reworking along historical lines.

[7] Perhaps its time to put the MEP Humpty Dumpty back together.

L2 Number Systems; Stream 1:
  a) First use of 0, $\infty$
  b) Taxonomy & Terminology of Numbers: $\pi_k \in \mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{I} \subset \mathbb{R} \subset \mathbb{C}$
  c) Fundamental Theorems of Mathematics:
    – Primes (Factoring numbers; density of primes; Number theory)
    – Algebra (FT Arith, Bèzout's Thm)
    – Scalar Calculus $\mathbb{R}^1$ Leibnez Integral Rule, $\mathbb{R}^2$ & $\mathbb{C}$ Complex
    – $\mathbb{R}^3$ Vector Calculus (Green's Theorems: Gauss', Stokes' Laws)

L3 The many reasons why integers $\mathbb{N}$ and primes $\mathbb{P} \in \mathbb{N}$ are important:
    – RSA (public/private key systems)
    – Physics: Eigenmodes in Mathematics, Physics (QM), Music and Acoustics (e.g., Strings, Chinese Bells & chimes)

## 1.1 Early Science and Mathematics

The first 5,000 years is not well document, but the basic record is clear, as outlined in Fig. 1.2. Thanks to Euclid and later Diophantus (c250 CE), we have some limited understanding of what they studied. For example, Euclid's formula (Fig. 2.7, Eq. 2.7) provides a method for computing all Pythagorean triplets (Stillwell, 2010, pp. 4-9).

Chinese Bells and stringed musical instruments were exquisitely developed in their tonal quality, as documented by ancient physical artifacts (Fletcher and Rossing, 2008). In fact this development was so rich that one must question why the Chinese failed to initiate the industrial revolution. Specifically, why did Europe eventually dominate with its innovation when it was the Chinese who did the extensive early invention?

According to Lin (1995) this is known as the *Needham question*:

> "Why did modern science, the mathematization of hypotheses about Nature, with all its implications for advanced technology, take its meteoric rise only in the West at the time of Galileo[, but] had not developed in Chinese civilization or Indian civilization?"

Needham cites the many developments in China:[8]

> "Gunpowder, the magnetic compass, and paper and printing, which Francis Bacon considered as the three most important inventions facilitating the West's transformation from the Dark Ages to the modern world, were invented in China." (Lin, 1995)

> "Needham's works attribute significant weight to the impact of Confucianism and Taoism on the pace of Chinese scientific discovery, and emphasizes what it describes as the 'diffusionist' approach of Chinese science as opposed to a perceived independent inventiveness in the western world. Needham held that the notion that the Chinese script had inhibited scientific thought was 'grossly overrated' " (Grosswiler, 2004).

Lin was focused on military applications, missing the importance of non-military applications. A large fraction of mathematics was developed to better understand the solar system, acoustics, musical instruments and the theory of sound and light. Eventually the universe became a popular topic, and still is today.

[8] `https://en.wikipedia.org/wiki/Joseph_Needham\#cite_note-11`

## Chronological history pre 16$^{th}$ century 1.1.2a

*200$^{th}$ BCE Chinese (Primes; quadratic equation; Euclidean algorithm (GCD))*

*180$^{th}$ BCE Babylonia (Mesopotamia/Iraq) (quadratic equation)*

*6$^{th}$ BCE Pythagoras (Thales) and the Pythagorean "tribe"*

*4$^{th}$ BCE Archimedes 300BCE; Euclid (quadratic equation)*

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*3$^{th}$ CE Diophantus c250CE;*

*4$^{th}$ CE Alexandria Library destroyed 391CE;*

*7$^{th}$ CE Brahmagupta (negative numbers; quadratic equation)*

*9$^{th}$ CE al-Khwārizmī (algebra) 830CE*

*15$^{th}$ Leonardo & Copernicus 1473-1543*

*16$^{th}$ Tartaglia (cubic eqs); Bombelli 1526-1572; Galileo Galilei 1564-1642*

## Time Line

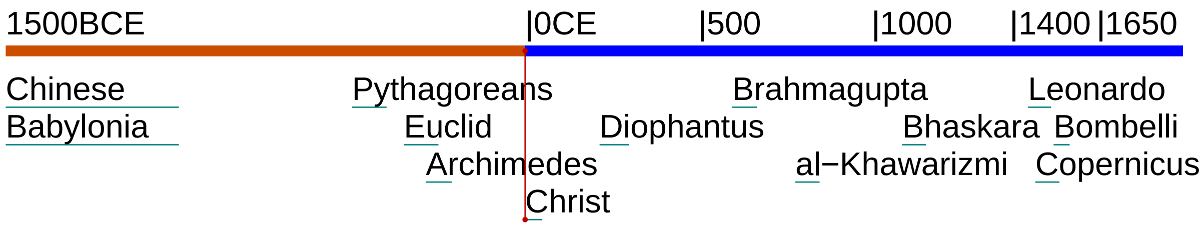


*Figure 1.2: Mathematical time-line between 1500* BCE *and 1650* CE*.*

### 1.1.1 Lec 1 The Pythagorean Theorem

While early Asian mathematics is not fully documented, it clearly defined the course for at least several thousand years. The first recorded mathematics was that of the Chinese (5000-1200 BCE) and the Egyptians (3,300 BEC). Some of the best early record were left by the people of Mesopotamia (Iraq, 1800 BEC). Thanks to Euclid's Elements (c323 BEC) we have an historical record, tracing the progress in geometry, as defined by the Pythagorean theorem *for any right triangle*

$$c^2 = a^2 + b^2. \tag{1.1}$$

having sides of lengths $(a, b, c)$, that are positive real numbers with $c > [a, b]$ and $a + b > c$. Such solutions were likely found by trial and error rather than by an algorithm.

figure

If $a, b, c$ are lengths, then $a^2, b^2, c^2$ are areas. Equation 1.1 says that the area $a^2$ of a square plus the area $b^2$ of a square equals the area $c^2$ of square. Today a simple way to prove this is to compute the magnitude of the complex number $c = a + b\jmath$, forcing the right angle

$$|c|^2 = (a + b\jmath)(a - b\jmath) = a^2 + b^2.$$

However, complex arithmetic was not an option for the Greek mathematicians, since complex numbers and algebra had yet to be invented.

Almost 700 years after Euclid's *Elements*, the Library of Alexandria was destroyed (391 EC) by fire, taking with it much of the accumulated Greek knowledge. Thus one of the best technical records may be Euclid's Elements, along with some sparse mathematics due to Archimedes (c300 BEC) on Geometrical series, computing the volume of a sphere and the area of the Parabola, and elementary hydrostatics. Additionally, a copy of a book by Diophantus *Arithmetic* was discovered by Bombelli (c1572) in the Vatican library (Stillwell, 2010, p. 51).

### 1.1.2 Pythagorean Triplets

Well before Pythagoras, the Babylonians had tables of *Pythagorean triplets* (PTs), integers $[a, b, c]$ that obey Eq. 1.1. An example is $[3, 4, 5]$. A stone tablet (Plimpton-322) dating back to 1800 [BCE] (Fig. 2.6) was found with integers for $[a, c]$. Given such sets of two numbers, which determined a third positive integer $b$ such that $b = \sqrt{c^2 - a^2}$, this table is more than convincing that the Babylonians were well aware of PTs, but less convincing that they had access to Euclid's Formula (Eq. 2.3).

It seems likely that Euclid's Elements was largely the source of the fruitful 6th century era due to the Greek Mathematician Diophantus (Fig. 1.2) who developed the concept of *discrete mathematics*, now known as Diophantine analysis.

The work of Diophantus was followed by a rich mathematical era, with the discovery of 1) early calculus (Brahmagupta, $7^{th}$ CE), 2) algebra (al-Khwārizmī, $9^{th}$ CE), and 3) complex arithmetic (Bombelli, $15^{th}$ CE). This period overlapped with the European middle (i.e., dark) ages. Presumably European intellectuals did not stop thinking during these many centuries, but what happened in Europe is presently unclear given the available records.[9]

### 1.1.3 What is mathematics?

Mathematics is a language, not so different from other languages. Today's mathematics is a written language with an emphasis on symbols and glyphs, biased toward Greek letters. The etymology of these symbols would be interesting to study. Each symbol is dynamically assigned a meaning, appropriate for the problem being described. These symbols are then assembled to make sentences. It is similar to Chinese in that the spoken and written version are different across dialects. In fact like Chinese, the sentences may be read out loud in the language (dialect) of your choice, while the mathematical sentence is universal.

**Math is a language:** It seems strange when people complain that they "can't learn math,"[10] but they claim to be good at languages, because math is a language, with the symbols taken from various languages, with a bias toward Greek, due to the popularity of Euclid's Elements. Learning a new language is fun because it opens doors to other cultures.

Math is different in the rigor of the rules of the language, along with the way it is taught (e.g., not as a language). A third difference between math and the romance languages is that math evolved from physics, with important technical applications. This was the concept behind the Pythagorean school, a band of followers called the *Pythagoreans*. Learning languages is an advanced social skill. Thus the social outcomes are very different between learning math and a romance language. A further problem is that pre-high-school, students confuse arithmetic with math. The two topics are very different, and students need to understand this difference. One does not need to be good at arithmetic to be good at math (but it doesn't hurt).

There are many rules that must be mastered. These rules are defined by algebra. For example the sentence $a = b$ means that the number $a$ has the same value as the number $b$. The sentence may be spoken as "a equals b." The numbers are nouns in this context and the equal sign says they are equivalent, playing the role of a verb, or action symbol. Following the rules of algebra, this sentence may be rewritten as $a - b = 0$. Here the symbols for minus and equal indicate two types of actions.

Sentences can become arbitrarily complex, such as the definition of the integral of a function, or a differential equation. But in each case, the mathematical sentence is written down, may be read out loud, has a well defined meaning, and may be manipulated according to the rules of algebra and calculus. This language of mathematics is powerful, with deep consequences, known as proofs.

The writer of an equation must translate (explicitly summarize the meaning of the expression), so the reader will not miss the main points. This is simply a matter of clear writing.

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[9]It might be interesting to search the archives of the monasteries, where the records were kept, to figure out what happened during this strange time.

[10]"It looks like Greek to me."

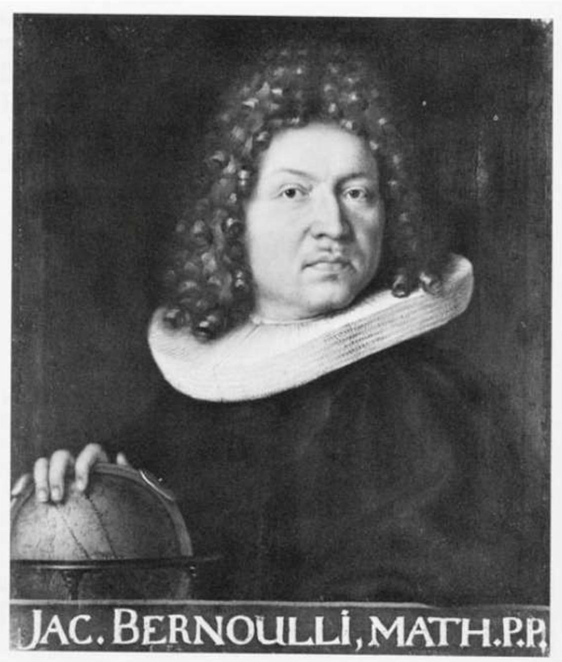


Figure 13.10: Portrait of Jakob Bernoulli by Nicholas Bernoulli

Figure 13.11: Johann Bernoulli

Figure 10.4: Leonhard Euler


Figure 1.3: *Above: Jakob (1655-1705) and Johann (1667-1748) Bernoulli; Below: Leonhard Euler (1707) and Jean le Rond d'Alembert (1717-1783). The figure numbers are from Stillwell (2010).*

Language can be thought of as mathematics (turning this idea on its head). To properly write correct English it can be necessary to understand the construction of the sentence. It is important to know the subject, verb, object, and various types of modifying phrases. If you wish to read about this, look up the distinction between the words *that* and *which*, which make a nice example of this concept. Most of us working directly with what we think sounds right, but if you're learning English as a second language, it is very helpful to understand these mathematical rules, which are arguably easier to master than the foreign phonemes (speech sounds).

### 1.1.4 Early Physics as Mathematics

Mathematics has many functions, but basically it summarizes an algorithm (a set of rules). It was clear to Pythagoras (and many others before him), that there was an important relationship between mathematics and the physical world. Pythagoras may have been one of the first to capitalize on this relationship, using science and mathematics to design and make things.[11] This was the beginnings of technology as we know it today, coming from the relationship between physics and math, impacting map making, tools, implements of war (the wheel, gunpowder), art (music), sound, water transport, sanitation, secure communication, food, . . . , etc.

Why is Eq. 1.1 called a *theorem*, and what exactly needs to be proved? We do not need to prove that $(a, b, c)$ obey this relationship, since this is a condition that is observed. We do not need to prove that $a^2$ is the area of a square, as this is the definition of the area of a square. What needs to be proved is that this relation only holds if the angle between the two shorter sides is 90°.

To appreciate the significance of this development it is helpful to trace the record back to before the time of the Greeks. The Pythagorean theorem (Eq. 1.1) did not begin with Euclid or Pythagoras. Rather Euclid and Pythagoras appreciated the importance of these ideas and documented them.

In the end the Pythagoreans were destroyed by fear. This may be the danger of mixing math with politics:

> "Whether the complete rule of number (integers) is wise remains to be seen. It is said that when the Pythagoreans tried to extend their influence into politics they met with popular resistance. Pythagoras fled, but he was murdered in nearby Mesopotamia in 497 BCE."
>
> –Stillwell (2010, p. 16)

### 1.1.5 The birth of modern mathematics

Modern mathematics (what we know today) was born in the 15-16$^{th}$ century, in the hands of Leonardo da Vinci, Bombelli, Galileo, Descartes, Fermat, and many others (Stillwell, 2010). Many of these early master were, like the Pythagoreans, secretive to the extreme about how they solved problems. They had no interest in sharing their ideas. This soon changed by Mersenne, Descartes and Newton, causing mathematics to blossom.

**The amazing Bernoulli family** The first individual that seems to have openly recognized the importance of mathematics, to actually teach it, was Jacob Bernoulli (Fig. 1.3). Jacob worked on what is now view as the standard package of analytic "circular" functions: $\sin(x)$, $\cos(x)$, $\exp(x)$, $\log(x)$. Eventually the full details were developed (for real variables) by Euler (Section 1.3.8 and 3.4.1).

From Fig. 1.4 we see that he was contemporary to Galileo, Mersenne, Descartes, Fermat, Huygens, Newton, and Euler. Thus it seems likely that he was strongly influenced by Newton, who in turn was influenced by Descartes,[12] and Vìte and Wallis (Stillwell, 2010, p. 175). With the closure of Cambridge University due to the plague of 1665, Newton returned home, Woolsthorpe-by-Colsterworth (95 [mi] north of London), to worked by himself for over a year.

[11] It is likely that the Chinese and Egyptians also did this, but this is more difficult to document.

[12] `https://en.wikipedia.org/wiki/Early_life_of_Isaac_Newton`

Discuss Newton and Euler along with $\log(x)$, $\exp(x)$, $\sin(x)$, $\cos(x)$ and $\zeta(x)$, with $x \in \mathbb{R}$, and the eventual transition to complex arguments $x \to x + y\jmath$.

Jacob Bernoulli, like all successful most mathematicians of the day, was largely self taught. Yet Jacob was in a new category as a mathematician because he was an effective teacher.Jacob taught his sibling Johann, who then taught his sibling Daniel, but most importantly, Leonhard Euler (Figs. 1.4, 1.3), the most prolific (thus influential) of all mathematicians. This resulted in an explosion of new ideas and understanding. It is most significant that all four mathematicians published their methods and findings. Much later Jacob studied with students of Descartes (Stillwell, 2010, p. 268-9).[13]

It is clear that Euler went far beyond all the Bernoulli family, Jacob, Johann and Daniel, (Stillwell, 2010, p. 315). A special strength of Euler was the degree to which he published. First he would master a topic and then he would publish. His papers continued to appear long after his death (Calinger, 2015).

Another individual of that time of special note, who also published extensively, was d'Alembert (Figs. 1.4, 1.3). Some of the most important tools were first proposed by d'Alembert. Unfortunately, and perhaps somewhat unfairly, his rigor was criticized by Euler and later by Gauss (Stillwell, 2010).

Once the tools were being openly published, mathematics grew exponentially. It was one of the most creative times in mathematics. Figure 1.4 shows the list of the many famous names, and their relative time-line. To aid in understand the time line, note that Leonhard Euler was a contemporary of Benjamin Franklin, James Clerk Maxwell of Abraham Lincoln. [14]

**Chronological history post $16^{th}$ century** **1.1.2b**

$17^{th}$ *Galieo 1564-1642, Kepler 1571-1630, Newton 1642-1727* Principia *1687; Mersenne; Huygen; Pascal; Fermat, Descartes (analytic geometry); Bernoullis Jakob, Johann & son Daniel*

$18^{th}$ *Euler 1748* Student of Johann Bernoulli*; d'Alembert 1717-1783; Kirchhoff; Lagrange; Laplace; Gauss 1777-1855*

$19^{th}$ *Möbius, Riemann 1826-1866, Galois, Hamilton, Cauchy 1789-1857, Maxwell, Heaviside, Cayley, von Helmholtz, Rayleigh*

$20^{th}$ *Hilbert; Einstein; ...*

**Time Line**

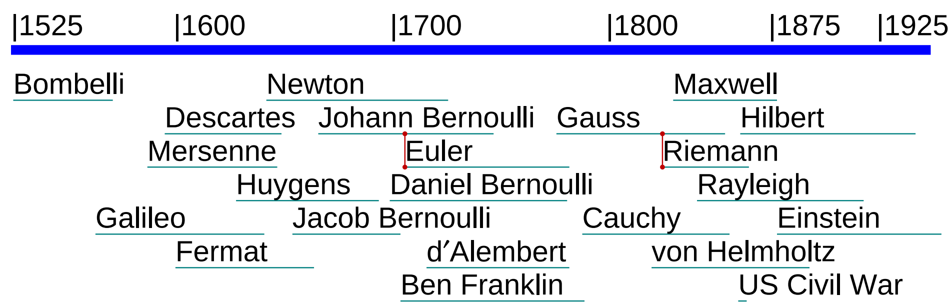


Figure 1.4: *Time-line of the four centuries from the $16^{th}$ and $20^{th}$* CE

### 1.1.6 Three Streams from the Pythagorean Theorem

From the outset of his presentation, Stillwell (2010, p. 1) defines "three great *streams* of mathematical thought: *Numbers, Geometry and Infinity,*" that flow from the Pythagorean theorem, as summarized in Table 1.1. Namely the Pythagorean theorem is the spring from which flow the three streams of *all* mathematics. This is a useful concept, based on reasoning not as obvious as one might think. Many factors are in play here. One of these was the strongly held opinion of Pythagoras that all

[13]It is clear that Descartes was also a teacher.

[14]Lincoln traveled through Mahomet IL (where I live) on his way to the Urbana Court house.

mathematics should be based on integers. The rest are tied up in the long, necessarily complex history of mathematics, as best summarized by the Fundamental Theorems, which are each discussed in detail in the appropriate chapter.

Stillwell's concept of three streams following from the Pythagorean theorem is the organizing principle behind the this book, organized by chapter:

1. *Introduction* (Chapter 1) A detailed overview of the fundamentals and the three streams are presented in Sections 1.2–1.5.

2. *Number Systems* (Chapter 2: Stream 1) Fundamentals of number systems, starting with prime numbers, through complex numbers, vectors and matrices.

3. *Algebraic Equations* (Chapter 3: Stream 2) Algebra and its development, as we know it today. The theory of real and complex equations and functions of real and complex variables. Complex impedance $Z(s)$ of complex frequency $s = \sigma + \omega \jmath$ is covered with some care, given its importance for engineering mathematics.

4. *Scalar Calculus* (Chapter 4: Stream 3a) Ordinary differential equations. Integral theorems. Acoustics.

5. *Vector Calculus*: (Chapter 5: Stream 3b) Vector Partial differential equations. Gradient, divergence and curl differential operators. Stokes, and Green's theorems. Maxwell's equations.

Table 1.1: *Three streams followed from Pythagorean theorem: Number Systems (Stream 1), Geometry (Stream 2) and Infinity (Stream 3).* **1.1.3**

- *The Pythagorean Theorem is the mathematical spring which bore the three streams.*
- *≈Several centuries per stream:*

*1)* **Numbers:**

| | |
|---|---|
| *$6^{th}$BC* | $\mathbb{N}$ *counting numbers,* $\mathbb{Q}$ *(Rationals),* $\mathbb{P}$ *Primes* |
| *$5^{th}$BC* | $\mathbb{Z}$ *Common Integers,* $\mathbb{I}$ *Irrationals* |
| *$7^{th}$CE* | zero $\in \mathbb{Z}$ |

*2)* **Geometry:** *(e.g., lines, circles, spheres, toroids, ...)*

| | |
|---|---|
| *$17^{th}$CE* | *Composition of polynomials (Descartes, Fermat)* |
| | *Euclid's Geometry + algebra ⇒ Analytic Geometry* |
| *$18^{th}$CE* | *Fundamental Theorem of Algebra* |

*3)* **Infinity:** *($\infty \to$ Sets)*

| | |
|---|---|
| *17-$18^{th}$CE* | $\mathbb{F}$ *Taylor series, Functions, Calculus (Newton)* |
| *$19^{th}$CE* | $\mathbb{R}$ *Real,* $\mathbb{C}$ *Complex 1851* |
| *$20^{th}$CE* | *Set theory* |

## 1.2 Stream 1: Number Systems

This era produced a new stream of Fundamental Theorems. A few of the individuals who played a notable role in this development, in chronological (birth) order, include Galileo, Mersenne, Newton, d'Alembert, Fermat, Huygens, Descartes and Helmholtz. These individuals were some of the first to develop the basic ideas, in various forms, that were then later reworked into the proofs, that today we acknowledge as *The Fundamental Theorems of mathematics*.

Number theory (discrete, i.e., integer mathematics) was a starting point for many key ideas. For example, in Euclid's geometrical constructions the Pythagorean theorem for $\{a, b, c\} \in \mathbb{R}$ was accepted as true, but the emphasis in the early analysis was on integer constructions, such as Euclid's formula for Pythagorean triplets (Eq. 2.3, Fig. 2.7)

As we shall see, the Pythagorean theorem is a rich source of mathematical constructions, such as composition of polynomials, and solutions of Pell's equation by eigenvector and recursive analysis methods. Recursive difference equation solutions predate calculus, at least going back to the Chinese (c2000 BCE). These are early (pre-limit) forms of differential equations, best analyzed using an eigenfunction expansion (Appendix B), a powerful geometrical concept from linear algebra, of an expansion in terms of an orthogonal set of normalized (unit-length) vectors.

**The first use of zero and $\infty$:** It is hard to imagine that one would not appreciate the concept of zero and negative numbers when using an abacus. If five beads are moved up, and one is moved down, then four are left. Then if four more are move down, that leaves zero. Taking away is the opposite of addition, and taking away from four to get zero beads, is no different than taking four away from zero, to get negative four beads. Subtraction, the inverse of addition, seems like an obvious idea, on an abacus.

However, understanding the concept of zero and negative numbers is not the same as having a symbolic notation. The Roman number system had no such symbols. The first recorded use of a symbol for zero is said to be by Brahmagupta in 628 CE.[15] Thus it does not take much imagination to go from counting numbers $\mathbb{N}$ to the set of all integers $\mathbb{Z}$, including zero, but apparently it takes 600 years to develop a terminology that represents these ideas. Defining the rules of subtraction required the creation of algebra c830 CE (Fig. 1.2). The concept that caused far more difficulty was $\infty$. Until Riemann's thesis in 1851 it was not clear if $\infty$ was a number, many numbers, or even definable.

### 1.2.1 Lec 2: The Taxonomy of Numbers: $\mathbb{P}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{I}, \mathbb{R}, \mathbb{C}$

First use of 0, $\infty$; Taxonomy of Numbers
Review of the Fund Theorems of Math: Primes, Factoring Scalar Calc, Vect Calc; disc of integers

Once symbols for zero and negative numbers were defined (and accepted), progress was made. In a similar manner, to fully understand numbers, a transparent notation is required. First one must differentiate between the different classes (genus) of numbers, providing a notation that defines each of these classes, along with their relationships. It is logical to start with the most basic *counting numbers*, which we indicate with the double-bold symbol $\mathbb{N}$. All the double-bold symbols and their genus are summarized in Appendix A.

**Counting numbers $\mathbb{N}$:** These are known as the "natural numbers" $\{1, 2, 3, \cdots\} \in \mathbb{N}$, denoted by the double-bold symbol $\mathbb{N}$. For increased clarity we shall refer to the natural numbers as *counting numbers*, to clarify that *natural* means *integer*. The mathematical sentence $2 \in \mathbb{N}$ is read as *2 is a member of the set of counting numbers*. The word *set* means the *sharing of a specific property*.

**Primes $\mathbb{P}$:** A prime number $\mathbb{P} \subset \mathbb{N}$ (set $\mathbb{P}$ is a subset of $\mathbb{N}$) is an integer that may not be factored, other than by 1 and itself. Since $1 = 1 \cdot 1$, $1 \notin \mathbb{P}$ as it is seen to violate this basic definition. Prime numbers $\mathbb{P}$ are a *subset* of the counting numbers, denoted by $\mathbb{P} \subset \mathbb{N}$. We shall use the convenient notation $\pi_n$ for the prime numbers, indexed by $n \in \mathbb{N}$. The first 12 primes $(n = 1 \ldots 12)$ are $\pi_n = 2, 3, 5,\ 7, 11, 13,\ 17, 19, 23,\ 29, 31, 37$. For example, $4 = 2^2$ and $6 = 2 \cdot 3$ may be factored. Thus $\{4, 6\} \notin \mathbb{P}$ (read as: *4 and 6 are not in the set of primes*). Given this definition, multiples of a prime, $n\pi_k \equiv [2, 3, 4, 5, ...] \times \pi_k$ of any prime $\pi_k$, cannot be prime. It follows that all primes except 2 must be odd and that every integer $N$ is unique in its factorization.

*Coprimes* are number whose factors are distinct (they have no common factors). Thus 4 and 6 are not coprime, since they have a common factor of 2, whereas $21 = 3 \cdot 7$ and $10 = 2 \cdot 5$ are coprime. By definition all distinct primes are coprime. The notation $m \perp n$ indicates that $m, n$ are coprime.

[15] The fall of the Roman Empire was Sept. 4, 476.

The *Fundamental Theorem of Arithmetic* states that all integers may be uniquely expressed as a product of primes. The *Prime Number Theorem* estimates the mean density of primes over $\mathbb{N}$.

**Integers $\mathbb{Z}$:** These include positive and negative counting numbers and zero. Notionally we might indicate this using *set notation* as $\mathbb{Z} : \{-\mathbb{N}, 0, \mathbb{N}\}$. Read this as *The integers are in the set composed of the negative of the natural numbers ($-\mathbb{N}$), zero, and counting numbers* $\mathbb{N}$. Note that $\mathbb{N} \subset \mathbb{Z}$.

**Rational numbers $\mathbb{Q}$:** These are defined as numbers formed from the ratio of two integers. Since the integers $\mathbb{Z}$ include 1, it follows that integers are a subset of rational numbers ($\mathbb{Z} \subset \mathbb{Q}$). For example the rational number $3/1 \in \mathbb{Z}$). The main utility of rational numbers is that that they can efficiently approximate any number on the real line, to any precision. For example $\pi \approx 22/7$ with a relative error of $\approx$0.04%. Of course, if the number is rational the error is zero.

**Fractional number $\mathbb{F}$ :** The utility of rational numbers is their power to approximate irrational numbers ($\mathbb{R} \not\subset \mathbb{Z}$). It follows that a subset of the rationals, that excludes the integers, has great value. We call these numbers *Fractional numbers* and assign them the symbol $\mathbb{F}$. They are defined as the subset of rationals that are not integers. From this definition $\mathbb{F} \not\subset \mathbb{Z}$, whereas $\mathbb{Z} \subset \mathbb{Q}$. Because of their approximating property, the fractional set $\mathbb{F}$ represent the most important (and the largest) portion of the rational numbers, dwarfing the size of the integers, another good reason for defining the two distinct subsets (i.e., $\mathbb{Q} = \mathbb{Z} \cup \mathbb{F}$ where $\mathbb{Z} \perp \mathbb{F}$).

Once factored and common terms are canceled, this subset $\mathbb{F} \subset \mathbb{Q}$ of rational numbers is always the ratio of two co-primes. For example $\pi \approx 22/7 = 11 \cdot 2/7 = 3 + 1/7$ with $22 \perp 7$, and $9/6 = 3/2 = 1 + 1/2$ with $3 \perp 2$.[16]

**Irrational numbers $\mathbb{I}$:** Every real number that is not rational ($\not\mathbb{Q}$) is *irrational* ($\mathbb{Q} \perp \mathbb{I}$). Irrational numbers include $\pi, e$ and the square roots of most integers (i.e., $\sqrt{2}$). These are decimal numbers that never repeat, thus requiring infinite precision in their representation.

Irrational numbers ($\mathbb{I}$) were famously problematic for the Pythagoreans, who incorrectly theorized that all numbers were rational. Like $\infty$, irrational numbers require a new and difficult concept before they may even be defined: They were not in the set of fractional numbers ($\mathbb{I} \not\subset \mathbb{F}$). It was easily shown, from a simple geometrical construction, that most, but not all of the square roots of integers are irrational. It was essential to understand the factorization of counting numbers before the concept of irrationals could be sorted out.

**Real numbers $\mathbb{R}$:** Reals are the union of rational and irrational numbers, namely $\mathbb{R} : \{\mathbb{I}, \mathbb{Q}\}$ ($\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$). Reals are the lengths in Euclidean geometry. Many people assume that *IEEE 754 floating point numbers* (c1985) are real (i.e., $\in \mathbb{R}$). In fact they are rational ($\mathbb{Q} : \{\mathbb{F} \cup \mathbb{Z}\}$) approximations to real numbers, designed to have a very large dynamic range. There can be no machine realization of irrational numbers, since such a number would require infinite precision ($\infty$ bits). The hallmark of fractional numbers ($\mathbb{F}$) is their power in making highly accurate approximations of any real number.

Using Euclid's compass and ruler methods, one can make line length proportionally shorter or longer, or (approximately) the same. A line may be made be twice as long, an angle bisected. However, the concept of an integer length in Euclid's geometry was not defined.[17] Nor can one construct an imaginary or complex line as all lines are assumed to be real.

Real numbers were first fully accepted only after set theory was developed by Cantor (1874) (Stillwell, 2010, pp. 461, 525...). It seems amazing, given how widely accepted real numbers are today. But in some sense they were accepted by the Greeks, as lengths of real lines.

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[16] HW problem: How to define $\mathbb{F}$ given two integers $(n, m) \subset Z$? Sol: Not sure how to approach this, but it seems like a fun problem. Here two simple methods that do *not* work: (1) One cannot define $\mathbb{F}$ as the ratio $x = n/m$, since given $m = 1, x \in \mathbb{Z}$. (2) One cannot define $\mathbb{F}$ as the ratio of two coprimes, since then $x = 1/m \notin \mathbb{F}$ (since $1 \not\mathbb{P}$).

[17] As best I know.

**Complex numbers $\mathbb{C}$:** Complex numbers are best defined as *ordered pairs of real numbers.*[18] They are quite special in engineering mathematics, since roots of polynomials having either real or complex coefficients may be complex. The best known example is the quadratic formula for the roots of a $2^d$ degree polynomial, with either real or complex coefficients.

The common way to write a complex number is using the common notation $z = a + b\jmath \in \mathbb{C}$, where $a, b \in \mathbb{R}$. Here $1\jmath = \sqrt{-1}$. We also define $1\imath = -1\jmath$ to account for the two possible signs of the square root. Accordingly $1\jmath^2 = 1\imath^2 = -1$.

Multiplication of complex numbers follows the rules of real algebra, similar to multiplying two polynomials. Multiplication of two first degree polynomials gives

$$(a + bx)(c + dx) = ac + (ac + bd)x + bdx^2$$

If we substitute $1\jmath$ for $x$, and use the definition $1\jmath^2 = -1$, we obtain the product of the two complex numbers

$$(a + b\jmath)(c + d\jmath) = ac - bd + (bc + ad)\jmath.$$

Thus multiplication of complex numbers obeys the accepted rules of algebra.

**Polar representation:** A alternative for complex multiplication is to work with polar coordinates. The polar form of complex number $z = a + b\jmath$ is written in terms of its magnitude $\rho = \sqrt{a^2 + b^2}$ and angle $\theta = \angle z = \tan^{-1}(z) = \arctan z$, as $z = \rho e^{\jmath\theta}$. From the definition of the complex natural log function

$$\ln \rho e^{\jmath\theta} = \ln \rho + j\theta,$$

which is useful in engineering calculations.[19]

**Matrix representation:** A second alternative and useful way to represent complex numbers is in terms of 2x2 matrices. This relationship is defined by the mapping from a complex number to a 2x2 matrix

$$a + jb \leftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

You might verify that

$$\frac{a + b\jmath}{c + d\jmath} = \frac{ab + bd + (bc - ad)\jmath}{c^2 + d^2} \longleftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \frac{1}{c^2 + d^2}.$$

By taking the inverse of the 2x2 matrix one can define the ratio of one complex number by another, Until you try out this representation, it may not seem obvious, or even that it could work.

This representation proves that $1\jmath$ is not necessary to define a complex number. What $1\jmath$ can do is simplify the algebra, both conceptually and for numerical results. It is worth your time to become familiar with the matrix representation, to clarify any possible confusions you might have about multiplication and division of complex numbers. This matrix representation can save you time, heartache and messy algebra. Once you have learned how to multiply two matrices, it's a lot simpler than doing the complex algebra. In many cases we will leave the results of our analysis in matrix form, to avoid the algebra altogether.[20] More on this topic may be found in Chapter 2.

---

[18] A polynomial $a + bx$ and a 2-vector $[a, b]^T = \begin{bmatrix} a \\ b \end{bmatrix}$ are also examples of ordered pairs.

[19] In Chapter 2 discuss the many ways to compute the phase (i.e., `arctan(`$e^{1\jmath\theta}$`)`, `arctan2(x,y)`), and the natural importance of the unwrapped phase. Give the example of $\delta(t - T_0)$.

[20] Sometimes we let the computer do the final algebra, numerically, as 2x2 matrix multiplications.

**Real versus complex numbers:** All numbers may be viewed as complex. Namely every real number is complex if we take the imaginary part to be zero (Boas, 1987). For example, $2 \in \mathbb{P} \subset \mathbb{C}$. Likewise every purely imaginary number (e.g., $0 + 1\jmath$) is complex with zero real part. It follows that $2\jmath \in \mathbb{P}\jmath$. Integers are a subset of reals, which are a subset of complex numbers[21] *Gaussian integers* are ordered pairs of complex integers ($\mathbb{I} \subset \mathbb{R} \subset \mathbb{C}$).[22] From the above discussion it should be clear that each of these different classes of number are nested in a hierarchy, in the following order

$$\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{I} \subset \mathbb{R} \subset \mathbb{C},$$

namely each genus (number classification) is contained within the next.

The integers $\mathbb{Z}$ and fractionals $\mathbb{F}$ split the rationals ($\mathbb{Q} : \mathbb{Z} \cup \mathbb{F}$, $\mathbb{Z} \perp \mathbb{F}$), each of which is a subset of the rationals ($\mathbb{Z} \in \mathbb{Q}$, $\mathbb{F} \subset \mathbb{Q}$). The rationals $\mathbb{Q}$ and irrationals $\mathbb{I}$ split the reals ($\mathbb{R} : \mathbb{Q} \cup \mathbb{I}$, $\mathbb{Q} \perp \mathbb{I}$), each of which is a subset of the reals ($\mathbb{Q} \in \mathbb{R}$, $\mathbb{I} \in \mathbb{R}$).

The roots of polynomials $x_k$ are complex ($x_k \in \mathbb{C}$), independent of the genus of the coefficients (e.g., real integer coefficients give rise to complex roots). Each genus plays an important role in algebra, with prime numbers at the bottom (root of the tree) and complex numbers at the top. We shall explore this further in Chaps. 2 and 3.

Finally, note that complex numbers $\mathbb{C}$ do not have order. For example it makes no sense to say that $\jmath > 1$ or $\jmath = 1$ (Boas, 1987).

**History of complex numbers:** It is notable how long it took for complex numbers to be accepted (1851), relative to when they were first introduced by Bombelli ($16^{th}$ century CE). In fact, complex integers (aka, *Gaussian integers*) were accepted before non-integral complex numbers. Apparently real numbers ($\mathbb{R}$) were not accepted (i.e., proved to exist, thus mathematically defined) until even later. It took the development of *set theory* in the late $19^{th}$ century to sort out a proper definition of the real number, due to the existence of irrational numbers.

**Computer Representations of $\mathbb{I}, \mathbb{R}, \mathbb{C}$:** When doing numerical work, one must consider how we may compute within the family of reals (i.e., irrationals). There can be no irrational number representation on a computers. IEEE floating point numbers, which are the international standard of computation, are actually rational approximations. The mantissa and the exponent are each integers, having sign and magnitude. The size of each integer depends on the precision of the number being represented. An IEEE floating-point number is rational because it has a mantissa integer multiplied by a base to the power of an exponent integer.

Floating point numbers contain irrational numbers, which must be approximate by rational numbers. This leads to the concept of *fractional representation*, which requires the definition of the *mantissa*, *exponent* and *base*. Numerical results must not depend on the base. For example, when using base ten[23] Make a table

$$\pi \cdot 10^5 \approx 314\,159.27\ldots = 3 \cdot 10^5 + 1 \cdot 10^4 + 4 \cdot 10^3 + \cdots + 9 \cdot 10^0 + 2 \cdot 10^{-1} \cdots .$$

According to Matlab's DEC2BIN() routine, the binary representation is ck 4 errors

$$\pi \cdot 2^{17} \approx 131072_{10} \cdot 22/7 = 110,0100,1001,0010,0101_2,$$

where 1 and 0 are multipliers of powers of 2, which are then added together as follows

$$2^{18} + 2^{17} + 2^{14} + 2^{11} + 2^8 + 2^5 + 2^2 + 2^0.$$

In base 16 (i.e, hexadecimal) $2^{17} \cdot 22/7 = 2^{18} \cdot 8_{16}/7_{16}$.

[21] The plural *cmplexs* (a double /s/) seems an unacceptable word in English.

[22] It follows that integers are a subset of Gaussian integers (the imaginary or real part of the Gaussian integer may be zero).

[23] Base 10 is the natural world-wide standard simply because we have 10 fingers.

One may keep track of the decimal point using the exponent, which in this case is a factor of $2^{17}$ = $131072_{10}$. The concept of a number having a decimal point is replaced by an integer, having the desired precision, and a scale factor of any base (radix). This scale factor may be thought of as moving the decimal point to the left.

Here is $x = 2^{17} \times 22/7$ at IEEE 754 full double precision, as computed by an IEEE-754 floating point converter[24] $x = 411940.5625_{10} = 2^{54} \times 1198372 = 010010, 001, 10010, 010010, 010010, 010010_2 = 0x48c92492_{16}$. The commas in the binary string of ones and zeros, are to help visualize the quasi-periodic nature of the bit-stream. The mantissa is $4793490_{10}$ and the exponent is $2^{18}$. The numbers are stored in a 32 bit format, with 1 bit for sign, 8 bits for the exponent and 23 bits for the mantissa. Perhaps a more instructive number is $x = 4793490.0 = 01, 001, 010, 100, 100, 100, 100, 100, 100, 100, 100_2$ $= 0x4a, 924, 924_{16}$ which has a repeating binary bit pattern of $((100))_3$, only broken by the scale factor $0x4a$. Another with even higher symmetry is $x = 6.344, 131, 191, 146, 9 \times 10^{-17} = 0x24, 924, 924_{16} =$ $00, 100, 100, 100, 100, 100, 100, 100, 100, 100, 100_2$. In this example the repeating pattern is clear in the Hex representation as a repeating $((942))_3$. As before, the commas are to help with readability, and have no other meaning.

There are other important types of representations. As pairs of reals, complex numbers have similar approximate representations. An important representations of complex numbers is $e^z = \cosh(z) + j\sinh(z)$, which includes the famous formula of Euler $e^{j\theta} = \cos\theta + j\sin\theta$. Some of these concepts can be generalized to include vectors, matrices and polynomials.

**Integers and the Pythagoreans** The integer is the corner stone of the Pythagorean doctrine, so much so that it caused a fracture within the Pythagoreans when it was discovered that not all numbers are rational. The famous example is the isosceles triangle $1, 1, \sqrt{2}$, which lead to the next triangle $[1, 2, \sqrt{3}]$, etc. This is known as the Spiral of Theodorus: the short side is 1 and the hypotenuse is extended by one, using a simple compass-ruler construction.

There are right-triangles with integral lengths, the best known being $[3, 4, 5]$. Such triplets of integers $[a, b, c]$ that satisfy Eq. 1.1 are denoted *Pythagorean triplets*, which may be verified using Euclid's formula (Eq. 2.3).

To form triangles with perfect 90° angles, the lengths need to satisfy Eq. 1.1. Such triangles are also useful in constructing buildings or roads made from of bricks uniform in size.

**Public-private key Security:** An important application of prime numbers is public-private key (RSA) encryption, essential for internet security applications (e.g., online banking). To send secure messages the security (i.e., utility) of the internet is dependent on key encryption.[25] Most people assume this is done by a personal login and passwords. Passwords are simply not secure, for many reasons. The proper method depends on factoring integers formed from products of primes having thousands of digits.[26] The security is based on the relative ease in multiplying large primes, but the virtual impossibility of factoring them.

When a computation is easy in one direction, but its inverse is impossible, it is called a *trap-door function*. We shall explore the reasons for this in Chapter 2. If everyone switched from passwords to public key encryption, the internet would be much more secure.

**Puzzles:** A third application of integers are imaginative problems that use integers. An example is the classic Chinese *Four stone problem:* "Find the weight of four stones that can be used with a scale to weigh any object (e.g., salt, gold) between 1 and 40 [lb]." As with the other problems, the answer

[24] http://www.h-schmidt.net/FloatConverter/IEEE754.html

[25] One might say this is either: i) a key application of primes, or ii) it is primary application of keys. Its a joke.

[26] It would seem that public key encryption could work by having two numbers with a common prime, and then by using Euclidean Algorithm, that GCD could be worked out. One of the integers could be the public key and the second could be the private key. Given the difficulty of factoring the numbers into their primes, and ease of finding the GCD using Euclidean Algorithm, a practical scheme may be possible. Ck this out.

is not as interesting as the method, since the problem may be easily recast into a related one.[27] This type of problem may be found in airline magazines as entertain on a long flight. The solution to this problem is best cast as a linear algebra problem, with integer solutions. Again, once you know the trick, it is "easy."[28]

### 1.2.2 Lec 3: The role of physics in mathematics

Physics: Eigenmodes, bells, music
Mathematics: The Fundamental Theorems

**Bells, chimes and Eigenmodes** Integers naturally arose in art, music and science. An example are the relations between musical notes, the natural eigenmodes (tones) of strings and other musical instruments. These relations were so common and well studied, it appeared that to understand the physical world (aka, the Universe), one needed to understand integers. This was a seductive view, but not actually correct. As will be discussed in Sections 1.3.1 and 3.1.1, it is best to view the relationship between acoustics, music and mathematics as historical, since these topics played such an important role in the development of mathematics. Also interesting is the role that integers seem to play in quantum mechanics.

Engineers are so accustomed to working with real (or complex) numbers, the distinction between real (i.e., irrational) and fractional numbers are rarely acknowledged. Integers on the other hand arise in many contexts. One cannot learn to program a computer without mastering integer, hexadecimal, octal, and binary representations. All numbers in a computer are represented in numerical computations in terms of rationals (i.e, either integers or fractionals).

As discussed in Section 1.2.1, the primary reason integers are so important is their absolute precision. Every integer $n \in Z$ is unique,[29] and $\mathbb{Z}$ has the indexing property, which is essential for making lists that are ordered so that later it can be naturally found. The alphabet also has this property (e.g., book's table of contents). Other than for hexadecimal numbers, which for notational reasons use the alphabet, letters are equivalent to integers.

Because of the integer's absolute precision, the digital computer overtook the analog computer, once it was practical to make logic circuits that were fast. The first digital computer was thought to be the *Eniac* at the University of Pennsylvania, but it turned out that the code-breaking effort in Bletchley Park, England, under the guidance of Alan Turing, created the first digital computer (The Colossus) to break the WWII German "Enigma" code. Due to the high secrecy of this war effort, the credit was only acknowledged in the 1970s when the project was declassified. Org?

There is zero chance of analog computing displacing digital computing, due to the importance of precision (and speed). But even with binary representation, there is a non-zero probability of error. To deal with this, error correcting codes have been developed, to reduce the error by several orders of magnitude. Today this is a science, and billions of dollars are invested to increase the density of bits per area, to increasingly larger factors. A few years ago a terabyte drive was unheard of; today the are the standard. In a few years petabyte drives will certainly become available. It is hard to comprehend how these will be used by individuals, but they are essential for on-line (cloud) computing.

#### Fundamental Theorems

Modern mathematics is build on a hierarchical construct of *Fundamental Theorems*, as summarized in Table 1.2. The importance of such theorems cannot be overemphasized. Every engineering student needs to fully appreciate the significance of these key theorems. If necessary, memorize them. But that will not do over the long run, as each and every theorem must be fully understood. Fortunately

[27] Hint: the first two stones have weight of 1 and 3.

[28] When ever someone tells you something is "easy," you should immediately appreciate that it is very hard, but there is a concept, that once you learn, the difficulty evaporates.

[29] Check out the history of $1729 = 1^3 + 12^3 = 9^3 + 10^3$.

Table 1.2: *The Fundamental Theorems of mathematics* 1.2.0

1. Fundamental Theorems of:
   (a) **Number Systems:**
      - Arithmetic
      - Prime Number
   (b) **Geometry:**
      - Algebra
      - Bèzout
   (c) **Calculus:**[a]
      - Leibniz $\mathbb{R}^1$
      - Complex $\mathbb{C} \subset \mathbb{R}^2$
      - vectors $\mathbb{R}^3, \mathbb{R}^n, \mathbb{R}^\infty$
2. Other key concepts:
   - Complex analytic functions (complex roots are finally accepted!)
     - Complex Taylor Series of complex functions
     - Region of convergence (ROC) of an infinite series
     - Laplace Transform, and its inverse
     - Complex frequency versus causal time
     - Cauchy Integral Theorem
     - Residue integration (i.e., Green's Thm in $\mathbb{R}^2$)
   - Riemann mapping theorem (Gray, 1994; Walsh, 1973)
   - Complex Impedance (Ohm's Law) Kennelly

[a] Flanders, Harley (June–July 1973). "Differentiation under the integral sign." *American Mathematical Monthly* 80 (6): 615-627. doi:10.2307/2319163. JSTOR 2319163.

most students already know several of these theorems, but perhaps not by name. In such cases, it is a matter of mastering the vocabulary.

The theorems are naturally organized, starting with two theorems on prime numbers (Table 1.2). They may also be thought of in terms of Stillwell's three streams. For Stream 1 there is the Fundamental Theorem of Arithmetic. For Stream 2 there is the Fundamental Theorem of Algebra and Bèzout's Theorem, while for Stream 3 there are a host of theorems on calculus, ordered by their dimensionality. Some of these theorems verge on the trivial (e.g., the *Fundamental Theorem of Arithmetic*). Others are more challenging, such as the *Fundamental Theorem of Vector Calculus* and *Green's Theorem.*

Complexity should not be confused with importance. Each of these theorems is, as stated, fundamental. Taken as a whole, they are a powerful way of thinking about mathematics.

### Stream 1: Prime Number theorems:

Discussion of Fundamental Theorem of Arithmetic & Prime Number Theorem

There are two fundamental theorems about primes,

1. *The Fundamental Theorem of Arithmetic*: This states that every counting number $n > 1 \in \mathbb{N}$ may be uniquely factored into prime numbers.

2. *The Prime Number Theorem:* One would like to know how many primes there are. That is easy: $|\mathbb{P}| = \infty$. The cardinality, or size of the set of primes, is infinite). The proper way of asking this questions is *What is the average density of primes, in the limit as $n \to \infty$?* This question was answered, for all practical purposes, by Gauss, who as a pastime computed the first million

primes by hand. He discovered that, to a good approximation, the primes are equally likely on a log scale. This is nicely summarized by the jingle attributed to the mathematician Pafnuty Chebyshev

> Chebyshev said, and I say it again: *There is always a prime between n and 2n.*

(Stillwell, 2010, p. 585)

#### Stream 2: Fundamental theorem of Algebra

This theorem states that every polynomial has at least one root. When that root is removed then the degree of the polynomial is reduced by 1. Thus when applied recursively, a polynomial of degree $N$ has $N$ roots.

Besides the Fundamental Theorem of Algebra, a second important theorem will be mentioned known as Bèzout's Theorem, which is a generalization of the Fundamental Theorem of Algebra will be briefly described. It states that when counting the $N$ roots of a polynomial of degree $N$, one must include the imaginary roots, double roots and roots at infinity, some of which may difficult to identify.

#### Stream 3: Fundamental theorems of calculus

Discuss FTs of integration: $\mathbb{R}, \mathbb{C}, \mathbb{R}^3$, e.g., Green's Thms, Helmholtz Thm of Vector Calculus

Picture of von Helmholtz here, with discussion of his paper

There are at least four theorems related to integral calculus:

1. *Leibnez Theorem* ($\mathbb{R}$) area under a real curve.
2. *Cauchy's Theorem* ($\mathbb{C}$) residue integration and analytic functions.

   *Gauss's Law* ($\mathbb{R}^2$) conservation of mass and charge crossing a close surface.
3. *Stoke's Theorem* ($\mathbb{R}^2$) relates line integrals to the rate of change of the flux crossing an open surface.
4. *Green's Theorem*, a generalization of the above theorems

In Sections 1.5.3, 1.5.5 and 5.1.3 we will deal with each of the theorems for Stream 3 where we consider the several Fundamental theorems of integration, starting with Leibniz's formula for integration on the real line ($\mathbb{R}$), then progressing to complex integration in the complex plane ($\mathbb{C}$) (*The Fundamental theorem of Complex calculus*), which is required for computing the inverse Laplace transform. Then we discuss Gauss' and Stokes' Laws for $\mathbb{R}^2$ with closed and open surfaces respectively and finally Green's theorems. One cannot understand Maxwell's equations, fluid flow, or acoustics without understanding these theorems. Any problem that deals with the wave equation in more than one dimension requires an understanding of these concepts. The derivation of the Kirchhoff voltage and current laws is based on these theorems.

#### Other Key Concepts

Besides the widely recognized Fundamental Theorems for the three streams, there are a number of equally important theorems that have not yet been labeled as "fundamental."[30]

The widely recognized *Cauchy Integral Theorem* is an excellent example since it is a stepping stone to the *Fundamental Theorem of Complex Integral Calculus*. In Chapter 4 we clarify the contributions of each of these special theorems.

Once these Fundamental theorems of integration (Stream 3) have been mastered, the student is ready for the *complex frequency domain*, which takes us back to Stream 2 and the *complex frequency*

[30] It is not clear what it takes to reach this more official sounding category.

*plane* ($s = \sigma + \omega\jmath \in \mathbb{C}$). While the Fourier and Laplace transforms are taught in Mathematics courses, typically few physical connections are made, accordingly the concept of *complex frequency* is rarely mentioned. The *complex frequency domain* and *causality* are fundamentally related and critical for the analysis of signals and systems.

Without the concept of time and frequency, one cannot develop an intuition for the Fourier and Laplace transform relationships, especially within the context of engineering and mathematical physics.

## WEEK 2 4.2.0

L 4 The two prime number theorems:

1. Fundamental Thm of Arith
   Brief discussion on Prime Numbers $\pi_k \in \mathbb{P}$;
   Prime sieves
2. Prime number theorem
   Density of prime numbers $\rho_\pi \in \mathbb{N}$; *Logrithmic integral* $Li(N)$ definition

L 5 Greatest Common Denominator (GCD) (Euclidean Algorithm) (Stillwell, 2010, p. 42)

1. Euclidean Algorithm for finding the GCD: $k = gcd(n, m)$
2. *Coprimes* $n, m$ have no common factors: $gcd(n, m) = 1 \Rightarrow n \perp m$

L 6 *Continued fractions* (extended Euclidean Algorithm) (e.g., $\pi \approx 22/7$)
Rational approximations:
Irrational numbers: e.g. $\sqrt{2} \approx 17/12$
Transcendental numbers: e.g. $\pi \approx 22/7$
Matlab's *rat(), rats()* commands.

### 1.2.3 Lec 4: Two theorems on primes

**Theorem 1: *Fundamental Theorem of Arithmetic***

**Factoring integers:** Every integer $n \in \mathbb{N}$ has a unique factorization (Stillwell, 2010, p. 43)

$$n = \prod_{k=1}^{K} \pi_k^{\beta_k}, \tag{1.2}$$

where $k = 1, \ldots, K$ indexes the integer's $K$ *prime factors* $\pi_k$ and their *multiplicity* $\beta_k$.

**Examples:** $2312 = 2^3 \cdot 17^2 = \pi_1^3\,\pi_7^2$ (i.e., $\pi_1 = 2, \beta_1 = 3; \pi_7 = 17, \beta_7 = 2$)
$2313 = 3^2 \cdot 257 = \pi_3^2\,\pi_{55}$ (i.e., $\pi_2 = 3, \beta_3 = 2; \pi_{55} = 257, \beta_{55} = 1$)

Integers 2312 and 2313 are said to be *coprime*, since they have no common factors. *Coprimes* may be identified via the *greatest common divisor*:

$$gcd(a, b) = 1$$

using the *Euclidean algorithm* (Stillwell, 2010, p. 41).

**Theorem 2: *Prime Number Theorem***

Gauss showed empirically that the average total number of primes less than $N$ is

$$\sum_{n=1}^{N} \delta_n \sim \frac{N}{\ln(N)}$$

based on hand calculations "as a pastime" in 1792-3 (Goldstein, 1973). Here $\delta(n) = 1$ if $\pi_n$ is a prime and zero otherwise. It follows that the *average density of primes* is $\rho_\pi(N) \sim 1/\ln n$, thus

$$\rho_\pi(N) \equiv \frac{1}{N}\sum_{n=1}^{N} \delta(n) \approx \frac{1}{N} Li(N), \equiv \frac{1}{N}\int_2^N \frac{d\xi}{\ln(\xi)},$$

where $Li(N)$ is the *logarithmic integral* (Stillwell, 2010, p. 585). The primes are distributed as $1/\ln(n)$ since the average total number of primes is proportional to the *logarithmic integral* $Li(n)$ (Goldstein, 1973; Fine, 2007).

From the Prime Number Theorem it is known that there are many primes (they are not scarce). As best I know there are no methods to find primes other than by the sieve method (Section 2.1.1). If there is any good news it is that they only need to be computed once, and saved. In practical applications this doesn't help much given their large number. In theory, given primes $\pi_n$ up to $n = N$, the density $\rho_\pi(N)$ could help one search for a particular prime of known size $N$, by estimating how many primes there are in the neighborhood of $N$.

Not surprisingly, playing with primes has been a popular pastime of mathematicians. Perhaps this is because those who have made inroads, providing improved understanding, have become famous.

### 1.2.4 Lec 5: Greatest common divisor (Euclidean Algorithm)

The Euclidean Algorithm is a method to find the greatest common divisor (GCD) $k$ between two integers $n, m$, denoted $k = gcd(n, m)$, where $n, m, k \in \mathbb{N}$. For example $15 = \gcd(30, 105)$ since when factored $(30, 105) = (2 \cdot 3 \cdot 5,\ 7 \cdot 3 \cdot 5) = 3 \cdot 5 \cdot (2, 7) = 15 \cdot (2, 7)$.

Finding the GCD is significantly less difficult than a full factoring (Fig. 2.2) but is useful in removing common factors when forming fractions of the two numbers. The GCD is important precisely because of the fundamental difficulty of factoring large integers into their primes. This utility surfaces when the two numbers are composed of very large primes. Euclidean Algorithm was known to the Chinese (i.e., not discovered by Euclid) (Stillwell, 2010, p. 41).

#### Euclidean Algorithm

The algorithm is best explained by a trivial example: Let the two numbers be $6, 9$. At each step the smaller number (6) is subtracted from the larger (9) and the difference and the smaller are stored. This process is continued until the two resulting numbers are equal, at which point the GCD equals that final number. For our example step 1 gives 9-6=3, leaving 6 and 3. Step 2 gives 6-3=3 and 3. Since the two numbers are the same, the GCD=3. We can easily verify this result since this example is easily factored (e.g., $3 \cdot 3, 3 \cdot 2) = 3(3, 2)$. The Matlab command for the GCD is `gcd(6,9)`, which returns 3.

In Chapter 2 we shall describe two methods for implementing this procedure using algebraic and matrix notation, and then explore the deeper implications.

#### Coprimes

Related to the prime numbers are *co-primes*, which are integers that when factored, have no common primes. For example $20 = 5 \cdot 2 \cdot 2$ and $21 = 7 \cdot 3$ have no common factors, thus they are coprime. Coprimes $[m, n]$ may be indicated with the "perpendicular" notation $n \perp m$, spoken as "n is perpendicular (perp) to m." One may use the GCD to determine if two numbers are coprime. When $\gcd(m, n) = 1$, $m$ and $n$ are coprime. For example since `gcd(21,20)=1` (i.e., $21 \perp 20$) the are coprime.

### 1.2.5 Lec 6: Continued Fraction Algorithm (CFA)

Lec 6: Continued fractions: rational approximations and irrational numbers

The Continued Fraction Algorithm was mentioned in Section 1.2.4 at the end of the discussion on the GCD. These two algorithms (CFA vs. GCD) are closely related, enough that Gauss referred to the CFA as the Euclidean Algorithm (i.e., the name of the GCD algorithm) (Stillwell, 2010, P. 48). This question of similarity needs some clarification, as it seems unlikely that Gauss would be confused about such a basic algorithm.

In its simplest form the CFA starts from a real decimal number and recursively expands it as a fraction. It is useful for finding rational approximations to any real number. The GCD uses the Euclidean algorithm on a pair of integers $m > n \in \mathbb{N}$ and finds their greatest common divisor $l \in \mathbb{N}$. At first glance it is not clear why Gauss would call the CFA the Euclidean algorithm. One must assume that Gauss had some deeper insight into the relationship. If so, it would be valuable to understand his insight.

In the following we refine the description of the CFA and give examples that go beyond the simple cases of expanding numbers. The CFA of any number, say $x_0$, is defined as follows:

1. Start with $n = 0$ and input target (starting value) $x_0 \in \mathbb{R}$.
2. If $|x_n| \geq 1/2$ define $a_n = \texttt{round}(x_n)$, which rounds to the nearest integer.
3. $r_n = x_n - a_n$ is the remainder. If $r_0 = 0$, the recursion terminates.
4. Define $x_{n+1} \equiv 1/r_n$ and return to step 2 (with $n = n + 1$).

An example: Let $x_0 \equiv \pi \approx 3.14159\ldots$. Thus $a_0 = 3$, $r_0 = 0.14159$, $x_1 = 7.065 \approx 1/r_0$, and $a_1 = 7$. If we were to stop here we would have

$$\widehat{\pi}_1 \approx 3 + \frac{1}{7 + 0.0625\ldots} \approx 3 + \frac{1}{7} = \frac{22}{7}. \tag{1.3}$$

This approximation of $\pi \approx 22/7$ has a relative error of 0.04%

$$\frac{22/7 - \pi}{\pi} = 4 \times 10^{-4}.$$

For the second approximation we continue by reciprocating the remainder $1/0.0625 \approx 15.9966$ which rounds to 16, resulting in the second approximation

$$\widehat{\pi}_2 \approx 3 + 1/(7 + 1/16) = 3 + 16/(7 \cdot 16 + 1) = 3 + 16/113 = 355/113.$$

Note that if we had truncated 15.9966 to 15, the remainder would have been much larger, resulting in a less accurate rational approximation. The recursion may continue to any desired accuracy as convergence is guaranteed.

**Notation:** Writing out all the fractions can become tedious. For example, expanding $e$ using the Matlab command `rat(exp(1))` gives the approximation

$$3 + 1/(-4 + 1/(2 + 1/(5 + 1/(-2 + 1/(-7))))).$$

A compact notation for this these coefficients of the CFA is $[3. -4, 2, 5, -2, -7]$. Note that the leading integer may be indicated by an optional semicolon to indicate the decimal point. Unfortunately Matlab does not support this notation.

If the process is carried further, the values of $a_n \in \mathbb{N}$ give increasingly more accurate rational approximations of $\pi = [3.\ 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, \ldots]$.

When the CFA is applied and the expansion terminates ($r_n = 0$), the target is rational. When the expansion does not terminate (which is not always easy to determine), the number is irrational.

***Rational approximation examples***

$$\begin{aligned}
\frac{22}{7} &= [3.\ 7] && \approx \pi + O(1.3 \times 10^{-3}) \\
\frac{355}{113} &= [3.\ 7, 16] && \approx \pi + O(2.7 \times 10^{-7}) \\
\frac{104348}{33215} &= [3.\ 7, 16, -249] && \approx \pi + O(3.3 \times 10^{-10})
\end{aligned}$$

Figure 1.5: The expansion of $\pi$ to various orders using the CFA, along with the order of the error of each rational approximation. For example 22/7 has an error $(22/7 - \pi)$ of about 0.13%.

Thus the CFA has important theoretical applications regarding irrational numbers. You may try this yourself using Matlab's `rats(pi)` command. Also try the Matlab command `rat(1+sqrt(2))`.

One of the useful things about the procedure, besides its being so simple, are its many generalizations, a few of which will be discussed in Section 1.2.5.

A *continued fraction expansion* can have a high degree of symmetry. For example, the CFA of

$$\frac{1+\sqrt{5}}{2} = 1 + \frac{1}{1 + \frac{1}{1+\cdots}} = 1.618033988749895\ldots,$$

Here the lead term in the fraction is *always* 1 ($a_n = [1; 1, 1, \cdots]$), thus the sequence will not terminate, proving that $\sqrt{5} \in \mathbb{I}$. A related example is `rat(1+sqrt(2))`, which gives $[2; 2, 2, 2, \ldots]$.

When expanding a target irrational number ($x_0 \in \mathbb{I}$), and the CFA is truncated, the resulting rational fraction approximates the irrational target. For the example above, if we truncate at three coefficients ($[1; 1, 1]$) we obtain

$$1 + \frac{1}{1 + \frac{1}{1+0}} = 1 + 1/2 = 3/2 = 1.5 = \frac{1+\sqrt{5}}{2} + 0.118 + \ldots.$$

Truncation after six steps gives

$$[1.\ 1, 1, 1, 1, 1, 1] = 13/8 \approx 1.6250 = \frac{1+\sqrt{5}}{2} + .0070\ldots.$$

Because all the coefficients are 1, this example converges very slowly, When the coefficients are large (i.e., remainder small), the convergence will be faster. The expansion of $\pi$ is an example of faster convergence.

**In summary:** Every rational number $m/n \in \mathbb{Q}$, with $m > n$, may be uniquely expanded as a continued fraction, with coefficients $a_k$ determined using the the CFA. When the target number is irrational ($p \in \mathbb{Q}$), the CFA will not terminate, thus each step produces a more accurate rational approximation to $p$, converging to $p$ in the limit as $n \to \infty$.

Thus the CFA expansion is a test that can, in theory, determine when the target is rational, but with an important caveat: one must determine if the expansion terminates. In cases where the expansion produces a repeating coefficient sequence, it is clear that the sequence cannot terminate. The fraction $1/3 = 0.33333\ldots$ is an example of such a target where the CFA will terminate.

**WEEK 3** **7.3.0**

— Labor Day (no class)

L 7 Definition of Pythagorean triplets (PT) $[a, b, c]$
Examples: $[3, 4, 5]$; Derivation of Euclid's formula for PTs;
Properties of Pythagorean triplets $[a, b, c]$;

L 8 *Pell's Equation:* $n^2 - Nm^2 = 1$ (i.e., $y^2 = Nx^2 + 1$)
1. Brahmagupta's solution by composition
2. The eigenvalue solution
3. Derivation of Euler-like solution of Pell's Equation (Chord & Tangent methods)

### 1.2.6 Labor day

### 1.2.7 Lec 7 Pythagorean triplets (Euclid's formula)

Lec 7: Pythagorean Triplets (Euclid's formula); Examples, Properties of

Euclid's formula is a method for finding three integer lengths $[a, b, c]$ that satisfy Eq. 1.1. It is important to ask "Which set are the lengths [a,b,c] drawn from?" There is a huge difference, both practical and theoretical, if they are from the real numbers $\mathbb{R}$, or the counting numbers $\mathbb{N}$. A well known example is the right triangle defined by the integers $[3, 4, 5] \in N$, having angles $[0.54, 0.65, \pi/2]$ [rad], which satisfies Eq. 1.1. As quantified by *Euclid's Formula* (Section 1.2.5). there are an infinite number of such *Pythagorean triplets* (PT). Furthermore the seemingly simple triangle, having angles of $[30, 60, 90] \in N$ [deg] (i.e., $[\pi/6, \pi/3, \pi/2] \in \mathbb{I}$ [rad]), has one irrational ($\mathbb{I}$) length ($[1, \sqrt{3}, 2]$).

The technique for proving Euclid's formula for PTs $[a, b, c] \in \mathbb{Q}$, derived in Fig. 2.7 of Section 2.1.3, is much more interesting than the PTs themselves.

The set from which the lengths $[a, b, c]$ are drawn from was not missed by the Indians, Chinese, Egyptians, Mesopotamians, Greeks, etc. Any equation whose solution is based on integers is called a *Diophantine equation*, named after the Greek mathematician Diophantus of Alexandria (c250 CE).

### 1.2.8 Lec 8: Pell's Equation (Euclid's Formula)

Lec 8: Pell's equation; Brahmagupta's solution, eigenvalue solution; tangent/chord solution

Pell's equation

$$a^2 - Nb^2 = 1, \tag{1.4}$$

with $N \in \mathbb{N}$ and where $a, b \in \mathbb{N}$ are to be determined, is related to the Pythagorean Theorem and Euclid's formula. For example, with $N = 2$, one solution is $a = 17, b = 12$ ($17^2 - 2 \cdot 12^2 = 1$).

An algorithmic known as a "composition based solution" of Pell's equation was used by the Pythagoreans to investigate the $\sqrt{2}$ using a 2x2 matrix recursions of the form

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} \tag{1.5}$$

(Stillwell, 2010, p. 44). This recursion must be initialized by a "trivial" solution (i.e., $a_0 = 1, b_0 = 0$) and has a geometrical interpretation in terms of meshed rectangles (Chapter 2).

Today we recognize this as a *difference equation,* which is an early (pre Stream 3) form of a differential equation. The Greek 2x2 form is an early precursor to $17^{th}$ and $18^{th}$ century developments in linear algebra. As discussed in Section 2.2.2, we now know that the Greek's recursive solution for the $\sqrt{2}$ is closely related to Brahmagupta's (620 CE) and Bhâskara's (1030 CE) solution of Pell's equation. Following the development of linear algebra c$19^{th}$ century, this could be evaluated by diagonalizing the matrix by a *unitary transformation* of the form[31]

$$\Lambda = E^{-1} A E, \tag{1.6}$$

[31] `https://en.wikipedia.org/wiki/Transformation_matrix#Rotation`

with $E$ and $\Lambda$ defined by the eigenvalue equation for $A$

$$Ae_k = \lambda_k e_k, \tag{1.7}$$

$E = [e_1, e_2]$ are formed from the eigenvectors $e_k$ and $\Lambda$ is a diagonal matrix of sorted eigenvalues $\lambda_k$. As an example if we take

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

then the eigenvalues are given by $(1 - \lambda_\pm)(1 + \lambda_\pm) = -1$, thus $\lambda_\pm = \sqrt{2}$. This method of eigen analysis is discussed in Section 2.3.1 and Appendix B.

The key idea of the 2x2 matrix solution, widely used in modern engineering, can be traced back to Brahmagupta's solution of Pell's equation, for arbitrary $N$. Brahmagupta's recursion, identical to that of the Pythagoreans' $N = 2$ case (Eq. 1.5), eventually lead to the concept of linear algebra, defined by the simultaneous solutions of many linear equations. The recursion by the Pythagoreans ($6^{th}$ BCE) predated the creation of algebra by al-Khwārizmī ($9^{th}$ CE century) (Fig. 1.2) (Stillwell, 2010, p. 88).

There are similarities between Pell's Equation (a hyperbola) and the Pythagorean Theorem (a circle). As we shall see in Chapter 2, Pell's equation is related to the geometry of a hyperbola just as the Pythagorean equation is related to the geometry of a circle. One might wonder if there is a Euclidean Formula for the solutions of Pell's Equations. After all, these are all conic sections with closely related geometry, in the complex plane.

#### Pell's Equation and Irrational numbers

Since the eigenvalues of Eq. 1.5 ($\lambda_\pm = 1 \mp \sqrt{N}$) must be restricted to irrational numbers ($\sqrt{N} \not\in \mathbb{N}$), solutions to Pell's equation raised the possibility that all numbers may not be integers, namely that irrational numbers must exist. This sparked the interest of the Pythagoreans. The early discovery of such irrational numbers forced the jarring realization that the Pythagorean dogma "all is integer" could be wrong. The problem was that the significance of irrational numbers were yet to be understood.

## WEEK 4 9.4.0

L 9 The *Fibonacci sequence* and its difference equation
Geometry and irrational numbers: $\sqrt{n} \in \mathbb{Q} \in \mathbb{R}$; Roots of $\sqrt{5}$

L10 In Class Exam I

### 1.2.9 Lec 9: Fibonacci sequence

Another classic problem, formulated by the Chinese, was the Fibonacci sequence, generated by the relation

$$f_{n+1} = f_n + f_{n-1}. \tag{1.8}$$

Here the next number $f_{n+1}$ is the sum of the previous two. If we start from $[0, 1]$, this difference equation leads to the Fibonacci sequence $f_n = [0, 1, 1, 2, 3, 5, 8, 13, \cdots]$.

It may be generated by the recursion of a 2x2 matrix equation, or by the generating function method, which today is also known in the engineering community as the z-transform. If we define $y_{n+1} = x_n$ then Eq. 2.4 is equivalent to

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} \tag{1.9}$$

It is easily verified that Eqs. 1.8 and 1.9 are the same. Starting with $[x_n, y_n]^T = [0, 1]^T$ we obtain for the first few steps

$$\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}, \quad \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}, \quad \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}, \quad \begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}, \quad \cdots$$

From the above $x_n = [0, 1, 1, 2, 3, 5, \ldots]$ is the Fibonacci sequence since the next number is the sum of the last two.

Note that this 2x2 equation is similar to Pell's equation, suggesting that an eigenfunction expansion of Eq. 1.6 may be used to analyze the sequence (Section 2.3.1) (Stillwell, 2010, 192).

### 1.2.10 Lec 10: Exam I (In class)

Lec 10: Exam I

## WEEK 4-AE 12.5.0

L 11 Stream 2: Algebra and geometry as physics (Physics drives early mathematics)
The first "algebra" (al-jabr) al-Khwarizmi ($9^{th}$CE)
Polynomial equations in one and two variables (Stillwell, 2010, Ch. 6, p. 87)
Solution of the Quadratic Equation; Taylor series
Composition and intersection of polynomials

AE-1 (HW4) for 9/16/16; Add convolution problem. Verify due date.

## 1.3 Algebraic Equations: Stream 2

### 1.3.1 Lec 11 Algebra and geometry as physics

Lec 11 Physics, Geometry and Algebra.
The first "algebra" (al-jabr) al-Khwarizmi ($9^{th}$CE)
Polynomial equations in one variable (Stillwell, 2010, Ch. 6, p. 87)
Solution of the Quadratic Equation; Taylor series;
Composition versus intersections of curves

Following Stillwell's history of mathematics, Stream 2 is geometry, which led to the merging of Euclid's geometrical methods and the $9^{th}$ century development of algebra by al-Khwarizmi (830 CE). This integration of ideas lead Descartes and Fermat to develop of *analytic geometry.* While not entirely a unique and novel idea, it was late in coming, given what was known at that time.

The mathematics up to the time of the Greeks, documented and formalized by Euclid, served students of mathematics for more than two thousand years. Algebra and geometry were, at first, independent lines of thought. When merged, the focus returned to the Pythagorean theorem, generalized as analytic conic sections rather than as geometry in Euclid's Elements. With the introduction of Algebra, numbers, rather than lines, could be used to represent a geometrical length. Thus the appreciation for geometry grew given the addition of the rigorous analysis using numbers. And as before,integers (i.e., numbers) are the precise representation.

**Physics inspires algebraic mathematics:** The Chinese used music, art, navigation to drive mathematics. With the invention of algebra this paradigm did not shift. A desire to understand motions of objects and planets participated many new discoveries. Galileo investigated gravity and invented the telescope. Kepler investigated the motion of the planets. While Kepler was the first to appreciate that the planets were described by ellipses, it seems he under-appreciate the significance of this finding, and continued with his epicycle models of the planets. Using algebra and calculus, Newton formalized the equation of gravity, forces and motion (Newton's three laws) and showed that Kepler's discovery of planetary elliptical motion naturally follows from these laws. With the discovery of Uranus "Kepler's theory was ruined ... in 1781." (Stillwell, 2010, p. 23).

It is somewhat amazing that to this day, we have failed to understand gravity significantly better than Newton. Perhaps this is too harsh, given the work of Einstein. Gravity waves were experimentally measured for the first time while I was formulating Chapter 3.

Once Newton proposed the basic laws of gravity, he proceed to calculate, for the first time, the speed of sound. This required some form of the *wave equation*, a key equation in mathematical physics

$$\frac{\partial^2}{\partial x^2}p(x,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}p(x,t). \tag{1.10}$$

Here $p(t,x)$ is the pressure as a function of time $t$ and position $x$ and $c = 343$ [m/s] is the speed of sound, which is a function of the density $\rho = 1.12$ [kg/m$^3$] and the dynamic stiffness $\eta P_0$ of air.[32]

While Newton's value for $c$ was incorrect by the $\sqrt{\eta}$, a problem that would take more than two hundred years to solved, his success was important since it quantified the physics behind the speed of sound and demonstrated that momentum $mv$ not mass $m$ was transported by the wave. His concept was correct, and his formulation using algebra and calculus represented a milestone in science.

Newton's *Principia* was finally published in 1687, and the general solution to Newton's wave equation [i.e., $p(x,t) = G(t \pm x/c)$], where $G$ is *any* function, was first published 60 years later by d'Alembert (c1747). This discovery proved, that for sounds of a single frequency, the wavelength $\lambda$ and frequency $f$ were related by

$$f\lambda = c.$$

Today d'Alembert's analytic wave solution is frequently written as

$$p(x,t) = e^{\jmath 2\pi(ft \pm kx)},$$

where $k = c/\lambda$ is the *wave number* (*propagation function*). This formulation led to the frequency domain concept of Fourier analysis, based on the *linearity* (i.e., superposition) property of the wave equation.

An analogous discovery of the formula for the speed of light was made 114 years later by Maxwell (c1861). This required great ingenuity, as it was necessary to hypothesize an experimentally unmeasured term in his equations, to get the mathematics to correctly predict the speed of light.

**The first Algebra:** Prior to the invention of algebra, people worked out problems as sentences using an obtuse description of the problem. Algebra solved this problem. It may be thought of as a compact language, where numbers are represented as abstract symbols (e.g., $x$ and $\alpha$). The most fundamental problems they wished to solve could be formulated in terms of sums of powers of smaller terms, the

[32] $\eta = c_p/c_v = 1.4$ is the ratio of two thermodynamic constants, and $P_0 = 10^5$ [Pa] is the barometric pressure of air.

most common being powers of some independent variable (i.e., time or frequency). Today we call such an expression a *polynomial* of *degree n*

$$P_n(x) \equiv a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 x^0 = \sum_{k=0}^{n} a_k x^k = \prod_{k=0}^{n} (x - x_k) \tag{1.11}$$

The most fundamental question is "What values of the $x = x_k$ result in $P_n(x_k) = 0$." In other words, what are the roots $x_k$ of the polynomial? The quest for the answer to this question consumed thousands of years, with intense efforts by many aspiring mathematicians. In the earliest attempts, it was a competition to evaluate mathematical acumen. Most of the results were held as a secret to the death bed. It would be fair to view this effort as an obsession. Today the roots of any polynomial may be found by numerical methods, to very high accuracy. There are also a number of important theorems.

Of particular interest was composing a circle with a line, for example when the line does not touch the circle, and finding the roots. There was no solution to this problem using geometry. We shall explore the answer in the assignments.[33]

**Analytic Series:** When the degree of the polynomial is infinite (i.e., $n=\infty$), $P_\infty(x)$, $x \in \mathbb{R}$ the series is called a *power series*. For values of $x$ where the power series converges it is said to be *analytic*. When the coefficients are determined by derivatives of $P(x)$ evaluated at $x = 0$, then it is called a *Taylor series*. These various series play a special role in mathematics, as the coefficients of the series uniquely determine a function (e.g., via the derivatives). Two well known examples are the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^2 + \ldots = \sum_{n=0}^{\infty} x^n$$

and the exponential

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3\cdot 2}x^3 + \frac{1}{4\cdot 3\cdot 2}x^4 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n.$$

When $|x| > 1$ the geometric series fails to converge. The exponential series converges for every finite value of $x$.

**Analytic functions:** Any function that has an analytic series representation is called an *analytic function*. Thus $1/(1-x)$ and $e^x$ are both analytic functions.

Because analytic functions are easily manipulated, the have important applications, including the solution of differential equations. It should be clear that the derivatives are easily computed, since they may be uniquely determined, term by term. Every analytic function has a differential equation, that is determined by the coefficients of the analytic power series. An example is the exponential, which has the property that it is the eigenfunction of the derivative operation

$$\frac{d}{dx}e^{ax} = ae^{ax}.$$

This relationship is a common definition of the exponential function, which is a very special function.

Likewise, analytic power series, and therefore analytic functions, may be easily integrated. Newton took full advantage of the properties of analytic functions in his work. To fully understand the theory of differential equations (DE) one needs to master the analytic function and the analytic power series. Newton used the analytic series (*Taylor series*) to solve many problems, especially for working out integrals. This allowed him to solve DEs.

[33] Be sure to include a problem that solves this problem. The answer of course is imaginary roots.

It was becoming understood that the significance of the DE is that it can characterize a law of nature at a single point in space and time. For example the law of gravity (first formulated by Galileo to explain the dropping to two objects of different masses) must obey conservation of energy. Newton (c1687) went on to show that there must be a gravitational potential between to masses $(m_1, m_2)$ of the form

$$\phi(r) = \frac{m_1 m_2}{r}, \tag{1.12}$$

where $r = |x_1 - x_2|$ is the distance between the two objects at vector locations $x_1$ and $x_2$. Note that this a power series, but with exponent of $-1$.

**Complex analytic functions:** A very delicate point, that seems to have been ignored for centuries, is that the roots of $P_n(x)$ are, in general, complex, namely $x_k \in \mathbb{C}$. It seems a mystery that complex numbers were not accepted once the quadratic equation was discovered, but they were not. Newton called complex roots *imaginary*, presumably in a pejorative sense. The algebra of complex numbers was first documented by Bombelli in 1575, more than 100 years before Newton. It is interesting however that Newton was using power series with fractional degree, thus requiring multi-valued solutions, much later to be known as *branch cuts* (c1851). These topics will be explored in Section 3.1.1.

When the argument is complex, the analytic function takes on an entirely new character. For example Euler's identity with $z = x + y\jmath \in \mathbb{C}$ results in $e^z \in \mathbb{C}$

$$e^z = e^x(\cos(y) + \jmath \sin(y)).$$

It should be clear that the complex analytic functions results in a new category of algebra, with no further assumptions beyond allowing the argument to be complex.

Prior to 1851 most of the analysis *assumed* that the roots $x_k \in \mathbb{R}$ even though there was massive evidence that $r_n \in \mathbb{C}$. Prior to 1851, everyone seemed to be looking for real roots. This is clearly evident in Newton's work (c1687): When he found a non-real root, he ignore it out. Euler (c1748) first derived the Zeta function as a function of real arguments $\zeta(x)$ with $\zeta, x \in \mathbb{R}$. Cauchy (c1814) broke this staid thinking with his analysis of complex analytic functions, but it was Riemann thesis (c1851), when working with Gauss (1777-1855), which had a several landmark breakthroughs. In this work Riemann introduced the *extended complex plane*, which explained the *point* at infinity. He also introduced *Riemann sheets* and *Branch cuts*, which finally allowed mathematics to better describe the physical world (Section 1.4.2).

Once the argument of an analytic function is complex, for example an impedance $Z(s)$, or the Riemann Zeta function $\zeta(s)$, The development of complex analytic functions led to many new fundamental theorems. Complex analytic functions have poles and zeros, branch cuts, Riemann sheets and can be analytic at the point at infinity. Many of these properties were first worked out by Augustin-Louis Cauchy (1789-1857), who drew heavily on the much earlier work of Euler, expanding Euler's ideas into the complex plane (Chapter 4).

**Impact on Physics:** The application of complex analytic functions to physics was dramatic, as may be seen in the six volumes on physics by Arnold Sommerfeld (1868-1951), from the productivity of his many (36) students (e.g., Debye, Lenz, Ewald, Pauli, Guillemin, Bethe, Heisenberg and Seebach, to name a few), notable coworkers (i.e., Leon Brillouin) and others (i.e., John Bardeen) upon whom he had a strong influence. Sommerfeld is known for having many students who were awarded the Nobel Prize in Physics, yet he was not (the prize is not awarded in Mathematics). Sommerfeld brought mathematical physics to a new level with the use of complex integration of analytic functions to solve otherwise difficult problems, thus following the lead of Newton who used real integration of Taylor series to solve differential equations. While much of this work is outside the scope of the present discussion, it is helpful to know who did what, and when, and how certain people and concepts are connected.

### Finding roots of polynomials

The problem of factoring polynomials has a history more than a millennium in the making. While degree $N = 2$ (quadratic) was solved by the time of the Babylonians (i.e., the earliest recorded history of mathematics), the cubic solution was finally published by Cardano in 1545. The same year, Cardano's student solved the quartic. In 1826 it was proved that the quintic could not be factored by analytic methods.

As a concrete example we begin with trivial case of the quadratic (2d degree) polynomial.

$$P_2(x) = ax^2 + bx + c. \tag{1.13}$$

The *roots* are those values of $x$ such that $P_2(x_k) = 0$. One of the first results (recorded by the Babylonians, c2000 BCE) was the factoring of this equation by *completing the square* (Stillwell, 2010, p. 93). One may rewrite Eq. 1.13 as

$$\frac{1}{a}P_2(x) = (x + b/2a)^2 - (b/2a)^2 + c/a, \tag{1.14}$$

which is easily verified by expanding the squared term and canceling $(b/2a)^2$

$$\frac{1}{a}P_2(x) = [x^2 + (b/a)x + \cancel{(b/2a)^2}] - \cancel{(b/2a)^2} + c/a.$$

Setting Eq. 1.14 to zero and solving for the two roots $x_\pm$, gives the *quadratic formula*[34]

$$x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1.15}$$

If $ac < 0$, then the two roots are real ($x_\pm \in \mathbb{R}$). Otherwise, they are complex.

No insight is gained by memorizing the quadratic formula (Eq. 1.15). On the other hand, an important concept is gained by learning Eq. 1.14, which can be very helpful when doing analysis. I suggest that instead of memorizing Eq. 1.15, memorize Eq. 1.14. Arguably, the factored form is easier to remember (or learn). Perhaps more importantly, the term $b/2a$ has significance $[P_2(-b/2a) = c/a - (b/2a)^2]$, the sign of which determines if the roots are real or complex.

In third grade I learned the trick[35]

$$9 \cdot n = (n - 1) \cdot 10 + (10 - n). \tag{1.16}$$

With this simple rule I did not need to depend on my memory for the 9 times tables. How one thinks about a problem can have great impact.

## WEEK 5

12.5.0

L 12 Examples of algebraic expressions in physics
Fundamental Thm of Algebra (d'Alembert, ≈1760)
Analytic Geometry: Algebra + Geometry (Euclid to Descartes)
Newton and power series; Taylor series & ROC Composition of polynomial equations in two variables.

[34] By direct substitution demonstrate that Eq. 1.15 is the solution of Eq. 1.13.

[35] E.G.: $9 \cdot 7 = (7 - 1) \cdot 10 + (10 - 7) = 60 + 3$ and $9 \cdot 3 = 2 \cdot 10 + 7 = 27$. As a check note that the two digits of the answer must add to 9.

L 13 Root classification for polynomials of Degree * = 1–4 (p.102);
*Convolution* of monomials gives polynomial construction; Work out convolution for cubic
Show that $a_{n-1}$ is sum of roots and $a_0$ is product of roots. Quintic (* = 5) cannot be solved

L 14 First Analytic Geometry (Fermat 1629; Descartes 1637) (p. 118) Descartes' insight: Composition of two polynomials of degrees (m,n $\to$ one of degree $m \cdot n$)
Examples: $x^4 \circ x^2 = x^8$. Discuss Composition vs. intersection of functions.

### 1.3.2 Lec 12 Physical equations quadratic in several variables

Lec 12: Examples of algebraic expressions in physics
Fundamental Thm of Algebra (d'Alembert, ≈1760)
Composition of polynomial equations in two variables.
Analytic Geometry: Algebra + Geometry (Euclid to Descartes)
Newton and power series
Taylor series & ROC

When planes and lines are defined, the equations are said to be *linear* in the independent variables. In keeping with this definition of *linear*, we say that the equations are *non-linear* when the equations have degree greater than 1 in the independent variables.

As an example consider the well known problem in geometry: the intersection of a plane with a cone, which leads to the four conic sections: the circle, hyperbola, ellipse and parabola, along with some degenerate case such as the intersection of two straight lines[36]. If we stick to such 3-dimensional cases, we can write equations in the three variables $[x, y, z]$, and be sure that they each represent some physical geometry. For example $x^2 + y^2 + z^2 = r_0^2$ is a sphere of radius $r_0$.

The geometry and the algebra do not always seem to agree. Which is correct? In general the geometry only looks at the real part of the solution, unless you know how to tease out the complex solutions. However the roots of any polynomial are from $\mathbb{C}$, so we cannot ignore the imaginary roots, as Newton did. There is an important related Fundamental Theorems, known as Bézout's Theorem.

### 1.3.3 Lec 13: Polynomial root classification by convolution

Lec 13: *Convolution* of monomials gives polynomial construction; Work out convolution for cubic
Show that $a_{n-1}$ is sum of roots and $a_0$ is product of roots. Quintic (degree 5) cannot be solved.
Root classification for degrees 1-4: Bèzout's Thm & complex roots

Following the exploration of algebraic relationships by Fermat and Descartes, the first theorem, not yet proved, was being formulated. The idea behind this theorem is that every polynomial of degree $N$Eq. 1.11 has at least one root, which may be written as the product of the root and a second polynomial of degree of $N-1$. By the recursive application of this concept, it is clear that every polynomial of degree $N$ has $N$ roots. This result is known as the *Fundamental Theorem of Algebra:*

> Every polynomial equation $p(z) = 0$ has a solution in the complex numbers. As Descartes observed, a solution $z = a$ implies that $p(z)$ has a factor $z - a$. The quotient
>
> $$q(z) = \frac{p(z)}{z-a}$$
>
> is then a polynomial of one lower degree. ... We can go on to factorize $p(z)$ into $n$ linear factors.
>
> —Stillwell (2010, p. 285).

[36]Such problems were first studied algebraically and Descartes (Stillwell, 2010, p. 118) and Fermat (c1637).

The ultimate expression of this theorem is given by Eq. 1.11, which indirectly states that an $n^{th}$ degree polynomial has $n$ roots.

Today this theorem is so widely accepted we rarely think about it. Certainly at the time that you learned the quadratic formula, you were prepared to understand the idea. The simple quadratic case may be extended to the more general and difficult higher degree polynomial. The Matlab command `roots(`$[a_3, a_2, a_1, a_0]$`)` will instantly provide the roots of the cubic equation defined by the four coefficients $a_3, \ldots, a_0$. I don't know the largest degree that can be accurately factored by Matlab, but I'm sure its well over $N = 1,000$. Today, finding the roots numerically is a solved problem.

**Factorization versus convolution:** The best way to gain insight into the polynomial *factorization* problem is to do the inverse operation, multiplication of monomials. Given the roots $x_k$ there is a simple algorithm for computing the coefficients $a_k$ of $P_n(x)$ for any $n$, no matter how large. This method is called *convolution*. Convolution is said to be a *trap-door* since it is easy, while the inverse, factoring (*deconvolution*), is hard, even analytically intractable for degree $N \geq 5$ (Stillwell, 2010, p. 102).

### Convolution of monomials

As outlined by Eq. 1.11, a polynomial has two descriptions, first as a series with coefficients $a_n$ and second in terms of its roots $x_r$. The question is "What is the relationship between the coefficients and the roots. The answer is that they are related by a procedure called *convolution*. Let us start with a simple example, the quadratic (two roots). In factored form the quadratic may be written as

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

where in vector notation $[-a, -b]$ are the the roots and $[1, a + b, ab]$ are the coefficients.

Let's work out the coefficients for the cubic ($N = 3$) to see how the result generalizes. Start by multiplying the following three factors to find the coefficients $a_k$

$$(x - 1)(x - 2)(x - 3) = (x^2 - 3x + 2)(x - 3) = x(x^2 - 3x + 2) - 3(x^2 - 3x + 2) = x^3 - 6x^2 + 11x - 6.$$

For this case the roots are $[1, 2, 3]$ then the coefficients of the polynomial are $[1, -6, 11, -6]$. To verify this substitute each of the three roots back into the polynomial and verify they give zero. For example $r_1 = 1$ is a root since $P_3(1) = 1 - 6 + 11 - 6 = 0$. As the degree increases, the algebra becomes more difficult; even a cubic becomes tedious. Imagine trying to work out the coefficients for $N = 100$. What is needed is an simple way of finding the coefficients from the roots. Fortunately, *convolution* keeps track of the book keeping by formalizing the procedure.

**Convolution of two vectors:** To get the coefficients by convolution, write the roots as two vectors $[1, a]$ and $[1, b]$. To find the coefficients we must convolve the root vectors, indicated by $[1, a] \star [1, b]$, where $\star$ denotes convolution. Convolution is a recursive operation. The convolution of $[1, a] \star [1, b]$ is done as follows: reverse one of the two monomials and padding unused elements with zeros. Next slide one monomial against the other, forming the local *dot product* (element-wise multiply and add):

$$\begin{array}{cccc} a & 1 & 0 & 0 \\ 0 & 0 & 1 & b \\ = & 0 & & \end{array} \qquad \begin{array}{ccc} a & 1 & 0 \\ 0 & 1 & b \\ = & 1 & \end{array} \qquad \begin{array}{ccc} a & 1 & 0 \\ 1 & b & 0 \\ = & a+b & \end{array} \qquad \begin{array}{ccc} 0 & a & 1 \\ 1 & b & 0 \\ = & ab & \end{array} \qquad \begin{array}{cccc} 0 & 0 & a & 1 \\ 1 & b & 0 & 0 \\ = & 0 & & \end{array}$$

producing a final result $[\cdots, 0, 0, 1, a + b, ab, 0, 0, \cdots]$.

As seen by the above example, the position of the first monomial coefficients are reversed, and then slid across the second set of coefficients, the dot-product is computed, and the result placed in the output vector. Outside the range shown all elements are zero. The first set of our numerical example gives

$$[1, -1] \star [1, -2] = [1, -1 - 2, 2] = [1, -3, 2].$$

The general case is

$$[a, b] \star [c, d] = [ac, bc + ad, bd],$$

Now convolve the third term $[1, -3]$ with the above, resulting in

$$[1, -3] \star [1, -3, 2] = [1, -3 - 3, 9 + 2, -6] = [1, -6, 11, -6],$$

which is identical to the cubic example we found by the traditional algebraic method.

By convolving one monomial factor at a time the overlap is always two elements, thus it is never necessary to compute more than two multiplies and an add for each output coefficient. This greatly simplifies the operations (they may be done in your head), and the final answer is more likely to be correct. Comparing this to the algebraic method, convolution has a clear advantage.

Each time we convolve a new monomial the degree of the polynomial increases by 1. Thus two monomials gives degree 2, three monomials degree 3, etc. In general the degree $l$ of the product of two polynomials of degree $n, m$ is the sum of the degrees. For our example, the degrees are each 1 ($n = m = 5$), then the output degree is $l = 10$. Simply put, the product of two polynomials of degree $m, n$ having $m$ and $n$ roots each gives a polynomial of degree $m + n$ having $m + n$ roots. Note that the degree is one less than the length of the vector of coefficients.

**Roots as a function of degree:** The roots are easily found numerically for any reasonable polynomial of any desired degree. While there is a way to factor the polynomial analytically from the coefficients for $N \leq 4$, factoring is not possible for $N \geq 5$, as famously proved by Galois during his development of *group theory* (Stillwell, 2010, p. 87). It may occur to you that factoring and convolutional relationship, as inverse operations, form a proof of the Fundamental Theorem of Algebra. These relationships will be explored in greater depth in Section 3.2.2 of Chapter 3.

### 1.3.4 Lec 14 Introduction to Analytic Geometry

Weak!

Analytic geometry was the natural consequence of Euclid's Geometry merged with the new tool called algebra. What algebra added to geometry was the ability to compute with numbers. For example, the length of a line was measured in Geometry with a ruler, but numbers played no role. Once algebra was available the length of the line could be computed from the coordinates of the two ends of the line. Many concepts in geometry could be made more precise, such as the concept of a vector, defined by three numbers and an origin. The dot product between two vectors took a new meaning and the triple product defined the volume of a parallelepiped.

The most obvious addition was to turn the conic section into algebra rather than drawings made with a compass and ruler. An example is the composition of the line and circle, a construction what was used many times over the history of mathematics. Now it could be done with formulas.

The first two mathematicians to do this were Fermat and Descartes (Stillwell, 2010, p. 111-115). Newton contributed to this effort as well (Stillwell, 2010, p. 115-117). Given the new methods some problems emerged. The complex solutions continued to appear, without any obvious physical meaning. This seem to have been viewed as more of an inconvenience that a problem. Newton's solution to this dilemma was to simply ignore the imaginary cases (Stillwell, 2010, p. 119). The resolution of this was eventually to be found in Bézout's theorem, which states the number of intersections determined by the composition of two functions is determined by the product of their degrees. This problem is described as the *construct of equations* (Stillwell, 2010, p. 118). It was not proved until 1779, by Bézout.

This entire section need serious work

## WEEK 6

18.7.0

L 15 Gaussian Elimination (upper-diagonal matrix); Permutation matrix method
Solution to $x^3 - Ny^3 = 1$ using chord and tangent methods
AE-2: Linear (& nonlinear) systems of equations

L 16 Composition and the Bilinear transformation (ABCD Transmission matrix method)

L 17 Riemann sphere and the extended plane ($3^d$ chord and tangent method)
Möbius Transformation (youtube video)
Closing the complex plane

### 1.3.5 Lec 15 Gaussian Elimination

The method for finding the intersection of equations is based on the recursive elimination of all the variables but one. This method, known as *Gaussian elimination*, works across a broad range of cases, but may be defined in a systematic procedure when the equations are linear in the variables.[37] Rarely do we even attempt to solve problems in several variables of degree greater than 1. But Gaussian eliminations can still work in such cases (Stillwell, 2010, p. 90).

In Appendix C the inverse of a 2x2 linear system of equations is derived. Even for a 2x2 case, the general solution requires a great deal of algebra. Working out a numeric example of Gaussian elimination is more instructive. For example, suppose we wish to find the intersection of the equations

$$x - y = 3$$
$$2x + y = 2.$$

This 2x2 system of equations is so simple that you may immediately see the solution: Adding the two equations, and the $y$ term is eliminated, giving $3x = 5$. But doing it this way takes advantage of the specific example, and we need a method for larger systems of equations. We need a generalized (algorithmic) approach. This general approach is *Gaussian elimination*.

Start by writing the equations in a standardized *matrix* format

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}. \tag{1.17}$$

Next, eliminate the lower left term ($2x$) using a scaled version of the upper left term ($x$). Specifically, multiply the first equation by -2, add it to the second equation, replacing the second equation with the result. This gives

$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 - 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}. \tag{1.18}$$

Note that the top equation did not change. Once the matrix is "upper triangular" (zero below the diagonal) you have the solution. Starting from the bottom equation, $y = -4/3$. Then the upper equation then gives $x - (-4/3) = 3$, or $x = 3 - 4/3 = 5/3$.

In principle Gaussian elimination is easy, but if you make a calculation mistake along the way, it is very difficult to find the error. The method requires a lot of mental labor, with a high probability of making a mistake. You do not want to apply this method every time. For example suppose the elements are complex numbers, or polynomials in some other variable such as frequency. Once the coefficients become more complicated, the seeming trivial problem becomes highly error prone. There is a much better way, that is easily verified, which puts all the numerics at the end in a single step.

The above operations may be automated by finding a carefully chosen upper-diagonalization matrix $U$ that does the same operation. For example let

$$U = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}. \tag{1.19}$$

[37] https://en.wikipedia.org/wiki/System_of_linear_equations

Multiplying Eq. 1.17 by $U$ we find

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \tag{1.20}$$

we obtain Eq. 1.18. With a little practice one can quickly and easily find a $U$ that does the job of removing elements below the diagonal.

In Appendix C the inverse of a general 2x2 matrix is summarized in terms of three steps: 1) swap the diagonal elements, 2) reverse the signs of the off diagonal elements and 3) divide by the determinant $\Delta = ab - cd$. Specifically

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{1.21}$$

There are very few things that you must memorize, but the inverse of a 2x2 is one of them. It needs to be in your tool-bag of tricks, as you did for the quadratic formula.

While it is difficult to compute the inverse matrix from scratch (Appendix C), it takes only a few seconds to verify it (steps 1 and 2)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & -ab + ab \\ cd - cd & -bc + ad \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}. \tag{1.22}$$

Finally, dividing by the determinant gives the 2x2 identity matrix. A good strategy, when you don't trust your memory, is to write down the inverse as best you can, and then verify.

Using 2x2 matrix inverse on our example, we find

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{1+2} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 \\ -6+2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -4/3 \end{bmatrix}. \tag{1.23}$$

If you use this method, you will rarely (never) make a mistake, and the solution is easily verified. Either you can check the numbers in the inverse, as was done in Eq. 1.22, or you can substitute the solution back into the original equation.

### 1.3.6 Lec 16: Transmission (ABCD) matrix composition method

Lec 16: Matrix ABCD composition

In this section we shall derive the method of composition of linear systems, known by several names as the *ABCD Transmission matrix method*, or in the mathematical literature as the Möbius (bilinear) transformation. By the application of the method of composition, a linear system of equations, expressed in terms of 2x2 matrices, can represent a large family of differential equation networks.

By the application of Ohm's Law to the circuit shown in Fig. 1.6, we can model a cascade of such cells. Since the CFA can also treat such circuits, as shown in Fig. 2.4 and Eq. 2.2, the two methods may be related to each other via the 2x2 matrix expressions.

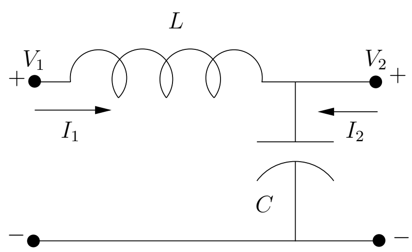



Figure 1.6: *This is a single LC segment of the transmission line show in Fig. 2.4. It may be modeled by the ABCD model as the product of two matrices, as show below.*

**Example of the use of the ABCD matrix composition:** In Fig. 1.6 we see the network is composed of a series inductor (mass) with an impedance $Z_l = sL$ and a shunt capacitor (compliance) with an impedance of $Z_c = 1/sC$. By Ohm's Law, each impedance is describe by a linear relation between the current and the voltage. Regarding the inductive impedance, applying Ohm's law we find

$$V_1 - V_2 = Z_l I_1.$$

Regarding the capacitive impedance, applying Ohm's law we find

$$V_2 = (I_1 + I_2) Z_c.$$

These two equations may be written in matrix form. The series inductor equation is

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & Z_l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_1 \end{bmatrix}, \tag{1.24}$$

while the shunt capacitor equation is

$$\begin{bmatrix} V_2 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}, \tag{1.25}$$

where $Y_c = 1/Z_c$.

When the second matrix equation for the capacitor is substituted into the inductor equation, we find the composite ABCD matrix for the cell, as the product of two matrices

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & sL \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \tag{1.26}$$

For each matrix the input voltage and current are on the left (e.g., $[V_1, I_1]^T$), while the output voltage and current is on the right (e.g., $[V_2, -I_2]^T$).

This is a composition because the output of the second matrix is the input of the first. The final equation (Eq. 1.26) completely characterizes the relations between the input and output of the cell of Fig. 1.6.

### 1.3.7 Lec 17: Riemann Sphere: $3^d$ extension of tangent/chord methods

Lec 17: Riemann Sphere and the extended complex plane and the point at $\infty$

Once algebra was formulated c830 CE, mathematics was able to expand beyond the limits placed on it by geometry on the real plane, and the verbose descriptions of each problem in prose. The geometry of Euclid's Elements had paved the way, but after 2000 years, the addition of the language of algebra would change everything. The analytic function was a key development that had served both Newton and Euler. Also the investigations of Cauchy made important headway with his work on complex variables. Of special note was integration and differentiation in the complex plane of complex analytic functions, the topic of stream 3.

Facts!

It was Riemann, working with Gauss, who made the breakthrough, with the concept of the *extended complex plane*. The idea was based on the composition of a line with the sphere, similar to the derivation of Euclid's formula for Pythagorean triplets. But the impact was unforeseen, and it changed analytic mathematics forever, and the physics that was supported by it, by simplifying integrals to the extreme. This idea is captured in the Fundamental Theorem of complex calculus (Sections 1.2.2 and 4.3.1).

The idea is outlined in Fig. 1.7. On the left is a circle and a line, the difference here is that the line starts at the north pole and ends on the real $x \in \mathbb{R}$ axis, at point $x$. At point $x'$ the line cuts through the circle. Thus the mapping from $x$ to $x'$ takes every point on the real line to a point on the circle. For example, the point $x = 0$ maps to the south pole (not indicated). To express $x'$ in terms of $x$ one must composition of the line and the circle, similar to the composition used in Fig. 2.7. The points on

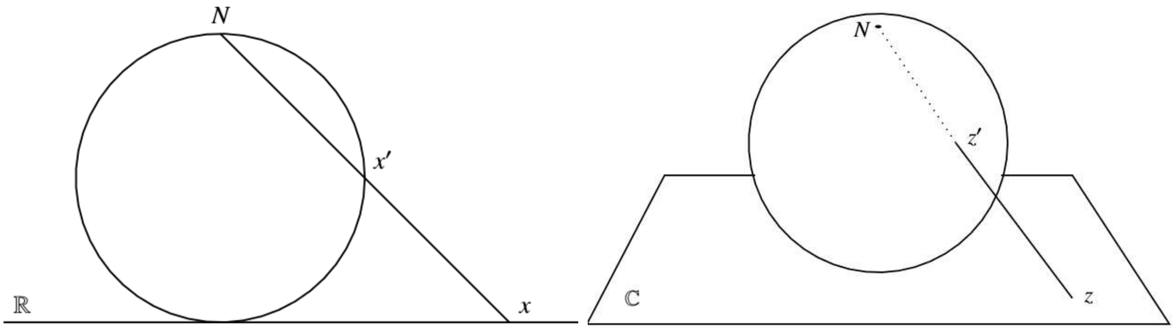



Figure 1.7: *The left panel shows how the real line may be composed with the circle. Each real $x$ value maps to a corresponding point $x'$ on on the unit circle. The point $x \to \infty$ then naturally maps to the north pole $N$. This simple idea may be extended with the composition of the complex plane with the unit sphere, thus mapping the plane onto the sphere. As with the circle, the point on the complex plane $z \to \infty$ maps onto the north pole $N$. This construction is important because while the plane is open (does not include $z \to \infty$), the sphere is analytic at the north pole. Thus the sphere defines the* closed extended plane. *Figure from Stillwell (2010, pp. 299-300).*

the circle, indicated here by $x'$, require a traditional polar coordinate system having a unit radius and an angle defined between the radius a vertical line going through the north pole. When $x \to \infty$ the point $x' \to N$, the north pole. The point at the north pole (on the circle) is called the *point at infinity.* But this idea must to go further, as shown on the right half of Fig. 1.7.

Here the real tangent line is replaced by the a tangent complex $z \in \mathbb{C}$ plane, and the puncture point $x'$ with a complex puncture point $z'$, in this case on the complex sphere, called the *extended complex plane.* This is a natural extension of the tangent/chord method on the left, but with significant consequences. The main difference between the complex plane $z$ and the extended complex plane, other than the coordinate system, is what happens at the north pole. On the plane the point at $|z| = \infty$ is not defined, whereas on the sphere the point at the north pole is simply another point, like every other point on the sphere.

Mathematically the plane is said to be an *open set* since the limit $z \to \infty$ is not defined, whereas on the sphere the $z'$ is a *closed set* since the the north pole is defined. The distinction between an open and closed set is important because the closed set allows the function to be analytic at the north pole, which it cannot be on the plane (since the point at infinity is not defined).

The $z$ plane may be replaced with another plane, say the $w = f(z) \in \mathbb{C}$ plane, where $w$ is some function $f$ of $z \in \mathbb{C}$. We shall limit ourselves to *complex analytic functions* of $z$, namely $w = u(x, y) + v(x, y)\jmath = f(z) = \sum_{n=0}^{\infty} z^n$. In summary, given a point $z = x + y\jmath$ on the open complex plane, we map this using the function $w = f(z) \in \mathbb{C}$ to the complex $w = u + v\jmath$ plane, and from there to the closed extended complex plane $w'(z)$. The point of doing this is that it allows us to allow the function $w'(z)$ to be analytic at the north pole, meaning it can have a convergent Taylor series at $z \to \infty$.

### Möbius bilinear transformation

In mathematics the *Möbious transformation* has special importance because it is linear in its action. In the engineering literature this transformation is known as the *bilinear transformation.* Since we are engineers we shall stick with the engineering terminology. But is you wish to read about this on the internet, be sure to also search for the mathematical term, which may be better supported..

When a point on the complex plane $z = x+y\jmath$ is composed with the bilinear transform $(a, b, c, d \in \mathbb{C})$, the result is $w(z) = u(x, y) + v(x, y)\jmath$

$$w = \frac{az + b}{cz + d} \tag{1.27}$$

the transformation is a cascade of four independent compositions

1. translation $(w = z + b)$
2. scaling $(w = |a|z)$

3. rotation ($w = \frac{a}{|a|}z$) and

4. inversion ($w = \frac{1}{z}$)

Each of these transformations are a special case of Eq. 1.27, with the inversion the most complicated. A wonderful video showing the effect of the bilinear (Möbius) transformation on the plane is available that I highly recommended you watch it: Low resolution: https://www.youtube.com/watch?v=0z1fIsUNhO4 High resolution: https://www.ima.umn.edu/~arnold/moebius/moebius-movie.mov

When the extended plane (Riemann sphere) is analytic at $z = \infty$, one may take the derivatives there, and one may meaningfully integrate through $\infty$. When the bilinear transformation rotates the Riemann sphere, the point at infinity is translated to a finite point on the complex plane, revealing normal characteristics. A second way to access the point at infinity is by inversion, which takes the north pole to the south pole, swapping poles with zeros. Thus a zero at infinity is the same as a pole at the origin, and vice-versa.

This construction of the Riemann sphere and the M̈bious (bilinear) transformation allow us to fully understand the point at infinity, and treat it like any other point. If you felt that you never understood the meaning of the point at $\infty$ (likely), then this should help.

## WEEK 7 18.7.0

L 18 Colorized plots of complex analytic functions (Matlab *zviz.m*)

L 19 Signals and Systems: Fourier vs. Laplace Transforms **AE-3**

L 20 Role of Causality and the Laplace Transform:
$u(t) \leftrightarrow 1/s$ (LT)
$2\tilde{u}(t) \equiv 1 + \mathrm{sgn}(t) \leftrightarrow 2\pi\delta(\omega) + 2/\jmath\omega$ (FT)

### 1.3.8 Lec 18: Complex analytic mappings (colorized plots)

One of the most difficult aspects of complex functions of a complex variable is understanding what's going on. For example, $w = \sin(s)$ is trivial when $s = \sigma + \omega\jmath$ is real, because $\sin(\sigma)$ is then real. But $w(s) = \sin(s) \in \mathbb{C}$ not so easily visualized when $s \in \mathbb{C}$, because such functions are mapping the $s = \sigma + \omega\jmath$ plane to the $w(\sigma, \omega) = u(\sigma, \omega) + v(\sigma, \omega)\jmath$ plane.

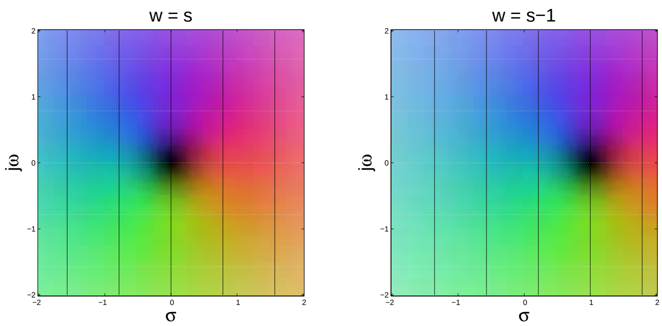



Figure 1.8: *On the left is a color map showing the definition of the complex $s$ plane, with hue (darkness) indicating magnitude and color indicating angle. On the left $w(s) = s$, $u = \sigma$ and $v = v$. On the right $w(s) = s - 1$, a simple shift of one unit in $\sigma$ is shown. Specifically $u = \sigma - 1$ and $v = \omega$. The color gives the phase of $w$ and hue (color saturation) the magnitude $|w|$, as discussed in the text.*

Every complex point from the $s$ plane is operated on by the function $F(s)$ to produce a new complex point $w(s) = F(s)$. This is typically difficult to understand the first time you see it, thus requires a visualizing method. Fortunately with computer software today, this problem can be solved by adding color to the graph. A Matlab script `zviz.m` was used to make these make the charts shown here.[38] By studying the function's color map, one can comprehend the complex mapping.

[38] URL for `zviz.m`: http://jontalle.web.engr.illinois.edu/uploads/298/zviz.m

We could look at $u(\sigma,\omega)$ and $v(\sigma,\omega)$ separately in black and white, but domain coloring allows us to display everything on one plot. Note that for this visualization we see the polar form of $w(s)$ rather than a rectangular $(u, v)$.

Before we can give an example we must explain the color code being used for the magnitude and phase of the complex plane. In Fig. 1.8 we show this code, as a 2x2 dimensional graph called "domain-coloring." The color allows us to visualize the magnitude and phase of the function. The color is used to represent the phase and hue (dark to light) to represent the magnitude. On the left is the reference condition given by the the identity mapping ($w = s$). Red is 0°, violet is 90°, blue is 135°, blue-green is 180° and sea-green is $-90°$ (or 270°). The hue (darkness) represents the magnitude. Since the function $w = s$ has a zero at $s = 0$ it is dark there, and becomes brighter as we move away from the origin. The figure on the right is $w = F(z-1)$, which moves the zero point to the right by 1. As one would predict, the zero has moved to the right by 1 unit, and the color has followed in line with the new location of the zero. Colorized plots can give you a clear picture of the complex analytic function mappings $w(x, y) = u(x, y) + v(x, y)\jmath = F(x + y\jmath)$.

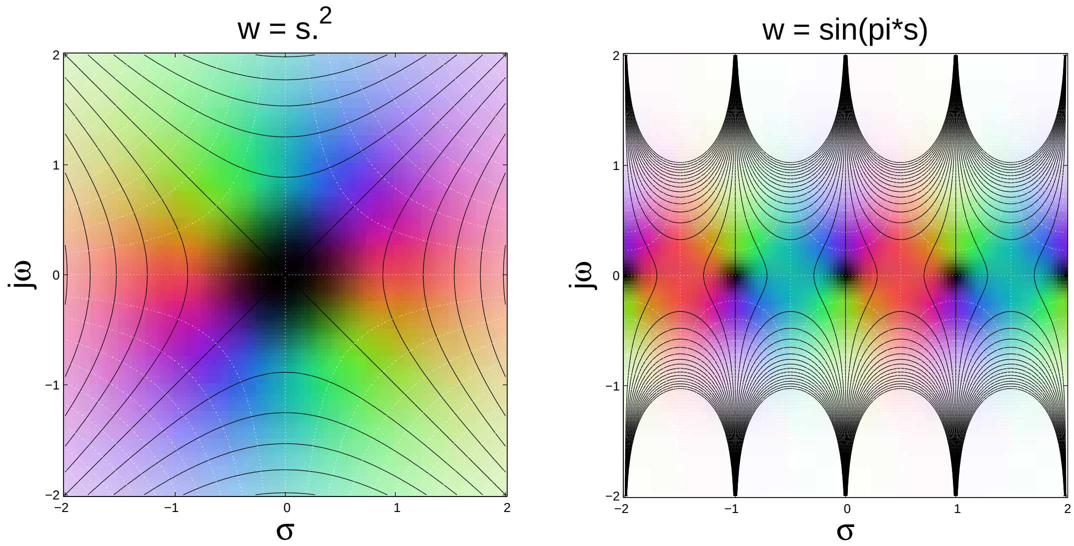



Figure 1.9: *On the left is the function $w(s) = s^2$ and on the right is $w(s) = \sin(\pi s)$. See the discussion in the text for an interpretation of these charts.*

Two more examples are given in Fig. 1.9 to help you interpret the two complex mappings $w = s^2$ (left) and $w = \sin(\pi s)$ (right). On the left there are two red regions because in polar coordinates $w(s) = |s|^2 e^{2\theta\jmath}$, thus the square causes the phase to rotate twice around for once around the $s$ plane. Namely the angle is doubled and the magnitude squared. Due to the faster changing phase in $w$ thus there are two red regions, one when $\theta = 0$ and the second at $\theta = \pi$ ($\angle w(s) = 2\theta$). The dark spot is larger because of the square on the magnitude, which expands the unit circle $|s| = 1$.

The right-hand plot of $w(s) = \sin(\pi s)$ is equally interesting. Along the $\sigma$ axis (real part of $s$) the function is the periodic $\sin(\sigma)$ function. The dark spots are at $\sigma = k\pi$, with $k \in \mathbb{Z}$. This is the normal $\sin(\pi\sigma)$ function with zeros at $0, \pm\pi, 2\pm\pi, \ldots$. When we stray off the $\omega\jmath = 0$ axis, the function either goes to zero (black) or $\infty$ (white). This behavior carries the same $2\pi$ periodicity as it has along the $\omega = 0$ line. These figure are worthy of careful study to develop an intuition for complex functions of complex variables. In Section 1.3.8 we shall explore more complex mappings, and in greater detail.

### 1.3.9 Lec 19: Signals: Fourier Transforms

Two heavily used transformations in engineering mathematics are the Fourier and the Laplace transforms, that are used for time–frequency domain analysis. They are not the same, but can be easily confused as being related. Here we will clarify the differences and similarities.

The Fourier and Laplace transforms take a (typically real) time domain signal $f(t) \in \mathbb{R}$ and transform it to the frequency domain $F(\omega) \in \mathbb{C}$, where it is typically complex. For the Fourier transform, both the time $-\infty < t < \infty$ and frequency $\infty < \omega < \infty$ are strictly real.

The Laplace transform takes signals that are strictly zero for negative time ($f(t) = 0$ for $t < 0$), and transforms them into complex functions of complex frequency $s = \sigma + \omega \jmath$. When a signal is zero for negative time $f(t < 0) = 0$ is is said to be *causal.* Any restriction on a function (e.g., real, causal, positive real part, etc.) is called a *symmetry property.* There are many forms of symmetry.

There is a very convenient notation for each of these two basic transformations, using a double-arrow: $f(t) \leftrightarrow F(\omega)$ and $f(t) \leftrightarrow F(s)$, where the first is the Fourier transform $t \in \mathbb{R}$, $\omega \in \mathbb{R}$ with a strictly real frequency, and the second is $t \geq 0 \in R$, $s = \sigma + \omega \jmath \in \mathbb{C}$, with complex *Laplace frequency.*

Besides these two basic types of time to frequency transforms, there are several variants that depend on the nature of the time and frequency representations. For example, when the time signal is sampled (discrete in time), the frequency response becomes periodic. And the time response become periodic, the frequency response is sampled (discrete in frequency). These two variants may be simply characterized as *periodic in time* $\Rightarrow$ discrete in frequency, and *periodic in frequency* $\Rightarrow$ discrete in time. In Section 3.4.2 we shall explain these concepts in greater detail, with examples.

**Definition of the Fourier Transform:** The definitions of the two transforms are similar, except the time response for the Laplace transform is restricted to be causal and the frequency response of the Fourier Transform is restricted to be real.

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-\jmath \omega t} dt \qquad \widehat{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{\jmath \omega t} d\omega$$

$$F(\omega) \leftrightarrow f(t) \qquad \widehat{f}(t) \leftrightarrow F(\omega)$$

Notes:

1. Both time $t$ and frequency $\omega$ are real.
2. When taking the forward transform (from time to frequency) the sign of the exponential is negative.
3. The limits on the integrals in both the forward and reverse FTs are $[-\infty, \infty]$.
4. When taking the inverse FT (IFT), the normalization factor of $1/2\pi$ is required to cancel the $2\pi$ in the differential of the integral $d\omega/2\pi = df$, where $f$ is frequency in [Hz], and $\omega$ is the radian frequency.
5. The Fourier step function may be defined by the use of superposition of 1 and $\text{sgn}(t) = t/|t|$ as

$$\widetilde{u}(t) \equiv \frac{1 + \text{sgn}(t)}{2} = \begin{cases} 1 & \text{if } t > 0 \\ 1/2 & t = 0 \\ 0 & \text{if } t < 0 \end{cases}.$$

The following is the derivation of this function assuming a delay of 1 [s]

$$\widetilde{U}(\omega) \equiv \int_{-\infty}^{\infty} \tilde{u}(t-1) e^{-j\omega t} dt \leftrightarrow \widehat{u}(t-1) = \left\{ \frac{1 - \text{sgn}(t-1)}{2} \right\} = \pi \tilde{\delta}(\omega) + \frac{e^{-j\omega}}{j\omega}$$

$$\neq \int_{1}^{\infty} e^{-j\omega t} dt = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{1}^{\infty} = \frac{e^{-j\omega} - e^{-j\omega\infty}}{j\omega} = \frac{e^{-j\omega}}{j\omega} - \frac{e^{-j\omega\infty}}{j\omega}$$

6. The convolution $\widetilde{u}(t) \star \widetilde{u}(t)$ has no meaning because $1 \star 1$ and $\widetilde{\delta}^2(\omega)$ have no meaning.

7. The inverse FT will have convergence problems whenever there is a discontinuity in the time response. This we indicate with a hat over the reconstructed time response. The error between the target time function and the reconstructed is zero in the root-mean sense, but not point-wise. Specifically, $\widehat{u}(t) \neq u(t)$ but $\int |\widehat{u}(t) - u(t)|^2 dt = 0$ near $t = 0$, the discontinuity point for the Fourier step function. At the point of the discontinuity the reconstructed function has Gibbs ringing (it does not converge at jumps). There are convergence issues with the IFT at jumps. More on this in Section 3.4.2.

### 1.3.10 Lec 20: Laplace Transforms

Lec 20: Signals (FT) versus Systems (LT): Fourier Transforms for signals versus Laplace transforms for systems; Causality

When dealing with engineering problems it is convenient to separate the signals we use from the systems that process them. We do this by treating signals, such as a music signal, differently from a system, such as a filter. In general signals may start and end at any time. The concept of causality has no physical meaning in signal space. Physical systems on the other hand obey very rigid rules (to assure that they remain physical). These Physical restrictions are described in terms of nine *Network Postulates*, which are discussed in some length in Lecture 1.3.11, and in greater detail in Section 3.5.1.

**Definition of the Laplace Transform:** The forward and inverse Laplace transforms are

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt \qquad\qquad f(t) = \frac{1}{2\pi\jmath}\int_{\sigma_0-\infty\jmath}^{\sigma_0+\infty\jmath} F(s)e^{st}ds$$

$$F(s) \leftrightarrow f(t) \qquad\qquad f(t) \leftrightarrow F(s)$$

Notes:

1. Time $t \in \mathbb{R}$. The complex Laplace frequency is defined as $s = \sigma + \omega\jmath$.

2. When taking the Forward transform (from time $t$ to frequency $s$), the sign of the exponential is negative. This is necessary to assure that the integral converges when the integrand $f(t) \to \infty$ as $t \to \infty$ (is diverging). For example, when $f(t) = e^t u(t)$ without the negative $\sigma$ exponent, the integral would not converge.

3. The target time function must be zero for negative time (causal).

   The time limits are $0^- < t < \infty$. Thus the integral must start from slightly below $t = 0$ to integrate over any delta functions at $t = 0$. For example if $f(t) = \delta(t)$, the integral must include both sides of the impulse. If you wish to include non-causal functions such as $\delta(t + 1)$ it is necessary to extend the lower limit to pick them up. In such cases simply let the lower limit be $-\infty$ and let the integrand determine the limits.

4. The limits on the integrals of the forward are $t : (0^-, \infty)$ and reverse FTs are $[\sigma_0 - \infty\jmath, \sigma_0 + \infty\jmath]$. These limits will be justified in Section 1.4.9.

5. When taking the inverse FT (IFT), the normalization factor of $1/2\pi\jmath$ is required to cancel the $2\pi\jmath$ in the differential $ds$ of the integral.

6. The frequency for the LT must be is complex, and in general $F(s)$ is complex analytic for $\sigma > \sigma_0$. For example The real and imaginary parts of $F(s)$ are related, and given one, it may be possible to find the other. More on this in Section 3.4.2.

7. To take the inverse Laplace transform, we must learn how to integrate in the complex $s$ plane. This will be discussed in Section 4.3.1.

8. The Laplace step function is defined as

$$u(t) = \begin{cases} 1 & \text{if } t > 0 \\ \text{NaN} & t = 0 \\ 0 & \text{if } t < 0 \end{cases}$$

   and not defined at $t = 0$.

9. It is easily shown that $u(t) \leftrightarrow 1/s$ since

$$F(s) = \int_0^\infty e^{-st} dt = -\frac{e^{-st}}{s}\bigg|_0^\infty = \frac{1}{s}.$$

   With the LT there is no Gibbs effect, as the step function, with a true discontinuity, is exactly represented by the inverse LT.

$$\begin{aligned} f(t) &\leftrightarrow F(s) \\ \delta(t) &\leftrightarrow 1 \\ u(t) &\leftrightarrow 1/s \\ u(t) \star u(t) = tu(t) &\leftrightarrow 1/s^2 \end{aligned}$$

10. Frequently the Laplace transform takes the form of a ratio of two polynomials. In such case the roots of the numerator polynomial are call the *zeros* while the roots of the denominator polynomial are called the *poles*. For example the LT of $u(t) \leftrightarrow 1/s$ has a pole at $s = 0$.

Disc relations between Fourier and Laplace delta and step functions

## WEEK 8

20.8.0

L 21 The 6 postulates of System (aka, Network) Theory; The important role of the Laplace transform re impedance

L 22 Exam II (Evening exam)

### 1.3.11 Lec 21: The 9 postulates of systems

Lec 21: The 9 postulates of Networks (systems)

Systems of differential equations, such as the wave equation, need a mathematical statement of underlying properties, which we present here in terms of nine *network postulates*:

(P1) *causality* (non-causal/acausal)

(P2) *linearity* (nonlinear)

(P3) *real* (complex) time response

(P4) *passive* (active)

(P5) *time-invariant* (time varying)

(P6) *reciprocal* (non-reciprocal)

(P7) *reversibility* (non-reversible)

(P8) *space-invariant* (space-variant)

(P9) *quasi-static* (multi-modal).

Each postulate has two (in one case three) categories. For example for (P2) a system is either linear or non-linear and for (P1) is either causal, non-causal or acausal. P6 and P9 only apply to 2-port networks (those having an input and an output. The others can apply to both 1 and 2 port networks.

Related forms of these postulates had been circulating in the literature for many years, widely accepted in the network theory literature (Van Valkenburg, 1964a,b; Ramo et al., 1965). But the first six of these were formally introduced Carlin and Giordano (1964), while (P7-P9) were added by Kim et al. (2016).

### 1.3.12 Lec 22: Exam II (Evening Exam)

Lec 22: Exam II

## WEEK 8 23.9.0

Week 8 Friday *Stream 3*

L 23 The amazing Bernoulli family; Fluid mechanics; airplane wings; natural logarithms
The transition from geometry $\rightarrow$ algebra $\rightarrow$ algebreic geometry $\rightarrow$ real analytic $\rightarrow$ complex analytic
From Bernoulii to Euler to Cauchy and Riemann

## 1.4 Stream 3: Scalar (Ordinary) Differential Equations

Stream 3 is $\infty$, a concept which inspires "very large," which takes us to calculus, where $\infty$ actually means "very small," since taking a limit requires small numbers. Taking the limit means you never reaching the target. This is a concept that the Greeks called *Zeno's paradox* (Stillwell, 2010, p. 76).

When speaking of the class of *ordinary* (versus *vector*) differential equations, the term *scalar* is preferable, since the term "ordinary" is vague.

There are a special subset of Fundamental theorems for scalar calculus, the first being Leibniz's Theorem. These will be discussed in Sections 1.4.3, 4.2.2 and 4.3.2.

### 1.4.1 Lec 23: Bernoulli to Euler and standard function package

Lec 23: Euler works out real analytic functions: sin, cos, log, tan, zeta function ...
Newton and Calculus
Bernoulli family
Euler standard periodic function package

The period of analytic discovery:
Coming out of the dark ages, from algebra, to analytic geometry, to calculus.
Starting with real analytic functions by Euler, we move to complex analytic functions with Cauchy.

Integration in the complex plane is finally conquered.
Beginning of real analytic functions. When do they converge? How are they used.

## WEEK 9 23.9.0

Week 9 Monday

L 24 Power series and integration of functions (ROC)
Fundamental Theorem of calculus (Leibniz theorm of integration)
$1/(1-x) = \sum_{k=0}^{\infty} x^k$ with $x \in \mathbb{R}$

L 25 Integration in the complex plane: Three theorems
Integration of $1/s$ on the unit circle, and on a unit circle centered about $s = 1 + i$.

L 26 Cauchy-Riemann conditions
Real and imaginary parts of analytic functions obey Laplace's equation.
Infinite power Series and analytic function theory; ROC

### 1.4.2 Lec 24: Complex Analytic functions and the ROC

Lec 24: Infinite analytic power series and complex analytic functions; ROC

To solve a differential equation, or integrate a function, Newton used the Taylor series to integrate one term at a time. However he only use real functions of a real variable due to the fundamental lack of understanding as to the meaning of a complex analytic series. This same method is the cornerstone of finding solutions to differential equations today, but in a modified form, that makes it less obvious how it works. Rather than working directly with the Taylor series, today we use the complex exponential. The reasoning is that the complex exponential is the eigenfunction of the derivative, namely

$$\frac{d}{dt}e^{st} = se^{st}.$$

Thus a linear differential equation in time may be simply transformed into a polynomial in complex frequency $s$ by looking for solutions of the form $A(s)e^{st}$. This substitution transforms the differential equation into a polynomial $A(s)$ in complex frequency. The roots of $A(s)$ are the eigenvalues of the original differential equation. Thus the keys to understanding the solutions of differential equations, both scalar and vector, is to work in the frequency domain.[39] The Taylor series has been replaced by $e^{st}$, transforming Newtons real Taylor series into the complex exponential eigenfunction of the derivative. In some sense, these are the same method.

This is heavily trodden soil, that every student now learns in their first course in scalar (ordinary) differential equations. However what the modern approach frequently ignores is the fundamental role of to the complex power series, that is the concept of the *complex analytic function* (Section 4.3.1. If a function $F(s)$ is complex analytic then it has a power series

$$F(s) = \sum_{0}^{\infty} c_k s^k.$$

If we take the term by term derivative we find

$$\frac{d}{ds}F(s) = \sum_{0}^{\infty} k c_k s^{k-1},$$

[39] Make explicit the connection between the roots of the polynomial $A(s)$ and the eigenvalues of the matrix of the vector form of the same equation.

which is also complex analytic. Thus if the series for $F(s)$ is valid (i.e., it converges), then its derivative is also valid, where it is convergent. This is a very powerful concept, fully exploited by Newton for real functions of a real variable, and later by Cauchy and Riemann for complex functions of a complex variable. The key here is "When does the series fail to converge?" in which case, the entire representation fails. This is the main message behind the *Fundamental Theorem of Complex Calculus*. The full power of this concept was first exploited by Bernard Riemann (1826-1866) in his PhD Thesis of 1851 at University of Göttingen, under the tutelage of Carl Friedrich Gauss (1777-1855), drawing heavily on the work of Cauchy.

The key definition of a complex analytic function is that it has a Taylor series representation over a region of the complex frequency plane $s = \sigma + j\omega$, that converges in a *region of convergence* (ROC), about each pole of that function. A surprising feature of an analytic function is that within the ROC, the inverse of that function also has an analytic expansion with its ROC. Thus given $w(s)$, one may also determine $s(w)$ to any desired accuracy, critically depending on the ROC.

This concept of analytic inverses becomes rich when the inverse function is multi-valued. For example, if $F(s) = s^2$ then $s(F) = \pm\sqrt{F}$. Riemann dealt with such extensions with the concept of a branch-cut with multiple planes, labeled by a branch number. Each branch describes an analytic function (Taylor series) that converges within some ROC, with a radius out to the nearest pole of that function. This explicitly dealt with the defining a unique inverse to multi-valued functions.

#### Complex impedance functions

One of the most obvious applications of complex functions of a complex variable an impedance. The impedance function $Z(s) = R(\sigma,\omega) + \jmath X(\sigma,\omega)$ has resistance $R$ and reactance $X$, as a function of complex frequency $s = \sigma + \jmath\omega$. The function $z(t) \leftrightarrow Z(s)$ are defined by a Laplace transform pair.

As an example, a resistor $R_0$ in series with an capacitor $C_0$ has an impedance

$$Z(s) = R_0 + 1/sC_0. \tag{1.28}$$

In mechanics a dash-pot (damper) and a spring have the same mechanical impedance. A resonant system has an inductor, resistor and a capacitor, with an impedance given by

$$Z(s) = R_0 + 1/sC_0 + sM_0 \tag{1.29}$$

which is a second degree polynomial in the complex frequency $s$. Thus it has two roots (eigenvalues). When $R_0 > 0$ these roots are in the left half $s$ plane.

Systems (networks) containing many elements, and transmission lines, can be much more complicated, yet still have a simple frequency domain representation. This is the key to understanding how these physical systems work, as will be described below.

### 1.4.3 Lec 25: Integration in the complex plane

Lec 25: Integration in the complex plane; Wave equation; Laplace's equation

Leibniz's formula gives the area under a curve as the difference in the integral between the two limits such that the area only depends on the end points

$$F(x) = F(0) + \int_0^x f(\xi)d\xi. \tag{1.30}$$

This is based on a one-dimensional integration of real function $f(x)$ along the real $x$ axis. As is well known,

$$\frac{d}{dx}F(x) = f(x)$$

because the total area only depends on the end points for real areas $F(x)$.

For the complex case of an impedance, we define

$$F(s,t) = Z(s)e^{st}, \tag{1.31}$$

and the integrate in the complex plane, we may write a relation similar to the one-dimensional case

$$f(s) = f(0) + \int_0^s Z(s)e^{zt}dz. \tag{1.32}$$

Compare this to the real integral of the area over the real line $x$ Eq. 1.30, Other than the limits, this formulas are the same as the Inverse Laplace transform. The integral can only dependent on the end points if

$$\frac{df}{ds} = F(s,t). \tag{1.33}$$

But what does it man to take the derivative of a function with respect to $s$?

In the 1 dimensional case (Leibniz formula) the area only depends on the end points. It is interesting to determine if, or when, this condition holds for complex integration. In the complex case the end points are in the complex plane, which for example is $z$ from $s = 0$ to $s$. Thus the condition is that if the integral of $F(z,t)$ only depends on the end points ($[0,s]$) then it must be independent of the path taken in the complex $z$ plane.

Many of these fundamental theorems of integration are closely related, in which case a teaching moment is near. The best example is the relationship between the *Fundamental Theorem of Calculus* (aka Leibniz formula) and the *Fundamental Theorem of Complex Calculus* (aka, the Cauchy integral theorem). The Leibniz formula Eq. 1.30 states that the area under a curve $f(x) \in \mathbb{R}$ only depends on the end points. Equation 1.33 follows.

Thus when the integral of $f(x)$ only depends on the limits, the function must be analytic. The same holds true for the complex analytic case. When $f(x)$ is not analytic (has no Taylor series) the derivative may not exist.

### 1.4.4 Lec 26: Cauchy-Riemann conditions

Lec 26: Cauchy-Riemann conditions

For path independence the value of the integral ($f(s,t)$) must be the same for a path holding either $\sigma$ or $\jmath\omega$ constant. This leads to a pair of equations called the *Cauchy-Riemann* conditions in terms of the real and imaginary parts $F(s) = R(\sigma,\omega) + \jmath X(\sigma,\omega)$ and $s = \sigma + \jmath\omega$:

$$\frac{\partial R(\sigma,\omega)}{\partial \sigma} = \jmath\frac{\partial X(\sigma,\omega)}{\partial \jmath\omega} \qquad \frac{\partial R(\sigma,\omega)}{\partial \jmath\omega} = \jmath\frac{\partial X(\sigma,\omega)}{\partial \sigma} \tag{1.34}$$

These are the necessary conditions that the integral of the function $F(s)$ is independent of the path, expressed in terms of the real and imaginary parts of the function and path. This assumption about the function is a very strong condition on $F(s)$ which requires that it may be written as a Taylor series in the complex argument $s$:

$$F(s) = F_0 + F_1 s + \frac{1}{2}F_2 s^2 + \cdots. \tag{1.35}$$

Any function that may be expressed as a *Taylor series* about a point is said to be *complex analytic* at that point. The series is said to converge within a *radius of convergence* (ROC). This a a highly restrictive conditions have significant physical consequences. For example, every impedance function $Z(s)$ obeys the CR conditions over large regions of the $s$ plane, including the entire right half plane (RHP), defined by $\sigma > 0$. When this conditions is generalize to volume integrals, it is called *Green's Theorem*, which is a special case of both Gauss's and Stokes's Laws, used heavily in the solution of boundary value problems in Engineering-Physics (e.g., solving EM problems that start from Maxwell's equations). The last third of this course shall depend heavily on this concept and various generalizations.

## WEEK 10

**26.10.0**

L 27 Differentiation in the complex plane: Fundamental Thm of complex calculus (FTCC);
Complex Analytic functions; ROC in the complex plane
$Z(s) = R(s) + \jmath X(s)$: real and imag parts obey Laplace]s Equation
Basic equations of mathematical Physics: Wave equation, Diffusion equation, Laplace's Equation
Motivation: Dispersion relation for the wave equation $\kappa \cdot \kappa = s^2/c_0^2$

L 28 Three Fundamental theorems of complex integral calculus
$\int_0^z = F(\zeta)d\zeta = F(z) - F(0)$: $dZ(s)/ds$ independent of direction
Integration in the complex plane; Integrals independent of limits
Cauchy-Riemann conditions

L 29 Inverse Laplace transform
Inverse Laplace transform: Poles and Residue expansions;
Application of the Fundamental Thm of Complex Calculus
The Inverse Laplace Transform (ILT); poles and the Residue expansion: The case for causality
ROC as a function of the sign of time in $e^{st}$ (How does causality come into play?)
Examples.

### 1.4.5 Lec 27: Differentiation in the Complex plane

Lec 27: Differentiation in the complex plane: Inv Lap Trans

### 1.4.6 Lec 28: Three complex integration theorems

Lec 28: Three complex integration theorems; Basic defs of complex integration: FTCompCalc

### 1.4.7 Lec 29: Inverse Laplace Transform (Cauchy residue theorem)

Lec 29: Inverse Laplace Transform (Cauchy residue theorem)
Fund Thm of complex Calculus: 3 thms (maybe 1?)

Use of the Residue theorem to evaluate the inverse Laplace Transform. Discuss causal and anti-causal cases. How does this relate to Green's theorem (in 2 dimensions).

## WEEK 11

**30.11.0**

L 30 Inverse Laplace transform & Cauchy Residue Theorem

L 31 Case for causality Closing the contour as $s \to \infty$; Role of $\Re st$
**DE-3**

L 32 Properties of the LT:
1) Modulation, 2) Translation, 3) convolution, 4) periodic functions
Tables of common LTs

### 1.4.8 Lec 30: Inverse Laplace Transform and the Cauchy Residue Thm

Inverse LT; Cauchy Residue Thm

### 1.4.9 Lec 31: Case for causality; closing the contour

Inv LT: Case for causality

### 1.4.10 Lec 32: Properties of the LT (e.g., modulation, translation, etc.)

Lec 32: Properties of the LT (e.g., modulation, translation, etc.)

## WEEK 12 33.12.0

L 33 Multi-valued functions in the complex plane; Branch cuts
The extended complex plane (regularization at $\infty$) (Stillwell, 2010, p. 280)
Complex analytic functions of Genus 1 (Stillwell, 2010, p. 343)

L 34 Euler's vs. Riemann's Zeta function $\zeta(s)$: Poles at the primes
colorized plot of $\zeta(s)$
??Sterling's formula??

L 35 Exam III

### 1.4.11 Lec 33: Multi-valued functions Branch cuts

Lec 33: *(Week 12)* Multi-valued functions: Branch Cuts; Extended complex plane (point at $\infty$

### 1.4.12 Lec 34: The Riemann zeta function

Lec 34: Riemann Zeta function $\zeta(s)$ with poles at the primes;
LT of $\zeta(s)$ The LT of the complex Riemann zeta function $\zeta(x)$ (Fig. 4.1), as introduced by Euler for real arguments. $x \in \mathbb{R}$ as his way of proving that the number of primes is infinite (Goldstein, 1973). In the end, this formulation provided detailed information about the structure of the primes. The zeta function depends explicitly on the primes, which is why it is interesting (Section 4.5.2).

One might wonder why Euler's zeta function is known as the Riemann zeta function. It is because Riemann showed its properties when the argument is complex, namely he extended $\zeta(s)$ into the complex plane ($s \in \mathbb{C}$) (Section 4.5.2). Given that $\zeta(s)$ is a function of complex (Laplace) frequency, one might naturally ask if $\zeta(s)$ has an inverse Laplace transform. There seems to be very little written on this topic,[40] but we shall explore this interesting question further (Table 4.1). Perhaps even more important is the taxonomy of $\zeta(s)$ is in question here, namely where are its poles and zeros? About this there are volumes written.

#### The Riemann Zeta function is analytic with poles at log-primes

Why does the zeta function have zeros? Perhaps this is some extension of the Euler function that has zeros, rather than zeta itself. Ask Andrew Odlyzko about this problem. Go to the Math dept first and find someone qualified to discuss this with.

[40] Cite book chapter on inverse LT of $\zeta(s)$.

### 1.4.13 Lec 35: Exam III (Evening Exam)

Lec 35: No class (Exam III)

## WEEK 13

36.13.0

L 36 Scaler wave equations and the Webster Horn equation; WKB method
A real-world example of large delay, where the branch-cut placement is critical

L 37 Partial differential equations of Physics
Scaler wave equation and its solution in 1 and 3 Dimensions
**VC-1**

L 38 Vector dot and cross products $A \cdot B, A \times B$
Gradient, divergence and curl

– Thanksgiving Holiday 11/19–11/27 2016

## 1.5 Vector Calculus (Stream 3b)

### 1.5.1 Lec 36: Scalar Wave Equation (Acoustics)

Lec 37: Scalar wave equation and its solution;
Causality and d'Alembert solutions to the wave equation. History about Newton & d'Alembert.

#### Acoustic waves; The scalar wave equation: scalar differential equation in the frequency domain

#### The Webster Horn equation

The effect of a spatial area functions for waves in horns (the horn equation).
Derivation of the Horn equation, starting from the basic equations of acoustics.
Development of the basic equations of acoustics: Conservation of mass and momentum.
Sound in a uniform tube.
Sound propagation away from a point source (Helmholtz's Equation)

$$\nabla^2\psi + k^2\psi = \delta(r).$$

### 1.5.2 Lec 37: Partial Diff Eqs of Physics

Partial diff eqs of physics; Fundamental theorem of Vector Calculus

### 1.5.3 Lec 38: Vector dot and cross products

Lec 38: Vector dot and cross products

### 1.5.4 Thanksgiving Holiday 11/19–11/27 2016

## WEEK 14 37.14.0

L 39 Gradient, divergence and curl: Gauss's (divergence) and Stokes's (curl) theorems

L 40 J.C. Maxwell unifies Electricity and Magnetism with the formula for the speed of light
Basic definitions of $E, H, B, D$
O. Heaviside's (1884) vector form of Maxwell's EM equations and the *vector wave equation*
How a loud-speaker works

L 41 *The Fundamental Thm of vector calculus*
*Incompressable* and *Irrotational* fluids and the two defining vector identities

### 1.5.5 Lec 39 $\nabla, \nabla\cdot, \nabla\times$ & Vector operators

Curl and Divergence and corresponding Thms: Gauss's and Stokes's Laws

There are three key vector differential operators that we need to understand Maxwell's equations. The *gradient* transforms a potential, such as a voltage $V(x, y, z)$ into a vector, such as the electric field vector $\mathbf{E}$. The *divergence* $\nabla \cdot \mathbf{E}(x, y, z)$ transforms a vector field into a scalar field. Finally the curl $\nabla \times \mathbf{A}(x, y, z)$ transforms a vector into a vector.

To define these three operations we first need to define scalar and vector fields. These are concepts that you already understand. It is the terminology that needs to be mastered, not a new concept. Think of a voltage field in space, say between two finite sized capacitor plates. In such a case, the voltage is given by a *scalar field* $V(x, y, z)$. A scalar field is also called a *potential.* Somewhat confusing is that one may also define *vector potentials* which is three scalar potentials turned into a vector. So this term is more than one use. It is therefore important to recognize the intended use of the filed. This can be gleaned from the units. Volts is a scalar field.

The simplest example of a scalar potential is the voltage between two very large (think $\infty$) conducting parallel planes, or plates (large so that we can ignore the edge effects). In this case the voltage varies linearly between the two plates. For example

$$V(x, y, z) = V_0(1 - x)$$

is a scalar potential, thus it is scalar field (i.e., potential). At $x = 0$ the voltage is $V_0$ and at $x = 1$ the voltage is zero. Between 0 and 1 the voltage varies linearly. Thus $V(x, y, z)$ defines a *scalar field.*

If the same setup were used but the two plates were 1x1 [cm$^2$], with a 1 [mm] air gap, there will be a small "fringe" effect at the edges that would slightly modify the ideal fields. The hope is that this effect can be made small so that it does not ruin the capacitor composed of the two plates. If we are given a set of three scalar fields, we define a *vector field.* If the three elements of the vector are potentials, then we have a vector potential.

#### Gradient operator $\nabla$

The gradient operator takes a scalar fields and outputs a vector field. This is exactly what the gradient does. Given any scalar field $V(x, y, z)$ it outputs a vector field[41]

$$\mathbf{E}(x, y, z) = [E_x(x, y, z), E_y(x, y, z), E_z(x, y, z)]^T = -\nabla V(x, y, z).$$

[41] As before vectors are columns, which take up space on the page, thus we write them as rows and take the transpose to properly format them.

To understand these three operations we therefore need to define the domain and range of their operation, as specified in Table 5.1.

### 1.5.6 Lec 40: Definitions of E, H, B, D and Maxwell's equations

Maxwell's Equations

#### Maxwell's Equations

Once you have mastered the three basic vector operations, the gradient, divergence and curl, you are able to understand Maxwell's equations. Like the vector operations, these equations may be written in integral or vector form. The notation is basically the same since the concept is the same. The only difference is that with Maxwell's equations we are dealing with well defined physical quantities. The scalar and vector fields take on meaning, and units. Thus to understand these important equations, one must master the units, and equally important, the names of the four fields that are manipulated by these equations.

We may now restate everything defined above in terms of two types of vector fields that decompose every vector field. Thus another name for the Fundamental Theorem of Vector Calculus is the *Helmholtz decomposition.* An *irrotational field* is define as one that is "curl free," namely the vector potential is a constant. An *incompressible field* is one that is "diverge free," namely the scalar potential is a constant. Just to confuse matters, the incompressible field is also called a *solenoidal field.* I recommend that you know this term (as it is widely used), but never use it. Rather use incompressible as a more meaningful and physical term. Once you learn the concept of a solenoid, you may wish to change your mind about this usage, but I predict you will not.

### 1.5.7 Lec 41 Fundamental Theorem of Vector calculus (Helmholtz Theorem)

#### The fundamental Theorem of Vector Calculus

Here we define the basic vector operations based on the $\nabla$ "operator," the *gradient*, *divergence* and the *curl.* These operations may be defined in terms of integral operations on a surface in 1, 2 or 3 dimensions, and then taking the limit as that surface goes to zero. These operators are required to understand Maxwell's Equations, the crown jewel of mathematical physics.

#### Incompressible and Irrotational vector fields

One of the most important fundamental theorems is that of vector calculus. This is also known as *Helmholtz theorem.* This theorem is very easily stated but less easily to appreciate. But a physical description of what is going on will help.

A vector field may be split into two parts, that are independent. Think of linear and angular momentum. They are also independent in that they represent different ways to store energy. An object with mass can be moving along a path and rotating at the same time. The two modes of motion define two different types of kinetic energy, transnational and rotational. In some real sense, Helmholtz theorem quantifies this independence.

**The Fundamental Theorem of Vector Calculus:** This theorem is also known as Helmholtz' Theorem. It states that every differentiable vector field may be written as the sum of two terms, a scalar part and a vector part expressed in terms of a scalar potential $\phi(x, y, z)$ (think voltage) and a vector potential (think magnetic vector potential). Specifically

$$\mathbf{E} = -\nabla\phi + \nabla \times \mathbf{A}. \tag{1.36}$$

Two show that this relationship splits the vector field $\mathbf{E}$ into two parts we need to add to the mix two key vector identities, that are always true (assuming they exist, i.e, that the fields are differentiable):

$$\nabla \times \nabla \phi(x, y, z) = 0, \tag{1.37}$$

or in words, the *curl of the divergence =0*, and

$$\nabla \cdot \nabla \times \mathbf{A} = 0 \tag{1.38}$$

or the *divergence of the gradient =0.* These identities are easily verified by working out a few examples based on the definitions of the three operators, for example in terms of their integral definitions. They also have an important physical meaning, as indicated above, that every vector field may be split into its transnational and rotational parts, as with our example of momentum.

Figure 1.10: *von Helmholtz portrait taken from the English translation of his 1858 paper "On integrals of the hydrodynamic equations that correspond to Vortex motions" (in German).*

If we apply these two identities to Helmholtz's Theorem (Eq. 1.36), we can appreciate the significance of the theorem. It is a form of proof actually, once you have satisfied yourself that the vector identities are true. In fact one can work backward using a physical argument, that rotational momentum and thus energy is independent from transnational momentum, thus energy. Again this all goes back to the definitions of rotation and transnational forces, hidden in the vector operations. Once these forces are made clear, the meaning of the vector operations all take on a very well defined meaning, and the mathematical constructions, centered around Helmholtz's theorem, begins to provide some common-sense meaning.

Specifically if we take the divergence of Eq. 1.36, and use the divergence vector identity

$$\nabla \cdot \mathbf{E} = \nabla \cdot \{-\nabla \phi + \nabla \times \mathbf{A}\} = -\nabla \cdot \nabla \phi = -\nabla^2 \phi.$$

since the divergence vector identity removes the vector potential $\mathbf{A}(x, y, z)$.

Likewise if we take the curl of Eq. 1.36, and use the curl vector identity

$$\nabla \times \mathbf{E} = \nabla \times \{-\nabla \phi + \nabla \times \mathbf{A}\} = \nabla \times \nabla \times \mathbf{A},$$

since using the curl vector identity, removes the scalar field $\phi(x, y, z)$.

There is a third vector identity that needs to be mentioned for later use

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.$$

The best way to think of this relationship is that it *defines* the vector Laplacian $\nabla^2 \mathbf{A}$. In other words, think of this identity the definition of the left hand side of

$$\nabla^2 \mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$$

## WEEK 15

**40.15.0**

L 42 Quasi-static approximation and applications:
The Kirchoff's Laws and the *Telegraph wave equation*, starting from Maxwell's equations The telegraph wave equation starting from Maxwell's equations
Quantum Mechanics

L 43 Last day of class: Review of Fund Thms of Mathematics:
Closure on Numbers, Algebra, Differential Equations and Vector Calculus,
The Fundamental Thms of Mathematics & their applications:
Theorems of Mathematics; Fundamental Thms of Mathematics (Ch. 9); Normal modes vs. eigenstates, delay and quasi-statics;

– Reading Day
VC-1 Due

Lec 42: Quasi-static approximation; Brune's Impedance;
Lec 43: Last lecture: Review of the fundamental thms of mathematics

### 1.5.8 Lec 42: Kirchhoff's Laws and the Quasi-static approximation

The term *quasi-statics* indicates a type of approximation that is widely used when reducing a problem based on partial differential equations to one of a scalar differential equation. It is important to understand the nature of this approximation so that it is not miss-applied. Quasi-statics is a way of reducing a three dimensional problem to a 1 dimensional problem. This approximation is at the heart of transmission line theory. Lets begin with an example: The acoustic wave equation describes how the scalar pressure $p(x, y, z, t$ propagates in three dimensions. If the wave propagation is restricted to a pipe, such as an organ pipe, or to a string, as in a guitar string, we do not need to worry about the transverse directions. What needs to be modeled by the equations is the wave propagation along the pipe or string. Thus we replace the three-dimensional wave with a one-dimensional wave, without further thought.

However if we wish to be more precise about this reduction in geometry, we need to consider the quasi-static approximation, as it makes some assumptions about what is happening in the other directions, and quantifies their effects. Taking the case of wave propagation in a tube, say the ear canal, there is the main wave direction, down the tube. But there is also wave propagation in the transverse direction, perpendicular to the direction of propagation. The key statement of the quasi-static approximation is that the wavelength in the transverse direction is much larger that the radius of the pipe. This is equivalent to saying that the radial wave reaches the walls and is reflected back, in a time that is small compared to the distance propagated down the pipe. Clearly the speed of sound down the pipe and in the transverse direction is the same if the medium is homogeneous (i.e., air or water). Thus the sound reaches the walls and is returned to the center line in a time that the axial wave traveled about 1 diameter along the pipe. So if the distance traveled is several diameters, the radial parts of the wave have time to come to equilibrium. So the question one must ask is, what are the conditions of such an equilibrium. The most satisfying answer to this is to look at the internal forces on the air, due to the gradients in the pressure.

The pressure $p(x, y, z, t$ is a potential, thus its gradient is a force density $\mathbf{f}(x, y, z, t) = -\nabla p(x, y, z, t)$. What this equation tells us is that as the pressure wave approaches that of a plane wave, the radial (transverse) forces go to zero. If the tube has a curvature, or a change in area, then there will be local forces that create radial flow. But after traveling a few diameters, these forces will come to equilibrium and the wave will return to a plane wave. The internal stress caused by a change is area must settle out very quickly. There is a very important caveat however: it is only at low frequencies that the plane wave can dominate. At frequencies such that the wavelength is very small compared to the diameter, the distance traveled between reflections is much greater than a few diameters. Fortunately the frequencies where this happens are so high that they play no role in frequencies that we care about. This effect is referred to as *cross-modes* which imply some sort of radial standing waves. In fact such modes exist in the ear canal, but on the eardrum where the speed of sound is much slower that that of air. Because of the slower speed, the ear drum has cross-modes, and these may be seen in the ear canal pressure. Yet they seem to have a negligible effect on our ability to hear sound with good fidelity. The point here is that the cross modes are present, but we call upon the quasi-static approximation as a justification for ignoring them, to get closer to the first-order physics.

**Breakdown of the quasi-static approximation at high frequencies:** If we wonder, for the sake of wonderment, what happens at high frequencies where the quasi-static approximation begins to break down, we need to consider other significant physics of the system. In acoustics there are two basic effects that have been ignored by assuming that wave propagation is dictated by the wave equation, viscosity and thermal effects. In fact, it turns out that these two loss mechanisms are related, but to understand why is quite difficult. However Helmholtz, with some help from Krichhoff, figured this out and published it between 1863 (Helmholtz, 1863b) and 1868 (Kirchhoff, 1868). Their theory was summarized by Lord Rayliegh (Rayleigh, 1896) and then experimentally verified to be correct by Warren P. Mason (Mason, 1928). The nature of the correction is that the wave number is extended to be of form $\kappa(s) = (s + \beta_0\sqrt{s})/c_0$, where the forwarded $P_-$ and backward $P_+$ pressure waves propagate as

$$P_\pm(s, x) = e^{\pm\kappa(s)x}.$$

The expression for $\kappa$ has a frequency where the loss term $\beta_0\sqrt{s}/c_0$ is equal to the loss-less wave number $s/c_0$. This is the frequency where the loss-less part equals the lossy part. This frequency is $s_0 + \beta_0\sqrt{s_0} = 0$, or $\sqrt{s_0} = \beta_0$ or $f_0 = \beta^2/2\pi$. Assuming air at 23.5° [C], $c_0 = \sqrt{\eta_0 P_0/\rho_0} \approx 344$ [m/s] is the speed of sound, $\eta_0 = c_p/c_v = 1.4$ is the ratio of specific heats, $\mu = 18.5\times10^{-6}$ [Pa-s] is the viscosity, $\rho_0 \approx 1.2$ [kgm/m$^2$] is the density, $P_0 = 10^5$ [Pa] (1 atm).

The constant $\beta_0 = P\eta'/2S\sqrt{\rho_0}$

$$\eta' = \sqrt{\mu}\left[1 + \sqrt{5/2}\left(\eta^{1/2} - \eta^{-1/2}\right)\right]$$

is a thermodynamic constant, P is the perimeter of the tube and S the area (Mason, 1928).

For a cylindrical tube having radius $R = 2S/P$, $\beta_0 = \eta_0'/R\sqrt{\rho}$. To get a feeling for the magnitude of $\beta_0$ consider a 7.5 [mm] tube (i.e., the average diameter of the adult ear canal). Then $\eta' = 6.6180\times10^{-3}$ and $\beta_0 = 0.8055$. Using these conditions the wave-number cutoff frequency is $0.8^2/2\pi = 0.1$ [Hz]. At 1 kHz the ratio of the loss over the propagation is $\beta_0/\sqrt{|s|} = 0.81/\sqrt{2\pi 10^3} \approx 1\%$. At 100 [Hz] this is a 3.2% effect. Mason shows that the wave speed drops from 344 [m/s] at 2.6 [kHz] to 339 [m/s] at 0.4 [kHz], which is a 1.5% reduction in the wave speed. In terms of the losses, this is much larger effect, since as the frequency decreases, the loss term goes to zero (it goes as $\beta_0/\sqrt{\omega}$), whereas at 1 [kHz] the loss is 1 [dB/m], which is $\infty$ compared to 0 [dB/m]. Note that the loss and the speed of sound vary inversely with the radius.

In Section 5.4.1 we shall look at some simple problems where we use the quasi-static effect and derive the Kirchhoff voltage and current equations, starting from Maxwell's equations.

### 1.5.9 Lec 43: Final Review for Final Exam

#### Summary

Physics and Mathematics evolved as tools to help us navigate our environment, not just physically around the globe, but how to solve daily problems such as food, water and waste management, understand the solar system and the stars, defend ourselves, use tools of war, etc. At first we used intuition by observing, but then we understood that *mathematics allows us to generalize these tools.*

Mathematics began as a simple way of keeping track of how many things there were.

Based on the historical record of the abacus, a memory tool used to assist in mental arithmetic, that went far beyond what one could do in their head, one can infer that people precisely understood the concept of counting, addition, subtraction and perhaps multiplication, which is recursive additions. However this knowledge did not seem to show up in the written number systems. The Roman numerals were not useful for doing calculations, which were done on the abacus. The final answer would then be expressed in terms of the Roman number system. All it was good for, it seems, is expressing the final answer $\mathbb{N}$ is converted to Roman numerals.

According to the known written record, the number zero (null) had no written symbol until the time of Brahmagupta (628 CE). One should not assume the concept of zero was not understood simply because there was no symbol for it in the Roman Numeral system. Negative numbers and zero would be obvious when using the abacus. Numbers between the integers would be represented as *rational numbers* $\mathbb{Q}$. Any number may be approximated with arbitrary accuracy using rations numbers.

There is some evidence that the abacus, commonly believe to be a Chinese invention, was introduced to the Chinese by the Romans, as it was needed in trade.

The abacus is a simple counting tool, formalizing the addition of very large numbers. Subtraction is a trivial generalization of addition, the opposite of "bringing together." Multiplication is also a generalization of addition when addition is repetitive. For example $10 + 10 + 10 = 3 \cdot 10$. Division, the inverse of multiplication (i.e., repetitive addition) is repetitive subtraction. For example $(10 + 10 + 10)/10 = 3$ is the same as $30 - 10 - 10 - 10 = 30 - 3 * 10 = 0$. Working with integers in this way, these ancient tools are simply common sense methods.

We are so used to multiplication and division, we can loose sight of what it really means. Lets try to above method on 31. Taking 10 away from 31 gives $31 - 10 - 10 - 10 = 31 - 3 * 10 = 1$. So in this case we have a remainder of 1. It follows that $31/10 = (30 + 1)/10 = 3 + 1/10$. It is easy to forget the basic principle, that division is based on repeated subtraction, having learned, to well, the rules of division.

Mathematics is the science of formalizing a repetitive method into a set of rules, and then generalizing it as much as possible. Generalizing the multiplication and division algorithm, to different types of numbers, becomes increasingly more complex as we move from integers to rational numbers, irrational numbers, real and complex numbers and ultimately, vectors and matrices. How do you multiply two vectors, or multiply and divide one matrix by another? Is it subtraction as in the case of two numbers? Multiplying and dividing polynomials (by long division) generalizes these operations even further. Linear algebra is further important generalization, fallout from the Fundamental Theorem of Algebra, and essential for solving the generalizations of the number systems.

The concept of a number evolved very slowly at first. Starting with the cardinal numbers $\mathbb{N}$, or *counting numbers*, rational numbers $\mathbb{Q}$, the ratio of integers, allowed a refined way of measuring with greater precision. For example: $\{3, 31/10, 157/50, 22/7, 1571/500, 355/113\}$ are increasingly better approximations to $\pi$.). The representation $22/7 = (21+1)/7 = 3+1/7$ jumps out due to its precision, having a relative error of 0.4% ($\approx 0.4 \times 10^{-3}$). The next rational approximation is given by $355/113 = 3 + 1/(7 + 1/16)$, with a relative error of $\approx 0.85 \times 10^{-7}$. These approximations were worked out by Chinese scholar Zu Chongzhi (429–500 AD). Note that 113 is prime but $355 = 5 \cdot 71$, thus they have no common factors. The next such approximation is 104348/33215, with a relative error of $10^{-10}$. These two integers are also not primes, but again have no common prime factors, which means they are *coprime*. This may have made early mathematicians wonder if this was the beginning of a pattern. An interesting question is "How might one test this hypothesis?"

Many of the concepts about numbers naturally evolved from music, where the length of a string (along with its tension) determined the pitch (Stillwell, 2010, pp. 11, 16, 153, 261). Cutting the string's length by half increased the frequency by a factor of 2. One forth of the length increases the frequency by a factor of 4. One octave is a factor of 2 and two octaves a factor of 4 while a half octave is $\sqrt{2}$. The musical scale was soon factored into rational parts. This scale almost worked, but did not generalize (sometimes known as *Pythagoreas' comma*[42]), resulting in today's *well tempered scale*, which is based on 12 equal geometric steps along one octave, or 1/12 octave ($\sqrt[12]{2} \approx 1.05946 \approx 18/17 = 1 + 1/17$).

But the concept of a *factor* was clear. Every number may be written as either a sum, or a product (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of a second degree polynomial was understood, which lead to a generalization of factoring, since the polynomial, a sum of terms, may be written in factored form. If you think about this a bit, it is sort of an amazing idea, that needed to be discovered (Stillwell, 2010, p. ). This concept lead to an important string of theorems on factoring polynomials, and how to numerically describe physical quantities. Newton was one of the first to master these tools with his proof that the orbits of the planets are ellipses, not circles. This lead him to expanding functions in terms of their derivatives and power series. Could these sums Way too dense.

[42] https://en.wikipedia.org/wiki/Pythagorean_comma

be factored? The solution to this problem led to calculus.

So mathematics, a product of the human mind, is a highly successful attempt to explain the physical world. All aspects of our lives were impacted by these tools. Mathematical knowledge is power. It allows one to think about complex problems in increasingly sophisticated ways. An equation is a mathematical sentence, expressing deep knowledge. Witnessed $E = mc^2$ and $\nabla^2\psi = \ddot{\psi}$.

**Reading List:** The above concepts come straight from mathematical physics, as developed in the $17^{th}$–$19^{th}$ centuries. Much of this was first developed in acoustics by Helmholtz, Stokes and Rayleigh, following in Green's footsteps, as described by Lord Rayleigh (1896). When it comes to fully appreciating Green's Theorem and reciprocity, I have found Rayleigh (1896) to be a key reference. If you wish to repeat my reading experience, start with Brillouin (1953), followed by Sommerfeld (1952); Pipes (1958). Second tier reading contains many items: Morse (1948); Sommerfeld (1949); Morse and Feshbach (1953); Ramo et al. (1965); Feynman (1970); Boas (1987). A third tier might include Helmholtz (1863a); Fry (1928); Lamb (1932); Bode (1945); Montgomery et al. (1948); Beranek (1954); Fagen (1975); Lighthill (1978); Hunt (1952). You must enter at a level that allows you to understand. Successful reading of these books critically depends on what you already know. A rudimentary (high school) level of math comprehension must be mastered first. Read in the order that helps you best understand the material.

Without a proper math vocabulary, mastery is hopeless. I suspect that one semester of college math can bring you up to speed. This book is my attempt to present this level of understanding.

# Chapter 2

# Number Systems: Stream 1

This chapter is devoted to *Number Systems* (Stream 1), starting with the counting numbers $\mathbb{N}$. In this chapter we delve more deeply into the details of the topics of Lectures 4-9.

## WEEK 2

4.2.0

L 4 The two prime number theorems:

1. Fundamental Thm of Arith
   Brief discussion on Prime Numbers $\pi_k \in \mathbb{P}$;
   Prime sieves
2. Prime number theorem
   Density of prime numbers $\rho_\pi \in \mathbb{N}$; *Logrithmic integral* $Li(N)$ definition

L 5 Greatest Common Denominator (GCD) (Euclidean Algorithm) (Stillwell, 2010, p. 42)

1. Euclidean Algorithm for finding the GCD: $k = gcd(n, m)$
2. *Coprimes* $n, m$ have no common factors: $gcd(n, m) = 1 \Rightarrow n \perp m$

L 6 *Continued fractions* (extended Euclidean Algorithm) (e.g., $\pi \approx 22/7$)
Rational approximations:
Irrational numbers: e.g. $\sqrt{2} \approx 17/12$
Transcendental numbers: e.g. $\pi \approx 22/7$
Matlab's *rat(), rats()* commands.

## 2.1 Week 2

In Section 1.2.3 we explore in more detail the two fundamental theorems of prime numbers, working out a sieve example, and explore the logarithmic integral $Li(N)$ which approximates the density of primes $\rho_k(N)$ up to prime $N$.

The topics of Section 1.2.4 consider the practical details of computing the *greatest common divisor* (GCD) of two integers $m, n$ (Matlab's routine `l=gcd(m,n)`), with detailed examples and comparing the algebraic and matrix methods. Homework assignments will deal with these two methods. Finally we discuss the relationship between coprimes and the GCD. In Section 1.2.5 we defined the Continued Fraction algorithm (CFA), a method for finding rational approximations to irrational numbers. The CFA and GCD are closely related, but the relation needs to be properly explained. In Section 1.2.7 we derive *Euclid's formula*, the solution for the Pythagorean triplets (PT), based on Diophantus' *chord/-tangent* method. This method is used many times throughout the course notes, first for computing Euclid's formula for the PTs, then for finding a related formula in Section 1.2.8 for the solutions to Pell's equation, and finally for finding the mapping from the complex plane to the extended complex plane (the Riemann sphere).

Finally in Section 1.2.9 the general properties of the *Fibonacci sequence* is discussed. This equation is a special case of the second order digital resonator (well known in digital signal processing), so it has both historical and practical application for engineering. The general solution of the Fibonacci is found by taking the Z-transform and finding the roots, resulting in an eigenvalue expansion (Appendix B).

### 2.1.1 Lec 4 Prime numbers

If someone came up to you and asked for a theory of counting numbers, I suspect you would look them in the eye with a blank stare, and start counting. It sounds like either a bad joke or a stupid question. Yet integers are rich topic, so the question is not even slightly dumb. It is somewhat amazing that even birds and bees can count. While I doubt birds and bees can recognize primes, cicadas and other insects only crawl out of the ground in multiples of prime years, (e.g., 13 or 17 year cycles). If you have ever witnessed such an event (I have), you will never forget it. Somehow they know. Finally, there is an analytic function, first introduced by Euler, based on his analysis of the Sieve, now known as the *Riemann zeta function* $\zeta(s)$, which is complex analytic, with its poles at the the logs of the prime numbers. The exact relationship between the primes and the poles will be discussed in Sections 1.4.12 and 4.5.2. The properties of this function are truly amazing, even fun. It follows that primes are fundamental properties of the counting numbers, that the theory of numbers (and primes) is an important topic of study. Many of the questions, and some of the answers, go back to at least the time of the Chinese (Stillwell, 2010).

The most relevant question at this point is "Why are integers so important?" We addressed this question in Section 1.2.9. First we count with them, so we can keep track of "how much." But there is much more to numbers than counting: We use integers for any application where absolute accuracy is essential, such as banking transactions (making change), and precise computing of dates (Stillwell, 2010, p. 70) or location (I'll meet you at location L$\in \mathbb{N}$ at time T$\in \mathbb{N}$), building roads or buildings out of bricks (objects of a uniform size). If you go to $34^{th}$ street and Madision and they are at 33th and Madison, that's a problem. To navigate we need to know how to predict the tides, the location of the moon and sun, etc. Integers are important because they are precise: Once a month there is a full moon, easily recognizable. The next day its slightly less than full.

#### Sieves

A recursive sieve method for finding primes was first devised by the Greek Eratosthenes (Fig. 2.1). One first writes down all the numbers from $2, \cdots, N$. Starting from the first prime $\pi_1 = 2$, one successively strikes out all the multiples of that prime. For example, starting from $\pi_1 = 2$ one strikes out $2{\cdot}2, 2{\cdot}3, 2{\cdot}4, 2{\cdot}5, \cdots, 2{\cdot}N$. By definition the multiples are products of the target prime (2 in our example) and every another integer ($n \ge 2$). All the even numbers are removed in the first iteration. One then considers the next integer not struck out (3 in our example), which is identified as the next (second) prime $\pi_2$. Then all the $(N-2)/2$ non-prime multiples of $\pi_2$ are struck out. The next number which has not been struck out is 5, thus is prime $\pi_3$. All remaining multiples of 5 are struck out ($\cancel{10}$, $\cancel{15}$, $\cancel{25}$, $\cdots$). This process is repeated until all the numbers on the starting list have been processed. As the word sieve implies, this sifting sifting process takes a heavy toll on the integers, rapidly pruning the non-primes. In four loops of the sieve algorithm, all the primes below $N = 50$ have identified, colored in bold-red. The final set of primes is displayed at the bottom of Fig. 2.1.

Once a prime greater than $N/2$ has been identified, we may stop, since twice that prime is greater than $N$, the maximum number under consideration. Once you have reached $\sqrt{N}$ all the primes have been struck out (this follows from the fact that the next prime $\pi_n$ is multiplied by an integer $n = 1, \ldots N$). Once this number $n\pi_n > N$ the list has been existed, which must be $n < \sqrt{N}$.

There are various schemes for making the sieve more efficient. For example the recursion $n\pi_k = (n-1)\pi_k + \pi_k$. could speed up the process, by replacing the multiply by an add by a known quantity. When using a computer, memory efficiency and speed are the main considerations.

### Finding the primes by the Sieve of Eratosthenes[a] 4.2.3

- *Write $N-1$ counting number from 2 to $N$ (List)*
- *Define a multiplier $n \in \mathbb{N}$ denoted $n := \{2, \cdots, N\}$.*
- *$k = 1$ is the loop index for the next prime $\pi_k$*
- *Identify in red each prime $\pi_k \in \mathbb{P}$*
- *Remove (Cross out) all multiples $n \cdot \pi_n$ of $\pi_k$*
  1. *The first element on List is a prime (e.g., for $k = 1$, $\pi_1 = 2$).*
  2. *Cross out $n \cdot \pi_n$: (e.g., for $k = 1$, cross out $n \cdot \pi_1 = 4, 8, 16, 32, \cdots$).*
  3. *Increment the loop index $k := k + 1$ and return to step 1*

*After the first step with $k = 1$ and $\pi_1 = 2$, we cross out $n\pi_k$ (all the even numbers):*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | **2** | 3 | ~~4~~ | 5 | ~~6~~ | 7 | ~~8~~ | 9 | ~~10~~ |
| 11 | ~~12~~ | 13 | ~~14~~ | 15 | ~~16~~ | 17 | ~~18~~ | 19 | ~~20~~ |
| 21 | ~~22~~ | 23 | ~~24~~ | 25 | ~~26~~ | 27 | ~~28~~ | 29 | ~~30~~ |
| 31 | ~~32~~ | 33 | ~~34~~ | 35 | ~~36~~ | 37 | ~~38~~ | 39 | ~~40~~ |
| 41 | ~~42~~ | 43 | ~~44~~ | 45 | ~~46~~ | 47 | ~~48~~ | 49 | ~~50~~ |

*Following the second loop $k = 2$, $\pi_2 = 3$, and we have removed $n\pi_k$ $(6, 9, 12, 15, \ldots)$:*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | **2** | **3** | ~~4~~ | 5 | ~~6~~ | 7 | ~~8~~ | ~~9~~ | ~~10~~ |
| 11 | ~~12~~ | 13 | ~~14~~ | ~~15~~ | ~~16~~ | 17 | ~~18~~ | 19 | ~~20~~ |
| ~~21~~ | ~~22~~ | 23 | ~~24~~ | 25 | ~~26~~ | ~~27~~ | ~~28~~ | 29 | ~~30~~ |
| 31 | ~~32~~ | ~~33~~ | ~~34~~ | 35 | ~~36~~ | 37 | ~~38~~ | ~~39~~ | ~~40~~ |
| 41 | ~~42~~ | 43 | ~~44~~ | ~~45~~ | ~~46~~ | 47 | ~~48~~ | 49 | ~~50~~ |

*Loops 3 and 4 result in primes $\pi_3 = 5$ (remove 25, 35) and $\pi_4 = 7$ (remove 49):*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | **2** | **3** | ~~4~~ | **5** | ~~6~~ | **7** | ~~8~~ | ~~9~~ | ~~10~~ |
| **11** | ~~12~~ | **13** | ~~14~~ | ~~15~~ | ~~16~~ | **17** | ~~18~~ | **19** | ~~20~~ |
| ~~21~~ | ~~22~~ | **23** | ~~24~~ | ~~25~~ | ~~26~~ | ~~27~~ | ~~28~~ | **29** | ~~30~~ |
| **31** | ~~32~~ | ~~33~~ | ~~34~~ | ~~35~~ | ~~36~~ | **37** | ~~38~~ | ~~39~~ | ~~40~~ |
| **41** | ~~42~~ | **43** | ~~44~~ | ~~45~~ | ~~46~~ | **47** | ~~48~~ | ~~49~~ | ~~50~~ |

*Thus $\Pi(50) = 15$ (i.e., 15 primes are $N \leq 50$): $\pi_k = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}$.*

*Figure 2.1: Sieve of Eratosthenes for the case of $N = 49$.*

[a] `https://en.wikipedia.org/wiki/Sieve_of_Eratosthenes\#Euler.27s_Sieve`

### The importance of prime numbers

Likely the first insight into the counting numbers starts with the sieve, shown in Fig. 2.1. A sieve answers the question "What is the definition of a prime number?" which is likely the first question to be asked. The answer comes from looking for irregular patterns in the counting numbers, by playing the counting numbers against themselves.

A prime is that subset of positive integers $\mathbb{P} \in \mathbb{N}$ that cannot be factored. The number 1 is not a prime, for some non-obvious reasons, but there is no pattern in it since it is always a (useless) factor of every counting number.

To identify the primes we start from the first candidate on the list, which is 2 (since 1 is not a prime), and strike out all multiples by the counting numbers greater than 1 $[(n + 1) \cdot 2 = 4, 6, 8, \cdots]$. While not obvious, this is our first result, that 2 is a prime, since it has no other factors but 1 and itself. This leaves only the odd numbers. We need a notation to indicate this result so we shall set $\pi_1 = 2$, as the first prime.[1]

[1] There is a potentially conflicting notation since $\pi(N)$ is commonly defined as the number of primes less than index $N$. Be warned that here we define $\pi_n$ as the $n^{th}$ prime, and $\Pi(N)$ as the number of primes $\leq N$, since having a convenient

**Two Fundamental Theorems of Primes:** Early theories of numbers revealed two fundamental theorems (there are many more than two), as discussed in Section 1.2.2. The first of these is the *Fundamental Theorem of Arithmetic*, which says that every integer greater than 1 may be uniquely factored into a product of primes (Eq. 1.2). Our demonstration of this is empirical, using Matlab's `factor(N)` routine, which delivers the prime numbers that compose $N$.[2] Typically the prime factors appear more than once, for example $4 = 2^2$. To make the notation compact we define the *multiplicity* $\beta_k$ of each prime factor $\pi_k$ (Eq. 1.2).

Each counting number is *uniquely* represented by a product of primes. There cannot be two integers with the same factorization. Once you multiply the factors out, the result is a unique $N$. Note that it's easy to multiply integers (e.g., primes), but nearly impossible to factor them. Factoring is not the same as dividing, as one needs to know what to divide by. Factoring means dividing by some integer and obtaining another integer with a zero remainder. This is what makes it so difficult (nearly impossible).

So the question remains: "What is the utility of the FTA?" which brings us to the topic of *internet security*. Unfortunately at this time I can not give you a proper summary of how it works. The full answer requires a proper course in number theory, beyond what is presented here.

The basic concept is that it is easy to construct the product of two primes, even very long primes having hundreds, or even thousands, of digits. It is very costly (but not impossible) to factor them. Why not use Matlab's `factor(N)` routine to find the factors? This is where *cost* comes in. The numbers used in RSA are too large for Matlab's routine to deliver an answer. In fact, even the largest computer in the world (such as the University of Illinois' super computer (NCSA Water) cannot do this computation. The reason has to do with the number of primes. If we were simply looking up a few numbers from a short list of primes, it would be easy, but the density of primes among the integers, is huge (see Section 1.2.3). This take us to the *Prime Number Theorem* (PNT). The security problem is the reason why these two theorems are so important: 1) Every integer has a unique representation as a product of primes, and 2) the number of primes is very dense (their are a very large number of them). Security reduces to the needle in the haystack problem, the cost of a search. A more formal way to measure the density is called the *entropy*, which is couched in terms of the probability of events, which in this case is "How often do you find a prime is a list of counting numbers?"

### Rational numbers $\mathbb{Q}$

The most important genus of numbers are the rational numbers since they maintain the utility of absolute precision, and they can approximate any irrational number (e.g., $\pi \approx 22/7$) to any desired degree of precision. However, the subset of rationals we really are interested in are the fractionals $\mathbb{F}$. Recall that $\mathbb{Q} : \mathbb{F} \cup \mathbb{Z}$ and $\mathbb{F} \perp \mathbb{Z}$. The fractionals are the numbers with the approximation utility, with arbitrary accuracy. Integers are equally important, but for a very different reason. All numerical computing today is done with $\mathbb{Q}$. Indexing uses integers $\mathbb{Z}$, while the rest of computing (flow dynamics, differential equations, etc.) is done with the fractionals $\mathbb{F}$. Computer scientists are trained on these topics, and engineers need to be at least conversant with them.

### Irrational numbers: The cardinality of numbers may be ordered: $|\mathbb{I}| \ggg |\mathbb{Q}| \gg |\mathbb{N}|$[2] $\gg |\mathbb{P}|$

mean?
ake sense?

The real line may be split into the irrationals and rationals. The rationals may be further split into the integers and the fractionals. Thus, all is *not* integer. As shown in Fig. **??** if a triangle has two unit sides then the hypotenuse must be irrational ($\sqrt{2} = \sqrt{1^2 + 1^2}$). This leads us to a fundamental question: "Are there integer solutions to Eq. 1.1?" We need not look further than the simple example $\{3, 4, 5\}$. In fact this example does generalize, and the formula for computing an infinite number of integer solutions is called *Euclid's Formula*, which we will discuss in Section 2.1.3.

---

notation for the $n^{th}$ prime is more important that for the number of primes less than $N$.

[2] If you wish to be a Mathematician, you need to learn how to prove theorems. If you're an Engineer, you are happy that someone else has already proved them, so that you can use the result.

However, the more important point is that the cardinality of the irrationals is much larger than any set other than the reals (i.e., complex numbers). Thus when we use computers to model physical systems, we are constantly needing to compute with irrational numbers. But this is impossible since every irrational numbers would require an infinite number of bits to represent it. Thus we must compute with rational approximations to the irrationals. This means we need to use the fractionals. In the end, we must work with the IEEE 754 floating point numbers,[3] which are fractionals, more fully discussed in Section 1.2.3.

**Greatest Common Divisor GCD(m,n)** **4.2.4**

- *The GCD(m,n) or* largest common factor *is the part that cancels in* $m/n > 1$)
  - $n, m \in \mathbb{Z}$
  - *Matlab only allows integers as arguments to* `gcd(m,n)` $m, n\mathbb{Z}$, *and ignores the sign.*
  - *The GCD may be extended to polynomials: E.G.,* $GCD(ax^2 + bx + c, \alpha x^2 + \beta x + \gamma)$
- *If you can factor the arguments, the GCD is "Obvious"*
- *Examples:*
  - *gcd(13,11) = 1 The gcd of two primes is always 1*
  - *gcd(13*5,11*5) = 5 The common 5 is the gcd*
  - *gcd(13*10,11*10) = 10 The gcd(130,110) = 10 = 2*5, is not prime*
  - *gcd(1234,1024) = 2 (1234=2*617,* $1024=2^{10}$*)*
  - *gcd*$((x-3)(x-4), (x-3)(x-5))$=$(x-3)$
  - *gcd*$(x^2 - 7x + 12, 3(x^2 - 8x + 15))$=$3(x-3)$
  - *gcd*$(x^2 - 7x + 12,\ 3x^2 - 24x + 45)$=$3(x-3)$
  - *gcd*$((x-2\pi)(x-4), (x-2\pi)(x-5))$=$(x-2\pi)$ *(Need long division, in general)*
- Co-primes *($a \perp b$) are numbers with no common factors (other than 1)*
  - *Example:* $a = 7*13, b = 5*19 \Rightarrow (7*13) \perp (5*19)$
  - *I.E.: If* $a \perp b$ *then* $gcd(a,b) = 1$
- $\frac{a}{\mathrm{gcd}(a,b)} \in \mathbb{Z}$

Figure 2.2: *Euclidean Algorithm for finding the GCD of two numbers is one of the oldest algorithms in mathematics, and is highly relevant today. It is both powerful and simple. It was used by the Chinese during the Han dynasty (Stillwell, 2010, p. 70) for reducing fractions. It may be used to find pairs of integers that are coprime (their gcd must be 1), and it is may be used to identifying factors of polynomials by long division. It has an important sister algorithm called the* continued fraction algorithm *(CFA), that is so similar in concept that Gauss referred to the Euclidean algorithm as the"continued fraction algorithm" (Stillwell, 2010, p. 48).*

### 2.1.2 Lec 5 Greatest common divisor (GCD)

2text: Fig 2.2

Multiplying two numbers together, or dividing one by the other, is very inexpensive on today's computer hardware. However, factoring an integer is very expensive. When the integers are large, it is so costly that it cannot be done in a lifetime, even with the very best computers. The *greatest common divisor* or $gcd(m,n)$ answers the question: "What is the largest common factor of two integers, $m, n$?"

In Matlab the GCD may be computed with `g=gcd(m,n)`. If the two integer are in factored form, the answer is trivial. For example $5 = gcd(5\cdot 13, 5\cdot 17)$, and $17 = gcd(17\cdot 53, 17\cdot 3\cdot 31)$. But what about $gcd(901, 1581)$? So the problem that computing the GCD solves is when the factors not unknown. Since $901 = 53*17$ and $1581 = 3*17*31$, $gcd(901, 1581) = 17$, which is not obvious.

The obvious question is "Can we find $l = gcd(m,n)$ without factoring $(m,n)$?" The answer is "yes," with the *Euclidean Algorithm*. Here is a Matlab code to find the gcd:

```
n = gcd(m,n)
 while m ~=0
```

[3]IEEE 754: `http://www.h-schmidt.net/FloatConverter/IEEE754.html`.

```
  A=m; B=n;
  m=max(A,B); n=min(A,B); %m>n
  m = m -n*floor(m/n);    %compute remainder after rounding down (i.e., floor)
 end
```

Below is 1-line vectorized code.

```
l = gcd(m,n) %entry point: input m,n; output l=gcd(m,n)
M=[abs(m),abs(n)]; %init M
 while M(2) ~=0 % < n*eps to ‘‘almost work’’ with irrational inputs
  M = [M(2) - M(1)*floor(M(2)/M(1)); M(1)]; %M = [M(1); M(2)] with M(1)<M(2)
 end
```

With a minor extension in the test for "end," this code can work with irrational inputs: e.g.: $(n\pi, m\pi)$.

The first step in the Euclidean Algorithm is to order the two integers so that $m > n$. The second step is to find the integer part of $m/n$ using the floor function, and then to remove that from $m$ to obtain the remainder, stored back in $m$. Note that the new value of $m$ is always less than $n$, and must remain greater or equal to zero. These two steps are then repeated until the remainder is 0. The gcd is then $n$ once $m = 0$. When using irrational numbers, this still works except the error is never exactly zero, due to IEEE 754 rounding. Thus the criterion must be that the error is within some factor times the smallest number (which in Matlab is the number `eps` $= 2.220446049250313 \times 10^{-16}$.

Thus without factoring the two numbers, the Euclidean Algorithm recursively finds the gcd simply by ordering the two numbers and then updating them with

$$\begin{bmatrix} m' \\ n' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor \frac{m}{n} \rfloor \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} \tag{2.1}$$

where $\lfloor x \rfloor$ finds the integer part of $x$. The method terminates when $m = n\lfloor m/n \rfloor$. The previous values of $m, n$ from the step before the final result are solutions to Bézout's identify $(ma + nb = 1)$ since the terminal state and the GCD of $a, b$ is $m - n\lfloor m/n \rfloor = 0$, for which $n = gcd(a, b)$. Perhaps this is best seen with some examples.

The GCD is an important and venerable method, useful in engineering and mathematics, but, as best I know, is not typically taught in the traditional engineering curriculum.

**Graphical meaning of the GCD:** The Euclidean Algorithm is actually very simple when viewed graphically. In Fig. 2.3 we show what is happening as one approaches the threshold. After reaching the threshold, the two number must be swapped, which is addressed by upper row of Eq. 2.1.

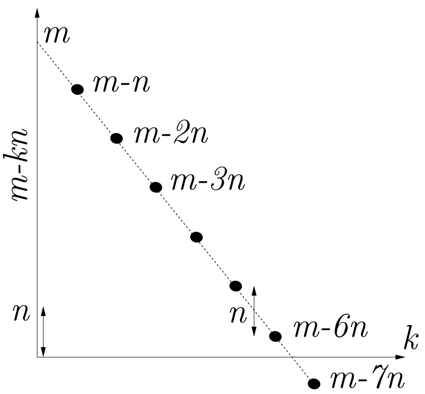



Figure 2.3: *The Euclidean Algorithm recursively subtracts $n$ from $m$ until the remainder $m - kn$ is either less than $n$ or zero. For the case depicted here the value of $k$ that renders the remainder less than $n$ is $k = 6$. If one more step were taken ($k = 7$) the remainder becomes negative. By linear interpolation we can find that $m - an = 0$ when $a = m/n$, which for this example is close to a=6.5. In this example $6 = floor(m/n) < n$.*

Multiplication is simply receptive addition, and finding the gcd takes advantage of this fact. Lets take a trivial example, (9,6). Taking the difference of the larger from the smaller, and writing multiplication as sums, helps one see what is going on. Since 6=3*2, this difference may be written two different ways

$$9 - 6 = (3 + 3 + 3) - (3 + 3) = 0 + 0 + 3 = 3$$

or

$$9 - 6 = (3 + 3 + 3) - (2 + 2 + 2) = 1 + 1 + 1 = 3$$

Written out the first way, it is 3 because subtracting (3+3) from (3+3+3) leaves 3. Written out in the second way, each 3 is matched with a -2, leaving 3 ones, which add to 3. Of course the two decomposition must yield the same result because $2 \cdot 3 = 3 \cdot 2$. Thus finding the remainder of the larger number minus the smaller yields the gcd of the two numbers.

**Coprimes:** When the gcd of two integers is 1, the only common factor is 1. This is of key importance when trying to find common factors between the two integers. When 1=gcd(m,n) they are said to be *coprime*, which is frequently written as $m \perp n$. By definition, the largest common factor of coprimes is 1. But since 1 is not a prime, they have no common primes.

**Why is the GCD important?** When two integers are coprime, their ratio is in *reduced form* (has no common factors). For example 4/2 is not reduced since 2=gcd(4,2). Canceling out the common factor 2, gives the reduced form 2/1=2. Thus if we wish to form the ratio of two integers, compute the gcd and remove it from the two numbers, then form the ratio. This assures the rational number is in its minimal form.

Example: Take two integers $[9, 6]$. In factored form these are $[3^2, 2 \cdot 3]$. Given the factors, we see that the largest common factor is 3. When we take the ratio of the two numbers this common factor cancels

$$\frac{a}{b} = \frac{9}{6} = \frac{3+3+3}{3+3} = \frac{3 \cdot \cancel{3}}{2 \cdot \cancel{3}} = \frac{3}{2}.$$

**Matrix method:** The GCD may be written a a matrix recursion, based on Eq. 2.1. The two starting numbers are given by the vector $(m_0, n_0)$. The recursion is then

$$\begin{bmatrix} m_{k+1} \\ n_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -I_k \end{bmatrix} \begin{bmatrix} m_k \\ n_k \end{bmatrix}$$

where $I_k = floor(m_k/n_k)$ is the integer part of $m_k/n_k$. This recursion continues until $m_{k+1}$ is zero. At that stage the GCD is $n_{k+1}$. Figure 2.3 along with the above matrix relation, give the deepest insight into the Euclidean Algorithm.

**Generalizations of GCD:** The GCD may be generalized in several significant ways. For example what is the GCD of two polynomials? To answer this question one must factor the two polynomials to identify common roots. This will be discussed in more detail in Section 1.3.4.

### 2.1.3 Lec 6 Continued Fraction Expansion (CFA)

**Continued Fractions and circuit theory:** One of the most powerful generalizations of the CFA seems to be the expansion of a function of a complex variable, such as the expansion of an impedance $Z(s)$, as a function of complex frequency $s$. This idea is described in Fig. 2.4 and Eq. 2.2. This is especially interesting in that it leads to a physical interpretation of the impedance in terms of a transmission line (horn), a structure well know in acoustics having a variable area $A(x)$ as function of the range variable $x$.

The CFA expansion is of great importance in circuit theory, where it is equivalent to an infinitely long segment of transmission line, composed of series and shunt impedance elements. Thus such a cascade network composed of 1 ohm resistors, has an input impedance of $(1+\sqrt{5})/2 \approx 1.6180$ [ohms].

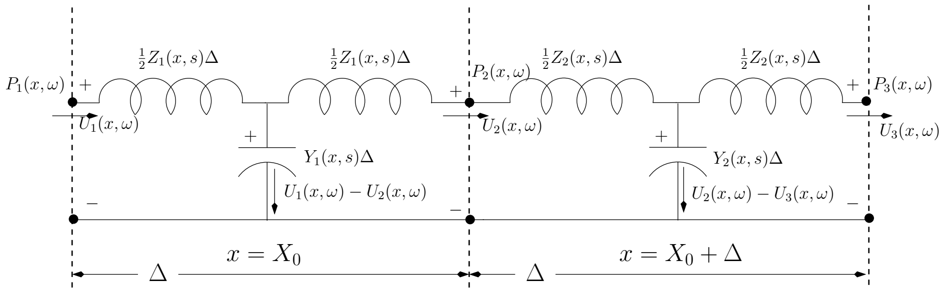



Figure 2.4: fig:LCTline *This transmission line is known as a* low-pass *filter wave-filter (Campbell, 1922). For long wavelengths it acts as a delay line, but as the wavelength approaches* $\Delta$*, the size of a section, the response becomes low-pass.*

The CFA may be extended to monomials in $s$. For example consider the input impedance of a cascade L-C transmission line as shown in Fig. 2.4. The input impedance of this transmission line is given by a continued fraction expansion of the form

$$Z_{in} = sL + \cfrac{1}{sC + \cfrac{1}{sL + \cfrac{1}{sC + \cfrac{1}{\cdots}}}} \qquad =: [sL, sC, sL, sC, \cdots].eq:CFA \tag{2.2}$$

where we have again used the bracket notation to describe the CFA coefficients, but without the period after the first term.

In some ways, Eq. 2.2 is reminiscent of the Taylor series expansion about $s = 0$, yet very different. In the limit, as the frequency goes to zero ($s \to 0$), the impedance of the inductors go to zero, and that of the capacitors go to $\infty$. In physical terms, the inductors become short circuits, while the capacitors become open circuits.

In terms of the transmission line, it becomes a long piece of wire, with a delay determined by the phase velocity. There are two basic parameters that characterize a transmission line, the characteristic resistance $r_0 = \sqrt{L/C}$ and the the wave number $\kappa = s/\sqrt{LC} = s/c$, which gives $c = \sqrt{LC}$. Each of these is a constant as $\Delta \to 0$, and in that limit the waves travel as

$$f(t \pm x/c) = e^{\pm\kappa x}e^{-st},$$

with a wave resistance ($r_0 = \sqrt{L/C}$. The total delay $T = L/c$ where $L$ is the line length and $c$ is the phase velocity of the line.

Since the CFA has a physical representation as a transmission line, as shown in Fig. 2.4, it can be of high utility for the engineer.[4] The theory behind this will be discussed in greater detail in Chapter 5. If you're ready to jump ahead, read the interesting book by Brillouin (1953) and the collected works of Campbell (1937).

## WEEK 3 7.3.0

— Labor Day (no class)

[4]Continued fraction expansions of functions are know in the circuit theory literature as a *Cauer* synthesis (Van Valkenburg, 1964b).

L 7 Definition of Pythagorean triplets (PT) $[a, b, c]$
Examples: $[3, 4, 5]$; Derivation of Euclid's formula for PTs;
Properties of Pythagorean triplets $[a, b, c]$;

L 8 *Pell's Equation:* $n^2 - Nm^2 = 1$ (i.e., $y^2 = Nx^2 + 1$)
1. Brahmagupta's solution by composition
2. The eigenvalue solution
3. Derivation of Euler-like solution of Pell's Equation (Chord & Tangent methods)

## 2.2 Week 3

### 2.2.1 Lec 7 Pythagorean triplets (PTs) and Euclid's formula ,

Pythagorean triplets (PTs) have many applications in architecture and scheduling, which explains why they are important and heavily studied. For example, if one wished to construct a triangle with a perfect 90° angle, then the materials need to be squared off as shown in Fig. 2.5. The lengths of the sides need to satisfy PTs.

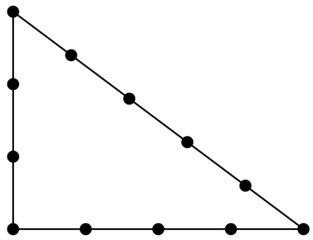


Figure 2.5: *Beads on a string form perfect right triangles when number of beads on each side satisfy Eq. 1.1.*

A stone tablet having the numbers engraved on it, as shown in Table 2.6 was discovered in Mesopotamia and from the $19^{th}$ century [BCE] and cataloged in 1922 by George Plimpton. These numbers are $a$ and $c$ pairs of PTs. Given this discovery, it is clear that the Pythagoreans were walking in the footsteps of those well before them. Recently a second similar stone, dating between 350 and 50 [BCE] has been reported, that indicates early calculus on the orbit of Jupiter's.[5]

**Derivation of Euclid's Formula:** The problem is to find integer solutions to the Pythagorean theorem (Eq. 1.1). The solution method, due to Diophantus, is call a *chord/tangent* method (Stillwell, 2010, p. 48). The method composes (Section 3.2.3) a line and a circle, where the line defines a chord within the circle (its not clear where the tangent line might go). The slope of the line is then taken to be rational, allowing one to determine integer solutions of the intersections points. This solution for *Pythagorean triplets* $[a, b, c]$ is known as *Euclid's Formula* (Stillwell, 2010, p. 4–9, 222)

$$a = p^2 - q^2, \qquad b = 2pq, \qquad c = p^2 + q^2. \tag{2.3}$$

This result is easily verified, since

$$[p^2 + q^2]^2 = [p^2 - q^2]^2 + [2pq]^2$$

or

$$p^4 + q^4 + \cancel{2p^2q^2} = p^4 + q^4 - \cancel{2p^2q^2} + \cancel{4p^2q^2}.$$

[5] http://www.nytimes.com/2016/01/29/science/babylonians-clay-tablets-geometry-astronomy-jupiter.html

### Pythagorean triplets: $b = \sqrt{c^2 - a^2} \in \mathbb{N}$ **5.2.2**

- *"Plimpton-322" is a stone tablet*[a b] *from 1800 BCE, displaying $a$ and $c$ values of the Pythagorean triplets $[a, b, c]$.*

**Exercises**

The integer pairs $(a, c)$ in Plimpton 322 are

| $a$ | $c$ |
|---|---|
| 119 | 169 |
| 3367 | 4825 |
| 4601 | 6649 |
| 12709 | 18541 |
| 65 | 97 |
| 319 | 481 |
| 2291 | 3541 |
| 799 | 1249 |
| 481 | 769 |
| 4961 | 8161 |
| 45 | 75 |
| 1679 | 2929 |
| 161 | 289 |
| 1771 | 3229 |
| 56 | 106 |

Figure 1.3: Pairs in Plimpton 322

**1.2.1** For each pair $(a, c)$ in the table, compute $c^2 - a^2$, and confirm that it is a perfect square, $b^2$. (Computer assistance is recommended.)

You should notice that in most cases $b$ is a "rounder" number than $a$ or $c$.

**1.2.2** Show that most of the numbers $b$ are divisible by 60, and that the rest are divisible by 30 or 12.

Figure 2.6: *Numbers $(a, c \in \mathbb{N})$, with the property $b = \sqrt{c^2 - a^2} \in \mathbb{N}$, known as* Pythagorean triplets, *were found carved on a stone tablet from the 19$^{th}$ century [BCE]. Several of the c values are primes, but not the a values. The stone is item 322 (item 3 from 1922) from the collection of George A. Plimpton. –Stillwell (2010, Exercise 1.2)*

[a] `http://www.nytimes.com/2010/11/27/arts/design/27tablets.html`
[b] `https://en.wikipedia.org/wiki/Plimpton_322`

Thus, its easy to prove given the solution. Next we derive the equations.

The derivation is outlined in Fig. 2.7. Starting from two integers $[p > q > 0] \in \mathbb{N}$, composing a line having a rational slope $t = p/q$, with a circle (Stillwell, 2010, p. 6), gives the formula for the Pythagorean triplets.

The construction starts with a circle and a line, which is terminated at the point $(-1, 0)$. The slope of the line is a free parameter $t$. By composing the circle and the line (i.e., solving for the intersection of the circle and line), the formula for the intersection point $(a, b)$ may be determined in terms of $t$, which will then be taken as the rational number $t =: p/q \in \mathbb{Z}$.

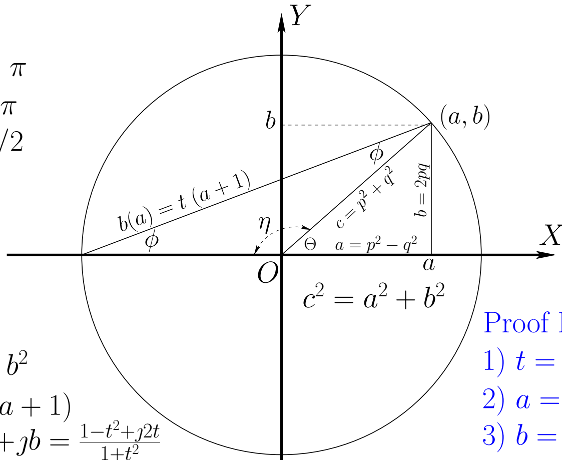
In Fig. 2.7 there are three panels labeled "Proofs" numbed I, II, III. Proof I shows the angle relationships of two triangles, the first an isosceles triangle formed by the chord, having slope $t$ and two equal sides formed from the radius of the circle, and a second right triangle having its hypotenuse as the radius of the circle and its right angle vertex at $(a, 0)$. As shown, it is this smaller right triangle that must satisfy Eq. 1.1.

Proof I shows the relations between the various angles for the two embedded triangles. The inner right triangle has its hypotenuse $c$ between the origin of the circle (O) to the point $(a, b)$. Side $a$ forms the $x$ axis and side $b$ forms the $y$ ordinate. Thus by construction Eq. 1.1 must be obeyed (Proof II).

Proof II gives the heart of the derivation, by a composition of the circle having radius $c$, with

**Euclid's Formula for Pythagorean triplets $[a, b, c]$** **5.2.4**

Proof I:
1) $2\phi + \eta = \pi$
2) $\eta + \Theta = \pi$
3) $\therefore \phi = \Theta/2$




Proof II:
1) $c^2 = a^2 + b^2$
2) $b(a) = t(a+1)$
3) $\zeta(t) \equiv a + \jmath b = \frac{1-t^2+\jmath 2t}{1+t^2}$
4) $\zeta = |c|e^{i\theta} = |c|\frac{1+it}{1-it} = |c|(\cos(\theta) + i\sin(\theta))$

Proof III:
1) $t = p/q \in \mathbb{Q}$
2) $a = p^2 - q^2$
3) $b = 2pq$
4) $c = p^2 + q^2$

Figure 2.7: *Derivation of Euclid's Formula for Pythagorean triplets $[a, b, c]$ based on composition of the line with a rational slope and a circle. PTs have applications in architecture and scheduling, and many other practical problems.*

$c^2 = a^2 + b^2$ and the line $b(a) = t(a + 1)$. Given some simple algebra, the coordinate $(a, b)$ is found to be circle

$$[a(t), b(t)] = \frac{[1 - t^2, 2t]}{1 + t^2}.$$

This pair of numbers may be written as a complex number $\zeta = a + b\jmath$ in terms of angle $\Theta(a, b)$, defined in Fig. 2.7.

The final step (Proof III) is to require that the slope parameter $t$ is rational, i.e., $t = p/q$ with $p, q \in \mathbb{Z}$. This results in the formula for the PTs $[a, b, c]$ as given by Eq. (2,3,4) of III.

We may form a complex number $\zeta(\Theta) \equiv a + b\jmath$ out of the lengths of the right triangle $[a, b]$, having a hypotenuse of length $|c|$

$$\zeta(\Theta) = |c|e^{\Theta\jmath} = |c|\left(\cos(\Theta) + \sin(\Theta)\right)\jmath.$$

It follows that

$$\zeta = |c|e^{\jmath\Theta(t)} = |c|\frac{1 - t^2 + 2t\jmath}{1 + t^2} = |c|\frac{\cancel{(1+\jmath t)}(1+\jmath t)}{\cancel{(1+t\jmath)}(1-t\jmath)} = (q + p\jmath)\sqrt{\frac{q + jp}{q - p\jmath}}$$

Examples of PTs include $a = 2^2 - 1^2 = 3$, $b = 2 \cdot 2 \cdot 1 = 4$, and $c = 2^2 + 1^2 = 5$, $3^2 + 4^3 = 5^2$.

If we redefine $p = q + N$ $(N \in \mathbb{N})$ then we obtain a slightly better parametric representation of the answers, as the pair $(q, N)$, behaved that $(p, q)$. General properties of the solutions are explored more naturally. Note that $b + c$ must always be a perfect square since $b + c = (p + q)^2 = (2q + N)^2$, as first summarized by Fermat Stillwell (2010, p. 212).

### 2.2.2 Lec 8 Pell's Equation

As discussed in the introduction (Section 1.2.8), the Pythagoreans were working with Pell's equation (Eq. 1.5) for the case of $N = 2$, to explore the rationality of the $\sqrt{2}$. According to Stillwell (2010, p. 37,55), they were using the Euclidean Algorithm on the matrix recurrence relation Eq. 1.5. It seems unlikely that the Pythagoreans were trying to solve Pell's equation.

Irrespective of what the Greeks had done, the first major advance toward integer solutions was made by Bramagupta (c628), who independently rediscovered the equation (Stillwell, 2010, p. 46). Bramagupta's novel solution introduced the *composition method* (Stillwell, 2010, p. 69). These integer solutions to Pell's equation were sparse (i.e., incomplete).

The matrix recursion method (used by the Pythagoreans to study $\sqrt{2}$), was first exploited by Bhâskara II (1150CE) to obtain integer solutions to Pell's equation (Stillwell, 2010, p.69). This seems to be the first matrix recursion (i.e., the first solution to Pell's equation by Bhâskara). This is the result we shall explore here, as summarized in Fig. 2.8.

As a first step let's review the Pythagorean/Bhâskara solution provided by the matrix recursion (Eq. 1.5). This is done in Fig. 2.8 for the case of $N = 2$ and again in Fig. B.2 for the case of $N = 3$. The math is very simple and can easily be verified manually. From the equations in the figure, we can see that Bhâskara's matrix recursion (Eq. 1.5) does give solutions to Pell's equation (Eq. 1.4). For $n = 0$, the solution is [1,0], the "trivial" solution, for $n = 2$ it is [-3,-2]. These are easily computed by this recursion, and easily checked on a hand calculator (or using Matlab). Without the $\jmath$ factor the sign would alternate; $\jmath$ fixes this so that every iteration yields a solution.

- Case of $N = 2$ & $[x_0, y_0]^T = [1, 0]^T$
  Note: $x_n^2 - 2y_n^2 = 1$, $\qquad x_n/y_n \underset{\infty}{\longrightarrow} \sqrt{2}$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \jmath \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \jmath^2 - 2 \cdot \jmath^2 = 1$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \jmath^2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \jmath \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad 3^2 - 2 \cdot 2^2 = 1$$

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \jmath^3 \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \jmath^2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad (7\jmath)^2 - 2 \cdot (5\jmath)^2 = 1$$

$$\begin{bmatrix} x_4 \\ y_4 \end{bmatrix} = \begin{bmatrix} 17 \\ 12 \end{bmatrix} = \jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \jmath^3 \begin{bmatrix} 7 \\ 5 \end{bmatrix} \qquad 17^2 - 2 \cdot 12^2 = 1$$

$$\begin{bmatrix} x_5 \\ y_5 \end{bmatrix} = \jmath \begin{bmatrix} 41 \\ 29 \end{bmatrix} = \jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 17 \\ 12 \end{bmatrix} \qquad (41\jmath)^2 - 2 \cdot (29\jmath)^2 = 1$$

Figure 2.8: *This summarizes the solution of Pell's equation due to the Pythagoreans using matrix recursion, for the special case of N=2. Of special interest is that $x_n/y_n \to \sqrt{2}$ as $n \to \infty$, which was what the Pythagoreans were pursuing.*

Thus it is clear that this 2x2 recursion is a method for generating a set of integer solutions to Pell's equation. This method was first attributed to Bhâskara II (1150 CE), but did not include the $\jmath$ factor. Without this factor, the solution is alternating in sign on the right hand side, so the generated solutions are for

$$x^2 - 2y^2 = \pm 1,$$

a minor variation on the Pell equation, fixed here.

At each step the ratio $x_n/y_n$ approaches $\sqrt{2}$ with increasing accuracy with each iteration. This explains the much earlier interest of the Pythagoreans in Pell's equation, due to their exploration of $\sqrt{2}$. The value of $41/29 \approx \sqrt{2}$ with a relative error of <0.03%.

The solution for $N = 3$ is discussed briefly at the end of Appendix B.

**Eigenvalue solution to Pell's equation:** To provide a full understanding of what was known to the Pythagoreans, it is helpful to provide the full solution to this recursive matrix equation, based on what we know today.

As shown in Fig. 2.8, $(x_n, y_n)$ may be written as a power series of the 2x2 matrix $A$. To find the powers of a matrix, the well know modern approach is to *diagonalize* the matrix. For the 2x2 matrix case, this is relatively simple. The final result written out in detail for the general solution $(x_n, y_n)$ is provided in Appendix B, as may be verified using Matlab.

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \jmath^n \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = E \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix} E^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The eigen-values are $\lambda_\pm = \jmath(1 \pm \sqrt{2})$ while the eigen-matrix and its inverse are

$$E = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.8165 & 0.8165 \\ 0.5774 & -0.5774 \end{bmatrix}, \qquad E^{-1} = \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & \sqrt{2} \\ 1 & -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0.6124 & 0.866 \\ 0.6124 & -0.866 \end{bmatrix}$$

The relative "weights" on the two eigen-solutions are equal, as determined by

$$E^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & \sqrt{2} \\ 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We still need to prove that

$$\frac{x_n}{y_n} \underset{\infty}{\rightarrow} \sqrt{N},$$

which follows intuitively from Pell's equation, since as $(x_n, y_n) \rightarrow \infty$, the difference between $x^2$ and $2y^2$, the $(\pm 1)$ becomes negligible.

## WEEK 4

9.4.0

L 9 The *Fibonacci sequence* and its difference equation
Geometry and irrational numbers: $\sqrt{n} \in \mathbb{Q} \in \mathbb{R}$; Roots of $\sqrt{5}$

L10 In Class Exam I

## 2.3 Week 4

### 2.3.1 Lec 9 Fibonacci Numbers

The Fibonacci sequence is famous in number theory. It is said that the sequence commonly appears in physical systems. Fibonacci numbers are related to the "golden ratio" $(1 + \sqrt{5})/2$, which could explain why these numbers appear in nature.

But from a mathematical point of view, they do not seem special. The Fibonacci sequence is generated by a linear recursion relationship

$$x_{n+1} = x_n + x_{n-1}, \tag{2.4}$$

namely the next number $x_{n+1}$ is the sum of the previous two. The term *linear* means that the principle of superposition holds.

Another similar linear recurrence relation would be that the next output be the average of the previous two

$$x_{n+1} = \frac{x_n + x_{n-1}}{2}.$$

In some ways this relationship might be more useful than the Fibonacci recursion, as it will remove oscillations in a sequence of the form $-1^n$ (it is a 2-sample *moving average*, a trivial form of low-pass filter).

In general recurrence relationships of the form

$$x_{n+1} = bx_n + cx_{n-:w1},$$

are more general, with constants $b, c \in \mathbb{R}$.

## General properties of the Fibonacci numbers[a]

$$x_n = x_{n-1} + x_{n-2}$$

- This is a 2-sample *moving average* difference equation with an unstable pole
- $x_n = [0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots]$, assuming $x_0 = 0, x_1 = 1$:
- Analytic solution (Stillwell, 2010, p. 194): $\sqrt{5}\, x_n \equiv \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \to \left(\frac{1+\sqrt{5}}{2}\right)^\infty$
  - $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1+\sqrt{5}}{2}$
  - Ex: $34/21 = 1.6190 \approx \frac{1+\sqrt{5}}{2} = 1.6180$ 0.10% error
- Matlab's `rat`$(1+\sqrt{5}) = 3 + 1/(4 + 1/(4 + 1/(4 + 1/(4)))) =: [3.\ 4, 4, 4, \cdots]$

[a] `https://en.wikipedia.org/wiki/Fibonacci_number` 8.3.5

Figure 2.9: Properties of the Fibonacci numbers (Stillwell, 2010, p. 28).

The first observation is that Eq. 2.4 may be written as a 2x2 matrix relationship. If we define $y_{n+1} = x_n$ then Eq. 2.4 is equivalent to

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}. \tag{2.5}$$

The first equation is $x_{n+1} = x_n + y_n$ while the second is $y_{n+1} = x_n$, which is the same as $y_n = x_{n-1}$. Note that the Pell 2x2 recursion is similar in form to the Fibonacci recursion. This removes the mystique from both equations.

In the matrix diagonalization of the Pell equation we found that the eigenvalues were $\lambda_\pm = 1 \mp \sqrt{N}$, and the two solutions turned out to be powers of the eigenvalues. The solution to the Fibonacci recursion may similarly be expressed in terms of a matrix. These two cases may thus be reduced by the same 2x2 eigenvalue solution method.

The eigenvalues of the Fibonacci matrix are given by

$$det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \lambda - 1 = (\lambda - 1/2)^2 - (1/2)^2 - 1 = (\lambda - 1/2)^2 - 5/4 = 0$$

thus $\lambda_\pm = \frac{1\pm\sqrt{5}}{2} = [1.618, -0.618]$.

### 2.3.2 Lec 10 Exam I

# Chapter 3

# Algebraic Equations: Stream 2

Topics to add: ABCD matrix method based on composition of Mobius transformations Taylor series of 1/(1-s) along with discussion of the ROC Work out some examples of polynomial composition and Bezout's Thm

Add intro to chapter that reviews what is here.

## WEEK 4-AE 12.5.0

L 11 Stream 2: Algebra and geometry as physics (Physics drives early mathematics)
The first "algebra" (al-jabr) al-Khwarizmi ($9^{th}$CE)
Polynomial equations in one and two variables (Stillwell, 2010, Ch. 6, p. 87)
Solution of the Quadratic Equation; Taylor series
Composition and intersection of polynomials

AE-1 (HW4) for 9/16/16; Add convolution problem. Verify due date.

## 3.1 Week 4

### 3.1.1 Lec 11 Algebra and geometry as physics

Physics drives analytic geometry
The first "algebra" (al-jabr) al-Khwarizmi ($9^{th}$CE)
Solution of the Quadratic Equation; Polynomial equations in one variable (Stillwell, 2010, Ch. 6, p. 87)
Composition vs. elimination (intersection) of curves
Gaussian Elimination in one variable
Composition of polynomial equations in two variables.
Gaussian Elimination in *two* variables (Ch. 6: Polynomials (p. 90))
Complex arithmetic rules documented (Bombelli, 1575)
Newton (1687) labels complex cubic roots impossible

Before Newton could work out his basic theories, algebra needed to be merged with Euler's early quantification of geometry. The key to putting geometry and algebra together is the Pythagorean theorem (Eq. 1.1), which is both geometry and algebra. To make the identification with geometry the sides of the triangle needed to be viewed as a length. This is done by recognizing that the area of a square is the square of a length. Thus a geometric proof requires one to show that the area of the square $A = a^2$ plus the area of square $B = b^2$ must equal the area of square $C = c^2$. There are many such constructions that show $A + B = C$ for the right triangle. It follows that in terms of coordinates of each vertex, the length of $c$ is given by

$$c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \tag{3.1}$$

with $a = x_2 - x_1$ and $b = y_2 - y_1$. Thus Eq. 1.1 is both an algebraic and a geometrical statement. This is not obvious.

Analytic geometry is based on coordinates of points, with the length given by Eq. 3.1. Geometry treats lines as lengths without specifying the coordinates. Algebra gave a totally new view to the quantification of geometrical lengths. We now explore the relationships between points represented as coordinates and the geometry behind them. We shall do this with simple examples from analytic geometry.

For example, in terms of the geometry, the intersection of two circles can occur at two points, and the intersection of two spheres gives a circle. These ideas may be verified using algebra.

For each of these problems, the lines and circles may intersect, or not, depending on how they are drawn. Yet we now know that even when they do not intersect on the sheet of paper, they still have an intersection, but the solution is complex. Finding such solutions require the use of algebra.

### Systems of equations

We don't need to restrict ourselves to polynomials in one variable. We can work with the equation for a circle having radius $r$

$$y^2 + x^2 = r^2,$$

which is quadratic in two variables. Solving for roots $y(x_r) = 0$ $(y^2(x_r) = r^2 - x_r^2 = 0)$ gives $(r - x_r)(r + x_r)$, which simply says that when the circle crosses the $y = 0$ line at $x_r = \pm r$.

This equation may also be factored as

$$(y - x\jmath)(y + x\jmath) = r^2,$$

Point being?

as is easily demonstrated by multiplying out the two monomials. This does not mean that a circle has complex roots. A root is defined by either $y(x_r) = 0$, or $x(y_r) = 0$.

Writing the circle in standard polynomial form we find

$$y^2(x) = ax^2 + bx + c.$$

Completing the square (Eq. 1.14) (verify)

$$\frac{1}{a}y^2(x) - \left(x + \frac{b}{2a}\right)^2 = \frac{c}{a} - \left(\frac{b}{2a}\right)^2.$$

This we see this is a hyperbola. For it to be a circle $a = -1$, $b = 0$ and $c = r^2$.

## WEEK 5

12.5.0

L 12 Examples of algebraic expressions in physics
Fundamental Thm of Algebra (d'Alembert, ≈1760)
Analytic Geometry: Algebra + Geometry (Euclid to Descartes)
Newton and power series; Taylor series & ROC Composition of polynomial equations in two variables.

L 13 Root classification for polynomials of Degree * = 1–4 (p.102);
*Convolution* of monomials gives polynomial construction; Work out convolution for cubic
Show that $a_{n-1}$ is sum of roots and $a_0$ is product of roots. Quintic (* = 5) cannot be solved

L 14 First Analytic Geometry (Fermat 1629; Descartes 1637) (p. 118) Descartes' insight: Composition of two polynomials of degrees (m,n $\rightarrow$ one of degree $m \cdot n$)
Examples: $x^4 \circ x^2 = x^8$. Discuss Composition vs. intersection of functions.

## 3.2 Week 5

### 3.2.1 Lec 12 Physics an complex analytic expressions: linear vs. nonlinear

Examples of algebraic expressions in physics
Composition of two polynomials of degrees (m,n $\rightarrow$ one of degree $m \cdot n$): Give examples.
First Analytic Geometry (Fermat 1629; Descartes 1637) (p. 118) Descartes' insight: Newton and power series; Taylor series & ROC

A relevant physical example comes from the solution of the wave equation (Eq. 1.10) in three dimensions. Such cases arise in wave-guide problems, semiconductors, plasma waves, or for acoustic wave propagation in crystals (Brillouin, 1960) and the earth's mantel (e.g., seismic waves, earthquakes, etc.). The solutions to these problems are based on the *eigenfunction* for the *vector wave equation* (see Chapter 5),

$$P(s, \mathbf{x}) = e^{st} e^{-\boldsymbol{\kappa} \cdot \mathbf{x}}, \tag{3.2}$$

where $\mathbf{x} = [x\hat{x} + y\hat{y} + z\hat{z}]$ is a vector pointing in the direction of the wave, $[\hat{x}, \hat{y}, \hat{z}]$ are unit vectors in the three dimensions and $s = \sigma + \omega \jmath$ [rad] is the Laplace frequency. The function $\boldsymbol{\kappa}(s)$ is the complex vector *wave number*, which describes the propagation of a plane wave of radian frequency $\omega$, in the $\mathbf{x}$ direction. The equation is linear in $\mathbf{x}$.

Just as the frequency $s = \sigma + \omega \jmath$ must be complex, it is important to allow the *wave number* function[1] $\boldsymbol{\kappa}(s)$ to be complex, because in general it will have a real part, to account for losses as the wave propagates. While it is common to assume there are no losses, in reality this assumption cannot be correct. In many cases it is an excellent approximation (e.g., even the losses of light in-vacuo are not zero) that gives realistic answers. But it is important to start with a notation that accounts for the most general situation, so that when losses must be accounted for, the notation need not change. With this in mind, we take the vector wave number to be complex

$$\boldsymbol{\kappa} = \mathbf{k}_r + \mathbf{k}\jmath,$$

where vector expression for the *lattice vector* is the imaginary part of $\boldsymbol{\kappa}$

$$\Im \boldsymbol{\kappa} = \mathbf{k} = \frac{2\pi}{\lambda_x}\hat{x} + \frac{2\pi}{\lambda_y}\hat{y} + \frac{2\pi}{\lambda_z}\hat{z}, \tag{3.3}$$

is the vector wave number for three dimensional solutions. The units of $|\boldsymbol{\kappa}|$ are reciprocal length [$\mathrm{m}^{-1}$]. When there are losses $\kappa_r(s) = \Re \kappa(s)$ must be a function of frequency, due to the physics behind these losses. In many important cases, such as loss-less wave propagation in semiconductors, $\boldsymbol{\kappa}(\mathbf{x})$ is a function of direction and position (Brillouin, 1960). We will not consider these more complex materials here, other than to acknowledge that they exist.

Add
ing meaning of [$\lambda$

When the eigenfunction Eq. 3.2 is applied to the wave equation, a quadratic (degree 2) algebraic expression results, known as the *dispersion relation*. The three dimensional dispersion relation

$$\left(\frac{s}{c}\right)^2 = \boldsymbol{\kappa} \cdot \boldsymbol{\kappa} \tag{3.4}$$

is a complex analytic algebraic relationship in four variables, frequency $s$ and the three complex *lattice wave numbers*.This represents a three-dimensional generalization of the well know relation between

[1]This function has many names in the literature, some of which are potentially confusing. It has been called the *wave number* and *propagation constant*, however its not a number nor is it constant.

wavelength and frequency $f\lambda = c$. For plane waves propagating in free space, assuming no loss, $|\boldsymbol{\kappa}(s)| = \pm|s/c|$, where the sign accounts for the direction of the plane wave.

This scalar relation ($f\lambda = c$) was first deduced by Galileo in the 16th century and was then explored further by Mersenne a few years later.[2] This relationship would have been important to Newton when formulating the wave equation, which he needed to estimate the speed of sound. We shall return to this in Chapters 4 and 5.

**Hilbert space:** Another important example of algebraic expressions in mathematics is Hilbert's generalization of Eq. 1.1, known as the Schwartz inequality, shown in Fig. 3.1. What is special about this generalization is that it proves that when the vertex is 90°, the length of the leg is minimum.

## Geometry: Hilbert space: David Hilbert 1900

- Define:
  1. Vectors $U, V = [v_1, v_2, \cdots, v_\infty])$ in an $\infty$ dimensional *inner product vector space*
  2. *Inner product* $U \cdot V = \sum_{k=1}^{\infty} u_k v_k$
  3. *Norm* $||U|| = \sqrt{U \cdot V} = \sqrt{\sum v_k^2}$ (the norm is the length of the vector)
- From these definitions we may define the minimum difference between the two vectors as the perpendicular from the end of one to the intersection of the second:
  $U \perp V$ may be found by minimizing the length of the vector difference:

$$\begin{aligned}\min_{\alpha} ||V - \alpha U||^2 = ||V||^2 + 2\alpha V \cdot U + \alpha^2 ||U||^2 > 0 \\ 0 = \partial_\alpha\, (V - \alpha U) \cdot (V - \alpha U) \\ = V \cdot U - \alpha^* ||U||^2 \\ \therefore\ \ \alpha^* = V \cdot U / ||U||^2.\end{aligned}$$

- The *Schwarz inequality* follows:

$$I_{\min} = ||V - \alpha^* U||^2 = ||V||^2 - \frac{|U \cdot V|^2}{||U||^2} > 0$$

$$0 \le |U \cdot V| \le ||U||\ ||V||$$

  Thus the direction cosine between the two vectors is

$$\cos(\theta) = \frac{U \cdot V}{||U||||V||}.$$

- Example:

$$U(\omega) = e^{-\omega_0 \jmath t} \qquad V(\omega) = e^{\omega \jmath t} \qquad U \cdot V = \int_\omega e^{\jmath \omega t} e^{-\jmath \omega_0 t} \frac{d\omega}{2\pi} = \delta(\omega - \omega_0)$$

Figure 3.1: *The* Schwartz inequality *is related to the shortest distance (length of a line) between the ends of the two vectors.* $||U|| = \sqrt{(U \cdot U)}$ *as the dot product of that vector with itself. This theory is widely used in quantum mechanics (Hilbert inner product spaces).*

It is a somewhat arbitrary requirement that $a, b, c \in \mathbb{R}$ for the Pythagorean theorem (Eq. 1.1). This seems natural enough since the sides are lengths. But, what if they are taken from the complex numbers, as for the lossy vector wave equation, or the lengths of vectors in $\mathbb{C}^n$? Then the equation generalizes to

$$c \cdot c = ||c||^2 = \sum_{k=1}^{n} |c_k|^2,$$

[2] Get this story straight.

where $||c||^2 = (c, c)$ is the inner (dot) product of a vector $c$ with itself where $||c|| = \sqrt{||c||^2}$ is called the *norm* of vector c, akin to a length, as assumed in Fig. 3.1.

### Power vs. power series, linear vs. nonlinear

Another place where equations of second degree appear in physical applications is in energy and power calculations. The electrical power is given by the product of the voltage $v(t)$ and current $i(t)$ (or in mechanics as the force times the velocity). For example if we define $\mathcal{P} = v(t)i(t)$ to be the power $\mathcal{P}$ [Watts], then the *total energy* [Joules] at time $t$ is (Van Valkenburg, 1964a, Chapter 14)

$$\mathcal{E}(t) = \int_0^t v(t)i(t)dt.$$

From this observe that the power is the rate of change of the total energy

$$\mathcal{P}(t) = \frac{d}{dt}\mathcal{E}(t).$$

**Ohm's Law and impedance:** The ratio of voltage over the current is call the *impedance* which has units of [Ohms]. For example given a resistor of $R = 10$ [ohms],

$$v(t) = R\, i(t).$$

Namely 1 amp flowing through the resistor would give 10 volts across it. Merging the linear relation due to Ohm's law with the definition of power, shows that the instantaneous power in a resistor is quadratic in voltage and current

$$\mathcal{P} = v(t)^2/R = i(t)^2 R.$$

Note that Ohm's law is linear in its relation between voltage and current whereas the power and energy are *nonlinear*.

Ohm's Law generalizes in a very important way, allowing the impedance (e.g., resistance) to be a linear complex analytic function of complex frequency $s = \sigma + \omega\jmath$ (Kennelly, 1893; Brune, 1931a). Impedance is a fundamental concept in many fields of engineering. For example:[3] Newton's second law $F = ma$ obeys Ohm's law, with mechanical impedance $Z(s) = sm$. Hooke's Law $F = kx$ for a spring is described by a mechanical impedance $Z(s) = k/s$. In mechanics a "resistor" is called a *dashpot* and its impedance is a positive and real constant.[4]

Force is a vector, voltage are a pote

**Kirchhoff's Laws KCL, KVL:** The laws of electricity and mechanics may be written down using Kirchoff's Laws current and voltage laws, (KCL, KVL), which lead to linear systems of equations in the currents and voltages (velocities and forces) of the system under study, with complex coefficients having positive real parts.

Points of major confusion are a number of terms that are misused, and overused, in the fields of mathematics, physics and engineering. Some of the most obviously abused terms are *linear/nonlinear*, *energy*, *power*, *power series*. These have multiple meanings, which can, and are, fundamentally in conflict.

**Transfer functions (Transfer matrix):** The only method that seems to work, to sort this out, is to cite the relevant physical application, in specific contexts. The most common touch point is a physical system that has an input $x(t)$ and an output $y(t)$. If the system is linear, then it may be represented by its *impulse response* $h(t)$. In such cases the system equation is

$$y(t) = h(t) \star x(t) \leftrightarrow Y(\omega) = H(s)|_{\sigma=0}\, X(\omega),$$

[3] In acoustics the pressure is a potential, like voltage. The force per unit area is given by $f = -\nabla p$ thus $F = -\int \nabla p\, dS$. Velocity is analogous to a current. In terms of the velocity potential, the velocity per unit area is $v = -\nabla\phi$.

[4] `https://en.wikipedia.org/wiki/Impedance_analogy`

namely the convolution of the input with the impulse response gives the output. From Fourier analysis this relation may be written in the real frequency domain as a product of the Laplace transform of the impulse response, evaluated on the $\omega \jmath$ axis and the Fourier transform of the input $X(\omega) \leftrightarrow x(t)$ and output $Y(\omega) \leftrightarrow y(t)$.

Mention ABCD Transfer matrix

If the system is nonlinear, then the output is not given by a convolution, and the Fourier and Laplace transforms have no obvious meaning.

The question that must be addressed is why is the power said to be nonlinear whereas a power series of $H(s)$ said to be linear. Both have powers of the underlying variables. This is massively confusing, and must be addressed. The question will be further addressed in Section 3.5.1 in terms of the system postulates of physical systems.

Whats going on? The domain variables must be separated from the codomain variables. In our example, the voltage and current are multiplied together, resulting in a nonlinear output, the power. If the frequency is squared, this is describing the degree of a polynomial. This is not nonlinear because it does not impact the signal output, it characterizes the Laplace transform of the system response.

### 3.2.2 Lec 13 Root classification of polynomials

Root classification for polynomials of Degree * = 1–4 (p.102);
Quintic (* = 5) cannot be solved: Why?
Fundamental Thm of Algebra (d'Alembert, ≈1760)

Add intro & merge convolution discussions.

#### Convolution

As we discussed in Chapter 1, given the roots, the construction of higher degree polynomials, is greatly assisted by the convolution method. This has physical meaning, and gives insight into the problem of factoring higher order polynomials. By this method we can obtain explicit relations for the coefficients of any polynomial in terms of its roots.

Extending the example of Section 1.3.3, let's find the relations for the cubic. For simplicity, assume that the polynomial has been normalized so that the lead $x^3$ term has coefficient 1. Then the cubic in terms of its roots $[a, b, c]$ is a convolution of three terms

$$[1, a] \star [1, b] \star [1, c] = [1, a + b, ab] \star [1, c] = [1, a + b + c, ab + c(a + b), abc].$$

Working out the coefficients for a *quartic* gives

$$[1, a+b+c, ab+c(a+b), abc] \star [1, d] = [1, a+b+c+d, d(a+b+c)+c(a+b)+ab, d(ab+ac+bc)+abc, abcd].$$

It is clear what is going on here. The coefficient on $x^4$ is 1 (by construction). The coefficient for $x^3$ is the sum over the roots. The $x^2$ term is the sum over all possible products of pairs of roots, The linear term $x$ is the sum over all triple products of the four roots, and finally the last term (a constant) is the product of the four roots.

In fact this is a well known, a frequently quoted result from the mathematical literature, and trivial to show given an understand of convolution. If one wants the coefficients for the quintic, it is not even necessary to use convolution, as the pattern (rule) for all the coefficients is now clear.

You can experiment with this numerically using Matlab's convolution routine `conv(a,b)`. Once we start studying Laplace and Fourier transforms, convolution becomes critically important because multiplying an input signal in the frequency domain by a transfer function, also a function of frequency, is the same a convolution of the time domain signal with the inverse Laplace transform of the transfer function. So you didn't need to learn how to take a Laplace transform, and then learn convolution. We have learned convolution first independent of the Fourier and Laplace transforms.

When the coefficients are real, the roots must appear as conjugate pairs. This is an important symmetry.

For the case of the quadratic we have the relations between the coefficients and the roots, found by completing the square. This required isolating $x$ to a single term, and solving for it. We then proceeded to find the coefficients for the cubic and quartic case, after a few lines of calculation. For the quartic

$$\begin{aligned}
a_4 &= 1\\
a_3 &= a+b+c+d\\
a_2 &= d(a+b+c)+c(a+b)+ba\\
a_1 &= d(ab+ac+bd)+abc\\
a_0 &= abcd
\end{aligned}$$

These relationships are algebraically nonlinear in the roots. From the work of Galois, for $N \geq 5$, this system of equations is impossible to invert. Namely, given $a_k$, one may not determine the four roots $[a,b,c,d]$ analytically. One must use numeric methods.

To gain some insight, let us look at the problem for $N = 2$, which has a closed form solution:

$$\begin{aligned}
a_2 &= 1\\
a_1 &= a+b\\
a_0 &= ab
\end{aligned}$$

We must solve for $[a,b]$ given twice the mean, $2(a+b)/2$, and the square of the geometric mean $\left(\sqrt{ab}\right)^2$. Since we already know the answer (i.e, the quadratic formula). The solution was first worked out by the Babylonians (2000 BCE) Stillwell (2010, p. 92). It is important to recognize that for physical systems, the coefficients $a_k$ are real. This requires that the roots come in conjugate pairs ($b = a^*$), thus $ab = |a|^2$ and $a + b = 2\Re a$, which makes the problem somewhat more difficult, due to the greater symmetry.

Once you have solved this problem, feel free to attempt the cubic case. Again, the answer is known, after thousands of years of searching. The solution to the cubic is given in (Stillwell, 2010, pp. 97-9), as discovered by Cardano in 1545. According to Stillwell "The solution of the cubic was the first clear advance in mathematics since the time of the Greeks." The ability to solve this problem required algebra, and the solutions were complex numbers. The denial of complex numbers was, in my view, the main stumbling block in the progress of these solutions. For example, how can two parallel lines have a solution? Equally mystifying, how can a circle and a line, that do not intersect, have intersections? From the algebra we know that they do. This was a basic problem that needed to be overcome. This story is still alive,[5] because the cubic solution is so difficult.[6] One can only begin to imagine how much more difficult the quartic is, solved by Cardano's student Ferrair, and published by Cardano in 1545. The impossibility of the quintic was finally resolved in 1826 by Able (Stillwell, 2010, p. 102).

Finally with these challenges behind them, Analytic Geometry, relating of algebra and geometry, via coordinate systems, was born.

### 3.2.3 Lec 14: Analytic Geometry

Lec 14: Early Analytic Geom (Merging Euclid and Descartes): Composition of degrees $n, m$ gives degree $m \cdot n$

Composition and Intersection (Gaussian elimination)

The first "algebra" (al-jabr) is credited to al-Khwarizmi (830 CE). Its invention advanced the theory of polynomial equations in one variable, Taylor series, and composition versus intersections of curves. The solution of the quadratic equation had been worked out thousands of year earlier, but with algebra

[5]M. Kac, *How I became a mathematician." American Scientist (72), 498–499.*

[6]`https://www.google.com/search?client=ubuntu&channel=fs&q=Kac+\%22how+I+became+a\%22+1984+pdf&ie=utf-8&oe=utf-8`

a general solution could be defined. The Chinese had found the way to solve several equations in several unknowns, for example, finding the values of the intersection of two circles. With the invention of algebra by al-Khwarizmi, a powerful tool became available to solve the difficult problems.

**Composition, Elimination and Intersection** In algebra there are two contrasting operations on functions: *composition* and *Elimination*, (aka *intersection*).

**Composition:** Composition is the merging of functions, by feeding one into the other. If the two functions are $f, g$ then their composition is indicated by $f \circ g$, meaning the function $y = f(x)$ is substituted into the function $z = g(y)$, giving $z = g(f(x))$.

**Examples:** Let $y = f(x) =: x^2 - 2$ and $z = g(y) =: y + 1$. Then

$$g \circ f = g(f(x)) = (x^2 - 2) + 1 = x^2 - 1. \tag{3.5}$$

In general composition does not commute (i.e., $f \circ g \neq g \circ f$), as is easily demonstrated. Swapping the order of composition for our example gives

$$f \circ g = f(g(y)) = z^2 - 2 = (y + 1)^2 - 2 = y^2 + 2y - 1. \tag{3.6}$$

**Intersection:** Complimentary to composition is *intersection* (i.e., decomposition) (Stillwell, 2010, pp. 119,149). For example, the intersection of two lines is defined as the point where they meet. This is not to be confused with finding roots. A polynomial of degree $N$ has $N$ roots, but the points where two polynomials intersect has nothing to do with the roots of the polynomials. The intersection is a function (equation) of lower degree, implemented with Gaussian elimination.

Not sure ref right.

**Intersection of two lines** Unless they are parallel, two lines meet at a point. In terms of linear algebra this is written as 2 linear equations (left) along with the intersection point $[x_1, x_2]^T$, given by the inverse of the 2x2 set of equations (right)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \tag{3.7}$$

where $\Delta = ab - cd$ is called the *determinant*. By substituting the right expression into the left, and do some minor matrix algebra, you will obtain an identity. If $\Delta = 0$ there can be no solution, in which case the two lines are parallel, thus meet at infinity.

Algebra can often give a solution when geometry cannot. When the two curves fail to intersect on the real plane, the solution still exists, but is complex valued. Thus geometry, which only considers the real solutions, fails. For example, when the coefficients $[a, b, c, d]$ are complex, the solution exists, but the determinant can be complex. Apparently algebra is more general than geometric, which fails due to the complex intersection.

Sarah comment: Illustrate this point with curves with real coeffs that have complex intersections, such as $y = x^2 + 1$ and $y = x$. $x_r = (1 \pm \sqrt{3})/2$

## WEEK 6

**18.7.0**

L 15 Gaussian Elimination (upper-diagonal matrix); Permutation matrix method
Solution to $x^3 - Ny^3 = 1$ using chord and tangent methods
AE-2: Linear (& nonlinear) systems of equations

L 16 Composition and the Bilinear transformation (ABCD Transmission matrix method)

L 17 Riemann sphere and the extended plane ($3^d$ chord and tangent method)
Möbius Transformation (youtube video)
Closing the complex plane

## 3.3 Week 6

### 3.3.1 Lec 15 Gaussian Elimination (Intersection)

**Toy problems in Gaussian Elimination:** Gaussian Elimination is valid for nonlinear systems of equations. Till now we have emphasized the reduction of linear systems of equations.

**Problem 1:** Two lines *in a plane* either intersect or are parallel, in which case they are said to meet at $\infty$. Does this make sense? The two equations that describe this may be written in matrix form as $Ax = b$, which written out as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{3.8}$$

The intersection point $x_0, y_0$ is given by the solution two these two equations

$$\begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \tag{3.9}$$

where $\Delta = a_{11}a_{22} - a_{12}a_{21}$ is the determinant of matrix $A$ (Matlab's `det(A)` function).

It is useful to give an interpretation of these two equations. Each row of the 2x2 matrix defines a line in the $(x, y)$ plane. The top row is

$$a_{11}x + a_{12}y = b_1.$$

Normally we would write this equation as $y(x) = \alpha x + \beta$, where $\alpha$ is the slope and $\beta$ is the *intercept* (i.e., $y(0) = \beta$). In terms of the elements of matrix A, the slope of the first equation is $\alpha = -a_{11}/a_{12}$ while the slope of the second is $\alpha = -a_{21}/a_{22}$. The two slopes are equal (the lines are parallel) when $-a_{11}/a_{12} = -a_{21}/a_{22}$, or written out

$$\Delta = a_{11}a_{22} - a_{12}a_{21} = 0.$$

Thus when the determinate is zero, the two lines are parallel and there is no solution to the equations.

This 2x2 matrix equation is equivalent to a $2^d$ degree polynomial. If we seek an eigenvector solution $[e_1, e_2]^T$ such that

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \lambda \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{3.10}$$

the 2x2 equation becomes singular, and $\lambda$ is one of the roots of the polynomial. One may proceed by merging the two terms to give

$$\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.11}$$

Clearly this new matrix has no solution, since if it did, $[e_1, e_2]^T$ would be zero, which is nonsense. If it has no solution, then the determinant of the matrix must be zero. Forming this determinate gives

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$

thus we obtain the following quadratic equation for the roots $\lambda_\pm$ (eigenvalues)

$$\lambda_\pm^2 - (a_{11} + a_{22})\lambda_\pm + \Delta = 0.$$

When $\Delta = 0$, one eigenvalue is zero while the other is $a_{11} + a_{22}$, which is known as the *trace* of the matrix.

Needs work.

**In summary:** Given a "linear" equation for the point of intersection of two lines, we see that there must be two points of intersection, as there are always two roots of the quadratic *characteristic polynomial*. However the two lines only intersect at one point. Whats going on? What is the meaning of this second root?

It takes some simple examples to see what is happening. The eigenvalues depend on the relative slopes of the lines, which in general can become complex. The intercepts are dependent on **b**. Thus when the RHS is zero, the eigenvalues are irrelevant. This covers the very simple examples. When one eigenvalue is real and the other is imaginary, more interesting things are happening since the slope of one line is real and the slope of the other is pure imaginary. The lines can intersect in the real plane, and again in the complex plane.

Lets try an example of two lines, one with a slope of 1, and the second with a slope of 2. Let

$$\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \tag{3.12}$$

Here the first equation is $y = x + a$ and the second is $y = 2x + b$.

The solution is

$$\begin{bmatrix} y_0 \\ x_0 \end{bmatrix} = -\begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a-b \\ a-b \end{bmatrix} \tag{3.13}$$

since $\Delta = -1$.

This seems to say (I don't understand) that two real lines having slopes of 1 and 2 and intercepts of a and b, meet $(x_0, y_0) = (a + b, a - b)$.

While there is a unique solution, there are two eigenvalues, given by the roots of

$$(1 - \lambda_\pm)(-2 - \lambda_\pm) + 1 = 0.$$

If we transfer the sign from one monomial to the other

$$(-1 + \lambda_\pm)(2 + \lambda_\pm) + 1 = 0$$

and reorder for simplicity

$$(\lambda_\pm - 1)(\lambda_\pm + 2) + 1 = 0$$

we obtain the quadratic for the roots

$$\lambda_\pm^2 + \lambda_\pm - 1 = 0.$$

Completing the square gives

$$(\lambda_\pm + 1/2)^2 = 3/4.$$

or

$$\lambda_\pm = -1/2 \pm \sqrt{3}/2.$$

The question is, what is the relationship between the eigenvalues and the final solution, if any? Maybe none. The solution $(x_0, y_0)$ is reasonable, and its not clear that the eigenvalues play any useful role here, other than to predict there is a second solution. I'm confused.

**Two lines in 3-space:** In three dimensions

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{3.14}$$

Each row of the matrix describes a plane, which is said to be linear in the unknowns $(x, y, z)$. Thus the system of linear equations represents three planes, which must intersect at one point. If two planes are parallel, there is no real solution. In this case the intersection by the third plane generates two parallel lines.

As in the 2x2 case, one may convert this linear equation into a cubic polynomial by setting the determinant of the matrix, with $-\lambda$ subtracted from the diagonal, equal to zero. That is, $det(A-\lambda I) = 0$. Here $I$ is the matrix with 1 on the diagonal and zero off the diagonal.

**Simple example:** As a simple example, let the first plane be $z = 0$ (independent of $x, y$), the second parallel plane be $z = 1$ (independent of $(x, y)$) and the third plane be $x = 0$ (independent of $y, z$). This results in the system of equations

$$\begin{bmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{3.15}$$

Writing out the three equations we find $a_{13}z = 0$, $a_{23}z = 1$, and $a_{31}x = 0$. Note that $det(A) = 0$ (we need to learn how to compute the 3x3 determinant). This means the three planes never intersect at one point. Use Matlab to find the eigenvalues.

### 3.3.2 Lec 16 Matrix composition: Bilinear and ABCD transformations

ABCD Transmission matrix (Bilinear transformation) AE-2: Linear systems of equations of various degrees
Classification of real roots for degree 1-4

#### The Transmission matrix

A transmission matrix is a 2x2 matrix that characterizes a 2-port circuit, one having an input and output voltage and current, as shown in Fig. 1.6. The input is the voltage and current $V_1, I_1$ and the output is the voltage and current $V_2, -I_2$, with the current always defined to flow into the port. For any such a linear network, the input-output relations may be written in a totally general way as

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}.$$

In Section 1.3.6 we showed that a cascade of such matrices is composition. We shall show below that the justification of this relationship is based on the composition of bilinear transformations.

Expanding Eq. 3.20 into its individual equations demonstrates the linear form of the relations

$$V_1 = A(s)V_2 - B(s)I_2 \qquad I_1 = C(s)V_2 - D(s)I_2,$$

quantifying the relationship between the input voltage and current to its output voltage and current.

Define $H(s) = V_2/V_1$ as the *transfer function*, as the ratio of the output voltage $V_2$ over the input voltage $V_1$, under the constraint that the output current $I_2 = 0$. From this definition $H(s) = 1/A(s)$.

In a similar fashion we may define the meaning of all four functions as

$$A(s) \equiv \left.\frac{V_1}{V_2}\right|_{I_2=0} \qquad B(s) \equiv -\left.\frac{V_1}{I_2}\right|_{V_2=0} \tag{3.16}$$

$$C(s) \equiv \left.\frac{I_1}{V_2}\right|_{I_2=0} \qquad D(s) \equiv -\left.\frac{I_1}{I_2}\right|_{V_2=0} \tag{3.17}$$

From Eq. 3.20 one may compute any desired quantity, specifically those quantities defined in Eq. 3.17, the open circuit voltage transfer function $(1/A(s))$, the short-circuit transfer current $(1/D(s))$ and the two transfer impedances $B(s)$ and $1/C(s)$.

In the engineering fields this matrix composition is called the *Transmission matrix*, also known as the ABCD method. It is a powerful method that is easy to learn and use, that gives important insights into transmission lines, and thus even the 1 dimensional wave equation.

### Derivation of ABCD matrix for example of Fig. 1.6.

The derivation is straight forward by the application of Ohm's Law, as shown in Section 1.3.6.

The convenience of the ABCD matrix method is that the output of one is identically the input of the next. Cascading (composing) the results for the series inductor with the shunt compliance leads to the 2x2 matrix form that precisely corresponds to the transmission line CFA shown in Fig. 2.4,

$$\begin{bmatrix} V_n(s) \\ In(s) \end{bmatrix} = \begin{bmatrix} 1 & sL_n \\ 1 & 0 \end{bmatrix} \begin{bmatrix} V_{n+1}(s) \\ -I_{n+1}(s) \end{bmatrix}. \tag{3.18}$$

This matrix relation characterizes the series mass term $sL_n$. A second equation maybe be used for the shunt capacitance term $sY_n(s)$

$$\begin{bmatrix} V_n(s) \\ In(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ sC_n & 0 \end{bmatrix} \begin{bmatrix} V_{n+1}(s) \\ -I_{n+1}(s) \end{bmatrix}. \tag{3.19}$$

The positive constants $L_n, C_n \in \mathbb{R}$ represent the series mass (inductance) and the shunt compliance (capacitance) of the mechanical (electrical) network. The integer $n$ indexes the series and shunt sections, that are composed one following the next.

### Matrix composition and the bilinear transform

Now that we have defined the composition of two functions, we will use it to define the *Möbius* or *bilinear* transformation. Once you understand how this works, hopefully you will understand why it is the unifying element in many important engineering problems.

The bilinear transformation is given by

$$w = \frac{a + bz}{c + dz}$$

This takes one complex number $z = x + iy$ and transforms it into another complex number $w = u + iv$. This transformation is *bilinear* in the sense that its linear in both the input and output side of the equation. This may be seen when written as

$$(c + dz)w = a + bz,$$

since this relation is linear in the coefficients $[a, b, c, d]$. An important example is the transformation between impedance $Z(s)$ and reflectance $\Gamma(s)$,

$$\Gamma(s) = \frac{Z(s) - r_0}{Z(s) + r_0},$$

which is widely used in transmission line problems. In this example $w = \Gamma, z = Z(s), a = -r_0, b = 1, c = r_0, d = 1$.

If we define a second bilinear transformation (this could be the transformation from reflectance back to impedance)

$$r = \frac{\alpha + \beta w}{\gamma + \delta w},$$

and then compose the two something astray wrt arguments

$$w \circ r = \frac{a + b\,r}{c + d\,r} = \frac{a(\gamma + \delta w) + b(\alpha + \beta w)}{c(\gamma + \delta w) + d(\alpha + \beta)w} = \frac{a\gamma + b\alpha + (a\delta + b\beta)w}{c\gamma + d\alpha + (c\delta + d\beta)w},$$

something surprising happens. The composition $w \circ r$ may be written in matrix form, as the product of two matrices that represents each bilinear transform. This may be seen as true by inspecting the coefficients of the composition $w \circ r$ (shown above) and the product of the two matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} (a\gamma + b\alpha) & (a\delta + b\beta) \\ (c\gamma + d\alpha) & (c\delta + d\beta) \end{bmatrix}.$$

The the power of this composition property of the bilinear transform may be put to work solving important engineering problems, using *transmission matrices.*

## Mapping the multi-valued square root of $w = \pm\sqrt{x+iy}$

- *This provides a deep (essential) insight to complex analytic functions*

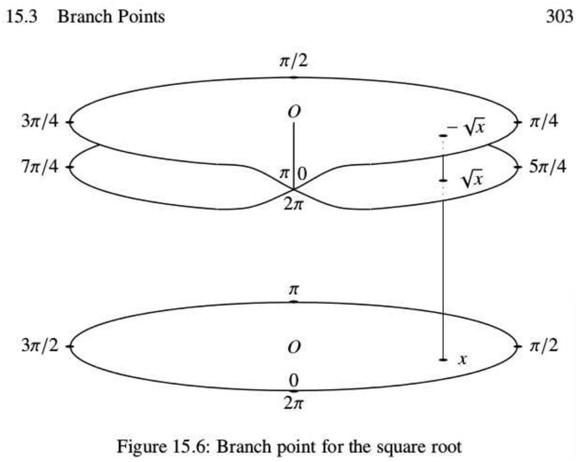



*Figure 3.2: Here we see the function* $w(z) = \pm\sqrt{z}$.

### 3.3.3 Lec 17 Introduction to the Riemann Sphere and infinity

Riemann sphere and the extended plane ($3^d$ chord and tangent method)
Möbius Transformation (youtube video)
Closing the complex plane

**WEEK 7** **18.7.0**

L 18 Colorized plots of complex analytic functions (Matlab *zviz.m*)

L 19 Signals and Systems: Fourier vs. Laplace Transforms **AE-3**

L 20 Role of Causality and the Laplace Transform:
$u(t) \leftrightarrow 1/s$ (LT)
$2\tilde{u}(t) \equiv 1 + \text{sgn}(t) \leftrightarrow 2\pi\delta(\omega) + 2/\jmath\omega$ (FT)

## 3.4 Week 7

### 3.4.1 Lec 18 Complex analytic mappings (colorized plots)

Colorized plots (Matlab *zviz.m*)

When one uses complex analytic functions it is helpful to understand their properties in the complex plane. In this sections we explore several well-known functions using colorized plots.

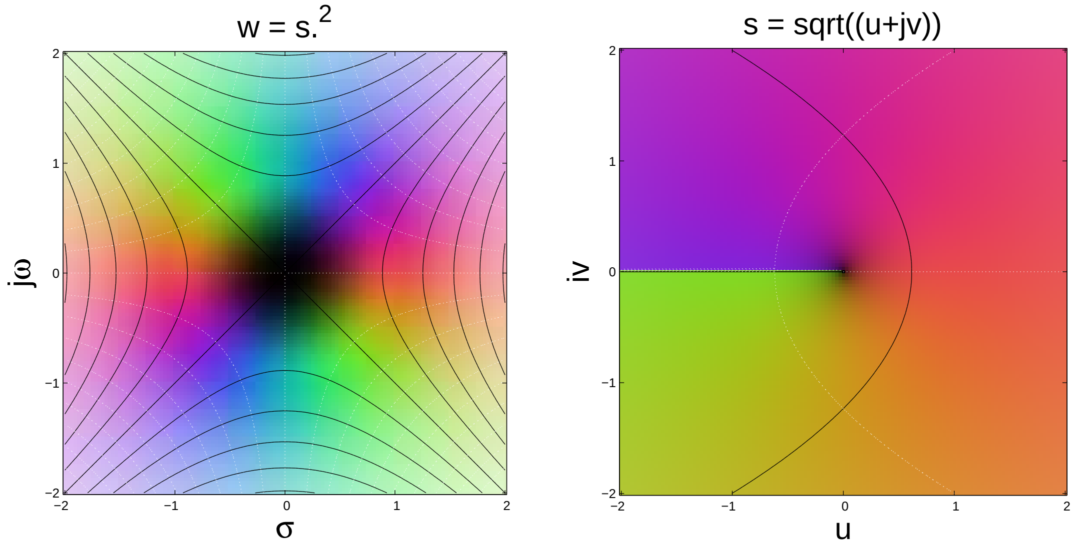



Figure 3.3: *On the left is the function $w(s) = s^2$ and on the right is $s = \sqrt{w}$.*

In the first example (Fig. 3.3) we look at $w(s) = s^2$ and its inverse $s(w) = \sqrt{w}$. On the left we see that the red region, corresponding to $0°$ [degrees] appears at both 0 and 180 in the $w$ plane. This is because in polar coordinates $s^2 = |s|^2 e^{2\theta\jmath}$ where $\theta$ is the angle of $s = |s|e^{2\theta\jmath}$. Note also that the black spot is dilated due to the squaring of the radius (expanding it). On the right the $\sqrt{w} = \sqrt{|w|}e^{\phi/2}$. Because the angle of $w$ is divided by two, it takes twice as much phase (in $w$) to cover the same angle. Thus the red region ($0°$) is expanded. We barely see the violet $90°$ and yellow $-90°$ angles. There is a *branch cut* running from $w = 0$ to $w = \infty$. As the branch cut is crossed, the function switches *Riemann sheets*, going from the top sheet (shown here) to the bottom sheet (not shown). Figure 3.2 in Section 3.3.3 depicts what is going on with these two sheets, and show the branch cut from the origin (point O) to $\infty$. In this depiction the first sheet $(+\sqrt{z})$ is on the bottom, while the second sheet $(\sqrt{z})$ is on top. For every value of $z$ there are two possible outcomes, $\pm\sqrt{z}$, represented by the two sheets.

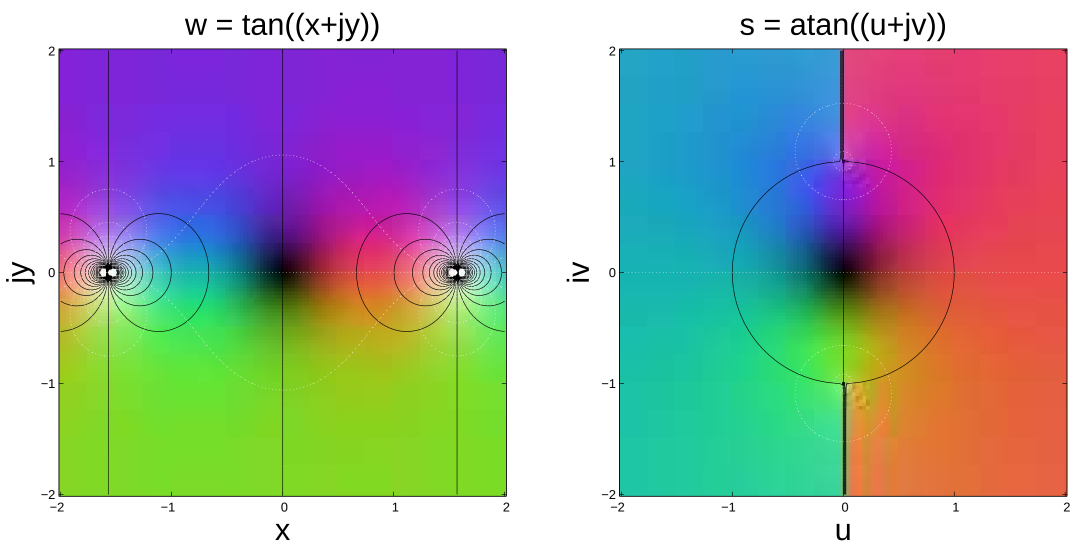



Figure 3.4: *On the left is the function $w(s) = \tan(z)$ and on the right is $w(s) = atan(\pi s)$.*

In the second example (Fig. 3.4) we show $w = \tan(z)$ and its inverse $z = \text{atan}(w)$. The tangent function has zeros where $\sin(z)$ has zeros (e.g., at $z = 0$) and poles where $\cos(z)$ is zero (e.g., at $\pm\pi/2$. The inverse function $s = \text{atan}(w)$ has a zero at $w = 0$ and branch cuts eliminating from $z = \pm\pi$.

The third example (Fig. 3.5) shows $w = e^s$ (left) and its inverse $s = \ln(w)$. The exponential is very easy to understand because $w = e^\sigma e^{\omega\jmath}$. The red region is where $\omega \approx 0$ in which case $w \approx e^\sigma$. As $\sigma$ becomes large and negative, $w \to 0$ so the entire field at the left becomes dark. The field is becoming

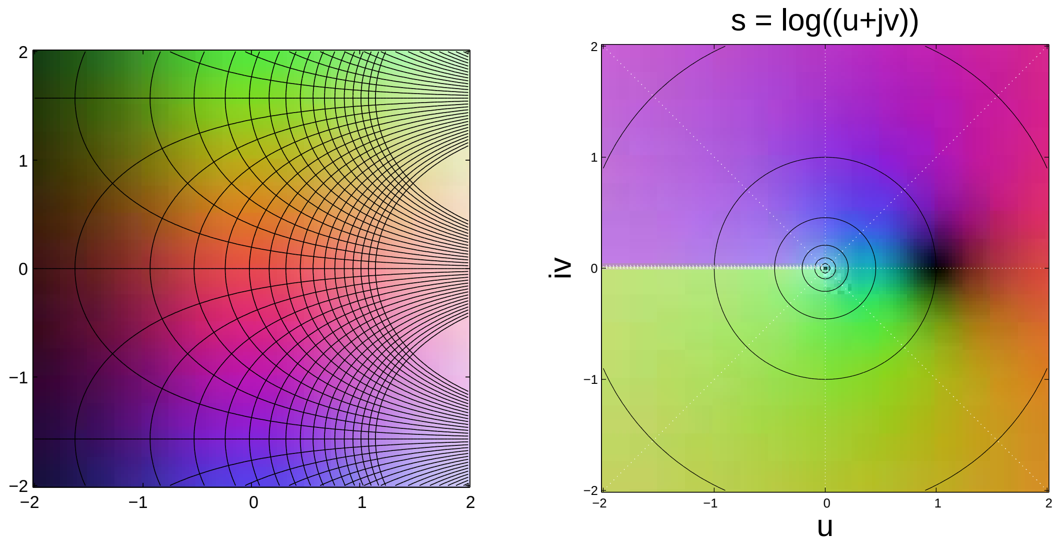



Figure 3.5: *On the left is the function $w(s) = e^s$ and on the right is $s = log(w)$.*

light on the right where $w = e^\sigma \to \infty$. If we let $\sigma = 0$ and look along the $\omega$ axis, we see that the function is changing phase, green -90° at the top and violet (90°) at the bottom.

A really important point is the zero in $\ln(w)$ at $w = 1$. A little algebra should solve this problem. If we solve for the root of the log function, $\log(s_r) = 0$. Since $\log(1) = 0$, we have that $s_r = 1$. More generally, express $w = |w|e^{\phi \jmath}$. Taking the log we find $s = log(|w|) + \phi\jmath$. Thus $s$ can only be zero when the angle of $w$ is zero ($\phi = 0$).

It becomes most interesting to study polynomials of degree 5 and 4, with one zero removed, to demonstrate the Fundamental Theorem of Algebra. Recall that degree 5 is not analytically tractable, and must be investigated numerically.

Discuss the branch cut.

### 3.4.2 Lec 19 Signals and Systems: Fourier vs. Laplace Transforms

Signals and Systems: Fourier vs. Laplace Transforms **AE-3**

### 3.4.3 Lec 20 Role of Causality and the Laplace Transform

Role of Causality and the Laplace Transform:
$u(t) \leftrightarrow 1/s$ (LT)
$2\tilde{u}(t) \equiv 1 + \text{sgn}(t) \leftrightarrow 2\pi\delta(\omega) + 2/\jmath\omega$ (FT)

## WEEK 8 **20.8.0**

L 21 The 6 postulates of System (aka, Network) Theory; The important role of the Laplace transform re impedance

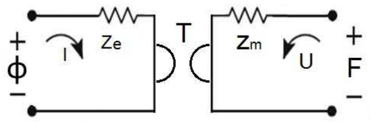
L 22 Exam II (Evening exam)

## 3.5 Week 8

### 3.5.1 Lec 21 The 6 postulates of System of algebraic Networks

The 6 postulates of System (aka, Network) Theory;
The important role of the Laplace transform to impedance

Taxonomy requires a proper statement of the laws of physics, which includes at least the nine basic network postulates described in Section 1.3.11. To describe each of the network postulates one must start from the Transmission matrix representation discussed in Section 3.3.2.




$$\begin{bmatrix}\Phi\\ I\end{bmatrix}=\begin{bmatrix}1 & Z_e\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & T\\ T^{-1} & 0\end{bmatrix}\begin{bmatrix}1 & z_m\\ 0 & 1\end{bmatrix}\begin{bmatrix}F\\ \text{-U}\end{bmatrix}$$

Figure 3.6: *A schematic representation of a 2-port ABCD electro-mechanic system using Hunt parameters $Z_e(s)$, $z_m(s)$, and $T(s)$: electrical impedance, mechanical impedances, and transduction coefficient (Hunt, 1952; Kim and Allen, 2013). Also $V(f)$, $I(f)$, $F(f)$, and $U(f)$ are the frequency domain voltage, current, force, and velocity respectively. Notice how the matrix method 'factors' the 2-port model into three 2×2 matrices. This allows one to separate the physical modeling from the algebra. It is a standard impedance convention that the flows $I(f), U(f)$ are always defined into the port. Thus it is necessary to apply a negative sign on the velocity $-U(f)$ so that it has an outward flow, to feed the next cell with an inward flow. Replace $\Phi$ with $V$.*

The 2-port *transmission matrix* for an acoustic transducer (loudspeaker) shown in Fig. 3.6 is defined
3.20,3.21← *F.3.6* as

$$\begin{bmatrix}\Phi_i\\ I_i\end{bmatrix}=\begin{bmatrix}A(s) & B(s)\\ C(s) & D(s)\end{bmatrix}\begin{bmatrix}F_l\\ -U_l\end{bmatrix}=\frac{1}{T}\begin{bmatrix}z_m(s) & z_e(s)z_m(s)+T^2\\ 1 & z_e(s)\end{bmatrix}\begin{bmatrix}F_l\\ -U_l\end{bmatrix}. \tag{3.20}$$

The input is electrical (voltage and current) $[\Phi_i, I_i]$ and the output (load) are the mechanical (force and velocity) $[F_l, U_l]$. The first matrix is the general case, expressed in terms of four unspecified functions $A(s), B(s), C(s), D(s)$, while the the second matrix is for the specific example of Fig. 3.6. The four entries are the electrical driving point impedance $Z_e(s)$, the mechanical impedance $z_m(s)$ and the transduction $T = B_0 l$ where $B_0$ is the magnetic flux strength and $l$ is the length of the wire crossing the flux. Since the transmission matrix is anti-reciprocal, its determinate $\Delta_T = -1$, as is easily verified.

Other common transduction examples of cross-modality transduction include current–thermal (thermoelectric effect) and force–voltage (piezoelectric effect). These systems are all reciprocal, thus the transduction has the same sign.

#### Impedance matrix

These nine postulates describe the properties of a system having an input and an output. For the case of an electromagnetic transducer (Loudspeaker) the system is described by the 2-port, as show in Fig. 3.6. P6 is inherently a 2-port network property, while P1-P5 also apply to 1-ports networks (e.g., a driving point impedance is a 1-port). For example the electrical input impedance of a loudspeaker is $Z_e(s)$, defined by

$$Z_e(s) = \left.\frac{V(\omega)}{I(\omega)}\right|_{U=0}.$$

Note that this *driving-point impedance* must be causal, thus it has a Laplace transform and therefore is a function of the complex frequency $s = \sigma + j\omega$, whereas the Fourier transforms of the voltage $V(\omega)$ and current $I(\omega)$ are functions of the real radian frequency $\omega$, since the time-domain voltage $v(t) \leftrightarrow V(\omega)$ and the current $i(t) \leftrightarrow I(\omega)$ are signals that may start and stop at any time (they are not typically causal).

The corresponding 2-port *impedance matrix* for Fig. 3.6 is

$$\begin{bmatrix}\Phi_i\\ F_l\end{bmatrix}=\begin{bmatrix}z_{11}(s) & z_{12}(s)\\ z_{21}(s) & z_{22}(s)\end{bmatrix}\begin{bmatrix}I_i\\ U_l\end{bmatrix}=\begin{bmatrix}Z_e(s) & -T(s)\\ T(s) & z_m(s)\end{bmatrix}\begin{bmatrix}I_i\\ U_l\end{bmatrix}. \tag{3.21}$$

The impedance matrix is an alternative description of the system but with generalized forces $[Phi_i, F_l]$ on the left and generalized flows $[I_i, U_l]$ on the right. A rearrangement of the equations allows one to go from one set of parameters to the other (Van Valkenburg, 1964b). Since the electromagnetic transducer is anti-reciprocal, $z_{12} = -z_{21} = T = B_0 l$. Such a description allows one to define *Thèvenin parameters*, a very useful concept used widely in circuit analysis and other network models from other modalities.

### Additional or modified postulates

Our definition of MMs must go beyond postulates P1-P7, since MMs result from the interaction of waves and a structured medium, along with other properties not covered by classic network theory (e.g., the quantum Hall effect). Assuming QS, the wavelength must be large relative to the medium's lattice constants. Thus the QS property must be extended for MM to three dimensions, and possibly to the cases of an-isotropic and random media.

**Causality: P1** As stated above, due to causality the negative properties (e.g., negative refractive index) of AMMs must be limited in bandwidth, as a result of the Cauchy Riemann conditions. However even causality needs to be extended to include the delay, as quantified by the d'Alembert solution to the wave equation, which means that the delay is proportional to the distance. Thus we generalize P1 to include the space dependent delay. When we wish to discuss this property we denote it *Einstein causality*, which says that the delay must be proportional to the distance $x$, with impulse response $\delta(t - x/c)$.

**Linearity: P2** The wave properties of MMs may be non-linear (P2). This is not restrictive as most physical systems are naturally nonlinear. For example, a capacitor is inherently nonlinear: as the charge builds up on the plates of the capacitor, a stress is applied to the intermediate dielectric due to the electrostatic force $F = qE$. In a similar manner, an inductor is nonlinear. Two wires carrying a current are attracted or repelled, due to the force created by the flux. The net force is the product of the two fluxes due to each current.

In summary, most physical systems are naturally nonlinear, it's simply a matter of degree. An important counter example is a amplifier with negative feedback, with very large open-loop gain. There are, therefore, many types of non-linear, instantaneous and those with memory (e.g., hysteresis). Given the nature of P1, even an instantaneous non-linearity may be ruled out. The linear model is so critical for our analysis, providing fundamental understanding that we frequently take this postulate for granted.

**Real time response: P3** The impulse response of every physical system is real, vs. complex. This requires that the Laplace Transform have conjugate-symmetric symmetry $H(s) = H^*(s^*)$, where the $*$ indicates conjugation (e.g., $R(\sigma, \omega) + X(\sigma, \omega) = R(\sigma, \omega) - X(\sigma, -\omega)$).

**Positive-realness: P4** Every physical system that has a positive-real part of the Laplace Transform of the system's impulse response $h(t)$ is called *positive-real* (Brune, 1931a). This requirement places a sever constraints on both the number of poles and zeros in $H(s) \leftrightarrow h(t)$ and their relative placement. When the input resistance of the impedance is real, the system is said to be *passive*, which means the system obeys conservation of energy. In a later chapter we shall describe the physics behind this constraint.

By building in the physics behind conservation of energy: Otto Brune's *positive-real* (PR) condition. Following up on the earlier work of his primary PhD thesis advisor Wilhelm Cauer (1900-1945), and working with Norbert Weiner and Vannevar Bush at MIT, Otto Brune mathematically characterized the properties of every PR 1-port driving point impedance (Brune, 1931b).

Given any PR impedance $Z(s) = R(\sigma, \omega) + jX(\sigma, \omega)$, having real part (resistance) $R(\sigma, \omega)$ and imaginary part (reactance) $X(\sigma, \omega)$, the impedance is defined as being PR (Brune, 1931a) if and only if

$$R(\sigma \geq 0, \omega) \geq 0. \tag{3.22}$$

That is, the real part of any PR impedance is non-negative everywhere in the right half $s$ plane ($\sigma \geq 0$). This is a very strong condition on the complex analytic function $Z(s)$ of a complex variable $s$. This condition is equivalent to any of the following statements: 1) There are no poles or zeros in the right half plane ($Z(s)$ may have poles and zeros on the $\sigma = 0$ axis). 2) If $Z(s)$ is PR then its reciprocal $Y(s) = 1/Z(s)$ is PR (the poles and zeros swap). 3) If the impedance may be written as the ratio of two polynomials (a limited case) having degrees $N$ and $L$, then $|N - L| \leq 1$. 4) The angle of the impedance $\theta \equiv \angle Z$ lies between $[-\pi \leq \theta \leq \pi]$. 5) The impedance *and its reciprocal* are *complex analytic* in the right half plane, thus they each obey the Cauchy Riemann conditions.

The PR (positive real or Physically realizable) condition assures that every impedance is *positive-definite* (PD), thus guaranteeing conservation of energy is obeyed (Schwinger and Saxon, 1968, p.17). This means that the total energy absorbed by any PR impedance must remain positive for all time, namely

$$\mathcal{E}(t) = \int_{-\infty}^{t} v(t)i(t)\, dt = \int_{-\infty}^{t} i(t)\star z(t)\; i(t)\, dt > 0,$$

where $i(t)$ is *any* current, $v(t) = z(t) \star i(t)$ is the corresponding voltage and $z(t)$ is the real causal impulse response of the impedance, e.g., $z(t) \leftrightarrow Z(s)$ are a Laplace Transform pair. In summary, if $Z(s)$ is PR, $\mathcal{E}(t)$ is PD.

As discussed in detail by Van Valkenburg, any rational PR impedance can be represented as a *rational polynomial fraction expansion* (residue expansion), which can be expanded into first-order poles as

$$Z(s) = K\frac{\Pi_{i=1}^{L}(s - n_i)}{\Pi_{k=1}^{N}(s - d_k)} = \sum_n \frac{\rho_n}{s - s_n} e^{j(\theta_n - \theta_d)}, \tag{3.23}$$

where $\rho_n$ is a complex scale factor (residue). Every pole in a PR function has only simple poles *and* zeros, requiring that $|L - N| \leq 1$ (Van Valkenburg, 1964a).

Whereas the PD property clearly follows P3 (conservation of energy), the physics is not so clear. Specifically what is the physical meaning of the specific constraints on $Z(s)$? In many ways, the impedance concept is highly artificial, as expressed by P1-P7.

When the impedance is not rational, special care must be taken. An example of this is the semi-inductor $M\sqrt{s}$ and semi-capacitor $K/\sqrt{s}$ due, for example, to the *skin effect* in EM theory and viscous and thermal losses in acoustics, both of which are frequency dependent boundary-layer diffusion losses. They remain positive-real but have a branch cut, thus are double valued in frequency.

**Rayleigh Reciprocity: P5** The meaning of *time-invariant* requires that the impulse response of a system does not change over time. This requires that the system coefficients of the differential equation describing the system are constant (independent of time).

**Rayleigh Reciprocity: P6** Reciprocity is defined in terms of the unloaded output voltage that results from an input current. Specifically

$$\begin{bmatrix} z_{11}(s) & z_{12}(s) \\ z_{21}(s) & z_{22}(s) \end{bmatrix} = \frac{1}{C(s)} \begin{bmatrix} A(s) & \Delta_T \\ 1 & D(s) \end{bmatrix}, \tag{3.24}$$

where $\Delta_T = A(s)D(s) - B(s)C(s) = \pm 1$ for the reciprocal and anti-reciprocal systems respectively. This is best understood in term of Eq. 3.21. The off-diagonal coefficients $z_{12}(s)$ and $z_{21}(s)$ are defined as

$$z_{12}(s) = \left.\frac{\Phi_i}{U_l}\right|_{I_i=0} \qquad\qquad z_{21}(s) = \left.\frac{F_l}{I_i}\right|_{U_l=0}$$

The these off-diagonal elements are equal [$z_{12}(s) = z_{21}(s)$] the system is said to obey *Rayleigh reciprocity.* If they are opposite in sign [$z_{12}(s) = -z_{21}(s)$], the system is said to be *anti-reciprocal.* If a network has neither of the reciprocal or anti-reciprocal characteristics, then we denote it as *non-reciprocal* (McMillan, 1946). The most comprehensive discussion of reciprocity, even to this day, is that of Rayleigh (1896, Vol. I). The reciprocal case may be modeled as an ideal transformer (Van Valkenburg, 1964a) while for the the anti-reciprocal case the generalized force and flow are swapped across the 2-port. An electromagnetic transducer (e.g., a moving coil loudspeaker or electrical motor) is anti-reciprocal (Kim and Allen, 2013; Beranek and Mellow, 2012), requiring a gyrator rather than a transformer, as shown in Fig. 3.6.

**Reversibility: P7** A second 2-port property is the *reversible/non-reversible* postulate. A reversible system is invariant to the the input and output impedances being swapped. This property is defined by the input and output impedances being equal.

Referring to Eq. 3.24, when the system is *reversible* $z_{11}(s) = z_{22}(s)$ or in terms of the transmission matrix variables $\frac{A(s)}{C(s)} = \frac{D(s)}{C(s)}$ or simply $A(s) = D(s)$.

An example of a non-reversible system is a transformer where the turns ratio is not one. Also an ideal operational amplifier (when the power is turned on) is non-reversible due to the large impedance difference between the input and output. Furthermore it is *active* (it has a power gain, due to the current gain at constant voltage) (Van Valkenburg, 1964b).

Generalizations of this lead to group theory, and *Noether's theorem.* These generalizations apply to systems with many modes whereas MMs operate below a cutoff frequency, meaning that like the case of the transmission line, there are no propagating transverse modes. While this assumption is never exact, it leads to highly accurate results because the non-propagating evanescent transverse modes are attenuated over a short distance, and thus, in practice, may be ignored (Montgomery et al., 1948; Schwinger and Saxon, 1968, Chap. 9-11).

We extend the Carlin and Giordano postulate set to include (P7) Reversibility, which was refined by Van Valkenburg (1964a). To satisfy the reversibility condition, the diagonal components in a system's impedance matrix must be equal. In other words, the input force and the flow are proportional to the output force and flow, respectively (i.e., $Z_e = z_m$).

| Measure | Domain |
|---|---|
| $ka < 1$ | Wavenumber constraint |
| $\lambda > 2\pi a$ | Wavelength constraint |
| $f_c < c/2\pi a$ | Bandwidth constraint |

Table 3.1: *There are several ways of indicating the quasi-static (QS) approximation. For network theory there is only one lattice constant a, which must be much less than the wavelength (wavelength constraint). These three constraints are not equivalent for MMs, when a is a 3-vector of lattice constants. where the object may be a larger structured medium, spanning many wavelengths, but with a cell structure size much less than the wavelength. For example, each cell could be a Helmholtz resonator, or an EM transducer. Thus MMs require a generalized QS definition.*

**Spatial invariant: P8** The characteristic impedance and wave number $\kappa(s, x)$ may be strongly frequency and/or spatially dependent, or even be negative over some limited frequency ranges. Due to *causality*, the concept of a negative group velocity must be restricted to a limited bandwidth (Brillouin, 1960). As is made clear by Einstein's theory of relativity, all materials must be strictly causal (P1), a view that must therefore apply to acoustics, but at a very different time scale. In the following we first discuss generalized postulates, expanding on those of Carlin and Giordano.

**The QS constraint: P9** When a system is described by the wave equation, delay is introduced between two points in space, that depends on the wave speed. An important property of MM is the use of the QS approximation, especially when the waves are guided or band limited. This property is not mentioned in the six postulates of Carlin and Giordano (1964), thus they need to be extended in some fundamental ways. Only when the dimensions of a cellular structure in the material are much less than the wavelength, can the QS approximation be valid. This effect can be viewed as a *mode filter* that suppresses unwanted (or conversely enhances the desired) modes. But a single number used to quantify the structure is not adequate for MMs, as it can be a 3 dimensional specification, as in a semiconductor lattice, or even a random variable matrix.

Formally, QS is defined as $ka < 1$ where $k = 2\pi/\lambda = \omega/c$ and $a$ is the cellular dimension or the size of the object ($k$ and $a$ can be vectors). Schelkunoff may have been the first to formalize this concept (Schelkunoff, 1943) (but not the first to use it, as exemplified by the Helmholtz resonator). George Ashley Campbell was the first to use the concept in the important application of a wave-filter, some 30 years before Schelkunoff Campbell (1903). These two men were 40 years apart, and both worked for the telephone company (after 1929, called AT&T Bell Labs) (Fagen, 1975).

There are alternative definitions of the QS approximation, and for the case of MMs these may not be equivalent, depending on the cell structure. The alternatives are outlined in Table 3.1.

#### Summary

A transducer converts between modalities. We propose the general definition of the nine system postulates, that include all transduction modalities, such as electrical, mechanical, and acoustical. It is necessary to generalize the concept of the QS approximation (P9) to allow for guided waves.

Given the combination of the important QS approximation, along with these space-time, linearity, and reciprocity properties, a rigorous definition and characterization a system can thus be established. It is based on a taxonomy of such materials, formulated in terms of material and physical properties and in terms of extended network postulates.

### 3.5.2 Lec 22 Exam II (Evening)

# Chapter 4

# Ordinary Differential Equations: Stream 3a

## WEEK 8 23.9.0

Week 8 Friday *Stream 3*

L 23 The amazing Bernoulli family; Fluid mechanics; airplane wings; natural logarithms
The transition from geometry → algebra → algebreic geometry → real analytic → complex analytic
From Bernoulii to Euler to Cauchy and Riemann

## 4.1 Week 8

### 4.1.1 Lec 23 Newton and early calculus & the Bernoulli Family

Newton and Calculus
Bernoulli family
Euler standard periodic function package

The period of analytic discovery:
Coming out of the dark ages, from algebra, to analytic geometry, to calculus.
Starting with real analytic functions by Euler, we move to complex analytic functions with Cauchy.
Integration in the complex plane is finally conquered.

### Lect DE 25.9 Stream 3: $\infty$ and Sets 25.9.1

The development of real representations proceeded at a deadly-slow pace:

- *Real numbers* $\mathbb{R}$*:* Pythagoras knew of irrational numbers ($\sqrt{2}$)
- *Complex numbers* $\mathbb{C}$*:* 1572 *"Bombelli is regarded as the inventor of complex numbers . . . "* http://www-history.mcs.st-andrews.ac.uk/Biographies/Bombelli.html http://en.wikipedia.org/wiki/Rafael_Bombelli & p. 258
- *Power Series*: Gregory-Newton interpolation formula c1670, p. 175
- Point at infinity and the Riemann sphere 1851
- Analytic functions p. 267 c1800; Impedance $Z(s)$ 1893

## *Stream 3* Infinity

- Infinity $\infty$ was not "understood" until $19^{th}$ CE
- $\infty$ is best defined in terms of a limit
- Limits are critical when defining calculus
- Set theory is the key to understanding Limits
- Open vs close sets determine when a limit exists (or not)
- Thus, to fully understand limits, one needs to understand set theory
- Related is the convergence of a series
- Every convergent series has a *Region of Convergence* (ROC)
- When the ROC is Complex:
  - Example of $\frac{1}{1-x}$ vs. $\frac{1}{i-x}$: The ROC is 1 for both cases
  - Why?
  - The case of the Heaviside step function $u(t)$ & the Fourier Transform

## Irrational numbers and limits (Ch. 4)

- How are irrational numbers interleaved with the integers?
- Between $n$ and $2n$ there is always an irrational number:

> Chebyshev said, and I say it again. There is always a prime between n and 2n. -p. $585_2$

- *Prime number theorem*: The number of of primes is approximately( the density of primes is $\rho_\pi(n) \propto 1/\ln(n)$.
- The number of primes less than $n$ is $n$ times the density, or

$$N(n) = n/\ln(n).$$

  - The formula for *entropy* is $\mathcal{H} = -\sum_n p_n \log p_n$.
    Could there be some hidden relationship lurking here?

## Stream 3: $\infty$ and Sets **25.9.2**

- Understanding $\infty$ has been a primary goals since Euclid
- The Riemann sphere solves this fundamental problem
- The point at $\infty$ simply "another point" on the Riemann sphere

**Open vs. closed sets**

## Influence of open vs. closed set 7.3.6

- Important example: LT vs. FT step function: Dirac step vs Fourier step:
- $u(t) \leftrightarrow \frac{1}{s}$ vs. $\tilde{u}(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{\jmath\omega}$

## WEEK 9 23.9.0

Week 9 Monday

L 24 Power series and integration of functions (ROC)
Fundamental Theorem of calculus (Leibniz theorm of integration)
$1/(1-x) = \sum_{k=0}^{\infty} x^k$ with $x \in \mathbb{R}$

L 25 Integration in the complex plane: Three theorems
Integration of $1/s$ on the unit circle, and on a unit circle centered about $s = 1 + i$.

L 26 Cauchy-Riemann conditions
Real and imaginary parts of analytic functions obey Laplace's equation.
Infinite power Series and analytic function theory; ROC

## 4.2 Week 9

### 4.2.1 Lec 24 Power series and complex analytic functions

L 24: Power series and complex analytic function

### 4.2.2 Lec 25 Integration in the complex plane

L 25: Integration in the complex plane; Infinite power Series and analytic function theory; ROC
Real and imaginary parts of analytic functions obey Laplace's equation.
Colorized plots of analytic functions. How to read the plots and what they tell us?

### 4.2.3 Lec 26 Cauchy Riemann conditions: Complex-analytic functions

L 26: Cauchy Riemann conditions: Complex-analytic functions

## WEEK 10 26.10.0

L 27 Differentiation in the complex plane: Fundamental Thm of complex calculus (FTCC);
Complex Analytic functions; ROC in the complex plane
$Z(s) = R(s) + \jmath X(s)$: real and imag parts obey Laplace]s Equation
Basic equations of mathematical Physics: Wave equation, Diffusion equation, Laplace's Equation
Motivation: Dispersion relation for the wave equation $\kappa \cdot \kappa = s^2/c_0^2$

L 28 Three Fundamental theorems of complex integral calculus
$\int_0^z = F(\zeta)d\zeta = F(z) - F(0)$: $dZ(s)/ds$ independent of direction
Integration in the complex plane; Integrals independent of limits
Cauchy-Riemann conditions

L 29 Inverse Laplace transform
Inverse Laplace transform: Poles and Residue expansions;
Application of the Fundamental Thm of Complex Calculus
The Inverse Laplace Transform (ILT); poles and the Residue expansion: The case for causality
ROC as a function of the sign of time in $e^{st}$ (How does causality come into play?)
Examples.

## 4.3 Integration and differentiation in the complex plane

### 4.3.1 Lec 27 Differentiation in the complex plane

L 27: Differentiation in the complex plane: CR conditions?
Motivation: Inverse Laplace transform
ROC in the complex plane
Basic equations of mathematical Physics: Wave equation, Diffusion equation, Laplace's Equation
Motivation: Dispersion relation for the wave equation $\kappa \cdot \kappa = s^2/c_0^2$

### 4.3.2 Lec 28 Three complex Integral Theorems

L 28: Integration in the complex plane: Basic definitions of Three theorems
Integration of $1/s$ on the unit circle, and on a unit circle centered about $s = 1 + i$.

Moved from Lec 3 (page 31)

**Set Theory:** Set theory is a topic that can be inadequately addressed in the undergraduate Engineering and Physics curriculum, and is relatively young to mathematics. The set that a number is drawn from is crucially important when taking limits.

### 4.3.3 Lec 29 Inverse Laplace Transform

L 29: Inverse Laplace transform: Poles and Residue expansions;
Application of the Fundamental Thm of Complex Calculus
Examples.

## *Stream 3:* Infinity and irrational numbers Ch 4 2.1.6

- *Limit points, open vs. closed sets are fundamental to modern mathematics*
- *These ideas first appeared with the discovery of $\sqrt{2}$, and $\sqrt{n}$* https://en.wikipedia.org/wiki/Spiral_of_Theodorus
*and related constructions (factoring the square, Pell's Eq. p. 44)*

**Infinity and irrational $\mathbb{Q}$ numbers**

## The fundamental theorem of calculus **2.1.7**

*Let $A(x)$ be the area under $f(x)$. Then*

$$\frac{d}{dx}A(x) = \frac{d}{dx}\int^{x} f(\eta)d\eta$$
$$= \lim_{\delta\to 0}\frac{A(x+\delta)-A(x)}{\delta}$$

*and/or*

$$A(b) - A(a) = \int_a^b f(\eta)d\eta$$

- *Stream 3 is about limits*
- *Integration and differentiation (Calculus) depend on limits*
- *Limits are built on open vs. closed sets*

## WEEK 11 **30.11.0**

L 30 Inverse Laplace transform & Cauchy Residue Theorem

L 31 Case for causality Closing the contour as $s \to \infty$; Role of $\Re st$
**DE-3**

L 32 Properties of the LT:
1) Modulation, 2) Translation, 3) convolution, 4) periodic functions
Tables of common LTs

## 4.4 Integration in the complex plane

### 4.4.1 Lec 30 Inverse Laplace Transform & Cauchy residue theorem

L30: The Inverse Laplace Transform (ILT); poles and the Residue expansion: The case for causality ROC as a function of the sign of time in $e^{st}$ (How does causality come into play?)

### 4.4.2 Lec 31 The case for causality

L31: Closing the contour as $s \to \infty$; Role of $\Re st$

### 4.4.3 Lec 32 Laplace transform properties: Modulation, time translation, etc.

L32: Detailed examples of the Inverse LT:
1) Modulation, 2) Translation, 3) convolution, 4) periodic functions
Tables of common LTs

## WEEK 12 **33.12.0**

L 33 Multi-valued functions in the complex plane; Branch cuts
The extended complex plane (regularization at $\infty$) (Stillwell, 2010, p. 280)
Complex analytic functions of Genus 1 (Stillwell, 2010, p. 343)

L 34 Euler's vs. Riemann's Zeta function $\zeta(s)$: Poles at the primes
colorized plot of $\zeta(s)$
??Sterling's formula??

L 35 Exam III

## 4.5 Complex plane concepts

### 4.5.1 Lec 33 Multi-valued complex functions, Branch Cuts, Extended plane

L33: Multi-valued functions in the complex plane; Branch cuts
The extended complex plane (regularization at $\infty$) (Stillwell, 2010, p. 280)
Complex analytic functions of Genus 1 (Stillwell, 2010, p. 343)

### 4.5.2 Lec 34 The Riemann Zeta function $\zeta(s)$

L34: Euler's vs. Riemann's Zeta function $\zeta(s)$: Poles at the primes
colorized plot of $\zeta(s)$
??Sterling's formula??

**Table 4.1: Physical meaning of each factor of $\zeta(s)$** **4.2.7**

- *Series expansion*

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad \textit{ROC: } |x| < 1$$

- *If time $T$ is a positive delay, then from the* Laplace transform

$$\delta(t-T) \leftrightarrow \int_0^\infty \delta(t-T)e^{st}dt = e^{-sT}$$

- *Each factor of $\zeta(s)$ is an $\infty$ sum of delays*
- *For example for $\pi_1 = 2$, ($T = \ln(2)$, thus $2^{-2} = e^{-s\ln 2}$)*

$$\sum_n \delta(t-nT) \leftrightarrow \frac{1}{1-2^{-s}} = 1 + e^{-sT} + e^{-s2T} + \cdots$$

*Table 4.1: Each prime number defines a delay $T_k = \ln(\pi_k)$, which in turn defines a pole in the complex $s$ plane. The series expansion of this pole is a train of delta functions that are one-sided periodic in the delta $T$. Thus each factor in the $\zeta(s)$ function defines a pole, having an incommensurate delay, since each pole is defined by a unique prime. Following this simple logic, we may interpret $\zeta(s)$ as being the Laplace transform of $Zeta(t)$, the cascade of quasi-periodic impulse responses, each with a recursive delay, determined by a prime. Note that $48100 = 10 \cdot (2 \cdot 5 \cdot 13 \cdot 37)$ is the sampling frequency [Hz] of modern CD players. This corresponds to the $20^{th}$ harmonic of the US line frequency (60 [Hz]).*[b]

[a] since $gcd(48100, 60) = 20$ and $gcd(48100, 50) = 50$.
[b] since $gcd(48100, 60) = 20$ and $gcd(48100, 50) = 50$.

#### Riemann Zeta Function $\zeta(s)$

This very important analytic function is the credible argument for true deeper understanding of the power to the analytic function. Just like the Pythagorean theorem is important to all mathematics, the zeta function is important to analysis, with many streams of analysis emanating from this form. For example the analytic Gamma function $\Gamma(s)$ is a generalization of the factorial by the relationship

$$n! = \Gamma(s-1).$$

Another important relationship is

$$\sum_{k=n}^{\infty} k = nu_n = u_n \star u_n$$

where the $\star$ represents convolution. If this is treated in the frequency domain the we obtain z-transforms of a very simple second-order pole[1]

$$nu_n \leftrightarrow \frac{2}{(z-1)^2}.$$

This follows from the geometric series

$$\frac{1}{1-z} = \sum_n z^n$$

with $z = e^s$, and the definition of convolution.

The Laplace transform does not require that the series converge, rather that the series have a region of convergence that is properly specified. Thus the non-convergent series $nu_n$ is perfectly well defined, just like

$$tu(t) = u(t) \star u(t) \leftrightarrow \frac{1!}{s^2}$$

is well defined, in the Laplace transform sense. More generally

$$t^n u(t) \leftrightarrow \frac{n!}{s^{n+1}}.$$

From this easily understood relationship we can begin to understand $\Gamma(s)$, as the analytic extension of the factorial. Its definition is simply related to the inverse Laplace transform, which is an integral. But to go there we must be able to think in the complex frequency domain. In fact we have the very simple definition for $\Gamma(p)$ with $p \in \mathbb{C}$

$$t^{p-1}u(t) \leftrightarrow \frac{\Gamma(p)}{s^p}$$

which totally explains $\Gamma(p)$. Thinking in the time domain is crucial for my understanding.

An example is a digital filter, which is linear. Such a system is shown in Fig. 4.3, where the two functions are second order digital filters. The input signal $x[n]$ enters from the left, is filtered by the first filter, producing output $y[n]$. This is then filtered again by the filter in the second box to produce signal $z[n]$. For this simple case of two linear filters the operation *commute*.

### 4.5.3 Lec 35 Exam III

L 35: Exam III

Thanksgiving Holiday 11/19–11/27 2016

[1] Need to verify the exact form of these relationships, not work from memory

## Riemann Zeta Function $\zeta(s)$ **4.2.5**

- *Integers appear as the "roots" (aka eigenmodes) of* $\zeta(s)$
- *Basic properties* $(s = \sigma + i\omega)$

$$\zeta(s) \equiv \sum_{1}^{\infty} \frac{1}{n^s} \qquad \sigma = \Re(s) > 0$$

  – *What is the* region of convergence (ROC)?

- *The amazing* Euler-Riemann Product formula *(Stillwell, 2010, Sect. 10.7:)*

$$\zeta(s) = \prod_k \frac{1}{1 - {\pi_k}^{-s}} = \prod_k \frac{1}{1 - \left(\frac{1}{\pi_k}\right)^s} = \prod_k \frac{1}{1 - \frac{1}{\pi_k^s}}$$

$$= \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdot \frac{1}{1-7^{-s}} \cdots \frac{1}{1-\pi_n^{-s}} \cdots$$

- *Euler c1750 assumed* $s \subset \mathbb{R}$. *Riemann c1850 extended* $s \subset \mathbb{C}$

Figure 4.1: *The zeta function arguably the most important of the special functions of analysis because it connects the primes to analytic function theory in a fundamental way.*

## Plot of $\angle\zeta(s)$ **4.2.6**

*Angle of Riemann Zeta function* $\angle\zeta(z)$ *as a function of complex* $z$

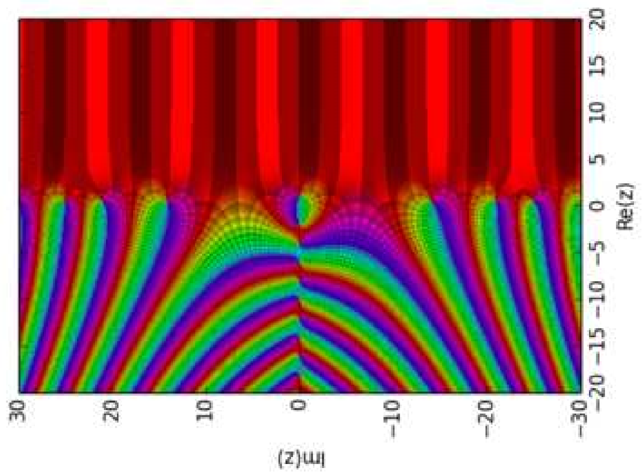



Figure 4.2: $\angle\zeta(z)$: Red $\Rightarrow \angle\zeta(z) < \pm\pi/2$

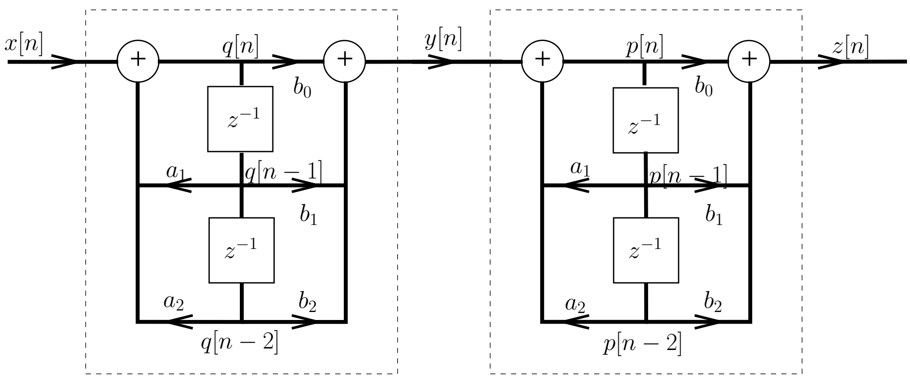



Figure 4.3: *Example of a signal flow diagram for the composition of signals* $z = g \circ f\ (x)$ *with* $y = f(x)$ *and* $z = g(y)$.

# Chapter 5

# Vector Calculus: Stream 3b

## WEEK 13 36.13.0

L 36 Scaler wave equations and the Webster Horn equation; WKB method
A real-world example of large delay, where the branch-cut placement is critical

L 37 Partial differential equations of Physics
Scaler wave equation and its solution in 1 and 3 Dimensions
**VC-1**

L 38 Vector dot and cross products $A \cdot B, A \times B$
Gradient, divergence and curl

– Thanksgiving Holiday 11/19–11/27 2016

## 5.1 Stream 3b

### 5.1.1 Lec 36 Scalar Wave equation

### 5.1.2 Lec 37 Partial differential equations of physics

Scalar wave equations and the Webster Horn equation; WKB method
Example of a large delay, where a branch-cut placement is critical (i.e., phase unwrapping)
L 37: Partial differential equations of Physics
Scalar wave equation and its solution in 1 and 3 Dimensions

### 5.1.3 Lec 38 Gradient, divergence and curl vector operators

L 38: Vector dot and cross products $A \cdot B, A \times B$
Gradient, divergence and curl vector operators

## WEEK 14 37.14.0

L 39 Gradient, divergence and curl: Gauss's (divergence) and Stokes's (curl) theorems

L 40 J.C. Maxwell unifies Electricity and Magnetism with the formula for the speed of light
Basic definitions of $E, H, B, D$
O. Heaviside's (1884) vector form of Maxwell's EM equations and the *vector wave equation*
How a loud-speaker works

L 41 *The Fundamental Thm of vector calculus*
*Incompressable* and *Irrotational* fluids and the two defining vector identities

## 5.2 Thanksgiving Holiday 11/19–11/27 2016

Thanksgiving Vacation: 1 week of rest

## 5.3 Vector Calculus

### 5.3.1 Lec 39 Geometry of Gradient, divergence and curl vector operators

Geometry of Gradient, divergence and curl vector operators

**Lec 39: Review of vector field calculus** **39.14.2**

- Review of last few lectures: Basic definitions
  - *Field*: i.e., Scalar & vector fields are functions of more than one variable
  - "Del:" $\nabla \equiv [\partial_x, \partial_y, \partial_z]^T$
  - Gradient: $\nabla\phi(x, y, z) \equiv [\partial_x\phi,\ \partial_y\phi,\ \partial_z\phi]^T$
- *Helmholtz Theorem:*
  Every vector field $\boldsymbol{V}(x, y, z)$ may be uniquely decomposed into *compressible* & *rotational* parts
  $$\boldsymbol{V}(x, y, z) = -\nabla\phi(x, y, z) + \nabla\times\boldsymbol{A}(x, y, z)$$
- Scalar part $\nabla\phi$ is *compressible* ($\nabla\phi = 0$ is *incompressible*)
- Vector part $\nabla\times\mathbf{A}$ is *rotational* ($\nabla\times\boldsymbol{A} = 0$ is *irrotational*)
- Key vector identities: $\nabla\times\nabla\phi = 0$; $\nabla\cdot\nabla\times\mathbf{A} = 0$
- Definitions of Divergence, Curl & Maxwell's Eqs;
- Closure: Fundamental Theorems of Integral Calculus

| Name | Input | Output | Operator |
|---|---|---|---|
| Gradient | Scalar | **Vector** | $-\nabla()$ |
| Divergence | **Vector** | Scalar | $\nabla\cdot()$ |
| Curl | **Vector** | **Vector** | $\nabla\times()$ |

Table 5.1: *The three vector operators manipulate scalar and vector fields, as indicated here. The gradient converts scalar fields into vector fields. The divergence eats vector fields and outputs scalar fields. Finally the curl takes vector fields into vector fields.*

**Gradient**

## Gradient: $\boldsymbol{E} = \nabla\phi(x, y, z)$ **39.14.3**

- Definition: $\mathbb{R}^1 \underset{\nabla}{\mapsto} \mathbb{R}^3$

$$\mathbf{E}(x, y, z) = [\partial_x,\, \partial_y,\, \partial_z]^T\, \phi(x, y, z) = \left[\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right]^T_{x,y,z}$$

- $\mathbf{E} \perp$ plane tangent at $\phi(x, y, z) = \phi(x_0, y_0, z_0)$
- Unit vector in direction of $\mathbf{E}$ is $\hat{\mathbf{n}} = \frac{\mathbf{E}}{||\mathbf{E}||}$, along *isocline*
- Basic definition

$$\nabla\phi(x, y, z) \equiv \lim_{|\mathcal{S}|\to 0} \left\{ \frac{\iiint \phi(x, y, z)\, \hat{\mathbf{n}}\, dA}{|\mathcal{S}|} \right\}$$

  $\hat{\mathbf{n}}$ is a unit vector in the direction of the gradient
  $\mathcal{S}$ is the surface area centered at $(x, y, z)$

**Divergence**

## Divergence: $\nabla\cdot\boldsymbol{D} = \rho$ **39.14.4a**

- Definition: $\mathbb{R}^3 \underset{\nabla\cdot}{\mapsto} \mathbb{R}^1$

$$\nabla\cdot\mathbf{D} \equiv [\partial_x,\, \partial_y,\, \partial_z] \cdot \mathbf{D} = \left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right] = \rho(x, y, z)$$

- Examples:
  - Voltage about a point charge $Q$ [SI Units of Coulombs]

$$\phi(x, y, z) = \frac{Q}{\epsilon_0\sqrt{x^2 + y^2 + z^2}} = \frac{Q}{\epsilon_0 R}$$

    $\phi$ [Volts]; $Q =$ [C]; Free space $\epsilon_0$ *permittivity* ($\mu_0$ *permeability*)
  - *Electric Displacement* (flux density) around a point charge ($\boldsymbol{D} = \epsilon_0\boldsymbol{E}$)

$$\mathbf{D} \equiv -\nabla\phi(R) = -Q\nabla\left\{\frac{1}{R}\right\} = -Q\,\delta(R)$$

## Divergence: The integral definition **39.14.4b**

- Surface integral definition of *incompressible* vector field

$$\nabla\cdot\mathbf{D} \equiv \lim_{|\mathcal{S}|\to 0} \left\{ \frac{\iint_{\mathcal{S}} \mathbf{D}\cdot\hat{\mathbf{n}}\, dA}{|\mathcal{V}|} \right\} = \rho(x, y, z)$$

  $\mathcal{S}$ must be a closed surface
  $\hat{\mathbf{n}}$ is the unit vector in the direction of the gradient
  - $\hat{\mathbf{n}}\cdot\mathbf{D} \perp$ surface differential $dA$

## Divergence: Gauss' Law **39.14.4c**

- General case of a *Compressible* vector field
- Volume integral over charge density $\rho(x, y, z)$ is total charge enclosed $Q_{enc}$
- $$\iiint_{\mathcal{V}} \nabla\cdot\boldsymbol{D}\, dV = \iint_{\mathcal{S}} \boldsymbol{D}\cdot\hat{\boldsymbol{n}}\, dA = Q_{enc}$$
- Examples
  - When the vector field is *incompressible*
    - $\rho(x, y, z) = 0$ [C/m$^3$] over enclosed volume
    - Surface integral is zero ($Q_{enc} = 0$)
  - Unit point charge: $D = \delta(R)$ [C/m$^2$]

**Curl**

## Curl: $\nabla\times\mathbf{H} = \mathbf{I}$ [amps/m$^2$] **39.14.5a**

- Definition: $\mathbb{R}^3 \underset{\nabla\times}{\mapsto} \mathbb{R}^3$

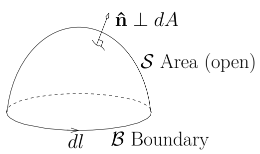
$$\nabla\times\mathbf{H} \equiv \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ H_x & H_y & H_z \end{vmatrix} = \mathbf{I}$$

- Examples:
  - Maxwell's equations: $\nabla\times\mathbf{E} = -\dot{\mathbf{B}}$, $\nabla\times\mathbf{H} = \sigma\mathbf{E} + \dot{\mathbf{D}}$,
  - $\mathbf{H} = -y\hat{x} + x\hat{y}$ then $\nabla\times\mathbf{H} = 2\hat{z}$ constant *irrotational*
  - $\mathbf{H} = 0\hat{x} + 0\hat{y} + z^2\hat{z}$ then $\nabla\times\mathbf{H} = \mathbf{0}$ is *irrotational*

**Stokes' Law**

## Curl: Stokes Law **39.14.5b**

- Surface integral definition of $\nabla\times\mathbf{H} = \mathbf{I}$ $\quad$ ($\boldsymbol{I} \perp$ rotation plane of $\boldsymbol{H}$)




$$\nabla\times\mathbf{H} \equiv \lim_{|\mathcal{S}|\to 0} \left\{ \frac{\iint_{\mathcal{S}} \hat{\mathbf{n}} \times \mathbf{H}\, dA}{|\mathcal{S}|} \right\} \tag{5.1}$$

$$\mathcal{I}_{enc} = \iint (\nabla\times\mathbf{H}) \cdot\hat{\boldsymbol{n}}\, dA = \oint_{\mathcal{B}} \mathbf{H}\cdot\boldsymbol{dl} \tag{5.2}$$

- Eq. (1): $\mathcal{S}$ must be an *open surface*
  with closed boundary $\mathcal{B}$
  $\hat{\mathbf{n}}$ is the unit vector $\perp$ to $dA$
  $\boldsymbol{H}\times\hat{\boldsymbol{n}} \in$ Tangent plane of $A$ (i.e., $\perp \hat{\boldsymbol{n}}$)
- Eq. (2): Stokes Law: Line integral of $\boldsymbol{H}$ along $\mathcal{B}$ is total current $\mathcal{I}_{enc}$

### 5.3.2 Lec: 40 Introduction to Maxwell's Equation

L 40: J.C. Maxwell unifies Electricity and Magnetism with the formula for the speed of light
Basic definitions of $E, H, B, D$
O. Heaviside's (1884) vector form of Maxwell's EM equations and the *vector wave equation*
How a loud-speaker works

### 5.3.3 Lec: 41 The Fundamental theorem of Vector Calculus

L 41: *The Fundamental Thm of vector calculus*
*Incompressible* and *Irrotational* fluids and the two defining vector identities

**WEEK 15** **40.15.0**

L 42 Quasi-static approximation and applications:
The Kirchoff's Laws and the *Telegraph wave equation*, starting from Maxwell's equations The telegraph wave equation starting from Maxwell's equations
Quantum Mechanics

L 43 Last day of class: Review of Fund Thms of Mathematics:
Closure on Numbers, Algebra, Differential Equations and Vector Calculus,
The Fundamental Thms of Mathematics & their applications:
Theorems of Mathematics; Fundamental Thms of Mathematics (Ch. 9); Normal modes vs. eigen-states, delay and quasi-statics;

– Reading Day
VC-1 Due

## 5.4 Kirchhoff's Laws

### 5.4.1 Lec 42: The Quasi-static approximation and applications

L 42: The Kirchhoff's Laws and the *Telegraph wave equation*, starting from Maxwell's equations
Quantum Mechanics

### 5.4.2 Lec 43: Last day of class: Review of Fund Thms of Mathematics

L 43: Closure on Numbers, Algebra, Differential Equations and Vector Calculus,
The Fundamental Thms of Mathematics & their applications:
Theorems of Mathematics; Fundamental Thms of Mathematics (Ch. 9)
Normal modes vs. eigen-states, delay and quasi-statics;

Reading Day

**Properties**

### Closure: Properties of fields of Maxwell's Equations 39.14.6

The variables have the following names and defining equations:

| Symbol | Equation | Name | Units |
|---|---|---|---|
| $\mathbf{E}$ | $\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$ | Electric Field strength | [Volts/m] |
| $\mathbf{D}$ | $\nabla \cdot \mathbf{D} = \rho$ | Electric Displacement (flux density) | [Col/m$^2$] |
| $\mathbf{H}$ | $\nabla \times \mathbf{H} = \dot{\mathbf{D}}$ | Magnetic Field strength | [Amps/m] |
| $\mathbf{B}$ | $\nabla \cdot \mathbf{B} = 0$ | Magnetic Induction (flux density) | [Weber/m$^2$] |

*In vacuo* $\boldsymbol{B} = \mu_0 \boldsymbol{H}$, $\boldsymbol{D} = \epsilon_0 \boldsymbol{E}$, $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ [m/s], $r_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377$ [Ω].

**Vector field properties**

## Closure: Summary of vector field properties 39.14.7

- Notation: $$\boldsymbol{v}(x,y,z) = -\nabla\phi(x,y,z) + \nabla\times\boldsymbol{w}(x,y,z)$$
- Vector identities: $$\nabla\times\nabla\phi = 0; \ \ \nabla\cdot\nabla\times\mathbf{w} = 0$$

| Field type | Generator: | Test (on $\boldsymbol{v}$): |
|---|---|---|
| Irrotational | $\boldsymbol{v} = \nabla\phi$ | $\nabla\times\boldsymbol{v} = 0$ |
| Rotational | $\boldsymbol{v} = \nabla\times\boldsymbol{w}$ | $\nabla\times\boldsymbol{v} = \boldsymbol{J}$ |
| Incompressible | $\boldsymbol{v} = \nabla\times\boldsymbol{w}$ | $\nabla\cdot\boldsymbol{v} = 0$ |
| Compressible | $\boldsymbol{v} = \nabla\phi$ | $\nabla\cdot\boldsymbol{v} = \rho$ |

- Source density terms: Current: $\boldsymbol{J}(x,y,z)$, Charge: $\rho(x,y,z)$
  - Examples: $\nabla\times\boldsymbol{H} = \dot{\boldsymbol{D}}(x,y,z)$, $\nabla\cdot D = \rho(x,y,z)$

**Fundamental Theorem of integral Calculus**

## Closure: Fundamental Theorems of integral calculus 39.14.8

1. $f(x) \in \mathbb{R}$ (Leibniz Integral Rule): $F(x) = F(a) + \int_a^x f(x)dx$
2. $f(s) \in \mathbb{C}$ (Cauchy's formula): $F(s) = F(a) + \int_a^s f(\zeta)d\zeta$
   –When integral is independent of path, $F(s) \in \mathbb{C}$ obeys CR conditions
   –Contour integration inverts causal Laplace transforms
3. $\boldsymbol{F} \in \mathbb{R}^3$ (Helmholtz Formula): $\boldsymbol{F}(x,y,z) = -\nabla\phi(x,y,z) + \nabla\times\boldsymbol{A}(x,y,z)$
   –Decompose $\boldsymbol{F}(x,y,z)$ as *compressible* and *rotational*
4. Gauss' Law (Divergence Theorem): $Q_{enc} = \iiint \nabla\cdot\boldsymbol{D}\,dV = \iint_{\mathcal{S}} \boldsymbol{D}\cdot\hat{\boldsymbol{n}}\,dA$
   –Surface integral describes enclosed compressible sources
5. Stokes' Law (Curl Theorem): $\mathcal{I}_{enc} = \iint(\nabla\times\mathbf{H})\cdot\hat{\boldsymbol{n}}\,dA = \oint_{\mathcal{B}} \mathbf{H}\cdot d\boldsymbol{l}$
   –Boundary vector line integral describes enclosed rotational sources
6. Green's Theorem ... Two-port boundary conditions
   –*Reciprocity* property (*Theory of Sound*, Rayleigh, J.W.S., 1896)

## Closure: Quasi-static (QS) approximation 39.14.9

- Definition: $ka \ll 1$ where $a$ is the size of object, $\lambda = c/f$ wavelength
- This is equivalent to $a \ll \lambda$ or
- $\omega \ll c/a$ which is a low-frequency approximation
- The QS approximation is widely used, but infrequently identified.
- All *lumped parameter models* (inductors, capacitors) are based on QS approximation as the lead term in a Taylor series approximation.

# Appendix A

# Notation

## A.1 Number systems

The notation used in this book is defined in this appendix so that it may be quickly accessed.[1] Where the definition is sketchy, Page numbers are provided where these concepts are fully explained, along with many other important and useful definitions. For example $\mathbb{N}$ may be found on page 24.

### A.1.1 Double-Bold notation

Table A.1 indicates the symbol followed by a page number and the name of the number type. For example $\mathbb{N}$ stands for the infinite set of *counting numbers* $\{1, 2, 3, \cdots\}$. From any counting number you may get the next one by adding 1.

**Summary of various number types:** Counting number ($\mathbb{N}$) are also know as the Cardinal numbers. The prime numbers ($\mathbb{P}$) cannot be further factored. The counter example of $-5 = -1 \cdot 5$ is questionable, as it could be included as a prime by a slight change in the definition. One may say that a real ($\mathbb{R}$) is a complex number ($\mathbb{C}$) with a zero imaginary part, thus real numbers are complex ($\mathbb{R} \subset \mathbb{C}$).

Table A.1: Double-bold notation for the types of numbers. (#) is a page number.

| Symbol (p. #) | Genus | Examples | Counter Examples |
|---|---|---|---|
| $\mathbb{N}$ (24) | Counting | 1,2,17,3, $10^{20}$ | 0, -10, 5j |
| $\mathbb{P}$ (24) | Prime | 2,17,3, $10^{20}$ | 0, 1, 4, $3^2$, 12, $-5$ |
| $\mathbb{Z}$ (25) | Integer | -1, 0, 17, 5j, -$10^{20}$ | 1/2,$\pi$, $\sqrt{5}$ |
| $\mathbb{Q}$ (25) | Rational | 2/1, 3/2, 1.5, 1.14 | $\sqrt{2}$, $3^{-1/3}$, $\pi$ |
| $\mathbb{F}$ (25) | Fractional | 1/2, 7/22 | 2/1, $1/\sqrt{2}$ |
| $\mathbb{I}$ (25) | Irrational | $\sqrt{2}$, $3^{-1/3}$, $\pi$ | Vectors |
| $\mathbb{R}$ (25) | Reals | $\sqrt{2}$, $3^{-1/3}$, $\pi$ | $2\pi j$ |
| $\mathbb{C}$ (109) | Complex | 1, $\sqrt{2}j$, $3^{-j/3}$, $\pi^j$ | Vectors |

Note that $\mathbb{R} : \mathbb{I} \cup \mathbb{Q}$, $\mathbb{I} \perp \mathbb{Q}$, $\mathbb{Q} : \mathbb{Z} \cup \mathbb{F}$.

We say that a number is in the set with the notation $3 \in \mathbb{N} \in \mathbb{R}$, which is read as "3 is in the set of counting numbers, which in turn in the set of real numbers," or in vernacular language "3 is a real counting number."

Each symbol defines an open (e.g., $\infty$) set, meaning that there are an infinite number of elements in the set. Closed sets may be defined for $\mathbb{Z}$ by returning to zero upon reaching some upper integer+1. This is called modular arithmetic. Any periodic function may be indicated using double-parentheses

[1] https://en.wikipedia.org/wiki/List_of_mathematical_symbols_by_subject#Definition_symbols

notation. For example function

$$f((t))_T = f(t) = f(t-T) = f(t+17T)$$

is periodic on $t \in \mathbb{R}$ with a period of $T$. This notation is useful when dealing with Fourier Transforms of periodic functions.

**Differential equations vs. Polynomials** A polynomial has *degree* $N$ defined by the largest power. A quadratic equation is degree 2, and a cubic has degree 3.

Differential equations have *order* rather than degree. A second order differential has degree 2 once it has been Laplace transformed. For example

$$a\frac{d^2}{dt^2}y(t) + b\frac{d}{dt}y(t) + y(t) = x(t) \leftrightarrow as^2Y(s) + bsY(s) + Y(s) = X(s)$$

**Vectors** Vectors are ordered sets of scalars. When we write then out, we use row notation, with the *transpose* symbol

$$[a, b, c]^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

*Vectors* are always columns. Row vectors are weights not vectors. A vector dot product is normally defined between weights and vectors, resulting in a real scalar. This is said to be a 3 *dimensional* vector. for example

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 + 2 + 3 = 6.$$

When the elements are complex, the transpose also takes the complex conjugate.

**Matrices** Unfortunately when working with matrices, the role of the weights and vectors can change, depending on the context. A useful way to view a matrix is as a set of column vectors, weighted by the elements of the column-vector of weights multiplied from the right. For example

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1M} \\ a_{21} & a_{22} & a_{32} & \cdots & a_{3M} \\ & & \ddots & & \\ a_{N1} & a_{N2} & a_{N3} & \cdots & a_{NM} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \cdots \\ w_M \end{bmatrix} = w_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{21} \\ \cdots \\ a_{N1} \end{bmatrix} + w_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \cdots \\ a_{N2} \end{bmatrix} \ldots w_M \begin{bmatrix} a_{1M} \\ a_{2M} \\ a_{3M} \\ \cdots \\ a_{NM} \end{bmatrix},$$

where the weights are $\begin{bmatrix} w_1, w_2, \ldots, w_M. \end{bmatrix}^T$

Another way to view the matrix is a set of row vectors of weights, each of which re applied to the vector $[w_1, w_2, \cdots, W_M]^T$.

Matlab's notational convention for a row-vector is $[a, b, c]$ and a column-vector is $[a; b; c]$. A prime on a vector takes the complex conjugate transpose. To suppress the conjugation, place a period before the prime. The : argument converts the array into a column vector, without conjugation. A tacit notation in Matlab is that *vectors* are columns and the index to a vector is a row vector. Matlab defines the notation 1:4 as the "row-vector" $[1, 2, 3, 4]$, which is unfortunate as it leads users to assume that the default vector is a row. This can lead to serious confusion later, as Matlab's default vector is a column. I have not found the above convention explicitly stated, and it took me years to figure this out for myself.

# Appendix B

# Matrix diagonalization

There is a standard method for finding powers of any matrix, by rotating the matrix to a diagonal form, which transforms the powers of the matrix, to powers of the eigenvalues. This greatly simplifies such an analysis. This Appendix will derive this transformation for the Pell matrix

$$A = \beta_0 \begin{bmatrix} 1 & N \\ 1 & 1 \end{bmatrix},$$

where $\beta_0$ is a constant to be determined. We wish to find powers of matrix $A$. Following the diagonalization of $A$, the powers are simply expressed by powers of the eigenvalues of $A$.

The general solution to Pell's equation may be expressed by an *eigenvalue analysis*, where the two eigenvalues $\lambda_\pm$ and the two eigenvectors $E_\pm$ are defined by

$$AE_\pm = \lambda_\pm E_\pm.$$

Since this matrix, $A - \lambda_\pm I_2 = 0$, which is equal to zero, must be singular. Thus the eigenvalues of $A$ may be found by solving the equation $det(A - \lambda I) = 0$,

$$\begin{aligned} \det \begin{bmatrix} \beta_0 - \lambda_\pm & N\beta_0 \\ \beta_0 & \beta_0 - \lambda_\pm \end{bmatrix} =& (\beta_0 - \lambda_\pm)^2 - N\beta_0^2 = 0 \\ & (\lambda_\pm - \beta_0)^2 = \beta_0^2 N \qquad \text{or} \\ & \lambda_\pm = \beta_0\,(1 \pm \sqrt{N}). \end{aligned}$$

For $N = 2$ $\beta_0 = 1\jmath$ thus the roots are $\lambda_\pm \approx [2.4142\jmath, -0.4142\jmath]$. We have assigned $\lambda_+$ as the larger of the two eigenvalues, by magnitude. For $N = 3, \beta_0 = 1\jmath/\sqrt{2}$ as discussed below.

For our case this results in a quadratic equation with roots in $\lambda_\pm = 1\jmath(1 \pm \sqrt{2})$. The solutions in terms of the eigenvalues is given at the bottom of the figure, in terms of powers of the eigenvalues.

Once the matrix has been diagonalized, one may compute powers of that matrix as powers of the eigenvalues. This results in the general solution given by

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = 1\jmath^n \begin{bmatrix} 1 & N \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The eigenvalue matrix $D$ is diagonal with the eigenvalues sorted, largest first. The Matlab command `[E,D]=eig(A)` is helpful to find $D$ and $E$ given any $A$. As we saw above,

$$D = 1\jmath \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix} \approx \begin{bmatrix} 2.414\jmath & 0 \\ 0 & -0.414\jmath \end{bmatrix}.$$

The matrix of *eigenvectors* $E$ must satisfy Eq. B, which has no solution because its determinate is zero. This should raise a red flag for any attempt of finding $E$. At least one of the eigenvectors will be under-determined, requiring the introduction of a free parameter to absorb that degree of freedom. Then the parameter is resolved by normalizing the eigenvectors to have unit length. To fully understand this you need to work out a 2x2 example, in full detail.

The matrix equation that does the diagonalization starts from $y = Ax$,

$$D = E^{-1}AE$$

or

$$A = EDE^{-1} = E\begin{bmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{bmatrix}E^{-1}.$$

where $y, x$ are 2x1 vectors and $A, E$ are 2x2 matrices. The job of diagonalization finds $E$ and $D$ given $A$ according to above eigenvalue specification. In a bit more detail, starting from $y = Ax$ we multiply by $E^{-1}$ to obtain $E^{-1}y = E^{-1}Ax$. Since $I = EE^{-1}$, this may be written as $E^{-1}y = (E^{-1}AE)E{-}1x$. The matrix E is found that $D = E^{-1}AE$ is diagonal, which is the same as $AE = ED$. Given $D$ by solving the quadratic equation (above) we then find $E$ such that $D = E^{-1}AE$. too complicated. Clean this up.

## Expression for $A^n$ by *Diagonalization* 9.4.4

- Matrix $A = \begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}$ may be written in *diagonal form* as follows:
  - $E$ represents the eigenvectors (Matlab command `[E,D]=eig(A)`)

    $$E = \frac{1}{\sqrt{3}}\begin{bmatrix}\sqrt{2} & -\sqrt{2}\\ 1 & 1\end{bmatrix} = \begin{bmatrix}0.8165 & 0.8165\\ 0.5774 & -0.5774.\end{bmatrix}$$

    The factor $\sqrt{3}$ normalizes each eigenvector to 1.
  - The inverse eigenvector matrix is

    $$E^{-1} = \frac{\sqrt{3}}{2\sqrt{2}}\begin{bmatrix}1 & \sqrt{2}\\ 1 & -\sqrt{2}\end{bmatrix} = \begin{bmatrix}0.6124 & 0.866\\ 0.6124 & -0.866\end{bmatrix}$$

  - The eigenvector matrix diagonalizes $A$ as follows

    $$(E^{-1}AE) = \Lambda \equiv \begin{bmatrix}\lambda_+ & 0\\ 0 & \lambda_-.\end{bmatrix}$$

- Next take the $n^{th}$ power, use $(E^{-1}AE)^n = E^{-1}A^nE$, and solving for $A^n$, gives

  $$A^n = \left(E\Lambda^n E^{-1}\right) = E\begin{bmatrix}\lambda_+^n & 0\\ 0 & \lambda_-^n\end{bmatrix}E^{-1},$$

  which is the desired expression for $A^n$. Note that $\lambda_+^n \to \infty$ and $\lambda_-^n \to 0$, with sign $(-1)^n$.

Figure B.1: Diagonalization of $A^n$.

## B.1 Case of N=3

### Pell's Equation $N = 3$

- Case of $N = 3$ & $[x_0, y_0]^T = [1, 0]^T$ , $\beta_0 = \jmath/\sqrt{2}$
  Note: $x_n^2 - 3y_n^2 = 1, \qquad x_n/y_n \underset{\infty}{\longrightarrow} \sqrt{3}$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \beta_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \beta_0 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (\beta_0)^2 - 3(\beta_0)^2 = 1$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \beta_0^2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \beta_0^2 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \left(4\beta_0^2\right)^2 - 3\left(2\beta_0^2\right)^2 = 1$$

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \beta_0^3 \begin{bmatrix} 10 \\ 6 \end{bmatrix} = \beta_0^3 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \qquad \left(10\beta_0^3\right)^2 - 3\left(6\beta_0^3\right)^2 = 1$$

$$\begin{bmatrix} x_4 \\ y_4 \end{bmatrix} = \beta_0^4 \begin{bmatrix} 28 \\ 16 \end{bmatrix} = \beta_0^4 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 6 \end{bmatrix} \qquad \left(28\beta_0^4\right)^2 - 3\left(16\beta_0^4\right)^2 = 1$$

$$\begin{bmatrix} x_5 \\ y_5 \end{bmatrix} = \beta_0^5 \begin{bmatrix} 76 \\ 44 \end{bmatrix} = \beta_0^5 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 28 \\ 16 \end{bmatrix} \qquad \left(76\beta_0^4\right)^2 - 3\left(44\beta_0^4\right)^2 = 1$$

Figure B.2: *This summarizes the solution of Pell's equation due to the Pythagoreans using matrix recursion, for the case of N=3. Of special interest is that $x_n/y_n \rightarrow \sqrt{3}$. Note that the solutions are not integers unless $n$ is even because beta has the factor $\sqrt{2}$, which would mean that the odd solutions would not be integral.*

The eigenvalues are given by $(1 - \lambda_\pm)^2 = N$ or $\lambda_\pm = 1 \mp \sqrt{N}$..

#### Eigenvector analysis

Out of place: Move to appendix If we start from an $N \times N$ system of linear equations, each row defines a $N$ dimensional hyper-plane (i.e, the extension of a plane in 3 dimensions, is called a N dimensional hyper-plane). If we form the characteristic polynomial by computing $det(A - \lambda I) = 0$, then the roots of this polynomial give $N$ independent solutions of matrix $A$. Each root has a corresponding *eigenvector* $[e_1, \ldots, e_N]^T$. In fact we may define an $E$ matrix by collecting all the $N$ eigenvectors corresponding to the $N$ eigenvalues found by factoring the characteristic polynomials. This is the same procedure that we used in Chapter 1 to find the solutions to the Fibonacci equations. Thus the matrix may be written in terms of $E$ and $\Lambda$ as

$$A = E \Lambda E^{-1},$$

as discussed in Appendix B. The Matlab command `[E,D]=eig(A)` will do this calculation for you so you can easily try this, for any matrix $A$.

# Appendix C

# Gaussian Elimination

We shall now apply Gaussian elimination to find the solution $[x_1, x_2]$ for the 2x2 matrix equation $Ax = y$ (Eq. 3.7, left). We assume to know $[a, b, c, d]$ and $[y_1, y_2]$. We wish to show that the intersection (solution) is given by the equation on the right.

Here we wish to prove that the left equation has an inverse given by the right equation:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad\qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Step 1: normalize the first column to 1. Step 2: subtract the top equation from the lower. Step 3: express result in terms of the determinate $\Delta = ad - bc$.

$$\begin{bmatrix} 1 & \frac{b}{a} \\ 1 & \frac{d}{c} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{a} y_1 \\ \frac{1}{c} y_2 \end{bmatrix} \qquad \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{d}{c} - \frac{b}{a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ -\frac{1}{a} & \frac{1}{c} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Step 4: These steps give the solution for $x_2 = -\frac{c}{\Delta} y_1 + \frac{a}{\Delta} y_2$. Step 5: Solve for $x_2$. Next the top equation may be solved for $x_1$. Still errors

$x_1 = \frac{1}{a} y_1 - \frac{b}{a} x_2 = x_1 = \frac{1}{a} y_1 - \frac{b}{a}[-\frac{c}{\Delta} y_1 + \frac{a}{\Delta} y_2]$. In matrix format, in terms of $\Delta = ab - cd$, the determinate of the matrix

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{a} - \frac{bc}{a\Delta} & \frac{b}{\Delta} \\ -\frac{c}{\Delta} & \frac{a}{\Delta} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \frac{\Delta - bc}{a} & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

This is a lot of algebra, that is why it is essential you learn that the last relationship is the inverse of the initial matrix. There is not enough time in life to re-derive this every time you need it. Here is the solution: *1) Swap the diagonal, 2) change the signs of the off-diagonal, and 3) divide by* $\Delta$.

### Residue expansions and the ROC

With the new tool of analytic functions came the concept of the *region of convergence* (ROC) that defines the regions in the complex plane where the infinite series is valid. In other words, the function $Z(s)$ and its analytic power series $\sum_0^\infty c_n s^n$, are equivalent over a region of $s$ that lies within the ROC. When the series fails to converge, it no longer represents $Z(s)$. A helpful example is the series

$$\frac{1}{1+x^2} = \frac{1}{(1+\jmath x)(1-\jmath x)} = \frac{1}{2(1+\jmath x)} + \frac{1}{2(1-\jmath x)} = \frac{1}{2}\sum_{n=0}^{\infty}(-\jmath x)^n + \frac{1}{2}\sum_{n=0}^{\infty}(+\jmath x)^n,$$

which is valid for $|x| < 1$. At face value this function seems fine at $x = 1$, where it is equal to $1/2$. In fact the series fails to converge at precisely this value (the ROC is 1 for this example). Until one views $x$ as complex, this behavior is not obvious.

# Index

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