

An Invitation to Mathematical Physics  
*and its History*

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## Abstract

An understanding of physics requires knowledge of mathematics. The contrary is not true. By definition, pure mathematics contains no physics. Yet historically, mathematics has a rich history filled with physical applications. Mathematics was developed by people with intent of making things work. In my view, as an engineer, I see these creators of early mathematics, as budding engineers. This book is an attempt to tell this story, of the development of mathematical physics, as viewed by an engineer.

The book is broken down into three topics, called streams, presented as five chapters: 1) Introduction, 2) Number systems, 3) Algebra Equations, 4) Scalar Calculus, and 5) Vector Calculus. The material is delivered as 40 “Lectures” spread out over a semester of 15 weeks, three lectures per week, with a 3 lecture time-out for administrative duties. Problems are provided for each week’s assignment. These problems are written out in L<sup>A</sup>T<sub>E</sub>X, with built in solutions, that may be expressed by un-commenting one line. Once the home-works are turned in, each student is given the solution. With regard to learning the material, the students rated these Assignments as the most important part of the course. There is a built in interplay between these assignments and the lectures. On many occasions I solved the homework in class, as motivation for coming to class. Four exams were given, one at the end of each of the three sections, and a final. Some of the exams were in class and some were evening exams, that ran over two hours. The final was two hours. Each of the exams, like the assignments, is provided as a L<sup>A</sup>T<sub>E</sub>X file, with solutions encoded with a one line software switch. The Exams are largely based on the Assignments. It is my philosophy that, in principle, the students could see the exam in advance of taking it.

## Author’s Personal Statement

An expert is someone who has made all the mistakes in a small field. I don’t know if I would be called an expert, but I certainly have made my share of mistakes. I openly state that “I love making mistakes, because I learn so much from them.” One might call that the “expert’s corollary.”

This book has been written out of both my love for the topic of mathematical physics, and a frustration for wanting to share many key concepts, and even new ideas on these basic concepts. Over the years I have developed a certain *physical sense* of math, along with a related mathematical sense of physics. While doing my research,<sup>1</sup> I have come across what I feel are certain conceptual holes that need filling, and sense many deep relationships between math and physics, that remain unidentified. While what we presently teach is not wrong, it is missing these relationships. What is lacking is an intuition for how math “works.” We need to start listening to the language of mathematics. We need to let mathematics guide us toward our engineering goals.

It is my strong suspicion that over the centuries many others have had similar insights, and like me, have been unable to convey this slight misdirection. I hope these views can be useful to open young minds.

This marriage of math and physics will help us make progress in understanding the physical world. I turn to mathematics and physics when trying to understand the universe. I have arrived in my views following a lifelong attempt to understand human communication. This research arose from my 32 years at Bell Labs in the Acoustics Research Department. There such lifelong pursuits were not only possible, they were openly encouraged. The idea was that if you were successful at something, take it

<sup>1</sup><http://auditorymodels.org/index.php/Main/Publications>

as far as you can. But on the other side, don't do something well that's not worth doing. People got fired for the latter. I should have left for University after a mere 20 years,<sup>2</sup> but the job was just too cushy.

In this text it is my goal to clarify some of the conceptual errors when telling the story about physics and mathematics. My views have been often inspired by classic works, as documented in the bibliography. This book was inspired by my careful reading of Stillwell (2002), through Chapter 21 (Fig. 2). Somewhere in Chapter 22 I stopped reading and switched to the third edition (Stillwell, 2010), where I saw there was much more to master. At that point I saw that teaching this material to sophomores would allow me to absorb the more advanced material at a reasonable pace, which led to this book.

## Back Cover Summary

This is foremost a math book, but not the typical math book. First, this book is for the engineering minded, for those who need to understand math to do engineering, to learn how things work. In that sense it is more about physics and engineering. Math skill are critical to making progress in building things, be it pyramids or computers, as clearly shown by the many great civilizations of the Chinese, Egyptians, Arabs (people of Mesopotamia), Greeks and Romans.

Second, this is a book about the math that developed to explain physics, to allow people to engineer complex things. To sail around the world one needs to know how to navigate. This requires a model of the planets and stars. You can only know where you are on earth if you understand where earth is, relative to the heavens. The answer to such a deep questions will depend on who you ask. The utility and accuracy of that answer depends critically on the depth of understanding of how the worlds and heavens work. Who is qualified to answer such question? It is best answered by those who study mathematics applied to the physical world.

Halley (1656–1742), the English astronomer, asked Newton (1643–1727) for the equation that describes the orbit of the planets. Halley was obviously interested in comets. Newton immediately answered “an ellipse.” It is said that Halley was stunned by the response (Stillwell, 2010, p. 176), as this was what had been experimentally observed by Kepler (c1619), and thus he knew Newton must have some deeper insight (Stillwell, 2010, p. 176).

When Halley asked Newton to explain how he knew this correct answer, Newton said he calculated it. But when challenged to show the calculation, Newton was unable to reproduce it. This open challenge eventually led to Newton's grand treatise, *Philosophiæ Naturalis Principia Mathematica* (July 5, 1687). It had a humble beginning, more as a letter to Halley, explaining how to calculate the orbits of the planets. To do this Newton needed mathematics, a tool he had mastered. It is widely accepted that Isaac Newton and Gottfried Leibniz invented calculus. But the early record shows that perhaps Bhāskara II (1114–1185 AD) had mastered this art well before Newton.<sup>3</sup>

Third, the main goal of this book is to teach engineering mathematics, in a way that it can be understood, remembered, and mastered, by anyone motivated to learn this topic. How can this near impossible goal be achieved? The answer is to fill in the gaps with “who did what, and when.” There is an historical story that may be told and mastered, by anyone serious about the science of making things.

One cannot be an expert in a field if they do not know the history of that field. This includes who the people were, what they did, and the credibility of their story. Do you believe the Pope or Galileo, on the topic of the relative position of the sun and the earth? The observables provided by science are clearly on Galileo's side. Who were those first engineers? They are names we all know: Archimedes, Pythagoras, Leonardo da Vinci, Galileo, Newton, . . . . All of these individuals had mastered mathematics. This book teaches the tools taught to every engineer. Do not memorize

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<sup>2</sup>I should have left when AT&T Labs was formed, c1997. I started around December 1970, fresh out of Graduate school, and retired in December 2002.

<sup>3</sup>[http://www-history.mcs.st-and.ac.uk/Projects/Pearce/Chapters/Ch8\\_5.html](http://www-history.mcs.st-and.ac.uk/Projects/Pearce/Chapters/Ch8_5.html)

complex formulas, rather make the equations “obvious” by teaching the simplicity of the underlying concept.

## Credits

Besides thanking my parents, I would like to credit John Stillwell for his constructive, historical summary of mathematics. My close friend and colleague Steve Levinson somehow drew me into this project, without my even knowing it. My brilliant graduate student Sarah Robinson was constantly at my side, grading home-works and exams, and tutoring the students. Without her, I would not have survived the first semester the material was taught. Her proof-reading skills are amazing. Thank you Sarah for your infinite help. Finally I would like to thank John D’Angelo for putting up with my many silly questions. When it comes to the heavy hitting, John was always there to provide a brilliant explanation that I could easily translate into Engineer’ese (Matheering?) (i.e., Engineer language).

To write this book I had to master the language of mathematics (John’s language). I had already mastered the language of engineering, and a good part of physics.<sup>4</sup> But we are all talking about the same thing. Via the physics and engineering, I already had a decent understanding of the mathematics, but I did not know that language. Hopefully, now I can get by.

Finally I would like to thank my wife (Sheau Feng Jeng aka Patricia Allen) for her unbelievable support and love. She delivered constant piece of mind, without which this project could never have been started, much less finish.

There are many others who played important roles, but they must remain anonymous, out of my fear of offending someone I forgot to mention.

–Jont Allen, Mahomet IL, Dec. 24, 2015

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<sup>4</sup>Each genre (i.e, group ) speaks their own dialect. One of my secondary goals is to bring down this scientific Tower of Bable.



# Preface

It is widely acknowledged that interdisciplinary science is the backbone of modern scientific investigation. This is embodied in the STEM (Science, Technology, Engineering, and Mathematics) programs. Contemporary research is about connecting different areas of knowledge, thus it requires an understanding of cross-disciplines. However, while STEM is being taught, interdisciplinary science is not, due to its inherent complexity and breadth. Furthermore there are few people to teach it. Mathematics, Engineering and Physics (MEP) are at the core of such studies.<sup>5</sup>

## STEM vs. MEP

Mathematics is based on the application rigor. Mathematicians specifically attend to the definitions of increasingly general concepts. Thus mathematics advances slowly, as these complex definitions must be collectively agreed upon. Mathematics shuns controversy, and embraces rigor, the opposite of uncertainty. Physics explores the fringes of uncertainty. Physicists love controversy. Engineering addresses the advancement the technology. Engineers, much like mathematicians, are uncomfortable with uncertainty, but are trained to deal with it.

To create such an interdisciplinary STEM program, a unified MEP curriculum is needed. In my view this unification could (should) take place based on a core mathematical training, from a historical perspective, starting with Euclid or before (i.e., Chinese mathematics), up to modern information theory and logic. As a bare minimum, the *fundamental theorems of mathematics* (arithmetic, algebra, calculus, vector calculus, etc.) need to be appreciated by every MEP student. The core of this curriculum is outlined in Table 1.1 (p. 21).

If, in the sophomore semester, students are taught a common MEP methodology and vocabulary, presented in terms of the history of mathematics, they will be equipped to

1. Exercise interdisciplinary science (STEM)
2. Communicate with other MEP trained (STEM) students and professors.

The goal is a *comprehensive understanding of the fundamental concepts of mathematics*, defined as those required for engineering. We assume that students with this deep understanding will end up being in the top 0.1% of Engineering. Time will tell if this assumption is correct.

The key tool is methodology. The traditional approach is a five to six course sequence: Calc I, II, III, DiffEq IV, Linear Algebra V and Complex Variables VI, over a time frame of three years (six semesters). This was the way I learned math. Following such a formal training regime, I felt I had not fully mastered the material, so I started over. I now consider myself to be self-taught. We need a more effective teaching method. I am not suggesting we replace the standard 6 semester math curriculum, rather I am suggesting replacing Calc I, II with this mathematical physics course, based on the historical thread, for those students who have demonstrated advanced ability. One needs more than a high school education to succeed in college engineering courses.

By teaching mathematics in the context of history, the student can fully appreciate the underlying principles. Including the mathematical history provides a uniform terminology for understanding the

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<sup>5</sup>I prefer MEP over STEM, as being better focused on the people that do the work, organized around their scientific point of view.

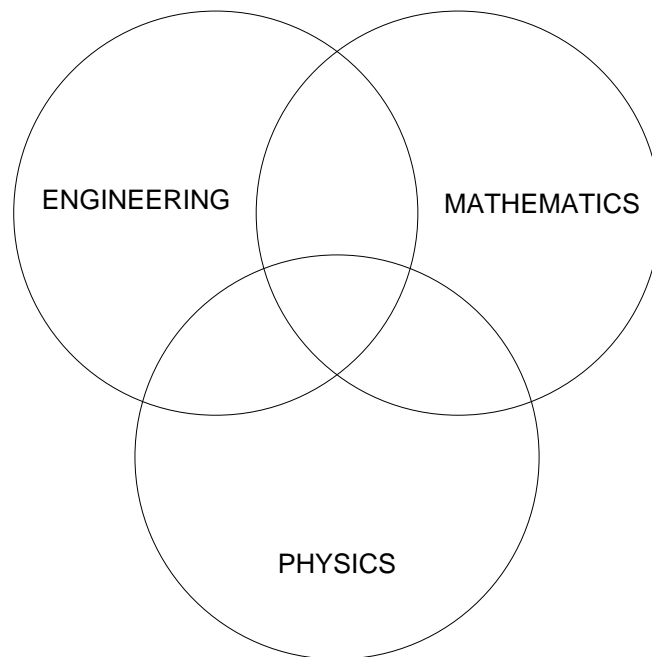


Figure 1: *There is a natural symbiotic relationship between Physics, Mathematics and Engineering, as depicted by this Venn diagram. Physics explores the boundaries. Mathematics provides the method and rigor. engineering transforms the method into technology. While these three disciplines work well together, there is poor communication due to a different vocabulary.*

fundamentals of mathematics. The present teaching method, using abstract proofs, with no (or few) figures or physical principles, by design removes intuition and the motivation that was available to the creators of these early theories. This present six semester approach does not function for many students, leaving them with a poor intuition.

## Mathematics and its History (Stillwell, 2002)

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Figure 2: Table of contents of Stillwell (2002)



# 1 Chapter 1

## 2 Introduction

3 Much of early mathematics centered around the love of art and music, due to our sensations of light  
4 and sound. Exploring our physiological senses required a scientific understanding of vision and hearing,  
5 as first explored by Newton (1687) and Helmholtz (1863a) (Stillwell, 2010, p. 261).<sup>1</sup> Our sense of color  
6 and musical pitch are determined by the frequencies of light and sound. The Chinese and Pythagoreans  
7 are well known for their early contributions to music theory. Pythagoras strongly believed that “all is  
8 integer,” meaning that every number, and every mathematical and physical concept, could be explained  
9 by integral relationships. It is likely that this belief was based on Chinese mathematics from thousands  
10 of years earlier. It is also known that his ideas about the importance of integers were based on what  
11 was known about music theory in those days. For example it was known that the relationships between  
12 the musical notes (pitches) obey natural integral relationships.

13 Other important modern applications of number theory are present with

- 14 • Public-private key encryption: which requires the computationally intensive factoring of large  
15 integers
- 16 • IEEE Floating point<sup>2</sup>

17 As acknowledged by Stillwell (2010, p. 16), the Pythagorean view is relevant today

18 *With the digital computer, digital audio, and digital video coding everything, at least*  
19 *approximately into sequences of whole numbers, we are closer than ever to a world in which*  
20 *“all is number.”*

21 Mersenne (1588-1647) contributed to our understanding of the relationship between the wavelength  
22 and the length of musical instruments. These results were extended by Galileo’s father, and then by  
23 Galileo himself (1564-1642). Many of these musical contributions resulted in new mathematics, such as  
24 the discovery of the wave equation by Newton (c1687), followed by its one-dimensional general solution  
25 by d’Alembert (c1747).

26 By that time there was a basic understanding that sound and light traveled at very different speeds  
27 (thus why not the velocities of different falling weights?).

28 Ole Rømer first demonstrated in 1676 that light travels at a finite speed (as opposed to  
29 instantaneously) by studying the apparent motion of Jupiter’s moon Io. In 1865, James  
30 Clerk Maxwell proposed that light was an electromagnetic wave, and therefore traveled at  
31 the speed  $c$  appearing in his theory of electromagnetism.<sup>3</sup>

32 Galileo famously conceptualized an experiment in 1589 where he suggested dropping two different  
33 weights from the Leaning Tower of Pisa, and showed that they must take the same time to hit the

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<sup>1</sup><https://en.wikipedia.org/wiki/Acoustics>

<sup>2</sup>[https://en.wikipedia.org/wiki/IEEE\\_floating\\_point\#Formats](https://en.wikipedia.org/wiki/IEEE_floating_point\#Formats)

<sup>3</sup>[https://en.wikipedia.org/wiki/Speed\\_of\\_light](https://en.wikipedia.org/wiki/Speed_of_light)

34 ground. Conceptually this is an important experiment, driven by a mathematical argument in which  
 35 he considered the two weights to be connected by an elastic cord. This resulted in the concept of  
 36 conservation of energy, one of the cornerstones of modern physical theory.

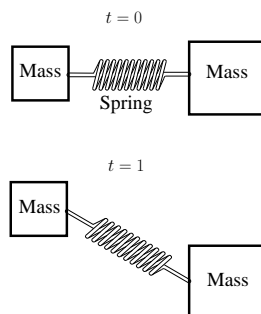


Figure 1.1: Depiction of the argument of Galileo (unpublished book of 1638) as to why weights of different masses (weight) must fall with identical velocity. By joining them with an elastic cord they become one. Thus if the velocity were proportional to the mass, the joined masses would fall even faster. This results in a logical fallacy. This may have been the first time that the principle of conservation of energy was clearly stated.

37 While Newton may be best known for his studies on light, he was the first to predict the speed of  
 38 sound. However his theory was in error by<sup>4</sup>  $\sqrt{c_p/c_v} = \sqrt{1.4} = 1.183$ . This famous error would not be  
 39 resolved for over two hundred years, awaiting the formulation of thermodynamics by Laplace, Maxwell  
 40 and Boltzmann, and others. What was needed was the concept of constant-heat, or *adiabatic process*.  
 41 For audio frequencies (0.02-20 [kHz]), the small temperature gradients cannot diffuse the distance of a  
 42 wavelength in one cycle (Pierce, 1981; Boyer and Merzbach, 2011), “trapping” the heat energy in the  
 43 wave. There were several other physical enigmas, such as the observation that sound disappears in a  
 44 vacuum and that a vacuum cannot draw water up a column by more than 34 feet.

45 There are other outstanding examples where physiology impacted mathematics. Leonardo da Vinci  
 46 is well known for his studies of the human body. Helmholtz’s theories of music and the percep-  
 47 tion of sound are excellent examples of under-appreciated fundamental mathematical contributions  
 48 (Helmholtz, 1863a). Lord Kelvin (aka William Thompson),<sup>5</sup> was one of the first true engineer-scientists,  
 49 equally acknowledged as a mathematical physicist and well known for his interdisciplinary research,  
 50 knighted by Queen Victoria in 1866. Lord Kelvin coined the term *thermodynamics*, a science more  
 51 fully developed by Maxwell (the same Maxwell of electrodynamics). Thus the interdisciplinary nature  
 52 of science has played many key roles in the development of thermodynamics.<sup>6</sup> Lord Rayleigh’s book on  
 53 the theory of sound (Rayleigh, 1896) is a classic text, read even today by anyone who studies acoustics.

54  
 55 It seems that we have detracted from this venerable interdisciplinary view of science by splitting the  
 56 disciplines into into smaller parts whenever we perceived a natural educational boundary. Reforging  
 57 these natural connections at some point in the curriculum is essential for the proper training of students,  
 58 both scientists and engineers.<sup>7</sup>

## 59 WEEK 1

60

<sup>4</sup>The square root of the ratio of the specific heat capacity at constant pressure to that at constant volume

<sup>5</sup>Lord Kelvin was one of half a dozen interdisciplinary mathematical physicists, all working about the same time, that made a fundamental change in our scientific understanding. Others include Helmholtz, Stokes, Green, Heaviside, Rayleigh and Maxwell.

<sup>6</sup>Thermodynamics is another example of a course that needs reworking along historical lines.

<sup>7</sup>Perhaps its time to put the MEP Humpty Dumpty back together.

## 61 1.1 Early Science and Mathematics

62 The first 5,000 years is not well document, but the basic record is clear, as outlined in Fig. 1.2. Thanks  
 63 to Euclid and later Diophantus (c250 CE), we have some limited understanding of what they studied.  
 64 For example, Euclid's formula (Fig. 2.3, Eq. 2.3) provides a method for computing all Pythagorean  
 65 triplets (Stillwell, 2010, pp. 4-9).

66 Chinese Bells and stringed musical instruments were exquisitely developed in their tonal quality, as  
 67 documented by ancient physical artifacts (Fletcher and Rossing, 2008). In fact this development was  
 68 so rich that one must question why the Chinese failed to initiate the industrial revolution. Specifically,  
 69 why did Europe eventually dominate with its innovation when it was the Chinese who did the extensive  
 70 early invention?

71 According to Lin (1995) this is known as the *Needham question*:

72 “Why did modern science, the mathematization of hypotheses about Nature, with all its  
 73 implications for advanced technology, take its meteoric rise only in the West at the time of  
 74 Galileo[, but] had not developed in Chinese civilization or Indian civilization?”

75 Needham cites the many developments in China:<sup>8</sup>

76 “Gunpowder, the magnetic compass, and paper and printing, which Francis Bacon consid-  
 77 ered as the three most important inventions facilitating the West's transformation from the  
 78 Dark Ages to the modern world, were invented in China.” (Lin, 1995)

79 “Needham's works attribute significant weight to the impact of Confucianism and Taoism on  
 80 the pace of Chinese scientific discovery, and emphasizes what it describes as the ‘diffusionist’  
 81 approach of Chinese science as opposed to a perceived independent inventiveness in the  
 82 western world. Needham held that the notion that the Chinese script had inhibited scientific  
 83 thought was ‘grossly overrated’ ” (Grosswiler, 2004).

84 Lin was focused on military applications, missing the importance of non-military applications. A  
 85 large fraction of mathematics was developed to better understand the solar system, acoustics, musical  
 86 instruments and the theory of sound and light. Eventually the universe became a popular topic, and  
 87 still is today.

### 88 1.1.1 Lec 1 The Pythagorean theorem

While early Asian mathematics is not fully documented, it clearly defined the course for at least several  
 thousand years. The first recorded mathematics was that of the Chinese (5000-1200 BCE) and the  
 Egyptians (3,300 BEC). Some of the best early record were left by the people of Mesopotamia (Iraq,  
 1800 BEC). Thanks to Euclid's Elements (c323 BEC) we have an historical record, tracing the progress  
 in geometry, as defined by the Pythagorean theorem *for any right triangle*

$$c^2 = a^2 + b^2, \tag{1.1}$$

89 having sides of lengths  $(a, b, c)$  that are positive real numbers with  $c > [a, b]$  and  $a + b > c$ . Solutions  
 90 were likely found by trial and error rather than by an algorithm.

If  $a, b, c$  are lengths, then  $a^2, b^2, c^2$  are areas. Equation 1.1 says that the area  $a^2$  of a square plus  
 the area  $b^2$  of a square equals the area  $c^2$  of square. Today a simple way to prove this is to compute  
 the magnitude of the complex number  $c = a + bj$ , which forces the right angle

$$|c|^2 = (a + bj)(a - bj) = a^2 + b^2.$$

<sup>8</sup>[https://en.wikipedia.org/wiki/Joseph\\_Needham#cite\\_note-11](https://en.wikipedia.org/wiki/Joseph_Needham#cite_note-11)

91 However, complex arithmetic was not an option for the Greek mathematicians, since complex numbers  
92 and algebra had yet to be invented.

93 **Almost 700 years after** Euclid's *Elements*, the Library of Alexandria was destroyed (391 EC) by  
94 fire, taking with it much of the accumulated Greek knowledge. Thus one of the best technical records  
95 may be Euclid's *Elements*, along with some sparse mathematics due to Archimedes (c300 BEC) on  
96 geometrical series, computing the volume of a sphere, and the area of the parabola, and elementary  
97 hydrostatics. Additionally, a copy of a book by Diophantus *Arithmetic* was discovered by Bombelli  
98 (c1572) in the Vatican library (Stillwell, 2010, p. 51).

## Chronological history pre 16<sup>th</sup> century

1.1.2a

200<sup>th</sup> BCE Chinese (Primes; quadratic equation; Euclidean algorithm (GCD))

180<sup>th</sup> BCE Babylonia (Mesopotamia/Iraq) (quadratic equation)

6<sup>th</sup> BCE Pythagoras (Thales) and the Pythagorean "tribe"

4<sup>th</sup> BCE Archimedes 300BCE; Euclid (quadratic equation)

3<sup>rd</sup> CE Diophantus c250CE;

4<sup>th</sup> CE Alexandria Library destroyed 391CE;

7<sup>th</sup> CE Brahmagupta (negative numbers; quadratic equation)

9<sup>th</sup> CE al-Khwārizmī (algebra) 830CE

10<sup>th</sup> CE Bhaskara (calculus) 1114-1183

15<sup>th</sup> Leonardo & Copernicus 1473-1543

16<sup>th</sup> Tartaglia (cubic eqs); Bombelli 1526-1572; Galileo Galilei 1564-1642

## Time Line

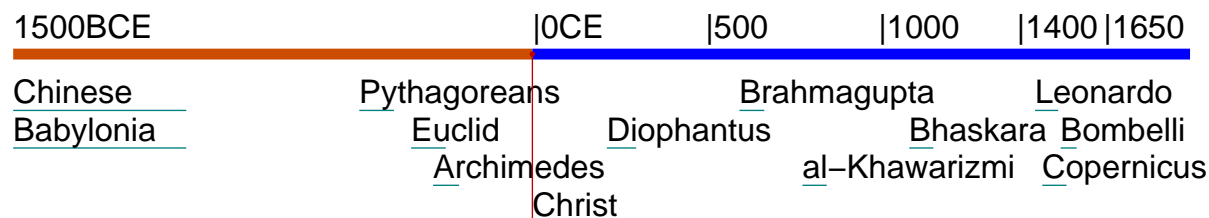


Figure 1.2: Mathematical time-line between 1500 BCE and 1650 CE.

### 99 1.1.2 Pythagorean Triplets

100 Well before Pythagoras, the Babylonians had tables of *Pythagorean triplets* (PTs), integers  $[a, b, c]$   
101 that obey Eq. 1.1. An example is  $[3, 4, 5]$ . A stone tablet (Plimpton-322) dating back to 1800 [BCE]  
102 (Fig. 1.9) was found with integers for  $[a, c]$ . Given such sets of two numbers, which determined a third  
103 positive integer  $b$  such that  $b = \sqrt{c^2 - a^2}$ , this table is more than convincing that the Babylonians were  
104 well aware of PTs, but less convincing that they had access to Euclid's formula (Eq. 1.3).

105 It seems likely that Euclid's *Elements* was largely the source of the fruitful 6th century era due to  
106 the Greek Mathematician Diophantus (Fig. 1.2), who developed the concept of *discrete mathematics*,  
107 now known as *Diophantine analysis*.

108 The work of Diophantus was followed by a rich mathematical era, with the discovery of 1) early cal-  
109 culus (Brahmagupta, 7<sup>th</sup> CE), 2) algebra (al-Khwārizmī, 9<sup>th</sup> CE), and 3) complex arithmetic (Bombelli,  
110 15<sup>th</sup> CE). This period overlapped with the European middle (i.e., dark) ages. Presumably European in-  
111 tellectuals did not stop thinking during these many centuries, but what happened in Europe is presently  
112 unclear given the available records.<sup>9</sup>

<sup>9</sup>It might be interesting to search the archives of the monasteries, where the records were kept, to figure out what



### 1.1.3 What is mathematics?

Mathematics is a language, not so different from other languages. Today's mathematics is a written language with an emphasis on symbols and glyphs, biased toward Greek letters. The etymology of these symbols would be interesting to study. Each symbol is dynamically assigned a meaning, appropriate for the problem being described. These symbols are then assembled to make sentences. It is similar to Chinese in that the spoken and written version are different across dialects. In fact, like Chinese, the sentences may be read out loud in the language (dialect) of your choice, while the mathematical sentence (like Chinese) is universal.

**Math is a language:** It seems strange when people complain that they “can't learn math,”<sup>10</sup> but they claim to be good at languages. Math is a language, with the symbols taken from various languages, with a bias toward Greek, due to the popularity of Euclid's Elements. Learning a new language is fun because it opens doors to other cultures.

Math is different due to the rigor of the rules of the language, along with the way it is taught (e.g., not as a language). A third difference between math and the romance languages is that math evolved from physics, with important technical applications. This was the concept behind the Pythagorean school, a band of followers called the *Pythagoreans*. Learning languages is an advanced social skill. Thus the social outcomes are very different between learning a romance language and math. A further problem is that pre-high-school, students confuse arithmetic with math. The two topics are very different, and students need to understand this. One does not need to be good at arithmetic to be good at math (but it doesn't hurt).

There are many rules that must be mastered. These rules are defined by algebra. For example the sentence  $a = b$  means that the number  $a$  has the same value as the number  $b$ . The sentence is spoken as “a equals b.” The numbers are nouns and the equal sign says they are equivalent, playing the role of a verb, or action symbol. Following the rules of algebra, this sentence may be rewritten as  $a - b = 0$ . Here the symbols for minus and equal indicate two types of actions.

Sentences can become arbitrarily complex, such as the definition of the integral of a function, or a differential equation. But in each case, the mathematical sentence is written down, may be read out loud, has a well defined meaning, and may be manipulated into equivalent forms following the rules of algebra and calculus. This language of mathematics is powerful, with deep consequences, known as proofs.

The writer of an equation should always translate (explicitly summarize the meaning of the expression), so the reader will not miss the main point. This is simply a matter of clear writing.

Language may be thought of as mathematics (turning this idea on its head). To properly write correct English it is necessary to understand the construction of the sentence. It is important to identify the subject, verb, object, and various types of modifying phrases. If you wish to read about this, look up the distinction between the words *that* and *which*, which make a nice example of this concept. Most of us work directly with what we think “sounds right,” but if you're learning English as a second language, it is very helpful to understand these mathematical rules, which are arguably easier to master than the foreign phones (i.e., speech sounds).

### 1.1.4 Early Physics as Mathematics

Mathematics has many functions, but basically it summarizes an algorithm (a set of rules). It was clear to Pythagoras (and many others before him), that there was an important relationship between mathematics and the physical world. Pythagoras may have been one of the first to capitalize on this relationship, using science and mathematics to design and make things.<sup>11</sup> This was the beginnings of technology as we know it, coming from the relationship between physics and math, impacting

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happened during this strange time.

<sup>10</sup>“It looks like Greek to me.”

<sup>11</sup>It is likely that the Chinese and Egyptians also did this, but this is more difficult to document.

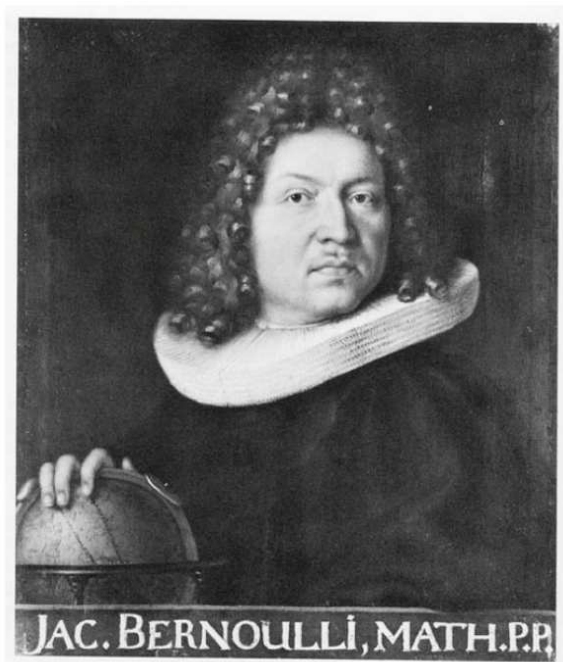


Figure 13.10: Portrait of Jakob Bernoulli by Nicholas Bernoulli



Figure 13.11: Johann Bernoulli



Figure 10.4: Leonhard Euler



Figure 1.3: Above: Jakob (1655-1705) and Johann (1667-1748) Bernoulli; Below: Leonhard Euler (1707) and Jean le Rond d'Alembert (1717-1783). The figure numbers are from Stillwell (2010).

158 map making, tools, implements of war (the wheel, gunpowder), art (music), sound, water transport,  
 159 sanitation, secure communication, food, . . . , etc.

160 Why is Eq. 1.1 called a *theorem*, and what exactly needs to be proved? We do not need to prove  
 161 that  $(a, b, c)$  obey this relationship, since this is a condition that is observed. We do not need to prove  
 162 that  $a^2$  is the area of a square, as this is the definition of the area of a square. What needs to be proved  
 163 is that this relation only holds if the angle between the two shorter sides is  $90^\circ$ .

164 To appreciate the significance of this development it is helpful to trace the record back to before  
 165 the time of the Greeks. The Pythagorean theorem (Eq. 1.1) did not begin with Euclid or Pythagoras.  
 166 Rather Euclid and Pythagoras appreciated the importance of these ideas and documented them.

167 In the end the Pythagoreans were destroyed by fear. This may be the danger of mixing technology  
 168 and politics:

169 “Whether the complete rule of number (integers) is wise remains to be seen. It is said that  
 170 when the Pythagoreans tried to extend their influence into politics they met with popular  
 171 resistance. Pythagoras fled, but he was murdered in nearby Mesopotamia in 497 BCE.”

172 –Stillwell (2010, p. 16)

### 173 1.1.5 The birth of modern mathematics

174 Modern mathematics (what we know today) was born in the 15-16<sup>th</sup> century, in the hands of Leonardo  
 175 da Vinci, Bombelli, Galileo, Descartes, Fermat, and many others (Stillwell, 2010). Many of these early  
 176 master were, like the Pythagoreans, secretive to the extreme about how they solved problems. They  
 177 had no interest in sharing their ideas. This soon changed by Mersenne, Descartes and Newton, causing  
 178 mathematics to blossom.

179 **The amazing Bernoulli family** The first individual that seems to have openly recognized the  
 180 importance of mathematics, to actually teach it, was Jacob Bernoulli (Fig. 1.3). Jacob worked on what  
 181 is now view as the standard package of analytic “circular” (i.e., periodic) functions:  $\sin(x)$ ,  $\cos(x)$ ,  
 182  $\exp(x)$ ,  $\log(x)$ .<sup>12</sup> Eventually the full details were developed (for real variables) by Euler (Section 1.3.8  
 183 and 3.4.1).

184 From Fig. 1.4 we see that he was contemporary to Galileo, Mersenne, Descartes, Fermat, Huygens,  
 185 Newton, and Euler. Thus it seems likely that he was strongly influenced by Newton, who in turn was  
 186 influenced by Descartes,<sup>13</sup> Viète and Wallis (Stillwell, 2010, p. 175). With the closure of Cambridge  
 187 University due to the plague of 1665, Newton returned home, Woolsthorpe-by-Colsterworth (95 [mi]  
 188 north of London), to worked by himself, for over a year.

189 Jacob Bernoulli, like all successful mathematicians of the day, was largely self taught. Yet Jacob  
 190 was in a new category of mathematicians, because he was an effective teacher. Jacob taught his sibling  
 191 Johann, who then taught his sibling Daniel. But most importantly, Johann taught Leonhard Euler  
 192 (Figs. 1.4, 1.3), the most prolific (thus influential) of all mathematicians. This resulted in an explosion  
 193 of new ideas and understanding. It is most significant that all four mathematicians published their  
 194 methods and findings. Much later, Jacob studied with students of Descartes<sup>14</sup> (Stillwell, 2010, p. 268-9).

195 Euler went far beyond all the Bernoulli family, Jacob, Johann and Daniel, (Stillwell, 2010, p. 315).  
 196 A special strength of Euler was the degree to which he published. First he would master a topic, and  
 197 then he would publish. His papers continued to appear long after his death (Calinger, 2015).

198 Another individual of that time of special note, who also published extensively, was d’Alembert  
 199 (Figs. 1.4, 1.3). Some of the most important tools were first proposed by d’Alembert. Unfortunately,  
 200 and perhaps somewhat unfairly, his rigor was criticized by Euler, and later by Gauss (Stillwell, 2010).

<sup>12</sup>The log and tan functions are related by  $\tan^{-1}(z) = -\frac{1}{2} \ln\left(\frac{1-z}{1+z}\right)$ .

<sup>13</sup>[https://en.wikipedia.org/wiki/Early\\_life\\_of\\_Isaac\\_Newton](https://en.wikipedia.org/wiki/Early_life_of_Isaac_Newton)

<sup>14</sup>It seems clear that Descartes was also a teacher.

201 Once the tools were being openly published, mathematics grew exponentially. It was one of the  
 202 most creative times in mathematics. Figure 1.4 shows the list of the many famous names, and their  
 203 relative time-line. To aid in understand the time line, note that Leonhard Euler was a contemporary  
 204 of Benjamin Franklin, James Clerk Maxwell of Abraham Lincoln.<sup>15</sup>

## Chronological history post 16<sup>th</sup> century 1.1.2b

17<sup>th</sup> Galileo 1564-1642, Kepler 1571-1630, Newton 1642-1727 Principia 1687; Merseune;  
 Huygen; Pascal; Fermat, Descartes (analytic geometry); Bernoullis Jakob, Johann &  
 son Daniel

18<sup>th</sup> Euler 1748 Student of Johann Bernoulli; d'Alembert 1717-1783; Kirchhoff; Lagrange;  
 Laplace; Gauss 1777-1855

19<sup>th</sup> Möbius, Riemann 1826-1866, Galois, Hamilton, Cauchy 1789-1857, Maxwell, Heavi-  
 side, Cayley, von Helmholtz, Rayleigh

20<sup>th</sup> Hilbert; Einstein; . . .

### Time Line

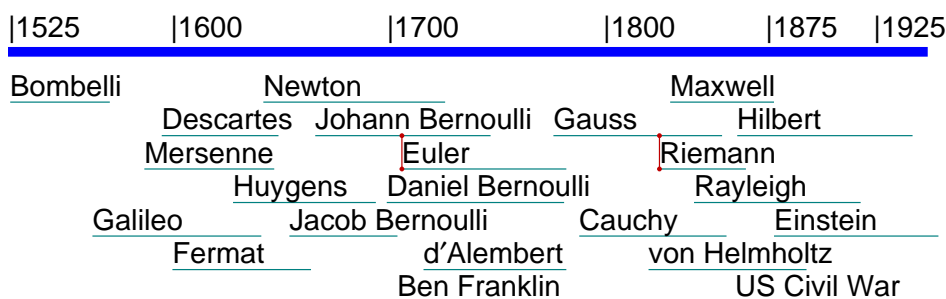


Figure 1.4: Time-line of the four centuries from the 16<sup>th</sup> and 20<sup>th</sup> CE

### 205 1.1.6 Three Streams from the Pythagorean theorem

206 From the outset of his presentation, Stillwell (2010, p. 1) defines “three great *streams* of mathematical  
 207 thought: *Numbers, Geometry and Infinity*,” that flow from the Pythagorean theorem, as summarized  
 208 in Table 1.1. Namely the Pythagorean theorem is the spring from which flow the three streams of  
 209 *all* mathematics. This is a useful concept, based on reasoning not as obvious as one might think.  
 210 Many factors are in play here. One of these was the strongly held opinion of Pythagoras that all  
 211 mathematics should be based on integers. The rest are tied up in the long, necessarily complex history  
 212 of mathematics, as best summarized by the Fundamental theorems, which are each discussed in detail  
 213 in the appropriate chapter.

214 Stillwell’s concept of three streams following from the Pythagorean theorem is the organizing prin-  
 215 ciple behind the this book, organized by chapter:

- 216 1. *Introduction* (Chapter 1) A detailed overview of the fundamentals and the three streams are  
 217 presented in Sections 1.2–1.5.
- 218 2. *Number Systems* (Chapter 2: Stream 1) Fundamentals of number systems, starting with prime  
 219 numbers, through complex numbers, vectors and matrices.
- 220 3. *Algebraic Equations* (Chapter 3: Stream 2) Algebra and its development, as we know it today.  
 221 The theory of real and complex equations and functions of real and complex variables. Complex  
 222 impedance  $Z(s)$  of complex frequency  $s = \sigma + \omega j$  is covered with some care, given its importance  
 223 for engineering mathematics.

<sup>15</sup>Lincoln traveled through Mahomet IL (where I live) on his way to the Urbana Court house.

- 224 4. *Scalar Calculus* (Chapter 4: Stream 3a) Ordinary differential equations. Integral theorems.  
225 Acoustics.
- 226 5. *Vector Calculus*: (Chapter 5: Stream 3b) Vector Partial differential equations. Gradient, diver-  
227 gence and curl differential operators. Stokes, and Green's theorems. Maxwell's equations.

Table 1.1: *Three streams followed from Pythagorean theorem: Number Systems (Stream 1), Geometry (Stream 2) and Infinity (Stream 3).*

- 
- *The Pythagorean Theorem is the mathematical spring which bore the three streams.*
  - *≈Several centuries per stream:*
    - 1) **Numbers:**
      - 6<sup>th</sup> BC*  $\mathbb{N}$  counting numbers,  $\mathbb{Q}$  (Rationals),  $\mathbb{P}$  Primes
      - 5<sup>th</sup> BC*  $\mathbb{Z}$  Common Integers,  $\mathbb{I}$  Irrationals
      - 7<sup>th</sup> CE* zero  $\in \mathbb{Z}$
    - 2) **Geometry:** (e.g., lines, circles, spheres, toroids, ...)
      - 17<sup>th</sup> CE* Composition of polynomials (Descartes, Fermat)  
Euclid's Geometry + algebra  $\Rightarrow$  Analytic Geometry
      - 18<sup>th</sup> CE* Fundamental Theorem of Algebra
    - 3) **Infinity:** ( $\infty \rightarrow$  Sets)
      - 17-18<sup>th</sup> CE*  $\mathbb{F}$  Taylor series, Functions, Calculus (Newton)
      - 19<sup>th</sup> CE*  $\mathbb{R}$  Real,  $\mathbb{C}$  Complex 1851
      - 20<sup>th</sup> CE* Set theory
- 

## 228 1.2 Stream 1: Number Systems

229 This era produced a new stream of fundamental theorems. A few of the individuals who played a  
230 notable role in this development, in chronological (birth) order, include Galileo, Mersenne, Newton,  
231 d'Alembert, Fermat, Huygens, Descartes and Helmholtz. These individuals were some of the first  
232 to develop the basic ideas, in various forms, that were then later reworked into the proofs, that today  
233 we acknowledge as *The fundamental theorems of mathematics*.

234 Number theory (discrete, i.e., integer mathematics) was a starting point for many key ideas. For  
235 example, in Euclid's geometrical constructions the Pythagorean theorem for  $\{a, b, c\} \in \mathbb{R}$  was accepted  
236 as true, but the emphasis in the early analysis was on integer constructions, such as Euclid's formula  
237 for Pythagorean triplets (Eq. 1.3, Fig. 2.3) k As we shall see, the Pythagorean theorem is a rich source  
238 of mathematical constructions, such as composition of polynomials, and solutions of Pell's equation by  
239 eigenvector and recursive analysis methods. Recursive difference equation solutions predate calculus, at  
240 least going back to the Chinese (c2000 BCE). These are early (pre-limit) forms of differential equations,  
241 best analyzed using an eigenfunction expansion (Appendix D), a powerful geometrical concept from  
242 linear algebra, of an expansion in terms of an orthogonal set of normalized (unit-length) vectors.

243 **The first use of zero and  $\infty$ :** It is hard to imagine that one would not appreciate the concept of  
244 zero and negative numbers when using an abacus. If five beads are moved up, and one is moved down,  
245 then four are left. Then if four more are move down, that leaves zero. Taking away is the opposite  
246 of addition, and taking away from four to get zero beads, is no different than taking four away from  
247 zero, to get negative four beads. Subtraction, the inverse of addition, seems like an obvious idea, on  
248 an abacus.

249 However, understanding the concept of zero and negative numbers is not the same as having a  
 250 symbolic notation. The Roman number system had no such symbols. The first recorded use of a  
 251 symbol for zero is said to be by Brahmagupta in 628 CE.<sup>16</sup> Thus it does not take much imagination  
 252 to go from counting numbers  $\mathbb{N}$  to the set of all integers  $\mathbb{Z}$ , including zero, but apparently it takes 600  
 253 years to develop a terminology that represents these ideas. Defining the rules of subtraction required  
 254 the creation of algebra c830 CE (Fig. 1.2). The concept that caused far more difficulty was  $\infty$ . Until  
 255 Riemann's thesis in 1851 it was not clear if  $\infty$  was a number, many numbers, or even definable.

### 256 1.2.1 Lec 2: The Taxonomy of Numbers: $\mathbb{P}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{I}, \mathbb{R}, \mathbb{C}$

257 Once symbols for zero and negative numbers were defined (and accepted), progress was made. In  
 258 a similar manner, to fully understand numbers, a transparent notation is required. First one must  
 259 differentiate between the different classes (genus) of numbers, providing a notation that defines each of  
 260 these classes, along with their relationships. It is logical to start with the most basic *counting numbers*,  
 261 which we indicate with the double-bold symbol  $\mathbb{N}$ . All the double-bold symbols and their genus are  
 262 summarized in Appendix A.

263 **Counting numbers  $\mathbb{N}$ :** These are known as the “natural numbers”  $\{1, 2, 3, \dots\} \in \mathbb{N}$ , denoted by  
 264 the double-bold symbol  $\mathbb{N}$ . For increased clarity we shall refer to the natural numbers as *counting*  
 265 *numbers*, to clarify that *natural* means *integer*. The mathematical sentence  $2 \in \mathbb{N}$  is read as *2 is a*  
 266 *member of the set of counting numbers*. The word *set* means the *sharing of a specific property*.

267 **Primes  $\mathbb{P}$ :** A prime number  $\mathbb{P} \subset \mathbb{N}$  (set  $\mathbb{P}$  is a subset of  $\mathbb{N}$ ) is an integer that may not be factored,  
 268 other than by 1 and itself. Since  $1 = 1 \cdot 1$ ,  $1 \notin \mathbb{P}$ , as it is seen to violate this basic definition of a prime.  
 269 Prime numbers  $\mathbb{P}$  are a *subset* of the counting numbers ( $\mathbb{P} \subset \mathbb{N}$ ). We shall use the convenient notation  
 270  $\pi_n$  for the prime numbers, indexed by  $n \in \mathbb{N}$ . The first 12 primes ( $n = 1, \dots, 12$ ) are  $\pi_n = 2, 3, 5, 7, 11,$   
 271  $13, 17, 19, 23, 29, 31, 37$ . Since,  $4 = 2^2$  and  $6 = 2 \cdot 3$  may be factored,  $\{4, 6\} \notin \mathbb{P}$  (read as: *4 and 6 are*  
 272 *not in the set of primes*). Given this definition, multiples of a prime, i.e.,  $n\pi_k \equiv [2, 3, 4, 5, \dots \text{times} \pi_k$   
 273 of any prime  $\pi_k$ , cannot be prime. It follows that all primes except 2 must be odd and every integer  
 274  $N$  is unique in its factorization.

275 *Coprimes* are number whose factors are distinct (they have no common factors). Thus 4 and 6 are  
 276 not coprime, since they have a common factor of 2, whereas  $21 = 3 \cdot 7$  and  $10 = 2 \cdot 5$  are coprime. By  
 277 definition all distinct primes are coprime. The notation  $m \perp n$  indicates that  $m, n$  are coprime.

278 The *Fundamental Theorem of Arithmetic* states that all integers may be uniquely expressed as a  
 279 product of primes. The *Prime Number Theorem* estimates the mean density of primes over  $\mathbb{N}$ .

280 **Integers  $\mathbb{Z}$ :** These include positive and negative counting numbers and zero. Notionally we might  
 281 indicate this using *set notation* as  $\mathbb{Z} : \{-\mathbb{N}, 0, \mathbb{N}\}$ . Read this as *The integers are in the set composed of*  
 282 *the negative of the natural numbers ( $-\mathbb{N}$ ), zero, and counting numbers  $\mathbb{N}$* . Note that  $\mathbb{N} \subset \mathbb{Z}$ .

283 **Rational numbers  $\mathbb{Q}$ :** These are defined as numbers formed from the ratio of two integers. Since  
 284 the integers  $\mathbb{Z}$  include 1, it follows that integers are a subset of rational numbers ( $\mathbb{Z} \subset \mathbb{Q}$ ). For example  
 285 the rational number  $3/1 \in \mathbb{Z}$ . The main utility of rational numbers is that that they can efficiently  
 286 approximate any number on the real line, to any precision. For example  $\pi \approx 22/7$  with a relative error  
 287 of  $\approx 0.04\%$ . Of course, if the number is rational the error is zero.

288 **Fractional number  $\mathbb{F}$ :** The utility of rational numbers is their power to approximate irrational  
 289 numbers ( $\mathbb{R} \not\subset \mathbb{Z}$ ). It follows that a subset of the rationals, that excludes the integers, has great value.  
 290 We call these numbers *Fractional numbers* and assign them the symbol  $\mathbb{F}$ . They are defined as the  
 291 subset of rationals that are not integers. From this definition  $\mathbb{F} \perp \mathbb{Z}$ ,  $\mathbb{F} \subset \mathbb{Q} = \mathbb{Z} \cup \mathbb{F}$ . Because of their

<sup>16</sup>The fall of the Roman Empire was Sept. 4, 476.

approximating property, the fractional set  $\mathbb{F}$  represent the most important (and the largest) portion of the rational numbers, dwarfing the size of the integers, another good reason for defining the two distinct subsets.

Once factored and common factors canceled, the subset  $\mathbb{F} \subset \mathbb{Q}$  of rational numbers is always the ratio of coprimes. For example  $\pi \approx 22/7 = 11 \cdot 2/7 = 3 + 1/7$  with  $22 \perp 7$ , and  $9/6 = 3/2 = 1 + 1/2$  with  $3 \perp 2$ .<sup>17</sup>

**Irrational numbers  $\mathbb{I}$ :** Every real number that is not rational ( $\mathbb{Q}$ ) is *irrational* ( $\mathbb{Q} \perp \mathbb{I}$ ). Irrational numbers include  $\pi, e$  and the square roots of most integers (i.e.,  $\sqrt{2}$ ). These are decimal numbers that never repeat, thus requiring infinite precision in their representation.

Irrational numbers ( $\mathbb{I}$ ) were famously problematic for the Pythagoreans, who incorrectly theorized that all numbers were rational. Like  $\infty$ , irrational numbers require a new and difficult concept before they may even be defined: They were not in the set of fractional numbers ( $\mathbb{I} \not\subset \mathbb{F}$ ). It was easily shown, from a simple geometrical construction, that most, but not all of the square roots of integers are irrational. It was essential to understand the factorization of counting numbers before the concept of irrationals could be sorted out.

**Real numbers  $\mathbb{R}$ :** Reals are the union of rational and irrational numbers, namely  $\mathbb{R} : \{\mathbb{I}, \mathbb{Q}\}$  ( $\mathbb{R} = \mathbb{Z} \cup \mathbb{F} \cup \mathbb{I}$ ). Reals are the lengths in Euclidean geometry. Many people assume that *IEEE 754 floating point numbers* (c1985) are real (i.e.,  $\in \mathbb{R}$ ). In fact they are rational ( $\mathbb{Q} : \{\mathbb{F} \cup \mathbb{Z}\}$ ) approximations to real numbers, designed to have a very large dynamic range. There can be no machine realization of irrational numbers, since such a number would require infinite precision ( $\infty$  bits). The hallmark of fractional numbers ( $\mathbb{F}$ ) is their power in making highly accurate approximations of any real number.

Using Euclid's compass and ruler methods, one can make line length proportionally shorter or longer, or (approximately) the same. A line may be made be twice as long, an angle bisected. However, the concept of an integer length in Euclid's geometry was not defined.<sup>18</sup> Nor can one construct an imaginary or complex line as all lines are assumed to be real.

Real numbers were first fully accepted only after set theory was developed by Cantor (1874) (Stillwell, 2010, pp. 461, 525. . .). It seems amazing, given how widely accepted real numbers are today. But in some sense they were accepted by the Greeks, as lengths of real lines.

**Complex numbers  $\mathbb{C}$ :** Complex numbers are best defined as *ordered pairs of real numbers*.<sup>19</sup> They are quite special in engineering mathematics, since roots of polynomials having either real or complex coefficients may be complex. The best known example is the quadratic formula for the roots of a  $2^d$  degree polynomial, with either real or complex coefficients.

The common way to write a complex number is using the common notation  $z = a + bj \in \mathbb{C}$ , where  $a, b \in \mathbb{R}$ . Here  $1j = \sqrt{-1}$ . We also define  $1i = -1j$  to account for the two possible signs of the square root. Accordingly  $1j^2 = 1i^2 = -1$ .

Multiplication of complex numbers follows the rules of real algebra, similar to multiplying two polynomials. Multiplication of two first degree polynomials gives

$$(a + bx)(c + dx) = ac + (ad + bc)x + bdx^2$$

If we substitute  $1j$  for  $x$ , and use the definition  $1j^2 = -1$ , we obtain the product of the two complex numbers

$$(a + bj)(c + dj) = ac - bd + (ad + bc)j.$$

<sup>17</sup>HW problem: How to define  $\mathbb{F}$  given two integers  $(n, m) \subset \mathbb{Z}$ ? Sol: Not sure how to approach this, but it seems like a fun problem. Here two simple methods that do *not* work: (1) One cannot define  $\mathbb{F}$  as the ratio  $x = n/m$ , since given  $m = 1, x \in \mathbb{Z}$ . (2) One cannot define  $\mathbb{F}$  as the ratio of two coprimes, since then  $x = 1/m \notin \mathbb{F}$  (since  $1 \perp \mathbb{P}$ ).

<sup>18</sup>As best I know.

<sup>19</sup>A polynomial  $a + bx$  and a 2-vector  $[a, b]^T = \begin{bmatrix} a \\ b \end{bmatrix}$  are also examples of ordered pairs.

327 Thus multiplication of complex numbers obeys the accepted rules of algebra.

**Polar representation:** A alternative for complex multiplication is to work with polar coordinates. The polar form of complex number  $z = a + bj$  is written in terms of its magnitude  $\rho = \sqrt{a^2 + b^2}$  and angle  $\theta = \angle z = \tan^{-1}(z) = \arctan z$ , as  $z = \rho e^{\theta j}$ . From the definition of the complex natural log function

$$\ln \rho e^{\theta j} = \ln \rho + \theta j,$$

328 which is useful in engineering calculations.<sup>20</sup>

**Matrix representation:** A second alternative and useful way to represent complex numbers is in terms of 2x2 matrices. This relationship is defined by the mapping from a complex number to a 2x2 matrix (along with several examples)

$$a + bj \leftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad 2 \leftrightarrow \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad -j \leftrightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad e^{\theta j} \leftrightarrow \begin{bmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix}$$

You might verify that

$$\frac{a + bj}{c + dj} = \frac{ab + bd + (bc - ad)j}{c^2 + d^2} \leftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \frac{1}{c^2 + d^2}.$$

329 By taking the inverse of the 2x2 matrix one can define the ratio of one complex number by another,  
330 Until you try out this representation, it may not seem obvious, or even that it could work.

331 This representation proves that  $1j$  is not necessary to define a complex number. What  $1j$  can  
332 do is simplify the algebra, both conceptually and for numerical results. It is worth your time to  
333 become familiar with the matrix representation, to clarify any possible confusions you might have  
334 about multiplication and division of complex numbers. This matrix representation can save you time,  
335 heartache and messy algebra. Once you have learned how to multiply two matrices, it's a lot simpler  
336 than doing the complex algebra. In many cases we will leave the results of our analysis in matrix form,  
337 to avoid the algebra altogether.<sup>21</sup> More on this topic may be found in Chapter 2.

**Real versus complex numbers:** All numbers may be viewed as complex. Namely every real number is complex if we take the imaginary part to be zero (Boas, 1987). For example,  $2 \in \mathbb{P} \subset \mathbb{C}$ . Likewise every purely imaginary number (e.g.,  $0 + 1j$ ) is complex with zero real part. It follows that  $2j \in \mathbb{P}j$ . Integers are a subset of reals, which are a subset of complex numbers<sup>22</sup> *Gaussian integers* are complex integers ( $\mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}$ ).<sup>23</sup> From the above discussion it should be clear that each of these different classes of number are nested in a hierarchy, in the following embeddings

$$\pi_k \in \mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Z} \cup \mathbb{F} = \mathbb{Q} \subset \mathbb{Q} \cup \mathbb{I} = \mathbb{R} \subset \mathbb{C}.$$

338 The integers  $\mathbb{Z}$  and fractionals  $\mathbb{F}$  split the rationals ( $\mathbb{Q} : \mathbb{Z} \cup \mathbb{F}, \mathbb{Z} \perp \mathbb{F}$ ), each of which is a subset of  
339 the rationals ( $\mathbb{Z} \in \mathbb{Q}, \mathbb{F} \subset \mathbb{Q}$ ). The rationals  $\mathbb{Q}$  and irrationals  $\mathbb{I}$  split the reals ( $\mathbb{R} : \mathbb{Q} \cup \mathbb{I}, \mathbb{Q} \perp \mathbb{I}$ ), each  
340 of which is a subset of the reals ( $\mathbb{Q} \in \mathbb{R}, \mathbb{I} \in \mathbb{R}$ ).

341 The roots of polynomials  $x_k$  are complex ( $x_k \in \mathbb{C}$ ), independent of the genus of the coefficients (e.g.,  
342 real integer coefficients give rise to complex roots). Each genus plays an important role in algebra, with  
343 prime numbers at the bottom (root of the tree) and complex numbers at the top. We shall explore this  
344 further in Chaps. 2 and 3.

<sup>20</sup>Chapter 2 discusses the definition of the phase, i.e., how is it computed (i.e.,  $\arctan(e^{\theta j})$ ,  $\arctan2(x,y)$ ), and the importance of the unwrapped phase, as in the example  $\delta(t - \tau) \leftrightarrow e^{-\tau j}$ .

<sup>21</sup>Sometimes we let the computer do the final algebra, numerically, as 2x2 matrix multiplications.

<sup>22</sup>The plural *complexs* (a double /s/) seems an unacceptable word in English.

<sup>23</sup>It follows that integers are a subset of Gaussian integers (the imaginary or real part of the Gaussian integer may be zero).



345 Finally, note that complex numbers  $\mathbb{C}$  do not have “order,” meaning one cannot be larger, smaller  
 346 or equal to another. It makes no sense to say that  $j > 1$  or  $j = 1$  (Boas, 1987). The real and imaginary  
 347 parts and the magnitude and phase have order.

348 **History of complex numbers:** It is notable how long it took for complex numbers to be accepted  
 349 (1851), relative to when they were first introduced by Bombelli (16<sup>th</sup> century CE). In fact, complex  
 350 integers (aka, *Gaussian integers*) were accepted before non-integral complex numbers. Apparently real  
 351 numbers ( $\mathbb{R}$ ) were not accepted (i.e., proved to exist, thus mathematically defined) until even later. It  
 352 took the development of *set theory* in the late 19<sup>th</sup> century to sort out a proper definition of the real  
 353 number, due to the existence of irrational numbers.

354 **Computer Representations of  $\mathbb{I}, \mathbb{R}, \mathbb{C}$ :** When doing numerical work, one must consider how we may  
 355 compute within the family of reals (i.e., irrationals). There can be no irrational number representation  
 356 on a computers. IEEE floating point numbers, which are the international standard of computation,  
 357 are actually rational approximations. The mantissa and the exponent are each integers, having sign  
 358 and magnitude. The size of each integer depends on the precision of the number being represented.  
 359 An IEEE floating-point number is rational because it has a mantissa integer multiplied by a base to  
 360 the power of an exponent integer.

Floating point numbers contain irrational numbers, which must be approximate by rational numbers. This leads to the concept of *fractional representation*, which requires the definition of the *mantissa*, *exponent* and *base*. Numerical results must not depend on the base. For example, when using base ten<sup>24</sup>

$$\pi \cdot 10^5 \approx 314159.27\dots = 3 \cdot 10^5 + 1 \cdot 10^4 + 4 \cdot 10^3 + \dots + 9 \cdot 10^0 + 2 \cdot 10^{-1} \dots$$

According to Matlab’s DEC2BIN() routine, the binary representation is

$$\pi \cdot 2^{17} \approx 131072_{10} \cdot 22/7 = 110,0100,1001,0010,0101_2,$$

where 1 and 0 are multipliers of powers of 2, which are then added together as follows

$$2^{18} + 2^{17} + 2^{14} + 2^{11} + 2^8 + 2^5 + 2^2 + 2^0.$$

361 In base 16 (i.e, hexadecimal)  $2^{17} \cdot 22/7 = 2^{18} \cdot 8_{16}/7_{16}$ .

362 One may keep track of the decimal point using the exponent, which in this case is a factor of  $2^{17}$   
 363  $= 131072_{10}$ . The concept of a number having a decimal point is replaced by an integer, having the  
 364 desired precision, and a scale factor of any base (radix). This scale factor may be thought of as moving  
 365 the decimal point to the right (larger number) or left (smaller number). The mantissa “fine-tunes” the  
 366 value about a scale factor (the exponent).

367 Here is  $x = 2^{17} \times 22/7$  at IEEE 754 full double precision, as computed by an IEEE-754 floating  
 368 point converter<sup>25</sup>  $x = 411940.5625_{10} = 2^{54} \times 1198372 = 010010,001,10010,010010,010010,010010_2 =$   
 369  $0x48c92492_{16}$ . The commas in the binary string of ones and zeros, are to help visualize the quasi-  
 370 periodic nature of the bit-stream. The mantissa is  $4793490_{10}$  and the exponent is  $2^{18}$ . The numbers  
 371 are stored in a 32 bit format, with 1 bit for sign, 8 bits for the exponent and 23 bits for the mantissa.  
 372 Perhaps a more instructive number is  $x = 4793490.0 = 01,001,010,100,100,100,100,100,100,100_2$   
 373  $= 0x4a,924,924_{16}$  which has a repeating binary bit pattern of  $((100))_3$ , only broken by the scale factor  
 374  $0x4a$ . Another with even higher symmetry is  $x = 6.344,131,191,146,9 \times 10^{-17} = 0x24,924,924_{16} =$   
 375  $00,100,100,100,100,100,100,100,100,100_2$ . In this example the repeating pattern is clear in the  
 376 Hex representation as a repeating  $((942))_3$ . As before, the commas are to help with readability, and  
 377 have no other meaning.

Make a table

ck 4 errors

<sup>24</sup>Base 10 is the natural world-wide standard simply because we have 10 fingers.

<sup>25</sup><http://www.h-schmidt.net/FloatConverter/IEEE754.html>

378 There are other important types of representations. As pairs of reals, complex numbers have similar  
 379 approximate representations. An important representations of complex numbers is  $e^z = \cosh(z) +$   
 380  $j \sinh(z)$ , which includes the famous formula of Euler  $e^{j\theta} = \cos \theta + j \sin \theta$ . Some of these concepts can  
 381 be generalized to include vectors, matrices and polynomials.

382 **Integers and the Pythagoreans** The integer is the corner stone of the Pythagorean doctrine, so  
 383 much so that it caused a fracture within the Pythagoreans when it was discovered that not all numbers  
 384 are rational. The famous example is the isosceles triangle  $1, 1, \sqrt{2}$ , which lead to the next triangle  
 385  $[1, 2, \sqrt{3}]$ , etc. This is known as the Spiral of Theodorus: the short side is 1 and the hypotenuse is  
 386 extended by one, using a simple compass-ruler construction.

387 There are right-triangles with integral lengths, the best known being  $[3, 4, 5]$ . Such triplets of  
 388 integers  $[a, b, c]$  that satisfy the Pythagorean formula (Eq. 1.1) are denoted *Pythagorean triplets*, which  
 389 may be verified using Euclid's formula (Eq. 1.3).

390 To form triangles with perfect  $90^\circ$  angles, the lengths need to satisfy Eq. 1.1. Such triangles are  
 391 also useful in constructing buildings or roads made from of bricks having a uniform size.

392 **Public-private key Security:** An important application of prime numbers is public-private key  
 393 (RSA) encryption, essential for internet security applications (e.g., online banking). To send secure  
 394 messages the security (i.e., utility) of the internet is dependent on key encryption.<sup>26</sup> Most people  
 395 assume this is done by a personal login and passwords. Passwords are simply not secure, for many  
 396 reasons. The proper method depends on factoring integers formed from products of primes having  
 397 thousands of bits.<sup>27</sup> The security is based on the relative ease in multiplying large primes, but the  
 398 virtual impossibility of factoring them.

399 When a computation is easy in one direction, but its inverse is impossible, it is called a *trap-door*  
 400 *function*. We shall explore the reasons for this in Chapter 2. If everyone switched from passwords to  
 401 public key encryption, the internet would be much more secure.

402 **Puzzles:** A third application of integers are imaginative problems that use integers. An example is  
 403 the classic Chinese *Four stone problem*: "Find the weight of four stones that can be used with a scale  
 404 to weigh any object (e.g., salt, gold) between 0, 1, 2, . . . , 40 [gm]." As with the other problems, the  
 405 answer is not as interesting as the method, since the problem may be easily recast into a related one.  
 406 This type of problem can be found in airline magazines as entertain on a long flight. The solution to  
 407 this problem is best cast as a linear algebra problem, with integer solutions. Again, once you know the  
 408 trick, it is "easy."<sup>28</sup>

### 409 1.2.2 Lec 3: The role of physics in mathematics

410 **Bells, chimes and Eigenmodes** Integers naturally arose in art, music and science. An example  
 411 are the relations between musical notes, the natural eigenmodes (tones) of strings and other musical  
 412 instruments. These relations were so common and well studied, it appeared that to understand the  
 413 physical world (aka, the Universe), one needed to understand integers. This was a seductive view, but  
 414 not actually correct. As will be discussed in Sections 1.3.1 and 3.1.1, it is best to view the relationship  
 415 between acoustics, music and mathematics as historical, since these topics played such an important  
 416 role in the development of mathematics. Also interesting is the role that integers seem to play in  
 417 quantum mechanics, for much the same reasons.

<sup>26</sup>One might say this is either: i) a key application of primes, or ii) it is primary application of keys. Its a joke.

<sup>27</sup>It would seem that public key encryption could work by having two numbers with a common prime, and then by using Euclidean algorithm, that GCD could be worked out. One of the integers could be the public key and the second could be the private key. Given the difficulty of factoring the numbers into their primes, and ease of finding the GCD using Euclidean algorithm, a practical scheme may be possible. Ck this out.

<sup>28</sup>When ever someone tells you something is "easy," you should immediately appreciate that it is very hard, but there is a concept, that once you learn, the difficulty evaporates.

418 Engineers are so accustomed to working with real (or complex) numbers, the distinction between  
 419 real (i.e., irrational) and fractional numbers are rarely acknowledged. Integers on the other hand  
 420 arise in many contexts. One cannot master programming computers without understanding integer,  
 421 hexadecimal, octal, and binary representations, since all numbers in a computer are represented in  
 422 numerical computations in terms of rationals ( $\mathbb{Q} = \mathbb{Z} \cup \mathbb{F}$ ).

423 As discussed in Section 1.2.1, the primary reason integers are so important is their absolute precision.  
 424 Every integer  $n \in \mathbb{Z}$  is unique,<sup>29</sup> and has the *indexing property*, which is essential for making lists that  
 425 are ordered, so that one can quickly look things up. The alphabet also has this property (e.g., a book's  
 426 index). Other than for hexadecimal numbers, which for notional reasons use the alphabet, letters are  
 427 equivalent to integers.

428 Because of the integer's absolute precision, the digital computer overtook the analog computer,  
 429 once it was practical to make logic circuits that were fast. The first digital computer was thought  
 430 to be the *Eniac* at the University of Pennsylvania, but it turned out that the code-breaking effort in  
 431 Bletchley Park, England, under the guidance of Alan Turing, created the first digital computer (The  
 432 Colossus) to break the WWII German "Enigma" code. Due to the high secrecy of this war effort, the  
 433 credit was only acknowledged in the 1970s when the project was declassified.

434 There is zero possibility of analog computing displacing digital computing, due to the importance  
 435 of precision (and speed). But even with binary representation, there is a non-zero probability of error,  
 436 for example on a hard drive, due to physical noise. To deal with this, error correcting codes have been  
 437 developed, to reduce the error by several orders of magnitude. Today this is a science, and billions of  
 438 dollars are invested to increase the density of bits per area, to increasingly larger factors. A few years  
 439 ago the terabyte drive was unheard of; today it is the standard. In a few years petabyte drives will  
 440 certainly become available. It is hard to comprehend how these will be used by individuals, but they  
 441 are essential for on-line (cloud) computing.

## 442 Fundamental theorems

443 Modern mathematics is build on a hierarchical construct of *fundamental theorems*, as summarized in  
 444 Table 1.2. The importance of such theorems cannot be overemphasized. Every engineering student  
 445 needs to fully appreciate the significance of these key theorems. If necessary, memorize them. But  
 446 that will not do over the long run, as each and every theorem must be fully understood. Fortunately  
 447 most students already know several of these theorems, but perhaps not by name. In such cases, it is a  
 448 matter of mastering the vocabulary.

449 The theorems are naturally organized, starting with two theorems on prime numbers (Table 1.2).  
 450 These may also be thought of in terms of Stillwell's three streams. For Stream 1 there is the *Fundamen-*  
 451 *tal Theorem of Arithmetic* and the *Prime Number Theorem*. For Stream 2 there is the *Fundamental*  
 452 *Theorem of Algebra* and Bézout's theorem, while for Stream 3 there are a host of theorems on calcu-  
 453 lus, ordered by their dimensionality. Some of these theorems verge on trivial (e.g., the Fundamental  
 454 Theorem of Arithmetic). Others are more challenging, such as the *Fundamental Theorem of Vector*  
 455 *Calculus* and *Green's theorem*.

456 Complexity should not be confused with importance. Each of these theorems is, as stated, funda-  
 457 mental. Taken as a whole, they are a powerful way of summarizing mathematics.

### 458 Stream 1: Prime Number theorems:

459 There are two fundamental theorems about primes,

- 460 1. *The Fundamental Theorem of Arithmetic*: This states that every counting number  $n > 1 \in \mathbb{N}$   
 461 may be uniquely factored into prime numbers. This raises the question of the meaning of *factor*  
 462 (split into a product).

---

<sup>29</sup>Check out the history of  $1729 = 1^3 + 12^2 = 9^3 + 10^3$ .

Table 1.2: *The Fundamental theorems of mathematics*

## 1. Fundamental theorems of:

(a) **Number Systems: Stream 1**

- Arithmetic
- Prime Number

(b) **Geometry: Stream 2**

- Algebra
- Bézout

(c) **Calculus: Stream 3<sup>a</sup>**

- Leibniz  $\mathbb{R}^1$
- Complex  $\mathbb{C} \subset \mathbb{R}^2$
- vectors  $\mathbb{R}^3, \mathbb{R}^n, \mathbb{R}^\infty$

## 2. Other key concepts:

- Complex analytic functions ([complex roots are finally accepted!](#))
  - Complex Taylor Series (complex analytic functions)
  - Region of convergence (ROC) of complex analytic series
  - Laplace transform, and its inverse
  - Causal time  $\implies$  complex frequency  $s$
  - Cauchy Integral Theorem
  - Residue integration (i.e., Green's Thm in  $\mathbb{R}^2$ )
- Riemann mapping theorem (Gray, 1994; Walsh, 1973)
- Complex Impedance (Ohm's Law) Kennelly

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<sup>a</sup>Flanders, Harley (June–July 1973). “Differentiation under the integral sign.” *American Mathematical Monthly* 80 (6): 615-627. doi:10.2307/2319163. JSTOR 2319163.

463 2. *The Prime Number Theorem*: One would like to know how many primes there are. That is easy:  
 464  $|\mathbb{P}| = \infty$ . (The *cardinality*, or size of the set of primes, is infinite). The proper way of asking  
 465 this questions is *What is the average density of primes, in the limit as  $n \rightarrow \infty$ ?* This question  
 466 was answered, for all practical purposes, by Gauss, who as a pastime computed the first million  
 467 primes by hand. He discovered that, to a good approximation, the primes are equally likely on  
 468 a log scale. This is nicely summarized by the jingle attributed to the mathematician Pafnuty  
 469 Chebyshev

470 Chebyshev said, and I say it again: *There is always a prime between  $n$  and  $2n$ .*

471 (Stillwell, 2010, p. 585)

472 When the ratio (interval) of two frequencies (pitch) is 2, the relationship is called an *octave*. Thus we  
 473 might say there is at least one prime per octave. This makes on wonder about the maximum number  
 474 of primes per octave. In modern music the octave is further divided into 12 intervals called *semitones*  
 475 (factors), equal to the  $\sqrt[12]{2}$ . The product of 12 semitones is an octave. Thus we must wonder how  
 476 many primes there is per semitone?

477 **Stream 2: Fundamental theorem of Algebra**

478 This theorem states that every polynomial has at least one root. When that root is removed then the  
 479 degree of the polynomial is reduced by 1. Thus when applied recursively, a polynomial of degree  $N$   
 480 has  $N$  roots.

481 Besides the Fundamental Theorem of Algebra, a second important theorem is Bézout's theorem,  
 482 which is a generalization of the Fundamental Theorem of Algebra. It says<sup>30</sup> that the *composition* of  
 483 two polynomials has degree equal to the product of the degrees of each polynomial. For example, if  
 484  $P_3(x) = x^3$  and  $P_5(x) = x^5$ , then  $P_3(P_5)(x) = (x^5)^3 = x^{15}$ . It further states that when counting the  
 485  $N$  roots of a polynomial of degree  $N$ , one must include the imaginary roots, double roots and roots at  
 486 infinity, some of which may difficult to identify.

487 **Stream 3: Fundamental theorems of calculus**

488 There are at least four theorems related to integral calculus:

Helmholtz image

Table 1.3: Summary of the fundamental theorems of integral calculus.

Name	Mapping	Description
<i>Leibnez</i>	$\mathbb{R}^1 \rightarrow \mathbb{R}^0$	Area under a real curve.
<i>Cauchy</i>	$\mathbb{C} \rightarrow \mathbb{C}^0$	Residue integration and analytic functions.
<i>Gauss's Law</i>	$\mathbb{R}^3 \rightarrow \mathbb{R}^2$	Conservation of mass and charge crossing a closed surface.
<i>Stokes</i>	$\mathbb{R}^3 \rightarrow \mathbb{R}^1$	Relates line integrals to the rate of change of the flux crossing an open surface.
<i>Green</i>	$\mathbb{R}^2 \rightarrow \mathbb{R}_0$	Special case of Stokes for a plane
<i>Helmholtz</i>		Every differentiable vector field may be decomposed into a dilatation and a rotation.

489 In Sections 1.5.3, 1.5.5 and 5.1.3 we will deal with each of the theorems for Stream 3, where we  
 490 consider the several Fundamental theorems of integration, starting with Leibniz's formula for integra-  
 491 tion on the real line ( $\mathbb{R}$ ), then progressing to complex integration in the complex plane ( $\mathbb{C}$ ) (*Cauchy's*  
 492 *theorem*), which is required for computing the inverse Laplace transform. Then we discuss Gauss' and  
 493 Stokes' Laws for  $\mathbb{R}^2$ , with closed and open surfaces. One cannot understand Maxwell's equations,  
 494 fluid flow, or acoustics without understanding these theorems. Any problem that deals with the wave  
 495 equation in more than one dimension requires an understanding of these concepts. The derivation of  
 496 the Kirchhoff voltage and current laws is based on these theorems.

497 **Other key concepts**

498 Besides the widely recognized fundamental theorems for the three streams, there are a number of  
 499 equally important theorems that have not yet been labeled as "fundamental."<sup>31</sup>

500 The widely recognized *Cauchy Integral Theorem* is an excellent example, since it is a stepping stone  
 501 to the *Fundamental Theorem of Complex Integral Calculus*. In Chapter 4 we clarify the contributions  
 502 of each of these special theorems.

503 Once these Fundamental theorems of integration (Stream 3) have been mastered, the student is  
 504 ready for the *complex frequency domain*, which takes us back to Stream 2 and the *complex frequency*  
 505 *plane* ( $s = \sigma + \omega j \in \mathbb{C}$ ). While the Fourier and Laplace transforms are taught in Mathematics courses,

<sup>30</sup>Statements of the theorem speak of *intersections* and constructions of curves, rather than compositions. I find this somewhat confusing. For example, how does intersection differ from elimination, or construction from composition (Stillwell, 2010, p. 119)?

<sup>31</sup>It is not clear what it takes to reach this more official sounding category.

506 typically few physical connections are made, accordingly the concept of *complex frequency* is rarely  
 507 mentioned. The *complex frequency domain* and *causality* are fundamentally related, and critical for  
 508 the analysis of signals and systems.

509 Without the concept of time and frequency, one cannot develop an intuition for the Fourier and  
 510 Laplace transform relationships, especially within the context of engineering and mathematical physics.

## 511 WEEK 2

512

### 513 1.2.3 Lec 4: Prime numbers

514 If someone came up to you and asked for a theory of counting numbers, I suspect you would look them  
 515 in the eye with a blank stare, and start counting. It sounds like either a bad joke or a stupid question.  
 516 Yet integers are rich topic, so the question is not even slightly dumb. It is somewhat amazing that  
 517 even birds and bees can count. While I doubt birds and bees can recognize primes, cicadas and other  
 518 insects only crawl out of the ground in multiples of prime years, (e.g., 13 or 17 year cycles). If you have  
 519 ever witnessed such an event (I have), you will never forget it. Somehow they know. Finally, there is  
 520 an analytic function, first introduced by Euler, based on his analysis of the Sieve, now known as the  
 521 *Riemann zeta function*  $\zeta(s)$ , which is complex analytic, with its poles at the logs of the prime numbers.  
 522 The exact relationship between the primes and the poles will be discussed in Sections 1.4.12 and 4.5.2.  
 523 The properties of this function are truly amazing, even fun. It follows that primes are fundamental  
 524 properties of the counting numbers, that the theory of numbers (and primes) is an important topic of  
 525 study. Many of the questions, and some of the answers, go back to at least the time of the Chinese  
 526 (Stillwell, 2010).

527 The most relevant question at this point is “Why are integers so important?” We addressed this  
 528 question in Section 1.2.9. First we count with them, so we can keep track of “how much.” But there  
 529 is much more to numbers than counting: We use integers for any application where absolute accuracy  
 530 is essential, such as banking transactions (making change), and precise computing of dates (Stillwell,  
 531 2010, p. 70) or location (I’ll meet you at location  $L \in \mathbb{N}$  at time  $T \in \mathbb{N}$ ), building roads or buildings  
 532 out of bricks (objects of a uniform size). If you go to 34<sup>th</sup> street and Madison and they are at 33th  
 533 and Madison, that’s a problem. To navigate we need to know how to predict the tides, the location of  
 534 the moon and sun, etc. Integers are important because they are precise: Once a month there is a full  
 535 moon, easily recognizable. The next day its slightly less than full.

### 536 Sieves

537 A recursive sieve method for finding primes was first devised by the Greek Eratosthenes,<sup>32</sup> and sum-  
 538 marized in Fig. 1.5.

- 539 1. Write  $N - 1$  counting number from 2 to  $N$  (List)
- 540 2. Define loop index  $k = 1$  and a multiplier  $n \in \mathbb{N}$  denoted  $n := \{2, \dots, N\}$ .
- 541 3. The next number on the list is prime  $\pi_k \in \mathbb{P}$
- 542 4. Remove (Cross out) all multiples  $n \cdot \pi_n$  of  $\pi_k$
- 543 5.  $k = k + 1$ : return to step 3.

544 Starting from the first prime  $\pi_1 = 2$ , one successively strikes out all the multiples of that prime.  
 545 For example, starting from  $\pi_1 = 2$  one strikes out  $2 \cdot 2, 2 \cdot 3, 2 \cdot 4, 2 \cdot 5, \dots, 2 \cdot (N/2)$ . By definition the

<sup>32</sup>[https://en.wikipedia.org/wiki/Sieve\\_of\\_Eratosthenes#Euler.27s\\_Sieve](https://en.wikipedia.org/wiki/Sieve_of_Eratosthenes#Euler.27s_Sieve)

546 multiples are products of the target prime (2 in our example) and every another integer ( $n \geq 2$ ). All  
 547 the even numbers are removed in the first iteration. One then considers the next integer not struck  
 548 out (3 in our example), which is identified as the next (second) prime  $\pi_2$ . Then all the  $(N - 2)/2$   
 549 non-prime multiples of  $\pi_2$  are struck out. The next number which has not been struck out is 5, thus is  
 550 prime  $\pi_3$ . All remaining multiples of 5 are struck out (~~10~~, ~~15~~, ~~25~~,  $\dots$ ). This process is repeated until  
 551 all the numbers on the starting list have been processed.

552 As the word sieve implies, this sifting process takes a heavy toll on the integers, rapidly pruning  
 553 the non-primes. In four loops of the sieve algorithm, all the primes below  $N = 50$  are identified in **red**.  
 554 The final set of primes is displayed at the bottom of Fig. 1.5.

1. Write  $N$  integers from 2 to  $N - 1$ . Let  $k = 1$ . The first element  $\pi_1 = 2$  is a prime. Cross out  $n \cdot \pi_n$ : (e.g.,  $n \cdot \pi_1 = 4, 8, 16, 32, \dots$ ).

	<b>2</b>	3	<del>4</del>	5	<del>6</del>	7	<del>8</del>	9	<del>10</del>
11	<del>12</del>	13	<del>14</del>	15	<del>16</del>	17	<del>18</del>	19	<del>20</del>
21	<del>22</del>	23	<del>24</del>	25	<del>26</del>	27	<del>28</del>	29	<del>30</del>
31	<del>32</del>	33	<del>34</del>	35	<del>36</del>	37	<del>38</del>	39	<del>40</del>
41	<del>42</del>	43	<del>44</del>	45	<del>46</del>	47	<del>48</del>	49	<del>50</del>

2. Let  $k = 2$ ,  $\pi_2 = 3$ . Cross out  $n\pi_k$  (6, 9, 12, 15,  $\dots$ ):

	<b>2</b>	<b>3</b>	<del>4</del>	5	<del>6</del>	7	<del>8</del>	<del>9</del>	<del>10</del>
11	<del>12</del>	13	<del>14</del>	<del>15</del>	<del>16</del>	17	<del>18</del>	19	<del>20</del>
<del>21</del>	<del>22</del>	23	<del>24</del>	25	<del>26</del>	<del>27</del>	<del>28</del>	29	<del>30</del>
31	<del>32</del>	<del>33</del>	<del>34</del>	35	<del>36</del>	37	<del>38</del>	<del>39</del>	<del>40</del>
41	<del>42</del>	43	<del>44</del>	<del>45</del>	<del>46</del>	47	<del>48</del>	49	<del>50</del>

3. Let  $k = 3$ ,  $\pi_3 = 5$ . Corss out  $n\pi_3$ . (Cross out 25, 35).

	<b>2</b>	<b>3</b>	<del>4</del>	<b>5</b>	<del>6</del>	<b>7</b>	<del>8</del>	<del>9</del>	<del>10</del>
<b>11</b>	<del>12</del>	<b>13</b>	<del>14</del>	<del>15</del>	<del>16</del>	<b>17</b>	<del>18</del>	<b>19</b>	<del>20</del>
<del>21</del>	<del>22</del>	<b>23</b>	<del>24</del>	<del>25</del>	<del>26</del>	<del>27</del>	<del>28</del>	<b>29</b>	<del>30</del>
<b>31</b>	<del>32</del>	<del>33</del>	<del>34</del>	<del>35</del>	<del>36</del>	<b>37</b>	<del>38</del>	<del>39</del>	<del>40</del>
<b>41</b>	<del>42</del>	<b>43</b>	<del>44</del>	<del>45</del>	<del>46</del>	<b>47</b>	<del>48</del>	<del>49</del>	<del>50</del>

4. Finally let  $k = 4$ ,  $\pi_4 = 7$ . Remove  $n\pi_4$ : (Cross out 49). Thus there are 15 primes less than  $N = 50$ :  $\pi_k = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}$ .

Figure 1.5: Sieve of Eratosthenes for the case of  $N = 50$ .

555 Once a prime greater than  $N/2$  has been identified, we may stop, since twice that prime is greater  
 556 than  $N$ , the maximum number under consideration. Once you have reached  $\sqrt{N}$  all the primes have  
 557 been struck out (this follows from the fact that the next prime  $\pi_n$  is multiplied by an integer  $n =$   
 558  $1, \dots, N$ ). Once this number  $n\pi_n > N$  the list has been exhausted, which must be  $n < \sqrt{N}$ .

559 There are various schemes for making the sieve more efficient. For example the recursion  $n\pi_k =$   
 560  $(n - 1)\pi_k + \pi_k$ . could speed up the process, by replacing the multiply by an add by a known quantity.  
 561 When using a computer, memory efficiency and speed are the main considerations.

## 562 The importance of prime numbers

563 Likely the first insight into the counting numbers starts with the sieve, shown in Fig. 1.5. A sieve  
 564 answers the question “What is the definition of a prime number?” which is likely the first question to

565 be asked. The answer comes from looking for irregular patterns in the counting numbers, by playing  
566 the counting numbers against themselves.

567 A prime is that subset of positive integers  $\mathbb{P} \in \mathbb{N}$  that cannot be factored. The number 1 is not a  
568 prime, for some non-obvious reasons, but there is no pattern in it since it is always a (useless) factor  
569 of every counting number.

570 To identify the primes we start from the first candidate on the list, which is 2 (since 1 is not a  
571 prime), and strike out all multiples by the counting numbers greater than 1  $[(n+1) \cdot 2 = 4, 6, 8, \dots]$ .  
572 While not obvious, this is our first result, that 2 is a prime, since it has no other factors but 1 and  
573 itself. This leaves only the odd numbers. We need a notation to indicate this result so we shall set  
574  $\pi_1 = 2$ , as the first prime.<sup>33</sup>

575 **Two Fundamental Theorems of Primes:** Early theories of numbers revealed two fundamental  
576 theorems (there are many more than two), as discussed in Section 1.2.2. The first of these is the  
577 *Fundamental Theorem of Arithmetic*, which says that every integer greater than 1 may be uniquely  
578 factored into a product of primes (Eq. 2.1). Our demonstration of this is empirical, using Matlab's  
579 `factor(N)` routine, which delivers the prime numbers that compose  $N$ .<sup>34</sup> Typically the prime factors  
580 appear more than once, for example  $4 = 2^2$ . To make the notation compact we define the *multiplicity*  
581  $\beta_k$  of each prime factor  $\pi_k$  (Eq. 2.1).

582 Each counting number is *uniquely* represented by a product of primes. There cannot be two integers  
583 with the same factorization. Once you multiply the factors out, the result is a unique  $N$ . Note that it's  
584 easy to multiply integers (e.g., primes), but nearly impossible to factor them. Factoring is not the same  
585 as dividing, as one needs to know what to divide by. Factoring means dividing by some integer and  
586 obtaining another integer with a zero remainder. This is what makes it so difficult (nearly impossible).

587 So the question remains: "What is the utility of the FTA?" which brings us to the topic of *internet*  
588 *security*. Unfortunately at this time I can not give you a proper summary of how it works. The full  
589 answer requires a proper course in number theory, beyond what is presented here.

590 The basic concept is that it is easy to construct the product of two primes, even very long primes  
591 having hundreds, or even thousands, of digits. It is very costly (but not impossible) to factor them.  
592 Why not use Matlab's `factor(N)` routine to find the factors? This is where *cost* comes in. The  
593 numbers used in RSA are too large for Matlab's routine to deliver an answer. In fact, even the largest  
594 computer in the world (such as the University of Illinois' super computer (NCSA Water) cannot do  
595 this computation. The reason has to do with the number of primes. If we were simply looking up a few  
596 numbers from a short list of primes, it would be easy, but the density of primes among the integers, is  
597 huge (see Section 2.1.1). This take us to the *Prime Number Theorem* (PNT). The security problem is  
598 the reason why these two theorems are so important: 1) Every integer has a unique representation as a  
599 product of primes, and 2) the number of primes is very dense (their are a very large number of them).  
600 Security reduces to the needle in the haystack problem, the cost of a search. A more formal way to  
601 measure the density is called the *entropy*, which is couched in terms of the probability of events, which  
602 in this case is "What is the probability of finding a prime between  $N$  and  $2N$ ?"

### 603 Rational numbers $\mathbb{Q}$

604 The most important genus of numbers are the rational numbers since they maintain the utility of  
605 absolute precision, and they can approximate any irrational number (e.g.,  $\pi \approx 22/7$ ) to any desired  
606 degree of precision. However, the subset of rationals we really are interested in are the fractionals  $\mathbb{F}$ .  
607 Recall that  $\mathbb{Q} : \mathbb{F} \cup \mathbb{Z}$  and  $\mathbb{F} \perp \mathbb{Z}$ . The fractionals are the numbers with the approximation utility,  
608 with arbitrary accuracy. Integers are equally important, but for a very different reason. All numerical

<sup>33</sup>There is a potentially conflicting notation since  $\pi(N)$  is commonly defined as the number of primes less than index  $N$ . Be warned that here we define  $\pi_n$  as the  $n^{\text{th}}$  prime, and  $\Pi(N)$  as the number of primes  $\leq N$ , since having a convenient notation for the  $n^{\text{th}}$  prime is more important than for the number of primes less than  $N$ .

<sup>34</sup>If you wish to be a Mathematician, you need to learn how to prove theorems. If you're an Engineer, you are happy that someone else has already proved them, so that you can use the result.



609 computing today is done with  $\mathbb{Q}$ . Indexing uses integers  $\mathbb{Z}$ , while the rest of computing (flow dynamics,  
 610 differential equations, etc.) is done with the fractionals  $\mathbb{F}$ . Computer scientists are trained on these  
 611 topics, and engineers need to be at least conversant with them.

612 **Irrational numbers: The cardinality of numbers may be ordered:**  $|\mathbb{I}| \ggg |\mathbb{Q}| \gg |\mathbb{N}| = |\mathbb{P}|$

613 The real line may be split into the irrationals and rationals. The rationals may be further split into  
 614 the integers and the fractionals. Thus, all is *not* integer. If a triangle has two integer sides, then the  
 615 hypotenuse must be irrational ( $\sqrt{2} = \sqrt{1^2 + 1^2}$ ). This leads us to a fundamental question: “Are there  
 616 integer solutions to Eq. 1.1?” We need not look further than the simple example  $\{3, 4, 5\}$ . In fact this  
 617 example does generalize, and the formula for computing an infinite number of integer solutions is called  
 618 *Euclid’s Formula*, which we will discuss in Section 2.1.3.

619 However, the more important point is that the cardinality of the irrationals is much larger than  
 620 any set other than the reals (i.e., complex numbers). Thus when we use computers to model physical  
 621 systems, we are constantly needing to compute with irrational numbers. But this is impossible since  
 622 every irrational numbers would require an infinite number of bits to represent it. Thus we must compute  
 623 with rational approximations to the irrationals. This means we need to use the fractionals. In the end,  
 624 we must work with the IEEE 754 floating point numbers,<sup>35</sup> which are fractionals, more fully discussed  
 625 in Section 2.1.1.

#### 626 1.2.4 Lec 5: Greatest common divisor (Euclidean algorithm)

627 The *Euclidean algorithm* is a method to find the greatest common divisor (GCD)  $k$  between two integers  
 628  $n, m$ , denoted  $k = \text{gcd}(n, m)$ , where  $n, m, k \in \mathbb{N}$ . For example  $15 = \text{gcd}(30, 105)$  since when factored  
 629  $(30, 105) = (2 \cdot 3 \cdot 5, 7 \cdot 3 \cdot 5) = 3 \cdot 5 \cdot (2, 7) = 15 \cdot (2, 7)$ . The Euclidean algorithm was known to the  
 630 Chinese (i.e., not discovered by Euclid) (Stillwell, 2010, p. 41).

631 **Why is the GCD important?** Computing the GCD is simple, whereas a full factoring is extremely  
 632 expensive. The GCD is important precisely because of the fundamental difficulty of factoring large  
 633 integers into their primes. This utility surfaces when the two numbers are composed of very large  
 634 primes. When two integers have no common factors they are said to be *coprime*, thus their GCD is 1.  
 635 The ratio of two integers which are coprime is automatically in *reduced form* (they have no common  
 636 factors).

637 For example  $4/2 \in \mathbb{Q}$  is not reduced since  $2 = \text{gcd}(4, 2)$ . Canceling out the common factor 2, gives  
 638 the reduced form  $2/1 = 2 \in \mathbb{N}$ . Thus if we wish to form the ratio of two integers, first compute the gcd  
 639 and remove it from the two numbers, then form the ratio. This assures the rational number is in its  
 640 reduced form. If the GCD were  $10^3$  digits it is obvious that the common factor must be removed before  
 641 any computation should proceed.

An example: Take the two integers  $[873, 582]$ . In factored form these are  $[\pi_{25} \cdot 3^2, \pi_{25} \cdot 3 \cdot 2]$ . Given  
 the factors, we see that the largest common factor is  $\pi_{25} \cdot 3 = 291$ . When we take the ratio of the two  
 numbers this common factor cancels

$$\frac{873}{582} = \frac{\cancel{\pi_{25}} \cdot \cancel{3} \cdot 3}{\cancel{\pi_{25}} \cdot \cancel{3} \cdot 2} = \frac{3}{2}.$$

642 Of course if we divide 582 into 873 this we will numerically obtain the answer  $1.5 \in \mathbb{R}$ . If the common  
 643 factor is large, a floating point number in  $\mathbb{F}$  is returned, since all floating point numbers are in  $\mathbb{F}$ .  
 644 But due to rounding errors, it may not be  $3/2$ . To obtain the exact answer, in  $\mathbb{F}$ , we need the GCD.  
 645 Removing large common factors, without actually factoring the two numbers, has obvious practical  
 646 utility.

<sup>35</sup>IEEE 754: <http://www.h-schmidt.net/FloatConverter/IEEE754.html>.

### Greatest Common Divisor: $k=\gcd(m,n)$

- Examples ( $m, n, k \in \mathbb{Z}$ ):
  - $\gcd(13 \cdot 5, 11 \cdot 5) = 5$  (The common 5 is the gcd)
  - $\gcd(13 \cdot 10, 11 \cdot 10) = 10$  (The  $\gcd(130, 110) = 10 = 2 \cdot 5$ , is not prime)
  - $\gcd(1234, 1024) = 2$  ( $1234=2 \cdot 617$ ,  $1024=2^{10}$ )
  - $\gcd(\pi_k \pi_m, \pi_k \pi_n) = \pi_k$
  - $k=\gcd(m,n)$  is the part that cancels in the fraction  $m/n \in F$
  - $m/\gcd(m,n) \in \mathbb{Z}$
- Co-primes ( $m \perp n$ ) are numbers with no common factors: i.e.,  $\gcd(m,n)=1$ 
  - The gcd of two primes is always 1:  $\gcd(13,11) = 1$ ,  $\gcd(\pi_m, \pi_n)=1$
  - $m = 7 \cdot 13, n = 5 \cdot 19 \Rightarrow (7 \cdot 13) \perp (5 \cdot 19)$
  - If  $m \perp n$  then  $\gcd(m,n) = 1$
  - If  $\gcd(m,n) = 1$  then  $m \perp n$
- The GCD may be extended to polynomials: e.g.,  $\gcd(ax^2 + bx + c, \alpha x^2 + \beta x + \gamma)$ 
  - $\gcd((x-3)(x-4), (x-3)(x-5)) = (x-3)$
  - $\gcd(x^2 - 7x + 12, 3(x^2 - 8x + 15)) = 3(x-3)$
  - $\gcd(x^2 - 7x + 12, (3x^2 - 24x + 45)) = 3(x-3)$
  - $\gcd((x-2\pi)(x-4), (x-2\pi)(x-5)) = (x-2\pi)$  (Needs long division)

Figure 1.6: The Euclidean algorithm for finding the GCD of two numbers is one of the oldest algorithms in mathematics, and is highly relevant today. It is both powerful and simple. It was used by the Chinese during the Han dynasty (Stillwell, 2010, p. 70) for reducing fractions. It may be used to find pairs of integers that are coprime (their gcd must be 1), and it may be used to identify factors of polynomials by long division. It has an important sister algorithm called the continued fraction algorithm (CFA), that is so similar in concept that Gauss referred to the Euclidean algorithm as the “continued fraction algorithm” (Stillwell, 2010, p. 48).

647 **Euclidean algorithm:** The algorithm is best explained by a trivial example: Let the two numbers  
 648 be 6, 9. At each step the smaller number (6) is subtracted from the larger (9) and the difference  
 649 (the remainder) and the smaller numbers are saved. This process continues until the two resulting  
 650 numbers are equal, at which point the GCD equals that final number. If we were to take one more  
 651 step, the final numbers would be the gcd and zero. For our example step 1 gives  $9-6=3$ , leaving 6 and  
 652 3. Step 2 gives  $6-3=3$  and 3. Since the two numbers are the same, the  $\text{GCD}=3$ . If we take one more  
 653 difference we obtain (3,0). We can easily verify this result since this example is easily factored (e.g.,  
 654  $3 \cdot 3, 3 \cdot 2) = 3(3, 2)$ . It may be numerically verified using the Matlab GCD command  $\text{gcd}(6, 9)$ , which  
 655 returns 3.

656 In Chapter 2, Section 2.1.2 (p. 79), we shall describe two methods for implementing this procedure  
 657 using matrix notation, and explore the deeper implications.

### 658 Coprimes

659 Related to the prime numbers are *co-primes*, which are integers that when factored, have no common  
 660 primes. For example  $20 = 5 \cdot 2 \cdot 2$  and  $21 = 7 \cdot 3$  have no common factors, thus they are coprime. Coprimes  
 661  $[m, n]$  may be indicated with the “perpendicular” notation  $n \perp m$ , spoken as “n is perpendicular (perp)  
 662 to m.” One may use the GCD to determine if two numbers are coprime. When  $\gcd(m, n) = 1$ ,  $m$  and  
 663  $n$  are coprime. For example since  $\gcd(21, 20)=1$  (i.e.,  $21 \perp 20$ ) they are coprime.

### 664 1.2.5 Lec 6: Continued fraction algorithm (CFA)

665 The Continued fraction algorithm was mentioned in Section 1.2.4 at the end of the discussion on  
 666 the GCD. These two algorithms (CFA vs. GCD) are closely related, enough that Gauss referred to  
 667 the Euclidean algorithm as the Continued fraction algorithm (i.e., the name of the CFA algorithm)

668 (Stillwell, 2010, P. 48). This question of similarity needs some clarification, as it seems unlikely that  
 669 Gauss would be confused about such a basic algorithm.

670 In its simplest form the CFA starts from a real decimal number and recursively expands it as  
 671 a fraction. It is useful for finding rational approximations to any real number. The GCD uses the  
 672 Euclidean algorithm on a pair of integers  $m > n \in \mathbb{N}$  and finds their greatest common divisor  $k \in \mathbb{N}$ .  
 673 At first glance it is not clear why Gauss would call the CFA the Euclidean algorithm. One must assume  
 674 that Gauss had some deeper insight into the relationship. If so, it would be valuable to understand.

675 In the following we refine the description of the CFA and give examples that go beyond the simple  
 676 cases of expanding numbers. The CFA of any number, say  $x_0$ , is defined as follows:

- 677 1. Start with  $n = 0$  and input target (starting value)  $x_0 \in \mathbb{R}$ .
- 678 2. If  $|x_n| \geq 1/2$  define  $a_n = \text{round}(x_n)$ , which rounds to the nearest integer.
- 679 3.  $r_n = x_n - a_n$  is the remainder. If  $r_n = 0$ , the recursion terminates.
- 680 4. Define  $x_{n+1} \equiv 1/r_n$  and return to step 2 (with  $n = n + 1$ ).

An example: Let  $x_0 \equiv \pi \approx 3.14159\dots$ . Thus  $a_0 = 3$ ,  $r_0 = 0.14159$ ,  $x_1 = 7.065 \approx 1/r_0$ , and  $a_1 = 7$ .  
 If we were to stop here we would have

$$\hat{\pi}_1 \approx 3 + \frac{1}{7 + 0.0625\dots} \approx 3 + \frac{1}{7} = \frac{22}{7}. \quad (1.2)$$

This approximation of  $\pi \approx 22/7$  has a relative error of 0.04%

$$\frac{22/7 - \pi}{\pi} = 4 \times 10^{-4}.$$

For the second approximation we continue by reciprocating the remainder  $1/0.0625 \approx 15.9966$  which  
 rounds to 16, resulting in the second approximation

$$\hat{\pi}_2 \approx 3 + 1/(7 + 1/16) = 3 + 16/(7 \cdot 16 + 1) = 3 + 16/113 = 355/113.$$

681 Note that if we had truncated 15.9966 to 15, the remainder would have been much larger, resulting  
 682 in a less accurate rational approximation. The recursion may continue to any desired accuracy as  
 683 convergence is guaranteed.

### *Rational approximation examples*

$$\begin{array}{ll} \frac{22}{7} = [3; 7] & \approx \pi + O(1.3 \times 10^{-3}) \\ \frac{355}{113} = [3; 7, 16] & \approx \pi + O(2.7 \times 10^{-7}) \\ \frac{104348}{33215} = [3; 7, 16, -249] & \approx \pi + O(3.3 \times 10^{-10}) \end{array}$$

Figure 1.7: The expansion of  $\pi$  to various orders using the CFA, along with the order of the error of each rational approximation. For example  $22/7$  has an absolute error ( $|22/7 - \pi|$ ) of about 0.13%.

**Notation:** Writing out all the fractions can become tedious. For example, expanding  $e$  using the Matlab command `rat(exp(1))` gives the approximation

$$3 + 1/(-4 + 1/(2 + 1/(5 + 1/(-2 + 1/(-7))))).$$

684 A compact notation for this these coefficients of the CFA is  $[3; -4, 2, 5, -2, -7]$ . Note that the leading  
 685 integer may be indicated by an optional semicolon to indicate the decimal point. Unfortunately Matlab  
 686 does not support the bracket notation.

687 If the process is carried further, the values of  $a_n \in \mathbb{N}$  give increasingly more accurate rational  
 688 approximations. If the floor rounding is used  $\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, \dots]$  whereas true  
 689 rounding gives  $\pi = [3; 7, 16, -294, 3, -4, 5, -15, \dots]$ , thus rounding introduces negative coefficients each  
 690 time a number rounds up.

691 When the CFA is applied and the expansion terminates ( $r_n = 0$ ), the target is rational. When  
 692 the expansion does not terminate (which is not always easy to determine), the number is irrational.  
 693 Thus the CFA has important theoretical applications regarding irrational numbers. You may try this  
 694 yourself using Matlab's `rats(pi)` command. Also try the Matlab command `rat(1+sqrt(2))`.

695 One of the useful things about the procedure, besides its being so simple, are its generalizations,  
 696 one of which will be discussed in Section 2.1.2 (p. 79).

A *continued fraction expansion* can have a high degree of symmetry. For example, the CFA of

$$\frac{1 + \sqrt{5}}{2} = 1 + \frac{1}{1 + \frac{1}{1 + \dots}},$$

697 Here the lead term in the fraction is *always* 1 ( $a_n = [1; 1, 1, \dots]$ ), thus the sequence will not terminate,  
 698 proving that  $\sqrt{5} \in \mathbb{I}$ . A related example is `rat(1+sqrt(2))`, which gives  $[2; 2, 2, 2, \dots]$ .

When expanding a target irrational number ( $x_0 \in \mathbb{I}$ ), and the CFA is truncated, the resulting rational fraction approximates the irrational target. For the example above, if we truncate at three coefficients ( $[1; 1, 1]$ ) we obtain

$$1 + \frac{1}{1 + \frac{1}{1+0}} = 1 + 1/2 = 3/2 = 1.5 = \frac{1 + \sqrt{5}}{2} + 0.118 + \dots$$

Truncation after six steps gives

$$[1, 1, 1, 1, 1, 1] = 13/8 \approx 1.6250 = \frac{1 + \sqrt{5}}{2} + .0070 \dots$$

699 Because all the coefficients are 1, this example converges very slowly. When the coefficients are large  
 700 (i.e., remainder small), the convergence will be faster. The expansion of  $\pi$  is an example of faster  
 701 convergence.

702 **In summary:** Every rational number  $m/n \in \mathbb{F}$ , with  $m > n > 1$ , may be uniquely expanded as a  
 703 continued fraction, with coefficients  $a_k$  determined using the CFA. When the target number is irrational  
 704 ( $x_0 \in \mathbb{Q}$ ), the CFA does not terminate, thus each step produces a more accurate rational approximation,  
 705 converging in the limit as  $n \rightarrow \infty$ .

706 Thus the CFA expansion is an algorithm that can, in theory, determine when the target is rational,  
 707 but with an important caveat: one must determine if the expansion terminates. In cases where the  
 708 expansion produces a repeating coefficient sequence, it is clear that the sequence cannot terminate.  
 709 The fraction  $1/3 = 0.33333\dots$  is an example of such a target where the CFA will terminate.<sup>36</sup>

## 710 WEEK 3

711

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<sup>36</sup>Taking the Fourier transform of the target number, represented as a sequence, could identify a periodic component. The number  $1/7 = [[1, 4, 2, 8, 5, 7]]_6$  has a 50 [dB] notch at  $0.8\pi$  [rad] due to its 6 digit periodicity, carried to 15 digits (Matlab precision), Hamming windows, and then zero padded to 1024 samples.

712 **1.2.6 Labor day**713 **1.2.7 Lec 7: Pythagorean triplets (Euclid's formula)**

*Euclid's formula* is a method for finding three integer lengths  $[a, b, c] \in \mathbb{N}$ , that satisfy Eq. 1.1. It is important to ask "Which set are the lengths  $[a, b, c]$  drawn from?" There is a huge difference, both practical and theoretical, if they are from the real numbers  $\mathbb{R}$ , or the counting numbers  $\mathbb{N}$ . Given  $p > q \in \mathbb{N}$ , the three lengths  $[a, b, c] \in \mathbb{N}$  of Eq. 1.1 are given by

$$a = p^2 - q^2, \quad b = 2pq, \quad c = p^2 + q^2. \quad (1.3)$$

This result may be directly verified, since

$$[p^2 + q^2]^2 = [p^2 - q^2]^2 + [2pq]^2$$

or

$$p^4 + q^4 + \cancel{2p^2q^2} = p^4 + q^4 - \cancel{2p^2q^2} + \cancel{4p^2q^2}.$$

714 Thus, this result is easily proven, given the solution. Construction the solution is more difficult.

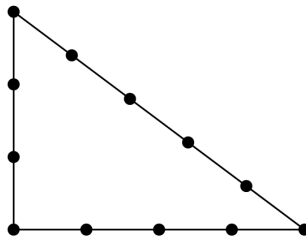


Figure 1.8: Beads on a string form perfect right triangles when number of beads on each side satisfy Eq. 1.1.

715 A well known example is the right triangle depicted in Fig. 1.8, defined by the integers  $[3, 4, 5] \in \mathbb{N}$ ,  
 716 having angles  $[0.54, 0.65, \pi/2]$  [rad], which satisfies Eq. 1.1. As quantified by Euclid's formula Eq. 1.3  
 717 (Section 2.2.1), there are an infinite number of *Pythagorean triplets* (PTs). Furthermore the seemingly  
 718 simple triangle, having angles of  $[30, 60, 90] \in \mathbb{N}$  [deg] (i.e.,  $[\pi/6, \pi/3, \pi/2] \in \mathbb{I}$  [rad]), has one irrational  
 719 ( $\mathbb{I}$ ) length  $([1, \sqrt{3}, 2])$ .

720 The technique for proving Euclid's formula for PTs  $[a, b, c] \in \mathbb{Q}$ , derived in Fig. 2.3 of Section 2.1.3,  
 721 is much more interesting than the PTs themselves.

722 The set from which the lengths  $[a, b, c]$  are drawn was not missed by the Indians, Chinese, Egyptians,  
 723 Mesopotamians, Greeks, etc. Any equation whose solution is based on integers is called a *Diophantine*  
 724 *equation*, named after the Greek mathematician Diophantus of Alexandria (c250 CE).

725 A stone tablet having the numbers engraved on it, as shown in Table 1.9 was discovered in  
 726 Mesopotamia and from the 19<sup>th</sup> century [BCE] and cataloged in 1922 by George Plimpton.<sup>37, 38</sup> These  
 727 numbers are  $a$  and  $c$  pairs from PTs  $[a, b, c]$ . Given this discovery, it is clear that the Pythagoreans were  
 728 walking in the footsteps of those well before them. Recently a second similar stone, dating between  
 729 350 and 50 [BCE] has been reported, that indicates early calculus on the orbit of Jupiter's.<sup>39</sup>

730 **1.2.8 Lec 8: Pell's Equation**

Pell's equation

$$x^2 - Ny^2 = 1, \quad (1.4)$$

<sup>37</sup><http://www.nytimes.com/2010/11/27/arts/design/27tablets.html>

<sup>38</sup>[https://en.wikipedia.org/wiki/Plimpton\\_322](https://en.wikipedia.org/wiki/Plimpton_322)

<sup>39</sup><http://www.nytimes.com/2016/01/29/science/babylonians-clay-tablets-geometry-astronomy-jupiter.html>

## EXERCISES

The integer pairs  $(a, c)$  in Plimpton 322 are

$a$	$c$
119	169
3367	4825
4601	6649
12709	18541
65	97
319	481
2291	3541
799	1249
481	769
4961	8161
45	75
1679	2929
161	289
1771	3229
56	106

Figure 1.3: Pairs in Plimpton 322

**1.2.1** For each pair  $(a, c)$  in the table, compute  $c^2 - a^2$ , and confirm that it is a perfect square,  $b^2$ . (Computer assistance is recommended.)

You should notice that in most cases  $b$  is a “rounder” number than  $a$  or  $c$ .

**1.2.2** Show that most of the numbers  $b$  are divisible by 60, and that the rest are divisible by 30 or 12.

Figure 1.9: “Plimpton-322” is a stone tablet from 1800 BCE, displaying  $a$  and  $c$  values of the Pythagorean triplets  $[a, b, c]$ . Numbers  $(a, c \in \mathbb{N})$ , with the property  $b = \sqrt{c^2 - a^2} \in \mathbb{N}$ , known as Pythagorean triplets, were found carved on a stone tablet from the 19<sup>th</sup> century [BCE]. Several of the  $c$  values are primes, but not the  $a$  values. The stone is item 322 (item 3 from 1922) from the collection of George A. Plimpton. –Stillwell (2010, Exercise 1.2)

731 with non-square  $N \in \mathbb{N}$  specified and  $a, b \in \mathbb{N}$  unknown, is related to the Euclidean algorithm (Stillwell,  
732 2010, 48). For example, with  $N = 2$ , one solution is  $a = 17, b = 12$  ( $17^2 - 2 \cdot 12^2 = 1$ ). This equation  
733 has a long history (Stillwell, 2010).

A 2x2 matrix recursion algorithmic, used by the Pythagoreans to investigate the  $\sqrt{2}$ ,

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}, \quad (1.5)$$

734 with  $[1, 0]^T$ , results in solutions of Pell’s equations (Stillwell, 2010, p. 44). Their approach was likely  
735 motivated by the Euclidean algorithm (GCD, p. 33), since  $y_n/x_n \rightarrow \sqrt{2}$  (Stillwell, 2010, p. 37,55).  
736 Note that this is a composition method, of 2x2 matrices, since the output of one matrix multiply is the  
737 input to the next.

738 **Asian solutions:** The first intended solutions of Pell’s was presented by Brahmagupta (c628), who  
739 independently discovered the equation (Stillwell, 2010, p. 46). Brahmagupta’s novel solution introduced  
740 a different *composition method* (Stillwell, 2010, p. 69), and like the Greek result, these solutions were  
741 incomplete.

742 Then in 1150CE, Bhâskara II obtained solutions using Eq. 1.5 (Stillwell, 2010, p.69). This is the  
743 solution method we shall explore here, as summarized in Fig. 1.10.

The best way to see how this recursion results in solutions to Pell's equation, is by example. Initializing the recursion with the trivial solution  $x_0 = [1, 0]^T$ , gives

$$\begin{aligned} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} && 1^2 - 2 \cdot 1^2 = -1 \\ \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} && 3^2 - 2 \cdot 2^2 = 1 \\ \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} &= \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} && (7)^2 - 2 \cdot (5)^2 = -1 \\ \begin{bmatrix} x_4 \\ y_4 \end{bmatrix} &= \begin{bmatrix} 17 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} && 17^2 - 2 \cdot 12^2 = 1 \\ \begin{bmatrix} x_5 \\ y_5 \end{bmatrix} &= \begin{bmatrix} 41 \\ 29 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 17 \\ 12 \end{bmatrix} && (41)^2 - 2 \cdot (29)^2 = -1 \end{aligned}$$

Thus the recursion results in a modified version of Pell's equation

$$x_n^2 - 2y_n^2 = (-1)^n,$$

744 thus only even values of  $n$  are solutions. This sign change had no effect on the Pythagoreans, who only  
 745 cared about  $y_n/x_n \rightarrow \sqrt{2}$ .

746 **Solution to Pell's equation:** By multiplying the matrix by  $1j$ , all the solutions to Pell's equation  
 747 are determined. This solution is shown in Fig. 1.10 for the case of  $N = 2$ , and again in Appendix D,  
 748 Eq. D.1, for  $N = 3$ . The math is straightforward and is easily verified using Matlab. From Fig. 1.10  
 749 we can see that every output this slightly modified matrix recursion gives solutions to Pell's equation  
 750 (Eq. 1.4).

751 For  $n = 0$  (the initial solution)  $[x_0, y_0]$  is  $[1, 0]$ ,  $[x_1, y_1] = j[1, 1]$ , and  $[x_2, y_2] = -[3, 2]$ . These are  
 752 easily computed by this recursion, and easily checked on a hand calculator (or using Matlab). Without  
 753 the  $j$  factor the sign would alternate; the  $1j$  factor corrects the alternation in sign, so every iteration  
 754 yields a solution.

- Case of  $N = 2$  &  $[x_0, y_0]^T = [1, 0]^T$   
 Note:  $x_n^2 - 2y_n^2 = 1$ ,  $x_n/y_n \xrightarrow{\infty} \sqrt{2}$

$$\begin{aligned} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} &= j \begin{bmatrix} 1 \\ 1 \end{bmatrix} = j \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} && j^2 - 2 \cdot j^2 = 1 \\ \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= j^2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = j \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} j \begin{bmatrix} 1 \\ 1 \end{bmatrix} && 3^2 - 2 \cdot 2^2 = 1 \\ \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} &= j^3 \begin{bmatrix} 7 \\ 5 \end{bmatrix} = j \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} j^2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} && (7j)^2 - 2 \cdot (5j)^2 = 1 \\ \begin{bmatrix} x_4 \\ y_4 \end{bmatrix} &= \begin{bmatrix} 17 \\ 12 \end{bmatrix} = j \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} j^3 \begin{bmatrix} 7 \\ 5 \end{bmatrix} && 17^2 - 2 \cdot 12^2 = 1 \\ \begin{bmatrix} x_5 \\ y_5 \end{bmatrix} &= j \begin{bmatrix} 41 \\ 29 \end{bmatrix} = j \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 17 \\ 12 \end{bmatrix} && (41j)^2 - 2 \cdot (29j)^2 = 1 \end{aligned}$$

Figure 1.10: This summarizes the solution of Pell's equation for  $N = 2$  using a slightly modified matrix recursion. Note that  $x_n/y_n \rightarrow \sqrt{2}$  as  $n \rightarrow \infty$ , which was what the Pythagoreans were pursuing.

At each iteration, the ratio  $x_n/y_n$  approaches  $\sqrt{2}$  with increasing accuracy, coupling it to the Euclidean algorithm (GCD). The value of  $41/29 \approx \sqrt{2}$ , with a relative error of  $<0.03\%$ . The solution for  $N = 3$  is discussed at the end of Appendix D.

**Relations to digital signal processing:** Today we recognize Eq. 1.5 as a *difference equation*, which is a pre-limit (pre Stream 3) form of differential equation. The Greek 2x2 form is an early precursor to 17<sup>th</sup> and 18<sup>th</sup> century developments in linear algebra. Thus the Greek's recursive solution for the  $\sqrt{2}$  and Bhâskara's (1030 CE) solution of Pell's equation, is an early precursor to discrete-time processing, as well as to calculus. Newton was fully aware of these developments as he reconstructed Diophantus chord/tangent method (Stillwell, 2010, p. 7, 49, 218).

Given the development of linear algebra c19<sup>th</sup> century, as discussed in Section 2.2.2 (page 85), this may be evaluated by eigenvector diagonalization.<sup>40</sup>

There are similarities between Pell's Equation and the Pythagorean theorem. As we shall see in Chapter 2, Pell's equation is related to the geometry of a hyperbola, just as the Pythagorean equation is related to the geometry of a circle. One might wonder if there is a Euclidean formula for the solutions of Pell's Equations. After all, these are all conic sections with closely related geometry, in the complex plane.

**Pell's Equation and irrational numbers:** Since the eigenvalues of Eq. 1.5 ( $\lambda_{\pm} = 1 \mp \sqrt{N} \notin \mathbb{N}$ ), solutions to Pell's equation raised the possibility that all numbers are not rational. This discovery of irrational numbers forced the jarring realization that the Pythagorean dogma "all is integer" was wrong. The significance of irrational numbers was far from understood.

## WEEK 4

### 1.2.9 Lec 9: Fibonacci sequence

Another classic problem, formulated by the Chinese, was the Fibonacci sequence, generated by the relation

$$f_{n+1} = f_n + f_{n-1}. \quad (1.6)$$

Here the next number  $f_{n+1}$  is the sum of the previous two. If we start from  $[0, 1]$ , this difference equation leads to the Fibonacci sequence  $f_n = [0, 1, 1, 2, 3, 5, 8, 13, \dots]$ . The solution may be generated by the recursion of a 2x2 matrix equation, or by the z-transform method. Alternatively, if we define  $y_{n+1} = x_n$ , then

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} \quad (1.7)$$

is equivalent to Eq. 1.6. The correspondence is easily verified. Starting with  $[x_n, y_n]^T = [0, 1]^T$  we obtain for the first few steps

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \dots$$

From the above  $x_n = [0, 1, 1, 2, 3, 5, \dots]$  is the Fibonacci sequence since the next number is the sum of the last two.

Note that this 2x2 equation is similar to Pell's equation, suggesting that an eigenfunction expansion of Eq. 1.7 may be used to analyze the sequence, as shown in Section 2.3.1 (p. 85) (Stillwell, 2010, 192).

<sup>40</sup>[https://en.wikipedia.org/wiki/Transformation\\_matrix#Rotation](https://en.wikipedia.org/wiki/Transformation_matrix#Rotation)



782 **1.2.10 Lec 10: Exam I (In class)**783 **WEEK 4-AE**

12.5.0

784

785 L 11 Stream 2: Algebra and geometry as physics (Physics drives early mathematics)

786 The first "algebra" (al-jabr) al-Khwarizmi (9<sup>th</sup>CE)

787 Polynomial equations in one and two variables (Stillwell, 2010, Ch. 6, p. 87)

788 Solution of the Quadratic Equation; Taylor series

789 Composition and intersection of polynomials

790 AE-1 (HW4) for 9/16/16; Add convolution problem. Verify due date.

791 **1.3 Algebraic Equations: Stream 2**792 **1.3.1 Lec 11 Algebra and geometry as physics**

793 Following Stillwell's history of mathematics, Stream 2 is geometry, which led to the merging of Euclid's  
 794 geometrical methods and the 9<sup>th</sup> century development of algebra by al-Khwarizmi (830 CE). This  
 795 integration of ideas lead Descartes and Fermat to develop of *analytic geometry*. While not entirely a  
 796 unique and novel idea, it was late in coming, given what was known at that time.

797 The mathematics up to the time of the Greeks, documented and formalized by Euclid, served  
 798 students of mathematics for more than two thousand years. Algebra and geometry were, at first, inde-  
 799 pendent lines of thought. When merged, the focus returned to the Pythagorean theorem, generalized as  
 800 analytic conic sections rather than as geometry in Euclid's Elements. With the introduction of Algebra,  
 801 numbers, rather than lines, could be used to represent a geometrical length. Thus the appreciation for  
 802 geometry grew given the addition of the rigorous analysis using numbers.

803 **Physics inspires algebraic mathematics:** The Chinese used music, art, navigation to drive math-  
 804 ematics. With the invention of algebra this paradigm did not shift. A desire to understand motions of  
 805 objects and planets precipitated many new discoveries. Galileo investigated gravity and invented the  
 806 telescope. Kepler investigated the motion of the planets. While Kepler was the first to appreciate that  
 807 the planets were described by ellipses, it seems he under-appreciate the significance of this finding, and  
 808 continued with his epicycle models of the planets. Using algebra and calculus, Newton formalized the  
 809 equation of gravity, forces and motion (Newton's three laws) and showed that Kepler's discovery of  
 810 planetary elliptical motion naturally follows from these laws. With the discovery of Uranus [in 1781]  
 811 "Kepler's theory was ruined." (Stillwell, 2010, p. 23).

Once Newton proposed the basic laws of gravity, he proceed to calculate, for the first time, the speed of sound. This required some form of the *wave equation*

$$\frac{\partial^2}{\partial x^2} p(x, t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} p(x, t), \quad (1.8)$$

812 a key equation in mathematical physics. The speed of sound is  $c = 343$  [m/s], which is a function of  
 813 the density  $\rho = 1.12$  [kg/m<sup>3</sup>] and the dynamic stiffness  $\eta P_0$  of air.<sup>41</sup>

If we substitute for the pressure

$$p(x, t) = e^{j2\pi(ft \pm kx)}, \quad (1.9)$$

814 where  $t$  is time and  $x$  is position, we find that  $(jk)^2 = s^2/c^2$ , or  $k = 2\pi/\lambda = 2\pi f/c$ .

815 While Newton's value for  $c$  was incorrect by the thermodynamic constant  $\sqrt{\eta}$ , a problem that would  
 816 take more than two hundred years to resolve, his success was important because it quantified the physics

verify eqs

<sup>41</sup> $c = \sqrt{\eta P_0/\rho}$ ,  $\eta = c_p/c_v = 1.4$  is the ratio of two thermodynamic constants, and  $P_0 = 10^5$  [Pa] is the barometric pressure of air.

817 behind the speed of sound, and demonstrated that momentum  $mv$ , not mass  $m$ , was transported by the  
 818 wave. His concept was correct, and his formulation using algebra and calculus represented a milestone  
 819 in science.

Newton's *Principia* was finally published in 1687, and the general solution to Newton's wave equation [i.e.,  $p(x, t) = G(t \pm x/c)$ ], where  $G$  is *any* function, was first published 60 years later by d'Alembert (c1747), which showed that for sounds of a single frequency, the wavelength  $\lambda$  and frequency  $f$  were related by

$$f\lambda = c.$$

820 Today d'Alembert's analytic wave solution is frequently written as Eq. 1.9 where  $k = c/\lambda$  is the *wave*  
 821 *number*. This formulation led to the frequency domain concept of Fourier analysis, based on the  
 822 *linearity* (i.e., superposition) property of the wave equation (Postulate P2: Lec. 1.3.11, p. 61).

823 The corresponding discovery for the formula for the speed of light was made 114 years later by  
 824 Maxwell (c1861). Maxwell's formulation also required great ingenuity, as it was necessary to hypothesize  
 825 an experimentally unmeasured term in his equations, to get the mathematics to correctly predict the  
 826 speed of light.

**The first Algebra:** Prior to the invention of algebra, people worked out problems as sentences using an obtuse description of the problem (Stillwell, 2010, p. 93). Algebra solved this problem. Algebra may be thought of as a compact language, where numbers are represented as abstract symbols (e.g.,  $x$  and  $\alpha$ ). The problems they wished to solve could be formulated in terms of sums of powers of smaller terms, the most common being powers of some independent variable (i.e., time or frequency). Today we call such an expression a *polynomial of degree  $n$*

$$P_n(z) \equiv z^n + a_{n-1}z^{n-1} + \cdots + a_0z^0 = \sum_{k=0}^n a_k z^k = \prod_{k=0}^n (z - z_k). \quad (1.10)$$

827 Here we have set  $a_n = 1$ . The coefficient  $a_n$  cannot be zero, or the polynomial would not have degree  
 828  $n$ . The solution is to force  $a_n = 1$ . This does not change the roots.

829 The key question was "What values of the  $z = z_k$  result in  $P_n(z_k) = 0$ ." In other words, what are  
 830 the roots  $z_k$  of the polynomial? The quest for the answer to this question consumed thousands of years,  
 831 with intense efforts by many aspiring mathematicians. In the earliest attempts, it was a competition  
 832 to evaluate mathematical acumen. Results were held as a secret to the death bed. It would be fair  
 833 to view this effort as an obsession. Today the roots of any polynomial may be found by numerical  
 834 methods, to very high accuracy. There are also a number of important theorems.

835 Of particular interest was composing a circle with a line, for example when the line does not touch  
 836 the circle, and finding the roots. There was no solution to this problem using geometry.

See assignments

837 **Finding roots of polynomials** The problem of factoring polynomials has a history more than  
 838 a millennium in the making. While the quadratic (degree  $N = 2$ ) was solved by the time of the  
 839 Babylonians (i.e., the earliest recorded history of mathematics), the cubic solution was finally published  
 840 by Cardano in 1545. The same year, Cardano's student solved the quartic ( $N = 4$ ). In 1826 it was  
 841 proved that the quintic ( $N = 5$ ) could not be factored by analytic methods.

As a concrete example we begin with trivial but important case of the quadratic

$$P_2(x) = ax^2 + bx + c. \quad (1.11)$$

The *roots* are those values of  $x$  such that  $P_2(x_k) = 0$ . One of the first results (recorded by the Babylonians, c2000 BCE) was the factoring of this equation by *completing the square* (Stillwell, 2010, p. 93). One may rewrite Eq. 1.11 as

$$\frac{1}{a}P_2(x) = (x + b/2a)^2 - (b/2a)^2 + c/a, \quad (1.12)$$

which is easily verified by expanding the squared term and canceling  $(b/2a)^2$

$$\frac{1}{a}P_2(x) = [x^2 + (b/a)x + (b/2a)^2] - (b/2a)^2 + c/a.$$

Setting Eq. 1.12 to zero and solving for the two roots  $x_{\pm}$ , gives the *quadratic formula*<sup>42</sup>

$$x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (1.13)$$

842 If  $b^2 + 4ac > 0$ , then the two roots are real ( $x_{\pm} \in \mathbb{R}$ ). Otherwise, they are complex. If  $a, b, c \in \mathbb{R}$ , which  
843 is typically the case, then this condition simplifies to  $ac < 0$  (i.e.,  $c/a > 0$ ).

844 No insight is gained by memorizing the quadratic formula (Eq. 1.13). On the other hand, an  
845 important concept is gained by learning Eq. 1.12, which can be very helpful when doing analysis.  
846 I suggest that instead of memorizing Eq. 1.13, memorize Eq. 1.12. Arguably, the factored form is  
847 easier to remember (or learn). Perhaps more importantly, the term  $b/2a$  has significance [ $P_2(-b/2a) =$   
848  $c/a - (b/2a)^2$ ], the sign of which determines if the roots are real or complex.

In third grade I learned the trick<sup>43</sup>

$$9 \cdot n = (n - 1) \cdot 10 + (10 - n). \quad (1.14)$$

849 With this simple rule I did not need to depend on my memory for the 9 times tables. How one thinks  
850 about a problem can have great impact.

**Analytic Series:** When the degree of the polynomial is infinite (i.e.,  $n = \infty$ ),  $P_{\infty}(x)$ ,  $x \in \mathbb{R}$  the series is called a *power series*. For values of  $x$  where the power series converges, it is said to be *analytic*. When the coefficients are determined by derivatives of  $P(x)$  evaluated at  $x = 0$ , then it is called a *Taylor series*. These various series play a special role in mathematics, as the coefficients of the series uniquely determine a function (e.g., via the derivatives). Two well known examples are the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^2 + \dots = \sum_{n=0}^{\infty} x^n$$

and exponential

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n.$$

**Region of convergence:** The set of values for which the series is analytic is called the *region of convergence*, or simply the ROC. Knowing how to determine the ROC for a given analytic function is quite important, and may not always be obvious. For the geometric series, the ROC is  $|x| < 1$ . The function  $1/(x^2 + 1)$  has the same ROC as the geometric series, since it may be written as (Section A.6, p. 130)

$$\frac{1}{x^2 + 1} = \frac{1}{(x + 1j)(x - 1j)} = \frac{1}{2j} \left( \frac{1}{x - 1j} - \frac{1}{x + 1j} \right).$$

851 Each term has an ROC of  $|x| < |1j| = 1$ . In other words, it is the sum of two geometric series, each  
852 having a pole at  $\pm 1j$ .

853 The exponential series converges for every finite value of  $x$  (the ROC is the entire open plane), thus  
854 the exponential is called an *entire function*.

<sup>42</sup>By direct substitution demonstrate that Eq. 1.13 is the solution of Eq. 1.11.

<sup>43</sup>E.G.:  $9 \cdot 7 = (7 - 1) \cdot 10 + (10 - 7) = 60 + 3$  and  $9 \cdot 3 = 2 \cdot 10 + 7 = 27$ . As a check note that the two digits of the answer must add to 9.

**Analytic functions:** Any function that has an analytic series representation is called an *analytic function*. Polynomials,  $1/(1-x)$  and  $e^{xj}$  are analytic functions. Because analytic functions are easily manipulated, term by term, they may be used to find solutions of differential equations. The derivatives are easily computed, since they may be uniquely determined, term by term. Every analytic function has a corresponding differential equation, that is determined by the coefficients of the analytic power series. An example is the exponential, which has the property that it is the eigenfunction of the derivative operation

$$\frac{d}{dx}e^{ax} = ae^{ax}.$$

855 This relationship is a common definition of the exponential function, which is a very special function.

856 Analytic functions may also be easily integrated, term by term. Newton took full advantage of  
857 these properties of analytic functions. To fully understand the theory of differential equations (DE),  
858 one needs to master single valued analytic functions and their analytic power series. Newton used the  
859 analytic series (*Taylor series*) to solve many problems, especially for working out integrals, allowing  
860 him to solve DEs.

During the 16th and 17th century, it had becoming clear that DEs can characterize a law of nature at a single point in space and time. For example the law of gravity (first formulated by Galileo to explain the dropping to two objects of different masses) must obey conservation of energy. Newton (c1687) went on to show that there must be a gravitational potential between two masses ( $m_1, m_2$ ) of the form

$$\phi(r) \propto \frac{m_1 m_2}{r}, \quad (1.15)$$

861 where  $r = |x_1 - x_2|$  is the Euclidean distance between the two point masses at locations  $x_1$  and  $x_2$ .  
862 Note that this is a power series, but with exponent of  $-1$ , which is called a *pole*.

863 **Single- vs. multi-valued functions:** Polynomials are single valued functions: for each  $x$  there is  
864 a single value of  $P_n(x)$ . The set of  $x$  values of a function are called the *domain* and the set of  $y(x)$   
865 values are called the *codomain*. The roles of the domain and codomain may be swapped, to obtain  
866 the inverse function, which is typically quite different in its properties compared to the function. For  
867 example  $y(x) = x^2 + 1$  has the inverse  $x = \pm\sqrt{y-1}$ , which is double valued, and complex when  $y < 1$ .  
868 Periodic functions such as  $y(x) = \sin(x)$  are more exotic, since  $x(y) = \arcsin(x) = \sin^{-1}(x)$  has an  $\infty$   
869 number of  $x(y)$  values for each  $y$ .

870 **Complex analytic functions:** When the argument of an analytic function is complex, that is,  
871  $x \in \mathbb{R}$  is replaced by  $z = x + yj \in \mathbb{C}$  (recall that  $\mathbb{R} \in \mathbb{C}$ ), the function is said to be a *complex analytic*.  
872 We shall delve further into this topic in Section 3.1.1 (p. 89).

When the argument of the exponential becomes complex, it is periodic since

$$e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t} e^{j\omega t} = e^{\sigma t} [\cos(\omega t) + j \sin(\omega t)],$$

thus

$$\cos(\omega) = \frac{e^{j\omega} + e^{-j\omega}}{2}.$$

873 It should be clear that the exponential and circular functions are fundamentally related. This exposes  
874 the family of *entire circular functions* [i.e.,  $e^s, \sin(s), \cos(s), \tan(s), \cosh(s), \sinh(s)$ ] and their inverses  
875  $[\ln(s), \arcsin(s), \arccos(s), \arctan(s), \cosh^{-1}(s), \sinh^{-1}(s)]$ , first fully elucidated by Euler (c1750). Thus  
876 there is a deep relationship between single vs. multi-valued functions, periodic functions, functions of  
877 a complex variable, and their inverse functions, all of which are complex analytic. We shall explore  
878 this in Lec 1.3.8 (p. 55).

879 Given a complex analytic function of a complex variable, one must resort to the *extended complex*  
880 *plane*, *Riemann sheets* and *branch cuts*, as discussed in Section 1.3.7 (p. 53). The extended complex  
881 plane is a tool that is needed throughout complex analytic functions, and into higher mathematics,

882 since at least the 16<sup>th</sup> century, and probably before. This topic is critically important in engineering  
 883 mathematics, and will be discussed in length in Sections 1.3.7-1.3.10 (pp. 53-59).

884 There seems to be some disagreement as to the status of multivalued functions: Are they functions,  
 885 or is a function strictly single valued? If so, then we are missing out on a host of possibilities, including  
 886 all the inverses of every complex analytic function.

887 **Impact on Physics:** It seems likely, if not obvious, that the success of Newton in mathematics was  
 888 his ability to find mathematics in physics. He was inventing new mathematics at the same time as  
 889 he was explaining new physics. The same might be said for Galileo. It seems likely that Newton was  
 890 extending the successful techniques and results of Galileo. Galileo died on Jan 8, 1642, and Newton  
 891 was born Jan 4, 1643, just short of one year later. Certainly Newton was well aware of Galileo's great  
 892 success, and naturally would have been influenced by them.

893 The application of complex analytic functions to physics was dramatic, as may be seen in the six  
 894 volumes on physics by Arnold Sommerfeld (1868-1951), and from the productivity of his many (36)  
 895 students (e.g., Debye, Lenz, Ewald, Pauli, Guillemin, Bethe, Heisenberg<sup>44</sup> and Seebach, to name a  
 896 few), notable coworkers (i.e., Leon Brillouin) and others (i.e., John Bardeen), upon whom he had a  
 897 strong influence. Sommerfeld is known for having many students who were awarded the Nobel Prize in  
 898 Physics, yet he was not (the prize is not awarded in Mathematics). Sommerfeld brought mathematical  
 899 physics (the merging of physical and experimental principles with mathematics) to a new level with  
 900 the use of complex integration of analytic functions to solve otherwise difficult problems, thus following  
 901 the lead of Newton who used real integration of Taylor series to solve differential equations, and later  
 902 Cauchy. While much of this work is outside the scope of the present discussion, it is helpful to know  
 903 who did what and when, and how people and concepts are connected.

## 904 WEEK 5

12.5.0

906 L 12 Examples of algebraic expressions in physics  
 907 Fundamental Thm of Algebra (d'Alembert,  $\approx 1760$ )  
 908 Analytic Geometry: Algebra + Geometry (Euclid to Descartes)  
 909 Newton and power series; Taylor series & ROC Composition of polynomial equations in two  
 910 variables.

912 L 13 Root classification for polynomials of Degree  $* = 1-4$  (p.102);  
 913 Convolution of monomials gives polynomial construction; Work out convolution for cubic  
 914 Show that  $a_{n-1}$  is sum of roots and  $a_0$  is product of roots. Quintic ( $* = 5$ ) cannot be solved

915 L 14 First Analytic Geometry (Fermat 1629; Descartes 1637) (p. 118) Descartes' insight: Composition  
 916 of two polynomials of degrees  $(m,n \rightarrow \text{one of degree } m \cdot n)$   
 917 Examples:  $x^4 \circ x^2 = x^8$ . Discuss Composition vs. intersection of functions.

### 918 1.3.2 Lec 12 Physical equations quadratic in several variables

When lines and planes are defined, the equations are said to be *linear* in the independent variables. In keeping with this definition of *linear*, we say that the equations are *non-linear* when the equations have degree greater than 1 in the independent variables. The term *bilinear* has a special meaning, in that

<sup>44</sup><https://www.aip.org/history-programs/niels-bohr-library/oral-histories/4661-1>

both the domain and codomain are linearly related by lines (or planes). As an example, an impedance is defined in frequency as the ratio of the voltage over the current

$$Z(s) = \frac{V(\omega)}{I(\omega)} = \frac{N(s)}{D(s)},$$

919 where  $Z(s)$  is the impedance and  $V$  and  $I$  are the voltage and current at radian frequency  $\omega$ . The  
 920 impedance is typically specified as the ratio of two polynomials,  $N(s)$  and  $D(s)$ , as functions of complex  
 921 Laplace frequency  $s = \sigma + j\omega$ . An example will be given in Section 1.3.6, Fig. 1.11. The bilinear  
 922 function may be written as  $D(s)V = N(s)I$ . Since  $D(s)$  and  $N(s)$  are both polynomials in  $s$ , this  
 923 is called bilinear. It comes from a corresponding scalar differential equation (Appendix A, Eq. A.1,  
 924 p. 129).

925 As an example consider the well known problem in geometry: the intersection of a plane with a  
 926 cone, which leads to the conic sections: the circle, hyperbola, ellipse and parabola, along with some  
 927 degenerate cases, such as the intersection of two straight lines<sup>45</sup>. If we stick to such 3-dimensional  
 928 objects, we can write equations in the three variables  $[x, y, z]$ , and be sure that they each represent  
 929 some physical geometry. For example  $x^2 + y^2 + z^2 = r_0^2$  is a sphere of radius  $r_0$ .

930 The geometry and the algebra do not always seem to agree. Which is correct? In general the  
 931 geometry only looks at the real part of the solution, unless you know how to tease out the complex  
 932 solutions. However the roots of any polynomial are from  $\mathbb{C}$ , so we may not ignore the imaginary roots,  
 933 as Newton did. There is an important related fundamental theorems, known as Bézout's theorem, that  
 934 address this case, as described next.

### 935 1.3.3 Lec 13: Polynomial root classification by convolution

936 Following the exploration of algebraic relationships by Fermat and Descartes, the first theorem was  
 937 being formulated by d'Alembert. The idea behind this theorem is that every polynomial of degree  $N$   
 938 (Eq. 1.10) has at least one root. This may be written as the product of the root and a second polynomial  
 939 of degree of  $N - 1$ . By the recursive application of this concept, it is clear that every polynomial of  
 940 degree  $N$  has  $N$  roots. Today this result is known as the *Fundamental Theorem of Algebra*:

Every polynomial equation  $p(z) = 0$  has a solution in the complex numbers. As Descartes observed, a solution  $z = a$  implies that  $p(z)$  has a factor  $z - a$ . The quotient

$$q(z) = \frac{p(z)}{z - a}$$

941 is then a polynomial of one lower degree. ... We can go on to factorize  $p(z)$  into  $n$  linear  
 942 factors.<sup>46</sup>

943 —Stillwell (2010, p. 285).

944 The ultimate expression of this theorem is given by Eq. 1.10 (p. 42), which indirectly states that an  
 945  $n^{\text{th}}$  degree polynomial has  $n$  roots.

946 Today this theorem is so widely accepted we may fail to appreciate it. Certainly about the time you  
 947 learned the quadratic formula, you were prepared to understand the concept. The simple quadratic case  
 948 may be extended a higher degree polynomial. The Matlab/Octave command `roots([a3, a2, a1, a0])` will  
 949 provide the roots of the cubic equation, defined by the four coefficients  $a_3, \dots, a_0$ . I don't know the  
 950 largest degree that can be accurately factored by Matlab/Octave, but I'm sure its well over  $N = 10^3$ .  
 951 Today, finding the roots numerically is a solved problem.

<sup>45</sup>Such problems were first studied algebraically and Descartes (Stillwell, 2010, p. 118) and Fermat (c1637).

<sup>46</sup>Look into expressing this in terms of complex 2x2 matrices, as on p. 26.

952 **Factorization versus convolution:** The best way to gain insight into the polynomial factorization  
 953 problem is through the inverse operation, multiplication of monomials. Given the roots  $x_k$ , there is  
 954 a simple algorithm for computing the coefficients  $a_k$  of  $P_n(x)$  for any  $n$ , no matter how large. This  
 955 method is called *convolution*. Convolution is said to be a *trap-door* since it is easy, while the inverse,  
 956 factoring (*deconvolution*), is hard, and analytically intractable for degree  $N \geq 5$  (Stillwell, 2010, p. 102).

### 957 Convolution of monomials

958 As outlined by Eq. 1.10, a polynomial has two descriptions, first as a series with coefficients  $a_n$  and  
 959 second in terms of its roots  $x_r$ . The question is “What is the relationship between the coefficients and  
 960 the roots?” The simple answer is that they are related by *convolution*.

Let us start with the quadratic

$$(x + a)(x + b) = x^2 + (a + b)x + ab,$$

961 where in vector notation  $[-a, -b]$  are the roots and  $[1, a + b, ab]$  are the coefficients.

To see how the result generalizes, we may work out the coefficients for the cubic ( $N = 3$ ). Multi-  
 plying the following three factors gives

$$(x - 1)(x - 2)(x - 3) = (x^2 - 3x + 2)(x - 3) = x(x^2 - 3x + 2) - 3(x^2 - 3x + 2) = x^3 - 6x^2 + 11x - 6.$$

962 When the roots are  $[1, 2, 3]$  the coefficients of the polynomial are  $[1, -6, 11, -6]$ . To verify, substitute  
 963 the roots into the polynomial, and show that they give zero. For example  $r_1 = 1$  is a root since  
 964  $P_3(1) = 1 - 6 + 11 - 6 = 0$ .

965 As the degree increases, the algebra becomes more difficult; even a cubic becomes tedious. Imagine  
 966 trying to work out the coefficients for  $N = 100$ . What is needed is a simple way of finding the  
 967 coefficients from the roots. Fortunately, *convolution* keeps track of the book-keeping, by formalizing  
 968 the procedure.

**Convolution of two vectors:** To get the coefficients by convolution, write the roots as two vectors  
 $[1, a]$  and  $[1, b]$ . To find the coefficients we must convolve the root vectors, indicated by  $[1, a] \star [1, b]$ ,  
 where  $\star$  denotes convolution. Convolution is a recursive operation. The convolution of  $[1, a] \star [1, b]$  is  
 done as follows: reverse one of the two monomials, padding unused elements with zeros. Next slide one  
 monomial against the other, forming the local *dot product* (element-wise multiply and add):

$$\begin{array}{cccc} a & 1 & 0 & 0 \\ 0 & 0 & 1 & b \\ = & 0 & & \end{array} \quad \begin{array}{ccc} a & 1 & 0 \\ 0 & 1 & b \\ = & x^2 & \end{array} \quad \begin{array}{ccc} a & 1 & 0 \\ 1 & b & 0 \\ = & (a + b)x & \end{array} \quad \begin{array}{cccc} 0 & a & 1 & \\ 1 & b & 0 & \\ = & abx^0 & & \end{array} \quad \begin{array}{cccc} 0 & 0 & a & 1 \\ 1 & b & 0 & 0 \\ = & 0 & & \end{array},$$

969 resulting in coefficients  $[\dots, 0, 0, 1, a + b, ab, 0, 0, \dots]$ .

970 By reversing one of the polynomials, and then taking successive dot products, all the terms in the  
 971 sum of the dot product correspond to the same power of  $x$ . This explains why convolution of the  
 972 coefficients gives the same answer as the product of the polynomials.

As seen by the above example, the position of the first monomial coefficients are reversed, and then  
 slid across the second set of coefficients, the dot-product is computed, and the result placed in the  
 output vector. Outside the range shown, all the elements are zero. In summary,

$$[1, -1] \star [1, -2] = [1, -1 - 2, 2] = [1, -3, 2].$$

In general

$$[a, b] \star [c, d] = [ac, bc + ad, bd],$$

Convolving a third term  $[1, -3]$  with  $[1, -3, 2]$  gives

$$[1, -3] \star [1, -3, 2] = [1, -3 - 3, 9 + 2, -6] = [1, -6, 11, -6],$$

973 which is identical to the cubic example, found by the algebraic method.

974 By convolving one monomial factor at a time, the overlap is always two elements, thus it is never  
 975 necessary to compute more than two multiplies and an add for each output coefficient. This greatly  
 976 simplifies the operations (i.e., they are easily done in your head). Thus the final result is more likely  
 977 to be correct. Comparing this to the algebraic method, convolution has the clear advantage.

978 **The product of monomial  $P_1(x)$  with a polynomial  $P_N(x)$  gives  $P_{N+1}(x)$ :** This statement is  
 979 another way of stating the *Fundamental Theorem of Algebra*. Each time we convolve a monomial with  
 980 a polynomial of degree  $N$ , we obtain a polynomial of degree  $N + 1$ . The convolution of two monomials  
 981 results in a quadratic (degree 2 polynomial). The convolution of three monomials give a cubic (degree  
 982 3). In general, the degree  $k$ , of the product of two polynomials, of degree  $n, m$ , is the sum of the degrees  
 983 ( $k = n + m$ ). For example, if the degrees are each 5 ( $n = m = 5$ ), then the resulting degree is 10.

984 In summary, the product of two polynomials of degree  $m, n$  having  $m$  and  $n$  roots, give a polynomial  
 985 of degree  $m + n$  having  $m + n$  roots. This is an analysis process, of merging polynomials, by coefficient  
 986 convolution. Multiplying polynomials is a merging process, into a single polynomial.

987 Note that the degree of a polynomial is one less than the length of the vector of coefficients. The  
 988 coefficient on the lead term should always be set to 1 since it cannot be zero, resulting in an illogical  
 989 result. *Always* normalize the lead term of  $P_n(x)$  to 1 (i.e.,  $a_N = 1$ ). This has no effect on the roots.

990 While you already know this theorem from high school algebra class, it is important to explicitly  
 991 state the *Fundamental theorem of algebra*.

### 992 1.3.4 Lec 14: Introduction to Analytic Geometry

993 *Analytic geometry* is the natural consequence of Euclid's Geometry (which deals with conical geometry  
 994 (e.g., points, lines, triangles, circles, spheres, ellipses, cones, etc., in two and three physical dimensions),  
 995 merged with algebra (which deals with simultaneous equations, roots of polynomials, analytic functions,  
 996 and ultimately, solutions of differential equations). The combination of Euclid's (323 BEC) geometry  
 997 and al-Khwarizmi's (830 CE) algebra provides a new powerful tool, *analytic geometry*.

There are many important relationships between Euclidean Geometry and 16th century algebra.  
 Important similarities include vectors, their Pythagorean lengths  $[a, b, c]$

$$c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \quad (1.16)$$

998  $a = x_2 - x_1$  and  $b = y_2 - y_1$ , and the angles. Geometry had no concept of coordinates of the vectors. This  
 999 is one of the main differences, the ability of algebra to compute with numbers. A detailed comparison  
 1000 is in order, attempted in Table 1.4.

1001 There are several basic concepts that merged under the development of analytic geometry

- 1002 1. Composition of functions: If  $y = f(x)$  and  $z = g(y)$  then the composition of functions  $f$  and  $g$  is  
 1003 denoted  $z(x) = g \circ f(x) = g(f(x))$ .
- 1004 2. Elimination: Given two functions  $f(x, y)$  and  $g(x, y)$ , elimination removes either  $x$  or  $y$ . This  
 1005 procedure, called *Gaussian Elimination*, was known to the Chinese.
- 1006 3. Vectors: Euclidean geometry includes the concept of a vector, as a line with a length and a slope.  
 1007 Analytic geometry defines a vector as an ordered set of points.
- 1008 4. Analytic geometry extends the ideas of geometry with the introduction of the product of two  
 1009 vectors, the dot product  $\mathbf{f} \cdot \mathbf{g}$  and the cross product  $\mathbf{f} \times \mathbf{g}$  (Appendix B).
- 1010 5. Intersection: While one may speak of the intersection of two lines, the term *intersection* is also a  
 1011 concept of set theory. To distinguish the very different uses of this important word, we shall coin  
 1012 a new spelling for the case of set theory: *intersetsion*. The intersetsion of two sets is indicate by  
 1013  $A \cap B$ .



Table 1.4: Comparison between Euclidean Geometry and Algebra. The concepts that are included in Euclidean geometry are under the column *Geometry*. Those under the column *Algebra* are unique to Algebra (not present in Geometry). The third column are uncertain.

Geometry	Algebra	Uncertain
Proof, Vector, Length, Point, Direction, $\leq 3$ dimensional, Intersection, Conic Section, Dot product, Square Root	$> 1$ dimensional, Numbers, Analytic series and functions, Composition, Elimination, Fund. Thm. Algebra, $\sin(\theta)$ , $\cos(\theta)$ , $e^{\theta j}$ , $\log(z)$ , Derivative, Calculus, Polynomial & Roots	Cross product, Recursion, Iteration (Newton's method)

What algebra added to geometry was the ability to compute with numbers. For example, the length of a line (Eq. 1.16) was measured in Geometry with a compass: numbers played no role. Once algebra was available, the line's Euclidean length could be computed from the coordinates of the two ends, defined by the *3-vector*

$$\vec{e} = x\hat{x} + y\hat{y} + z\hat{z},$$

1014 which represents a point at  $(x, y, z) \in \mathbb{R}^3 \subset \mathbb{C}^3$  in three dimensions, having *direction*, from the origin  
 1015  $(0, 0, 0)$  to  $(x, y, z)$ . A point, by itself, has no direction, thus is a scalar. An alternative matrix notation  
 1016 is  $\vec{e} = [x, y, z]^T$ , a column vector of three numbers. These two notations are very different ways of  
 1017 representing exactly the same thing. View them as equivalent concepts.

1018 By defining the vector, analytic geometry allows Euclidean geometry to become quantitative, be-  
 1019 yond the physical drawing of an object (e.g., a sphere, triangle or line). With analytic geometry we  
 1020 have the Euclidean concept of a vector, a line having a magnitude and direction (a slope), but in terms  
 1021 of physical coordinates (i.e., numbers). The difference between two vectors defines a length, a concept  
 1022 already present in Euclidean geometry, but not quantified other than by the spread of the compass  
 1023 points placed at the two ends of the line.

**Dot product of two vectors:** When using algebra, many concepts in geometry are made precise. There are many examples of how algebra extends Euclidean geometry, the most basic being the *dot product* between two vectors

$$\begin{aligned} \vec{x} \cdot \vec{\zeta} &= (x\hat{x} + y\hat{y} + z\hat{z}) \cdot (\alpha\hat{x} + \beta\hat{y} + \gamma\hat{z}) \\ &= \alpha x + \beta y + \gamma z. \end{aligned}$$

The dot product takes two vectors in  $\mathbb{R}^3$  and returns a real positive scalar function of the coordinates of the two vectors in  $\mathbb{R}$ , which represents a length. In matrix notation the dot product is written as

$$\vec{x} \cdot \vec{\zeta} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = [x, y, z] \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \alpha x + \beta y + \gamma z.$$

1024 Recall that in matrix notation a vector, by default, is always a column. The transpose is a row-vector.

**Norm of a vector:** The *vector*  $\vec{e} \in \mathbb{C}^3$  encompasses the scalar point  $(x, y, z) \in \mathbb{C}$  along with the scalar origin  $(0, 0, 0)$ , to define a the vector, having both a *magnitude*

$$||\vec{e}|| = \sqrt{\vec{e} \cdot \vec{e}} = \sqrt{x^2 + y^2 + z^2},$$

and *direction* (i.e., three angles). The magnitude of the vector (its length) is given by the Pythagorean theorem, in  $\mathbb{R}^3$ . Notionally,  $\|\vec{e}\|$  is called the *norm* of the vector and is defined as the positive square root of the dot product of the vector with itself. This is the generalization of a length, denoted the *norm* of  $\vec{e}$ , is

$$\|\vec{e}\| \equiv +\sqrt{\vec{e} \cdot \vec{e}}.$$

1025 The sign of the square-root must be positive as the length of a vector is not a concept from complex  
1026 analytic functions. Rather the length is a concept of Euclidean geometry, and it must always be positive  
1027 and real. A zero-length vector is a point, which has no direction (thus it is not a vector).

More generally, the Euclidean length of a line is given as the *norm* of the difference between two vectors

$$\begin{aligned} \|\vec{e}_1 - \vec{e}_2\|^2 &= (\vec{e}_1 - \vec{e}_2) \cdot (\vec{e}_1 - \vec{e}_2) \\ &= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2, \end{aligned}$$

1028 which is the Euclidean length between the two vectors. This length is not an analytic concept. Thus  
1029 the minus sign on the root makes no sense in this context.

1030 **Cross product of two vectors:** The second type of product between two vectors is the cross-  
1031 product  $\vec{e}_1 \times \vec{e}_2$ . While the dot product results in a scalar, the dot product results in a third vector,  
1032 having a third component, not in the plane of the two being multiplied. The cross product results in a  
1033 vector that points out of the plane defined by  $\vec{e}_1$  and  $\vec{e}_2$ . For example, if the two vectors are in  $\hat{x}$  and  
1034  $\hat{y}$ , then the cross-product has a component in  $\hat{z}$ . It is strictly in  $\hat{z}$  if the two vectors are perpendicular  
1035 to each other (i.e.,  $\hat{z} = \hat{x} \times \hat{y}$ ).

The formula for computing the cross product is

$$\vec{a} \times \vec{b} = (a_1\hat{x} + a_2\hat{y} + a_3\hat{z}) \times (b_1\hat{x} + b_2\hat{y} + b_3\hat{z}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

1036 The cross product of a vector with itself (or the difference between two vectors) is zero.

The dot product of a third vector  $\vec{c}$  with the cross product  $\vec{x} \times \vec{z}$  is computed from

$$\vec{c} \cdot (\vec{a} \times \vec{b}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

1037 This computes the volume of a parallelepiped (sort of a squashed sugar cube).

1038 **Impact of Analytic Geometry:** The most obvious development of algebra, given the creation  
1039 of analytic geometry, was a detailed analysis of the conic section, using algebra, rather than using  
1040 drawings made with a compass and ruler. A useful example is the composition of the line and circle, a  
1041 construction what was used many times over the history of mathematics. Once algebra was invented  
1042 the composition could be done using formulas.

1043 The first two mathematicians to appreciate this mixture of Euclid's geometry and the new algebra  
1044 were the French, Fermat and Descartes (Stillwell, 2010, p. 111-115); soon Newton contributed to this  
1045 effort, by the addition of physics (calculations in acoustics, orbits of the planets, and the theory of  
1046 gravity and light (Stillwell, 2010, p. 115-117).

1047 Given these new methods, many new solutions to problems emerged. The complex roots of poly-  
1048 nomials continued to appear, without any obvious physical meaning. This seem to have been viewed  
1049 as more of an inconvenience than a problem. Newton's solution to this dilemma was to simply ignore  
1050 the imaginary cases (Stillwell, 2010, p. 119). The resolution of this was eventually to be found in  
1051 Bézout's theorem, which states the number of roots of composition of two functions is determined by

1052 the product of their degrees. This problem is described as the *construct of equations* (Stillwell, 2010,  
 1053 p. 118). It was finally proved much later, by Bézout (1779), who showed how one needs to count all  
 1054 the roots, including multiple roots, and roots at infinity (Stillwell, 2010, p. 295).

### 1055 1.3.5 Lec 15 Gaussian Elimination

1056 The method for finding the intersection of equations is based on the recursive elimination of all the  
 1057 variables but one. This method, known as *Gaussian elimination*, works across a broad range of cases,  
 1058 but may be defined in a systematic procedure when the equations are linear in the variables.<sup>47</sup> Rarely  
 1059 do we even attempt to solve problems in several variables of degree greater than 1. But Gaussian  
 1060 eliminations can still work in such cases (Stillwell, 2010, p. 90).

In Appendix B the inverse of a 2x2 linear system of equations is derived. Even for a 2x2 case,  
 the general solution requires a great deal of algebra. Working out a numeric example of Gaussian  
 elimination is more instructive. For example, suppose we wish to find the intersection of the equations

$$\begin{aligned}x - y &= 3 \\2x + y &= 2.\end{aligned}$$

1061 This 2x2 system of equations is so simple that you may immediately see the solution: Adding the  
 1062 two equations, and the  $y$  term is eliminated, giving  $3x = 5$ . But doing it this way takes advantage of  
 1063 the specific example, and we need a method for larger systems of equations. We need a generalized  
 1064 (algorithmic) approach. This general approach is *Gaussian elimination*.

Start by writing the equations in a standardized *matrix* format

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}. \quad (1.17)$$

Next, eliminate the lower left term ( $2x$ ) using a scaled version of the upper left term ( $x$ ). Specifically,  
 multiply the first equation by  $-2$ , add it to the second equation, replacing the second equation with the  
 result. This gives

$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 - 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}. \quad (1.18)$$

1065 Note that the top equation did not change. Once the matrix is “upper triangular” (zero below the  
 1066 diagonal) you have the solution. Starting from the bottom equation,  $y = -4/3$ . Then the upper  
 1067 equation then gives  $x - (-4/3) = 3$ , or  $x = 3 - 4/3 = 5/3$ .

1068 In principle Gaussian elimination is easy, but if you make a calculation mistake along the way, it  
 1069 is very difficult to find the error. The method requires a lot of mental labor, with a high probability  
 1070 of making a mistake. You do not want to apply this method every time. For example suppose the  
 1071 elements are complex numbers, or polynomials in some other variable such as frequency. Once the  
 1072 coefficients become more complicated, the seeming trivial problem becomes highly error prone. There  
 1073 is a much better way, that is easily verified, which puts all the numerics at the end in a single step.

The above operations may be automated by finding a carefully chosen lower-diagonalization matrix  
 $L$  that does the same operation. For example let

$$L = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}. \quad (1.19)$$

Multiplying Eq. 1.17 by  $U$  we find

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \quad (1.20)$$

<sup>47</sup>[https://en.wikipedia.org/wiki/System\\_of\\_linear\\_equations](https://en.wikipedia.org/wiki/System_of_linear_equations)

1074 we obtain Eq. 1.18. With a little practice one can quickly and easily find a  $U$  that does the job of  
1075 removing elements below the diagonal.

In Appendix B the inverse of a general 2x2 matrix is summarized in terms of three steps: 1) swap the diagonal elements, 2) reverse the signs of the off diagonal elements and 3) divide by the determinant  $\Delta = ab - cd$ . Specifically

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \quad (1.21)$$

1076 There are very few things that you must memorize, but the inverse of a 2x2 is one of them. It needs  
1077 to be in your tool-bag of tricks, as you once did for the quadratic formula.

While it is difficult to compute the inverse matrix from scratch (Appendix B), it takes only a few seconds to verify it (steps 1 and 2)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & -ab + ab \\ cd - cd & -bc + ad \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}. \quad (1.22)$$

1078 Finally, dividing by the determinant gives the 2x2 identity matrix. A good strategy, when you don't  
1079 trust your memory, is to write down the inverse as best you can, and then verify.

Using 2x2 matrix inverse on our example, we find

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{1+2} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 \\ -6+2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -4/3 \end{bmatrix}. \quad (1.23)$$

1080 If you use this method, you will rarely (never) make a mistake, and the solution is easily verified.  
1081 Either you can check the numbers in the inverse, as was done in Eq. 1.22, or you can substitute the  
1082 solution back into the original equation.

### 1083 1.3.6 Lec 16: Transmission (ABCD) matrix composition method

1084 In this section we shall derive the method of composition of linear systems, known by several names  
1085 as the *ABCD Transmission matrix method*, or in the mathematical literature as the Möbius (bilin-  
1086 ear) transformation. By the application of the method of composition, a linear system of equations,  
1087 expressed in terms of 2x2 matrices, can represent a large family of differential equation networks.

1088 By the application of Ohm's law to the circuit shown in Fig. 1.11, we can model a cascade of such  
1089 cells. Since the CFA can also treat such circuits, as shown in Fig. 2.2 and Eq. 2.3, the two methods  
1090 are related to each other via the 2x2 matrix expressions.

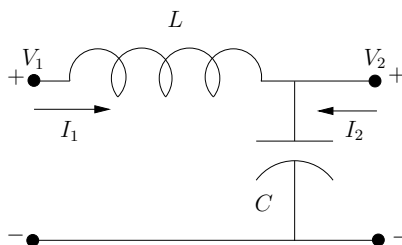


Figure 1.11: This is a single LC segment (cell) of the transmission line show in Fig. 2.2. It may be modeled by the ABCD model as the product of two matrices, as show below. A more detailed analysis shows that the wave number is  $\kappa = \sqrt{ZY} = \sqrt{sL \cdot sC} = s\sqrt{LC}$ . It follows that the wave velocity is  $c = 1/\sqrt{LC}$ . The characteristic resistance is  $r_c = \sqrt{Z/Y} = \sqrt{sL/sC} = \sqrt{L/C}$ .

**Example of the use of the ABCD matrix composition:** In Fig. 1.11 we see the network is composed of a series inductor (mass) having an impedance  $Z_l = sL$ , and a shunt capacitor (compliance) having an impedance  $Z_c = 1/sC$ , where  $s$  is the Laplace frequency. By Ohm's Law, each impedance

is describe by a linear relation between the current and the voltage. For the inductive impedance, applying Ohm's law, we find

$$V_1 - V_2 = Z_l I_1$$

where  $Z_l$  is the impedance of the inductor. For the capacitive impedance, applying Ohm's law we find

$$V_2 = (I_1 + I_2)Z_c,$$

where  $Z_c$  is the impedance of the capacitor. Each of these linear impedance relations may be written in matrix form. The series inductor equation gives

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & Z_l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_1 \end{bmatrix}, \quad (1.24)$$

while the shunt capacitor equation yields

$$\begin{bmatrix} V_2 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}, \quad (1.25)$$

1091 where  $Y_c = 1/Z_c$  is called the *admittance*.

When the second matrix equation for the shunt admittance (Eq. 1.25) is substituted into the series impedance equation (Eq. 1.24), we find the composed form ABCD matrix for the cell, as the product of two matrices

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & Z_l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} 1 + Z_l Y_c & Z_l \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \quad (1.26)$$

1092 This equation characterizes the relations between the input and output voltage and current of the cell  
1093 of Fig. 1.11 (p. 52).

For example, the ratio of the output to input voltage with the output unloaded ( $I_2 = 0$ )

$$\left. \frac{V_2}{V_1} \right|_{I_2=0} = \frac{1}{A} = \frac{1}{1 + Z_l Y_c} = \frac{Z_c}{Z_c + Z_l}$$

1094 is known as the *voltage divider equation*. Can you derive the formula for the current divider equation?  
1095 Hint:  $V_2 = 0$  For each matrix the input voltage and current are on the left (e.g.,  $[V_1, I_1]^T$ ), while the  
1096 output voltage and current is on the right (e.g.,  $[V_2, -I_2]^T$ ). The sign of the output current must be  
1097 negative, so that it is into the input node of the adjoining cell. In this way we may chain these matrices  
1098 together. On the left the input current is into the node. The sign of current  $I_2$  must be reversed in  
1099 sign, so that it will be into the node of the adjacent cell to the right (every current is defined to be into  
1100 each node).

**Matrix composition:** Matrix multiplication represents a composition of 2x2 matrices, because the output of the second matrix is the input of the first (this follows from the definition of composition). Thus the ABCD matrix is also known as the *chain matrix*. The general expression for a "chain matrix" is

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}, \quad (1.27)$$

1101 which is discussed in more detail in Section 3.3.2 (p. 99).

### 1102 1.3.7 Lec 17: Riemann Sphere: $3^d$ extension of chord and tangent method

1103 Once algebra was formulated c830 CE, mathematics was able to expand beyond the limits placed on it  
1104 by geometry on the real plane, and the verbose descriptions of each problem in prose (Stillwell, 2010,  
1105 p. 93). The geometry of Euclid's Elements had paved the way, but after 2000 years, the addition of the

1106 language of algebra would change everything. The analytic function was a key development, heavily  
 1107 used by both Newton and Euler. Also the investigations of Cauchy made important headway with his  
 1108 work on complex variables. Of special note was integration and differentiation in the complex plane of  
 1109 complex analytic functions, which is the topic of stream 3.

1110 It was Riemann, working with Gauss, who made the breakthrough, with the concept of the *extended*  
 1111 *complex plane*. The idea was based on the composition of a line with the sphere, similar to the derivation  
 1112 of Euclid's formula for Pythagorean triplets (Fig. 2.3, p. 83). But the impact was unforeseen, and it  
 1113 changed analytic mathematics forever, and the physics that was supported by it, by simplifying many  
 1114 important integrals, to the extreme. This idea is captured by the *Fundamental Theorem of Complex*  
 1115 *Integral Calculus* (Table 1.3 p. 29) and 4.3.1, p. 114).

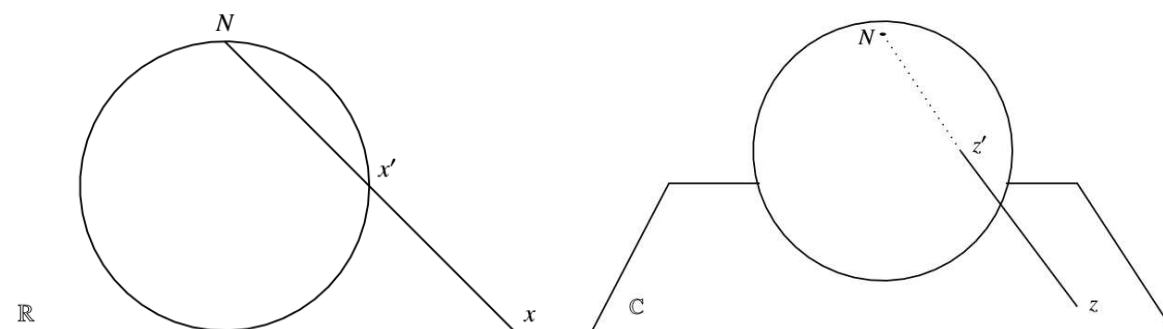


Figure 1.12: The left panel shows how the real line may be composed with the circle. Each real  $x$  value maps to a corresponding point  $x'$  on on the unit circle. The point  $x \rightarrow \infty$  then naturally maps to the north pole  $N$ . This simple idea may be extended with the composition of the complex plane with the unit sphere, thus mapping the plane onto the sphere. As with the circle, the point on the complex plane  $z \rightarrow \infty$  maps onto the north pole  $N$ . This construction is important because while the plane is open (does not include  $z \rightarrow \infty$ ), the sphere is analytic at the north pole. Thus the sphere defines the closed extended plane. Figure from Stillwell (2010, pp. 299-300).

1116 The idea is outlined in Fig. 1.12. On the left is a circle and a line. The difference between this  
 1117 case and the derivation of the Pythagorean Triplets is, that the line starts at the north pole, and ends  
 1118 on the real  $x \in \mathbb{R}$  axis at point  $x$ . At point  $x'$ , the line cuts through the circle. Thus the mapping  
 1119 from  $x$  to  $x'$  takes every point on  $\mathbb{R}$  to a point on the circle. For example, the point  $x = 0$  maps to  
 1120 the south pole (not indicated). To express  $x'$  in terms of  $x$  one must composition of the line and the  
 1121 circle, similar to the composition used in Fig. 2.3 (p. 83). The points on the circle, indicated here by  
 1122  $x'$ , require a traditional polar coordinate system, having a unit radius and an angle defined between  
 1123 the radius and a vertical line passing through the north pole. When  $x \rightarrow \infty$  the point  $x' \rightarrow N$ , known  
 1124 as the *point at infinity*. But this idea must go much further, as shown on the right half of Fig. 1.12.

1125 Here the real tangent line is replaced by the a tangent complex  $z \in \mathbb{C}$ , and the complex puncture  
 1126 point  $z' \in \mathbb{C}$ , in this case on the complex sphere, called the *extended complex plane*. This is a natural  
 1127 extension of the chord/tangent method on the left, but with significant consequences. The main  
 1128 difference between the complex plane  $z$  and the extended complex plane, other than the coordinate  
 1129 system, is what happens at the north pole. On the plane the point at  $|z| = \infty$  is not defined, whereas  
 1130 on the sphere the point at the north pole is simply another point, like every other point on the sphere.  
 1131

1132 Mathematically the plane is said to be an *open set* since the limit  $z \rightarrow \infty$  is not defined, whereas  
 1133 on the sphere the point  $z'$  is a member of a *closed set*, since the north pole *is* defined. The distinction  
 1134 between an open and closed set is important because the closed set allows the function to be analytic  
 1135 at the north pole, which it cannot be on the plane (since the point at infinity is not defined).

1136 The  $z$  plane may be replaced with another plane, say the  $w = F(z) \in \mathbb{C}$  plane, where  $w$  is some  
 1137 function  $F$  of  $z \in \mathbb{C}$ . We shall limit ourselves to *complex analytic functions* of  $z$ , namely  $w = F(z) =$   
 1138  $u(x, y) + v(x, y)j = \sum_{n=0}^{\infty} z^n$ .

1139 In summary, given a point  $z = x + yj$  on the open complex plane, we map it to  $w = F(z) \in \mathbb{C}$ , the  
 1140 complex  $w = u + vj$  plane, and from there to the closed extended complex plane  $w'(z)$ . The point of

1141 doing this is that it allows us to allow the function  $w'(z)$  to be analytic at the north pole, meaning it  
 1142 can have a convergent Taylor series at the point at infinity  $z \rightarrow \infty$ .

### 1143 Möbius bilinear transformation

1144 In mathematics the *Möbius transformation* has special importance because it is linear in its action. In  
 1145 the engineering literature this transformation is known as the *bilinear transformation*. Since we are  
 1146 engineers we shall stick with the engineering terminology. But if you wish to read about this on the  
 1147 internet, be sure to also search for the mathematical term, which may be better supported.

When a point on the complex plane  $z = x + yj$  is composed with the bilinear transform ( $a, b, c, d \in \mathbb{C}$ ),  
 the result is  $w(z) = u(x, y) + v(x, y)j$

$$w = \frac{az + b}{cz + d} \quad (1.28)$$

1148 the transformation is a cascade of four independent compositions

- 1149 1. translation ( $w = z + b$ )
- 1150 2. scaling ( $w = |a|z$ )
- 1151 3. rotation ( $w = \frac{a}{|a|}z$ ) and
- 1152 4. inversion ( $w = \frac{1}{z}$ )

1153 Each of these transformations are a special case of Eq. 1.28, with the inversion the most complicated.  
 1154 A wonderful video showing the effect of the bilinear (Möbius) transformation on the plane is available  
 1155 that I highly recommended you watch it: <https://www.youtube.com/watch?v=0z1fIsUNh04>

1156  
 1157 When the extended plane (Riemann sphere) is analytic at  $z = \infty$ , one may take the derivatives  
 1158 there, and meaningfully integrate through  $\infty$ . When the bilinear transformation rotates the Riemann  
 1159 sphere, the point at infinity is translated to a finite point on the complex plane, revealing normal  
 1160 characteristics. A second way to access the point at infinity is by inversion, which takes the north pole  
 1161 to the south pole, swapping poles with zeros. Thus a zero at infinity is the same as a pole at the origin,  
 1162 and vice-versa.

1163 This construction of the Riemann sphere and the Möbius (bilinear) transformation allow us to fully  
 1164 understand the point at infinity, and treat it like any other point. If you felt that you never understood  
 1165 the meaning of the point at  $\infty$  (likely), then this should help.

### 1166 1.3.8 Lec 18: Complex analytic mappings (Domain coloring)

1167 One of the most difficult aspects of complex functions of a complex variable is understanding what's  
 1168 going on. For example,  $w = \sin(\sigma)$  is trivial when  $\omega = 0$  (i.e.,  $s$  is real), because  $\sin(\sigma)$  is then real.  
 1169 But the general case,  $w(s) = \sin(s) \in \mathbb{C}$ , is not so easily visualized, because with argument  $s \in \mathbb{C}$ , the  
 1170 function  $w \in \mathbb{C}$ , since the mapping is from the  $s = \sigma + \omega j$  plane to the  $w(\sigma, \omega) = u(\sigma, \omega) + v(\sigma, \omega)j$   
 1171 plane, which in general is complicated. Fortunately with computer software today, this problem can  
 1172 be solved by adding color to the graph. A Matlab script `zviz.m` has been used to make these make  
 1173 the charts shown here.<sup>48</sup> This tool is known as *Domain coloring*. Rather than plotting  $u(\sigma, \omega)$  and  
 1174  $v(\sigma, \omega)$  separately, domain coloring allows us to display the entire function on one plot. Note that for  
 1175 this visualization we see the complex polar form of  $w(s)$ , rather than a rectangular  $(u, v)$  graph.

1176 In Fig. 1.13 we show this color code as a 2x2 dimensional domain-coloring graph. The color (hue)  
 1177 represents the phase, and intensity (dark to light) the magnitude. On the left is the reference condition,  
 1178 the identity mapping ( $w = s$ ). Red is  $0^\circ$ , violet is  $90^\circ$ , blue is  $135^\circ$ , blue-green is  $180^\circ$  and sea-green  
 1179 is  $-90^\circ$  (or  $270^\circ$ ). The hue (darkness) represents the magnitude. Since the function  $w = s$  has a dark

<sup>48</sup>URL for `zviz.m`: <http://jontalle.web.engr.illinois.edu/uploads/298/zviz.m>

1180 spot (a zero) at  $s = 0$ , becoming brighter as we move radially away from the zero. The figure on the  
 1181 right is  $w = F(z - 1)$ , which moves the zero to the right by 1. In summary, domain coloring gives the  
 1182 full picture of the complex analytic function mappings  $w(x, y) = u(x, y) + v(x, y)j$ .

1183 Two examples are given in Fig. 1.14  
 1184 to help you interpret the two complex  
 1185 mappings  $w = e^s$  (left) and its in-  
 1186 verse  $s = \ln(w)$ . The exponential is  
 1187 relatively easy to understand because  
 1188  $w = e^\sigma e^{j\omega}$ . The red region is where  
 1189  $\omega \approx 0$  in which case  $w \approx e^\sigma$ . As  $\sigma$   
 1190 becomes large and negative,  $w \rightarrow 0$  so  
 1191 the entire field at the left becomes dark.  
 1192 The field is becoming light on the right  
 1193 where  $w = e^\sigma \rightarrow \infty$ . If we let  $\sigma = 0$   
 1194 and look along the  $\omega$  axis, we see that  
 1195 the function is changing phase, green ( $-$   
 1196  $90^\circ$ ) at the top and violet ( $90^\circ$ ) at the  
 1197 bottom.

1198 Note the zero in  $\ln(w)$  at  $w = 1$ .  
 1199 The root of the log function is  $\log(s_r) =$   
 1200  $0$  is  $s_r = 1$ , since  $\log(1) = 0$ . More generally, taking the log of  $w = |w|e^{j\phi}$  gives  $s = \log(|w|) + j\phi$ . Thus  
 1201  $s$  can only be zero when the angle of  $w$  is zero (i.e.,  $\phi = 0$ ).

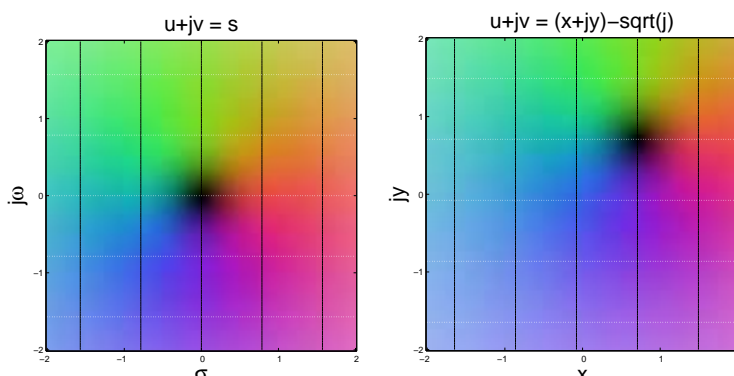


Figure 1.13: On the left is a color map showing the definition of the complex mapping from the  $s = \sigma + \omega j$  plane to the  $w(s) = u(\sigma, \omega) + v(\sigma, \omega)j$  plane. This mapping from one complex plane  $s = \sigma + \omega j$  to another  $w = u(\sigma, \omega) + v(\sigma, \omega)j$ , is visualized by the use of intensity (light/dark) to indicate magnitude, and color (hue) to indicate angle (phase). On the left  $w(s) = s = \sigma + \omega j$  (i.e.,  $u = \sigma$  and  $v = v$ ). On the right is  $w(z) = z - \sqrt{j}$ , a shift to the right and up by  $\sqrt{2}/2 = 0.707$ .

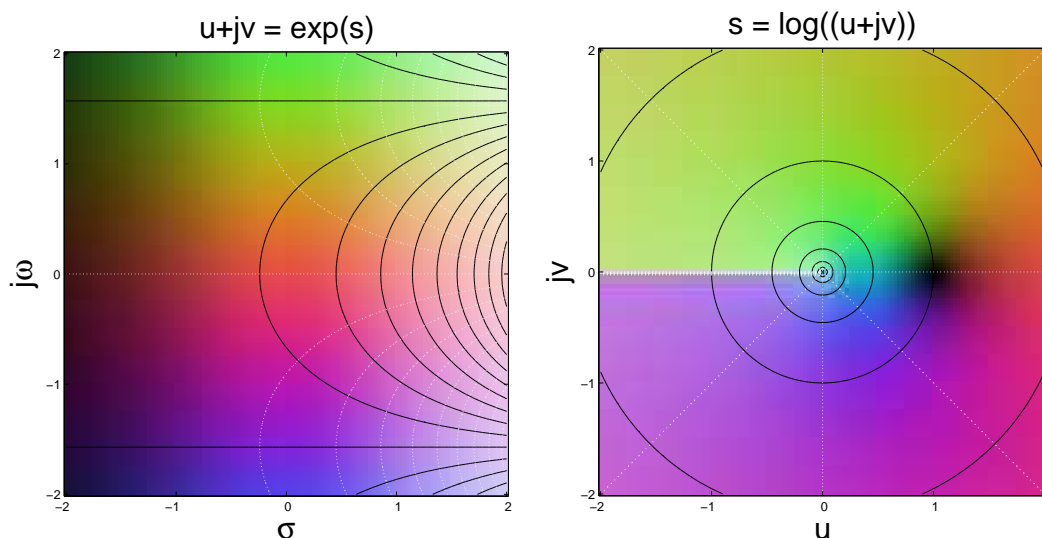


Figure 1.14: On the left is the function  $w(s) = e^s$  and on the right is  $s = \log(w)$ .

1202 **1.3.9 Lec 19: Signals: Fourier transforms**

1203 Two basic transformations in engineering mathematics are the Fourier and the Laplace transforms,  
 1204 which deal with time–frequency analysis.

The Fourier transform takes a time domain signal  $f(t) \in \mathbb{R}$  and transforms it to the frequency domain  $\omega \in \mathbb{R}$ , where it is complex ( $F(\omega) \in \mathbb{C}$ ). For the Fourier transform, both the time  $-\infty < t < \infty$  and frequency  $-\infty < \omega < \infty$  are strictly real. The time vs. frequency relationship, between  $f(t)$  and its transform  $F(\omega)$ , is indicated by the double arrow symbol  $\leftrightarrow$

$$f(t) \leftrightarrow F(\omega).$$



1205 Since the FT obeys superposition, it is possible to define a FT of a complex time function. This is  
 1206 useful in certain applications (i.e, Hilbert envelope, Hilbert transforms), but because the FT is not  
 1207 complex analytic, can be error prone if your not fully in charge of the method.

1208 The Laplace transform takes real signals  $f(t) \in \mathbb{R}$  as a function of real time  $t \in \mathbb{R}$ , that are strictly  
 1209 zero for negative time ( $f(t) = 0$  for  $t < 0$ ), and transforms them into complex functions  $F(s) \in \mathbb{C}$   
 1210 of complex frequency  $s = \sigma + \omega j$ . As for the Fourier transform, there is the convenient notation  
 1211  $f(t) \leftrightarrow F(s)$ .

1212 When a signal is zero for negative time  $f(t < 0) = 0$ , it is said to be *causal*, and the resulting  
 1213 transform  $F(s)$  is then complex analytic over significant regions of the  $s$  plane. For a function of time  
 1214 to be causal, time *must* be real, since if it were complex, it would lose the order property.

1215 Restriction on a function (e.g., real, causal, periodic, positive real part, etc.) are called a *symmetry*  
 1216 property. There are many forms of symmetry (Section 1.3.11, p. 61). The concept of symmetry is very  
 1217 general and widely used in both mathematics and physics, where it is more generally known as *Group*  
 1218 *theory*. We shall restrict ourselves to a few very basic symmetries.

1219 **Periodic signals:** Besides these two basic types of time–frequency transforms, there are several  
 1220 variants that depend on the symmetry in time and frequency. For example, when the time signal  
 1221 is sampled (discrete in time), the frequency response becomes periodic, leading to the *Discrete-time*  
 1222 *Fourier transform* (DTFT). When a time response is periodic, the frequency response is sampled (dis-  
 1223 crete in frequency), leading to the *Fourier Series*. These two symmetries may be simply characterized  
 1224 as *periodic in time*  $\Rightarrow$  discrete in frequency, and *periodic in frequency*  $\Rightarrow$  discrete in time. In Section  
 1225 3.4.2 we shall explain these concepts in greater detail, with examples. When a function is both dis-  
 1226 crete in time and frequency, it is necessarily periodic in time and frequency, leading to the *Discrete*  
 1227 *Fourier Transform* (DFT). The DFT is typically computed with an algorithm called the *Fast Fourier*  
 1228 *Transform* (FFT), which can dramatically speed up the calculation when the data is composite (e.g.,  
 1229 a power of 2 in length).

1230 A very important case is that of functions that are both causal (in time) and periodic (in frequency).  
 1231 The best know example is the class of signals that have *z transforms*. These are causal (one-sided in  
 1232 time), discrete-time signals. The harmonic series is the z-transform step function, discrete and one-sided  
 1233 in time, and analytic within the ROC, in the complex frequency ( $z$ ) domain.

1234 **Summary:** The definitions of the FT and LT transforms are superficially similar. The key difference  
 1235 is that the time response of the Laplace transform is causal, leading to complex analytic frequency  
 1236 responses. The frequency response of the Fourier transform is real, thus typically not analytic. These  
 1237 are not superficial differences. The concept of symmetry is helpful in understanding the many different  
 1238 types of time-frequency transforms. Two fundamental types of symmetry are causality and periodicity.  
 1239

1240 **Definition of the Fourier transform:** The forward transform takes  $f(t)$  to  $F(\omega)$  while the inverse  
 1241 transform takes  $F(\omega)$  to  $\tilde{f}(t)$ . The tilde symbol indicates that in general recovered inverse transform  
 1242 signal can be slightly different from  $f(t)$ . We will give examples of this below.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt \qquad \hat{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega \qquad (1.29)$$

$$F(\omega) \leftrightarrow f(t) \qquad \hat{f}(t) \leftrightarrow F(\omega). \qquad (1.30)$$

1243 **Properties of Fourier Transforms:**

- 1244 1. Both time  $t$  and frequency  $\omega$  are real.
- 1245 2. For the forward transform (time to frequency), the sign of the exponential is negative.

- 1246 3. The limits on the integrals in both the forward and reverse FTs are  $[-\infty, \infty]$ .
- 1247 4. When taking the inverse FT (IFT), the normalization factor of  $1/2\pi$  is required to cancel the  $2\pi$
- 1248 in the differential of the integral  $d\omega/2\pi = df$ , where  $f$  is frequency in [Hz], and  $\omega$  is the radian
- 1249 frequency [rads].
5. The Fourier step function may be defined by the use of superposition of 1 and  $\text{sgn}(t) = t/|t|$  as

$$\tilde{u}(t) \equiv \frac{1 + \text{sgn}(t)}{2} = \begin{cases} 1 & \text{if } t > 0 \\ 1/2 & t = 0 \\ 0 & \text{if } t < 0 \end{cases} .$$

Taking the FT of a delayed step function

$$\tilde{u}(t - T_0) \leftrightarrow \frac{1}{2} \int_{-\infty}^{\infty} [1 - \text{sgn}(t - T_0)] e^{-j\omega t} dt = \pi \tilde{\delta}(\omega) + \frac{e^{-j\omega T_0}}{j\omega}$$

1250 Thus the FT of the step function has the term  $\pi\delta(\omega)$  due to the 1 in the definition of the Fourier

1251 step. This term introduces a serious flaw with the FT of the step function: While it appears to

1252 be causal, it is not.

1253 6. The convolution  $\tilde{u}(t) \star \tilde{u}(t)$  is not defined because both  $1 \star 1$  and  $\tilde{\delta}^2(\omega)$  do not exist (and cannot

1254 be defined).

1255 7. The inverse FT has convergence problems whenever there is a discontinuity in the time response.

1256 This we indicate with a hat over the reconstructed time response. The error between the target

1257 time function and the reconstructed is zero in the root-mean sense, but not point-wise.

1258 Specifically, at the discontinuity point for the Fourier step function ( $t = 0$ ),  $\tilde{u}(t) \neq u(t)$ , yet

1259  $\int |\tilde{u}(t) - u(t)|^2 dt = 0$ . At the point of the discontinuity the reconstructed function displays *Gibbs*

1260 *ringing* (it oscillates around the step, hence does not converge at the jump).<sup>49</sup> More about this

1261 in Section 3.4.2.

1262 8. The FT is not always analytic in  $\omega$ , as in this example of the step function. The step function

1263 cannot be expanded in a Taylor series about  $\omega = 0$ , because  $\tilde{\delta}(\omega)$  is not analytic in  $\omega$ .

1264 9. The Fourier  $\delta$  function is denoted  $\tilde{\delta}(t)$ , to differentiate it from the Laplace delta function  $\delta(t)$ .

1265 They differ because the step functions differ, due to the convergence problem described above.

10. One may define

$$\tilde{u}(t) = \int_{-\infty}^t \tilde{\delta}(t) dt,$$

and define the somewhat questionable notation

$$\tilde{\delta}(t) =: \frac{d}{dt} \tilde{u}(t),$$

1266 since the Fourier step function is not analytic.

11. The  $\text{rec}(t)$  function is defined as

$$\text{rec}(t) = \frac{\tilde{u}(t) - \tilde{u}(t - T_0)}{T_0} = \begin{cases} 0 & \text{if } t > 0 \\ 1/T_0 & 0 < t < T_0 \\ 0 & \text{if } t < 0 \end{cases} .$$

1267 It follows that  $\tilde{\delta}(t) = \lim_{T_0 \rightarrow 0}$ . Like  $\tilde{\delta}(t)$ , the  $\text{rec}(t)$  has unit area.

<sup>49</sup>[https://en.wikipedia.org/wiki/Gibbs\\_phenomenon](https://en.wikipedia.org/wiki/Gibbs_phenomenon)

Table 1.5: The following table provides a short table of important Fourier Transforms. Note  $a > 0 \in \mathbb{R}$  has units [rad/s]. To flag this necessary condition, we use  $|a|$  to assure this condition will be met. The other constant  $T_0 \in \mathbb{R}$  [s] has no restrictions, other than being real. Complex constants may not appear as the argument to a delta function, since complex numbers do not have the order property.

$f(t) \leftrightarrow F(\omega)$	Name
$\tilde{\delta}(t) \leftrightarrow 1(\omega)$	Dirac
$1(t) \leftrightarrow 2\pi\tilde{\delta}(\omega)$	Dirac
$\text{sgn}(t) = \frac{t}{ t } \leftrightarrow \frac{2}{j\omega}$	
$\tilde{u}(t) = \frac{1(t) + \text{sgn}(t)}{2} \leftrightarrow \pi\tilde{\delta}(\omega) + \frac{1}{j\omega}$	step
$\tilde{\delta}(t - T_0) \leftrightarrow e^{-j\omega T_0}$	delay
$\tilde{\delta}(t - T_0) \star f(t) \leftrightarrow F(\omega)e^{-j\omega T_0}$	delay
$\tilde{u}(t)e^{- a t} \leftrightarrow \frac{1}{j\omega +  a }$	exp
$\text{rec}(t) = \frac{1}{T_0} [\tilde{u}(t) - \tilde{u}(t - T_0)] \leftrightarrow \frac{1}{T_0 s} (1 - e^{-j\omega T_0})$	pulse
$\tilde{u}(t) \star \tilde{u}(t) \leftrightarrow \tilde{\delta}^2(\omega)$ Not defined	NaN
FT Properties	
$\frac{d}{dt}v(t) \leftrightarrow j\omega V(\omega)$	deriv
$f(t) \star g(t) \leftrightarrow F(\omega)G(\omega)$	conv

1268 **1.3.10 Lec 20: Laplace transforms**

1269 When dealing with engineering problems it is convenient to separate the signals we use from the systems  
 1270 that process them. We do this by treating signals, such as a music signal, differently from a system,  
 1271 such as a filter. In general signals may start and end at any time. The concept of causality has no  
 1272 mathematical meaning in signal space. Systems, on the other hand, obey very rigid rules (to assure that  
 1273 they remain physical). These physical restrictions are described in terms of nine *Network Postulates*,  
 1274 which are briefly discussed in Lecture 1.3.11, and again in greater detail in Section 3.5.1.

**Definition of the Laplace transform:** The forward and inverse Laplace transforms are

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st} dt \qquad f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - \infty j}^{\sigma_0 + \infty j} F(s)e^{st} ds \qquad (1.31)$$

$$F(s) \leftrightarrow f(t) \qquad f(t) \leftrightarrow F(s) \qquad (1.32)$$

1275 1. Time  $t \in \mathbb{R}$  and Laplace frequency is defined as  $s = \sigma + \omega j \in \mathbb{C}$ .

Table 1.6: The following table provides a brief table of important Laplace Transforms. Assume that:  $f(t), \delta(t), u(t), \text{rec}(t), T_0, e, \in \mathbb{R}$  and  $F(s), G(s), s, a \in \mathbb{C}$ .

$f(t) \leftrightarrow F(s)$	Name
$\delta(t) \leftrightarrow 1$	Dirac
$\delta(t - T_0) \leftrightarrow e^{-sT_0}$	delay
$\delta(t - T_0) \star f(t) \leftrightarrow F(s)e^{-sT_0}$	-
$u(t) \leftrightarrow \frac{1}{s}$	step
$u(t) \star u(t) = tu(t) \leftrightarrow 1/s^2$	ramp
$u(t) \star u(t) \star u(t) = \frac{1}{2}t^2u(t) \leftrightarrow 1/s^3$	-
$e^{at}u(t) \leftrightarrow \frac{1}{s - a}$	exp
$\text{rec}(t) = \frac{1}{T_0} [u(t) - u(t - T_0)] \leftrightarrow \frac{1}{T_0s} (1 - e^{-sT_0})$	pulse
LT Properties	
$\frac{d}{dt}f(t) \leftrightarrow sF(s)$	deriv
$f(t) \star g(t) \leftrightarrow F(s)G(s)$	conv
$u(t) \star f(t) = \int_{0^-}^t f(t)dt \leftrightarrow \frac{F(s)}{s}$	conv

- 1276 2. When taking the forward transform ( $t \rightarrow s$ ), the sign of the exponential is negative. This is  
 1277 necessary to assure that the integral converges when the integrand  $f(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . For  
 1278 example, if  $f(t) = e^t u(t)$  (i.e., without the negative  $\sigma$  exponent), the integral does not converge.
- 1279 3. The target time function  $f(t < 0) = 0$  (i.e., it must be causal). The time limits are  $0^- < t < \infty$ .  
 1280 Thus the integral must start from slightly below  $t = 0$  to integrate over a delta functions at  $t = 0$ .  
 1281 For example if  $f(t) = \delta(t)$ , the integral must include both sides of the impulse. If you wish to  
 1282 include non-causal functions such as  $\delta(t + 1)$  it is necessary to extend the lower limit to include  
 1283 it. In such cases simply set the lower limit to  $-\infty$ , and let the integrand to determine the limits.
- 1284 4. The limits on the integrals of the forward transform are  $t : (0^-, \infty) \in \mathbb{R}$ , and the reverse LTs are  
 1285  $[\sigma_0 - \infty, \sigma_0 + \infty] \in \mathbb{C}$ . These limits will be further discussed in Section 1.4.9 (p. 68).
- 1286 5. When taking the inverse FT (IFT), the normalization factor of  $1/2\pi j$  is required to cancel the  
 1287  $2\pi j$  in the differential  $ds$  of the integral.
- 1288 6. The frequency for the LT must be is complex, and in general  $F(s)$  is complex analytic for  $\sigma > \sigma_0$ .  
 1289 It follows that the real and imaginary parts of  $F(s)$  are related. Given  $\Re F(s)$  it is possible to  
 1290 find  $\Im F(s)$  (Boas, 1987). More on this in Section 3.4.2 (p. 102).

1291 7. To take the inverse Laplace transform, we must learn how to integrate in the complex  $s$  plane.  
1292 This will be explained in Section 1.4.6 (p. 68).

8. The Laplace step function is defined as

$$u(t) = \int_{-\infty}^t \delta(t) dt = \begin{cases} 1 & \text{if } t > 0 \\ \text{NaN} & t = 0 \\ 0 & \text{if } t < 0 \end{cases} .$$

1293 Alternatively one could define  $\delta(t) = du(t)/dt$ .

9. It is easily shown that  $u(t) \leftrightarrow 1/s$  by direct integration

$$F(s) = \int_0^{\infty} u(t) e^{-st} dt = -\frac{e^{-st}}{s} \Big|_0^{\infty} = \frac{1}{s} .$$

1294 With the LT there is no Gibbs effect, as the step function, with a true discontinuity, is exactly  
1295 represented by the inverse LT.

10. In many physical applications, the Laplace transform takes the form of a ratio of two polynomials. In such case the roots of the numerator polynomial are called the *zeros* while the roots of the denominator polynomial are called the *poles*. For example the LT of  $u(t) \leftrightarrow 1/s$  has a pole at  $s = 0$ , which represents integration, since

$$u(t) \star f(t) = \int_{-\infty}^t f(\tau) d\tau \leftrightarrow \frac{F(s)}{s} .$$

1296

1297 11. The LT is quite different from the FT in terms of its analytic properties. For example, the step  
1298 function  $u(t) \leftrightarrow 1/s$  is complex analytic everywhere, except at  $s = 0$ . The FT of  $1 \leftrightarrow 2\pi\tilde{\delta}(\omega)$  is  
1299 not analytic anywhere.

1300 Once complex integration in the complex plane has been defined (Section 1.4.3, p. 113), we can  
1301 justify the definition of the inverse LT (Eq. 1.31).

### 1302 1.3.11 Lec 21: The 9 postulates of systems

1303 Systems of differential equations, such as the wave equation, need a mathematical statement of under-  
1304 lying properties, which we present here in terms of nine *network postulates*:

1305 (P1) *causality* (non-causal/acausal): Causal systems respond when acted upon. Virtually all physical  
1306 systems obey causality. An example of a causal system is an integrator, which has a response of  
1307 a step function. Filters are also examples of causal systems. Signals represent acausal responses.  
1308 They do not have a clear beginning or end, such as the sound of the wind or traffic noise.

1309 (P2) *linearity* (nonlinear): Linear systems obey superposition. If two signals  $x(t)$  and  $y(t)$  are the  
1310 inputs to a linear system, producing outputs  $x'(t)$  and  $y'(t)$ , then if the inputs are presented  
1311 together as  $ax(t) + by(t)$ , with weights  $a, b \in \mathbb{R}$ , then the output will be  $ax'(t) + by'(t)$ . If either  
1312  $a, b$  are zero, that signal is removed from the output.

1313 Nonlinear systems mix two inputs, thereby producing other signals not present in the input. For  
1314 example, if the inputs to a nonlinear system are two sine waves, the output will contain distortion  
1315 components, having frequencies not present at the input. An example of a nonlinear system is  
1316 one that multiplies the two inputs. A second is a diode, which rectifies a signal, letting current  
1317 flow only in one direction. Most physical systems have some degree of nonlinear response, but  
1318 this is not always desired. Other systems are designed to be nonlinear, such as the diode example.

- 1319 (P3) *real* (complex) time response : Typically systems are “real in, real out.” They do not naturally  
 1320 have complex responses (real and imaginary parts). While a Fourier transform takes real inputs  
 1321 and produces complex outputs, this is not an example of a complex time response. P3 is a  
 1322 characterization of the input signal, not its Fourier transform.
- 1323 (P4) *passive* (active): An active system has a power source, such as a battery while a passive system  
 1324 has no power source. While you may consider a transistor amplifier to be active, it is only so  
 1325 when connected to a power source.
- 1326 (P5) *time-invariant* (time varying): For a system to be time varying system its properties must depend  
 1327 on when the signal starts and stops. If the output, relative to the input, is independent of the  
 1328 starting time, then the system is time-invariant.
- 1329 (P6) *reciprocal* (anti-reciprocal): In many ways this is the most difficult property to understand. It  
 1330 is best characterized by the ABCD matrix. If  $B = C$  it is said to be reciprocal. If  $B = -C$   
 1331 it is said to be anti-reciprocal. The loudspeaker is anti-reciprocal, which is why it is modeled by  
 1332 the gyrator rather than a transformer. All non-reciprocal systems are modeled by such gyrator,  
 1333 which swap the force and flow variables. In some ways this property is beyond the scope of this  
 1334 book.
- 1335 (P7) *reversibility* (non-reversible): If the system can be flipped, between input and output, then it is  
 1336 said to be reversible. In terms of the ABCD parameters, if  $A = D$  it is reversible.
- 1337 (P8) *space-invariant* (space-variant): If a system operates independently as a function of where it  
 1338 physically is in space, then it is space-invariant. When the parameters that characterize the  
 1339 system depend on position, it is space-variant.
- 1340 (P9) *quasi-static* (multi-modal): Quasi-static follows from systems that are small compared to the  
 1341 wavelength. This is a very general assumption, that must be false when the frequency is raised  
 1342 and the wavelength becomes short. Thus this is also known as the *long-wavelength* approximation.  
 1343 It is a very powerful tool in modeling systems such as transmission lines.

1344 Each postulate has two (in one case three) categories. For example for (P2) a system is either linear  
 1345 or non-linear and for (P1) is either causal, non-causal or acausal. P6 and P9 only apply to 2-port  
 1346 networks (those having an input and an output). The others can apply to both a 2- or 1-port networks  
 1347 (e.g., an impedance is a 1-port).

1348 **Include several examples of different types of systems.**

1349 Related forms of these postulates had been circulating in the literature for many years, widely  
 1350 accepted in the network theory literature (Van Valkenburg, 1964a,b; Ramo et al., 1965). The first six  
 1351 of these were formally introduced Carlin and Giordano (1964), and (P7-P9) were added by Kim et al.  
 1352 (2016).

### 1353 1.3.12 **Lec 22: Exam II (Evening Exam)**

## 1354 1.4 Stream 3: Scalar (Ordinary) Differential Equations

1355 Stream 3 is  $\infty$ , a concept which inspires “very large.” In the case of calculus,  $\infty$  means “very small,”  
 1356 since taking a limit requires small numbers. Taking the limit means you never reaching the target.  
 1357 This is a concept that the Greeks called *Zeno’s paradox* (Stillwell, 2010, p. 76).

1358 When speaking of the class of *ordinary* (versus *vector*) differential equations, the term *scalar* is  
 1359 preferable, since the term “ordinary” is vague. There are a special subset of Fundamental theorems for  
 1360 scalar calculus, all of which are about integration, and the first being Leibniz’s theorem. These will be  
 1361 discussed below, and in Sections 1.4.3, 4.2.2 and 4.3.2 (p. 66 and 113-114).

1362 **The beginning of modern Mathematics**

1363 The birth of mathematics as we know it today, occurred during the 16th to 18th centuries, perhaps  
 1364 starting with Galileo, Descartes, Fermat, Newton, the Bernoulli family, and Euler, as outlined in  
 1365 Fig. 1.4. Galileo was formative due to his fame, fortune, and his stance against the powerful Catholic  
 1366 church and its establishment. His creativity in scientific circles was certainly well known due to his  
 1367 many skills and accomplishments. Fermat and Descartes were at the forefront of merging algebra and  
 1368 geometry, but while Fermat kept meticulous notebooks, he did not publish what he knew, and tended to  
 1369 be secretive. Such secret behavior dramatically changed with the Bernoulli family. The oldest brother,  
 1370 Jacob, taught his much younger brother Johann, who then taught his son Daniel, and Johann's star  
 1371 pupil, Euler. Euler first mastered all the tools and then published, with prolific intensity previously  
 1372 unseen.

1373 **1.4.1 Lec 23: Newton, the Bernoullis, Euler, and circular functions**

The first task was to master the family of circular functions,  $e^x$ ,  $\sin(x)$ ,  $\cos(x)$ , and the log function, a task begun by the Bernoulli family, but mastered by Euler. Euler sought relations between all these functions, some of which are not thought of as being related, such as the log and sin functions. The connection that may "easily" be made is through the Taylor series. By the manipulation of the series representations, the tight relationship between  $e^x$ , and the sin and cos were captured with Euler's formula

$$e^{jx} = \cos(x) + j \sin(x).$$

1374 This formula not only relates the exponential function to the circular functions, but implicates  $j = \sqrt{-1}$   
 1375 as well. While every high school student has these relations drilled into their head, it is doubtful (i.e.,  
 1376 rare) that they fully appreciate their meaning. While Euler found these relationships with the  
 1377 use of the power series, namely analytic functions, neither he, nor those who followed, appreciated the  
 1378 importance of  $j$  and its role. Newton famously ignored imaginary numbers, and called them imaginary,  
 1379 in a disparaging (pejorative) way. Given Newton's prominence, that certainly must have attenuated  
 1380 any interest in any complex algebra, even though it had been previously quantified by Bombelli in  
 1381 1525, likely based on his amazing finding in the Vatican library of Diophantus' book *Arithmetic*.

1382 Much was yet to be done, by Newton and Leibniz, with the development of calculus, again based on  
 1383 the analytic function concept, by integration of functions based on their power series representation,  
 1384 term by term. This was a powerful technique, but as stated, incomplete, because the series is only  
 1385 valid within the ROC, but more importantly, Newton (and others) failed to recognize the powerful  
 1386 generalization to complex analytic series, and their functions.

In some real sense, the story of calculus truly begins with the *Fundamental theorem of calculus*, also known generically as *Leibniz's formula*. Simply stated, the area under a real curve  $x$ ,  $f(x) \in \mathbb{R}$ , only depends on the end points

$$F(x) = F(a) + \int_a^x f(\xi) d\xi.$$

1387 This makes sense intuitively, by a simple drawing of  $f(x)$  (or of  $f(t)$ ), where  $x$  represents a spatial  
 1388 dimension [meters], and  $t$  a temporal variable [seconds]. As before, when we start from a physical  
 1389 intuition, to create a key mathematical concept, the area under a function agrees with common sense.  
 1390 For example, if we travel from point  $a$  to  $x = b$ , along a straight line measured by  $\xi$ , the distance  
 1391 traveled depends on the length of the line. This is reminiscent of Eq. 1.1 (p. 15), which is in  $\mathbb{R}^2$ ,  
 1392 since it is about lengths and areas.

1393 It is rarely stated that the variable that we are integrating over, either  $x$  (space) or  $t$  (time), is real  
 1394  $x, t \in \mathbb{R}$ , since that obvious fact is implicit, due to the physical nature of the formulation of the integral.  
 1395 But this intuition must change once complex numbers are included with  $s \in \mathbb{C}$ , where  $s = \sigma + \omega j$ , as  
 1396 later first developed by Cauchy (1789-1857) and Gauss (1777-1855).

The fact that time and space are real variables is more than an assumption, rather it is a requirement, due to the *order property*. Real numbers have order. For example if  $t = 0$  is now (the present),

then  $t < 0$  is the past and  $t > 0$  is the future. The order property of time and space allows one to order these along a real axis. If the axis were complex, as in frequency  $s$ , the order property is lost. It follows that if we desire order, time and space must be real ( $t, x \in \mathbb{R}$ ). Interestingly, it has been forcefully and successfully argued that time and space are related and are even the same thing, namely that  $\vec{x} \in \mathbb{R}^3$  and  $t \in \mathbb{R}$  may be combined as

$$\zeta = jct \pm \vec{x},$$

where  $c$  [m/s] is the phase velocity of the waves. The d'Alembert solution to the wave equation (c1747), describing waves on a string under tension, is

$$u(x, t) = F(ct - \vec{x}) + F(ct + \vec{x}),$$

which describes the transverse velocity (or displacement) of two independent waves on the string, the forward and backward traveling waves. It naturally follows that the plane-wave eigen-functions of the wave equation are given by

$$\Psi_{\pm}(\vec{x}, t) = e^{st \pm j\vec{k} \cdot \vec{x}}.$$

1397 where  $s = \sigma + \omega j$  and  $\vec{k} = j2\pi/\vec{\lambda}$  is called the *wave number*, and  $\vec{\lambda}$  is the wavelength.

## 1398 1.4.2 Lec 24: Complex Analytic functions and the ROC

1399 Lec 24: Power series and integration of functions (ROC)

1400 Fundamental Theorem of calculus (Leibniz theorem of integration)

1401  $1/(1-x) = \sum_{k=0}^{\infty} x^k$  with  $x \in \mathbb{R}$

To solve a differential equation, or integrate a function, Newton used the Taylor series to integrate one term at a time. However he only use real functions of a real variable due to the fundamental lack of appreciation of the complex analytic series. This same method is how one finds solutions to differential equations today, but in a “plug-and-chug” approach, that makes the solution less transparent. Rather than working directly with the Taylor series, today we use the complex exponential. The reasoning given is that the complex exponential is the eigenfunction of the derivative, namely

$$\frac{d}{dt}e^{st} = se^{st}.$$

Of course  $e^{st}$  may be expressed as a Taylor series having coefficients  $c_n = 1/n!$ , so in some real sense this is doing what Newton did. Thus every linear constant coefficient differential equation in time may be simply transformed into a polynomial in complex Laplace frequency  $s$ , by looking for solutions of the form  $A(s)e^{st}$ , transforming the differential equation into a polynomial  $A(s)$  in complex frequency. For example

$$\frac{d}{dt}f(t) + af(t) \leftrightarrow (s+a)F(s).$$

The root of  $A(s_r) = s_r + a = 0$  is the eigenvalue of the differential equation. The key to understanding the solutions of differential equations, both scalar and vector, is to work in the Laplace frequency domain. The Taylor series has been replaced by  $e^{st}$ , transforming Newton's real Taylor series into the complex exponential eigenfunction. In some sense, these are the same method, since

$$e^{st} = \sum_{n=0}^{\infty} \frac{(st)^n}{n!}.$$

If we take the term by term derivative we find

$$\frac{d}{dt}e^{st} = se^{st} = s \sum_{n=0}^{\infty} \frac{(st)^n}{n!}.$$



which is also complex analytic. Thus if the series for  $F(s)$  is valid (i.e., it converges), then its derivative is also valid. This was a very powerful concept, fully exploited by Newton for real functions of a real variable, and later by Cauchy and Riemann for complex functions of a complex variable. The key here is “Where does the series fail to converge?” in which case, the entire representation fails. This is the main message behind the *Fundamental Theorem of Complex Calculus* (Greenberg, 1988, p. 1197).

$$F(s) = F(s_0) + \int_{s_0}^s f(\zeta)d\zeta.$$

This result is formally the same as the Fundamental theorem of calculus (Leibniz formula), the very big difference being that the argument of the integrand  $s \in \mathbb{C}$  is complex. Thus this integration is a line integral in the complex plane. One would naturally assume that the value of the integral depend on the path of integration. But behold, accord to this theorem, it does not. In fact is it is indistinguishable from its simple cousin the Fundamental theorem of calculus. And the reasoning is the same. If  $f(s) = dF(s)/ds$  is complex analytic (i.e., has a power series  $f(s) = \sum_k c_k s^k$ , with  $f(s), c_k, s \in \mathbb{C}$ ), then it may be integrated, and *the integral does not depend on the path*. This is sort of amazing. The key is that  $F(s)$  and  $f(s)$  must be complex analytic, which means they are differentiable, since from the Taylor series formula for the coefficients of the complex analytic series,

$$c_n = \frac{1}{n!} \frac{d}{ds} f(s)|_{s=0}.$$

1402 But what does it mean to take the derivative with respect to a complex variable  $s = \sigma + \omega j$ ? We shall  
 1403 show that the derivative must be independent of the direction, and that this requirement has eminent  
 1404 consequences.

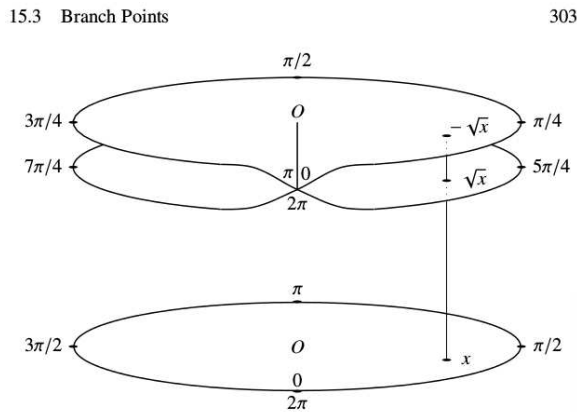


Figure 15.6: Branch point for the square root

Figure 1.15: Here we see the mapping for the square root function  $w(z) = \pm\sqrt{z}$  which has two single-valued sheets, corresponding to the two signs of the square root. The lower sheet is  $+\sqrt{z}$ , and the upper sheet is  $-\sqrt{z}$ . The location of the branch cut may be moved by rotating the  $z$  coordinate system. For example,  $w(z) = \pm j\sqrt{z}$  and  $w(z) = \pm\sqrt{-z}$  have a different branch cuts, as may be easily verified using the Matlab commands `j*zviz(z)` and `zviz(-z)`. A function is analytic on the branch cut, since the cut may be moved. If a Taylor series is formed on the branch cut, it will describe the function on the two different sheets. Thus the Taylor series is agnostic to the location of the cut, as it only describes the function uniquely (as a single valued function), valid in its local region of convergence. Figure from Stillwell (2010, p. 303).

1405 The full power of this concept was first exploited by Bernard Riemann (1826-1866) in his PhD  
 1406 Thesis of 1851 at University of Göttingen, under the tutelage of Carl Friedrich Gauss (1777-1855),  
 1407 drawing heavily on the work of Cauchy.

1408 The key definition of a complex analytic function is that it has a Taylor series representation over  
 1409 a region of the complex frequency plane  $s = \sigma + j\omega$ , that converges in a *region of convergence* (ROC)  
 1410 about the expansion point, with a radius determined by the nearest pole of the function. A further  
 1411 surprising feature of all analytic functions is that within the ROC, the inverse of that function also has

1412 a complex analytic expansion. Thus given  $w(s)$ , one may also determine  $s(w)$  to any desired accuracy,  
 1413 critically depending on the ROC.

1414 This concept of analytic inverses becomes rich when the inverse function is multi-valued. For  
 1415 example, if  $F(s) = s^2$  then  $s(F) = \pm\sqrt{F}$ . Riemann dealt with such extensions with the concept of a  
 1416 *branch-cut* with multiple *sheets*, labeled by a sheet number. Each sheet describes an analytic function  
 1417 (Taylor series), that converges within some ROC, with a radius out to the nearest pole of that function.  
 1418 This Riemann's branch cut and sheets explicitly deal with the need to define a unique single valued  
 1419 inverses to multi-valued functions.

1420 **Branch Cuts:** The concept of a *branch cut* allows one to manipulate (and visualize) multi-valued  
 1421 functions, by breaking each region into a single valued *sheets*. The concepts of the branch cut and the  
 1422 sheets, along with the *extended plane*, were first devised by Riemann, working with Gauss (1777-1855),  
 1423 first described in his thesis of 1851. Of course it was these three mathematical constructions that  
 1424 provide the deep insight to complex analytic functions, supplementing the important earlier work of  
 1425 Cauchy (1789-1857), on the calculus of complex analytic functions.

1426 The branch cut is a line that separates the various single valued parts of a multi-valued function. For  
 1427 example, in Fig. 1.15 we see the double-valued mapping of  $w(z) = \pm\sqrt{z}$ . Since the square root function  
 1428 has two overlapping regions, corresponding to the  $\pm$  of the radical, there must be two connected regions,  
 1429 sort of like mathematical Siamese-twins, distinct, yet the same.

### 1430 Complex impedance functions

1431 One of the most obvious applications of complex functions of a complex variable is impedance. The  
 1432 impedance function  $Z(s) = R(\sigma, \omega) + jX(\sigma, \omega)$  has resistance  $R$  and reactance  $X$  as a function of  
 1433 complex frequency  $s = \sigma + j\omega$ . The function  $z(t) \leftrightarrow Z(s)$  are defined by a Laplace transform pair.  
 1434 From the causality postulate (P1) of Section 3.5.1,  $z(t < 0) = 0$ .

As an example, a resistor  $R_0$  in series with an capacitor  $C_0$  has an impedance

$$Z(s) = R_0 + 1/sC_0. \quad (1.33)$$

In mechanics a dash-pot (damper) and a spring have the same mechanical impedance. A resonant system has an inductor, resistor and a capacitor, with an impedance given by

$$Z(s) = R_0 + 1/sC_0 + sM_0, \quad (1.34)$$

1435 which is a second degree polynomial in the complex frequency  $s$ . Thus it has two roots (eigenvalues).  
 1436 When  $R_0 > 0$  these roots are in the left half  $s$  plane.

1437 Systems (networks) containing many elements, and transmission lines, can be much more compli-  
 1438 cated, yet still have a simple frequency domain representation. This is the key to understanding how  
 1439 these physical systems work, as will be described below.

### 1440 1.4.3 Lec 25: Integration in the complex plane

1441 **Lec 25: Integration in the complex plane: Three theorems**

1442 **Integration of  $1/s$  on the unit circle, and on a unit circle centered about  $s = 1 + i$ .**

1443

Leibniz's formula gives the area under a real analytic function  $F(x)$ ,  $f(x)$ ,  $x \in \mathbb{R}$ ,

$$f(x) = f(0) + \int_0^x F(\xi) d\xi. \quad (1.35)$$

the integral only depends only on the end points. It follows (Leibnitz Rule) that

$$\frac{d}{dx} f(x) = F(x).$$

1444 The integral is also known as the *anti-derivative*.

Given a complex analytic impedance  $Z(s)$ , due to the Fundamental theorem of complex calculus, we may integrate  $Z(s)e^{st}$  in the complex plane

$$\zeta(s) = \zeta(s_0) + \int_{s_0}^s Z(s)e^{st} ds. \quad (1.36)$$

Compare this to (Eq. 1.35), the real integral of the area over the real line  $x$ , other than the limits, this formula is similar to the Inverse Laplace transform (Section 1.3.10, p. 59). The integral can only dependent on the end points if it is complex analytic, in which case

$$\frac{d}{ds}\zeta(s) = Z(s)e^{st}. \quad (1.37)$$

1445 But what does it mean to take the derivative of a function with respect to  $s$ ? If one were to form the  
 1446 partial derivative with respect to  $\sigma$  would that not be very different than the partial derivative with  
 1447 respect to  $\omega$ ? For complex analytic functions, the integral is independent of the path in the  $s$  plane.  
 1448 Thus the derivative must be independent of the direction at every point. This directly follows from  
 1449 the Fundamental theorem of complex calculus and the *Cauchy-Riemann conditions*.

#### 1450 1.4.4 Lec 26: Cauchy-Riemann conditions

1451 Lec 26: Cauchy-Riemann conditions

1452 Real and imaginary parts of analytic functions obey Laplace's equation.

1453 Infinite power Series and analytic function theory; ROC

1454

For the integral of  $Z(s) = R(\sigma, \omega) + X(\sigma, \omega)j$  to be independent of the path, the derivative of  $Z(s)$  must be independent of the direction of the derivative. As we show next, this leads to a pair of equations known as the *Cauchy-Riemann conditions*. Taking partial derivatives of  $Z(s)$  with respect to  $\sigma$  and  $\omega$ , and the equating the real and imaginary parts (i.e., forcing the derivative of  $Z(s)$  to be independent of direction), we obtain the pair of equations

$$\frac{\partial R(\sigma, \omega)}{\partial \sigma} = j \frac{\partial X(\sigma, \omega)}{\partial j\omega} \quad \frac{\partial R(\sigma, \omega)}{\partial j\omega} = j \frac{\partial X(\sigma, \omega)}{\partial \sigma}. \quad (1.38)$$

These are the necessary conditions that the integral, and thus the derivative, of the function  $Z(s)$  be independent of the path, expressed in terms of conditions on the real and imaginary parts of  $Z$ . This condition on  $Z(s)$  is a very strong condition, and requires that  $Z(s)$  may be written as a Taylor series in  $s$

$$Z(s) = Z_0 + Z_1s + \frac{1}{2}Z_2s^2 + \dots. \quad (1.39)$$

1455 Every function that may be expressed as a *Taylor series* about a point  $s$  is said to be *complex*  
 1456 *analytic* at that point. The series is said to converge within a *radius of convergence* (ROC). This  
 1457 highly restrictive conditions has significant physical consequences. For example, every impedance  
 1458 function  $Z(s)$  obeys the CR conditions over large regions of the  $s$  plane, including the entire right  
 1459 half plane (RHP), defined by  $\sigma > 0$ . When this conditions is generalize to volume integrals, it is  
 1460 called *Green's theorem*, which is a special case of both Gauss's and Stokes's Laws, used heavily in the  
 1461 solution of boundary value problems in Engineering-Physics (e.g., solving EM problems that start from  
 1462 Maxwell's equations). The last third of this course shall depend heavily on this concept and various  
 1463 generalizations.

- 1464 **WEEK 10** **26.10.0**  
 1465
- 1466 L 27 Differentiation in the complex plane: Fundamental Thm of complex calculus (FTCC);  
 1467 Complex Analytic functions; ROC in the complex plane  
 1468  $Z(s) = R(s) + jX(s)$ : real and imag parts obey Laplace's Equation  
 1469 Basic equations of mathematical Physics: Wave equation, Diffusion equation, Laplace's Equation  
 1470 Motivation: Dispersion relation for the wave equation  $\kappa \cdot \kappa = s^2/c_0^2$
- 1471 L 28 Three Fundamental theorems of complex integral calculus  
 1472  $\int_0^z F(\zeta)d\zeta = F(z) - F(0)$ :  $dZ(s)/ds$  independent of direction  
 1473 Integration in the complex plane; Integrals independent of limits  
 1474 Cauchy-Riemann conditions
- 1475 L 29 Inverse Laplace transform  
 1476 Inverse Laplace transform: Poles and Residue expansions;  
 1477 Application of the Fundamental Thm of Complex Calculus  
 1478 The Inverse Laplace Transform (ILT); poles and the Residue expansion: The case for causality  
 1479 ROC as a function of the sign of time in  $e^{st}$  (How does causality come into play?)  
 1480 Examples.
- 1481 **1.4.5 Lec 27: Differentiation in the Complex plane**  
 1482 **1.4.6 Lec 28: Three complex integration theorems**  
 1483 **1.4.7 Lec 29: Inverse Laplace transform (Cauchy residue theorem)**  
 1484 Use of the Residue theorem to evaluate the inverse Laplace transform. Discuss causal and anti-causal  
 1485 cases. How does this relate to Green's theorem (in 2 dimensions).
- 1486 **WEEK 11** **30.11.0**  
 1487
- 1488 L 30 Inverse Laplace transform & Cauchy Residue Theorem
- 1489 L 31 Case for causality Closing the contour as  $s \rightarrow \infty$ ; Role of  $\Re st$   
 1490 **DE-3**
- 1491 L 32 Properties of the LT:  
 1492 1) Modulation, 2) Translation, 3) convolution, 4) periodic functions  
 1493 Tables of common LTs
- 1494 **1.4.8 Lec 30: Inverse Laplace transform and the Cauchy Residue Thm**  
 1495 **1.4.9 Lec 31: Case for causality: closing the contour**  
 1496 **1.4.10 Lec 32: Properties of the LT (e.g., modulation, translation, etc.)**

- 1497 **WEEK 12** **33.12.0**  
 1498

1499 L 33 Multi-valued functions in the complex plane; Branch cuts  
 1500 The extended complex plane (regularization at  $\infty$ ) (Stillwell, 2010, p. 280)  
 1501 Complex analytic functions of Genus 1 (Stillwell, 2010, p. 343)

1502 L 34 Euler's vs. Riemann's Zeta function  $\zeta(s)$ : Poles at the primes  
 1503 colorized plot of  $\zeta(s)$   
 1504 ??Sterling's formula??

1505 L 35 Exam III

1506 **1.4.11 Lec 33: Multi-valued functions Branch cuts**

1507 **1.4.12 Lec 34: The Riemann zeta function**

1508 The LT of the complex Riemann zeta function  $\zeta(x)$  (Fig. 4.1), as introduced by Euler for real arguments.  
 1509  $x \in \mathbb{R}$  as his way of proving that the number of primes is infinite (Goldstein, 1973). In the end, this  
 1510 formulation provided detailed information about the structure of the primes. The zeta function depends  
 1511 explicitly on the primes, which is why it is interesting (Section 4.5.2).

1512 One might wonder why Euler's zeta function is known as the Riemann zeta function. It is because  
 1513 Riemann showed its properties when the argument is complex, namely he extended  $\zeta(s)$  into the  
 1514 complex plane ( $s \in \mathbb{C}$ ) (Section 4.5.2). Given that  $\zeta(s)$  is a function of complex (Laplace) frequency,  
 1515 one might naturally ask if  $\zeta(s)$  has an inverse Laplace transform. There seems to be very little written  
 1516 on this topic,<sup>50</sup> but we shall explore this interesting question further (Table 4.1). Perhaps even more  
 1517 important is the taxonomy of  $\zeta(s)$  is in question here, namely where are its poles and zeros? About  
 1518 this there are volumes written.

1519 **The Riemann Zeta function is analytic with poles at log-primes**

1520 Why does the zeta function have zeros? Perhaps this is some extension of the Euler function that has  
 1521 zeros, rather than zeta itself. Ask Andrew Odlyzko about this problem. Go to the Math dept first and  
 1522 find someone qualified to discuss this with.

1523 **1.4.13 Lec 35: Exam III (Evening Exam)**

1524 **WEEK 13**

**36.13.0**

1525

1526 L 36 Scaler wave equations and the Webster Horn equation; WKB method  
 1527 A real-world example of large delay, where the branch-cut placement is critical  
 1528

1529 L 37 Partial differential equations of Physics  
 1530 Scaler wave equation and its solution in 1 and 3 Dimensions  
 1531 **VC-1**

1532 L 38 Vector dot and cross products  $A \cdot B, A \times B$   
 1533 Gradient, divergence and curl

1534 – Thanksgiving Holiday 11/19–11/27 2016

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<sup>50</sup>Cite book chapter on inverse LT of  $\zeta(s)$ .

## 1535 1.5 Vector Calculus (Stream 3b)

### 1536 1.5.1 Lec 36: Scalar Wave Equation (Acoustics)

1537 Acoustic waves; The scalar wave equation: scalar differential equation in the frequency  
1538 domain

#### 1539 The Webster Horn equation

1540 The effect of a spatial area functions for waves in horns (the horn equation).

1541 Derivation of the Horn equation, starting from the basic equations of acoustics.

1542 Development of the basic equations of acoustics: Conservation of mass and momentum.

1543 Sound in a uniform tube.

Sound propagation away from a point source (Helmholtz's Equation)

$$\nabla^2\psi + k^2\psi = \delta(r).$$

### 1544 1.5.2 Lec 37: Partial Diff Eqs of Physics

### 1545 1.5.3 Lec 38: Vector dot and cross products

### 1546 1.5.4 Thanksgiving Holiday 11/19–11/27 2016

## 1547 WEEK 14

37.14.0

1548

1549 L 39 Gradient, divergence and curl: Gauss's (divergence) and Stokes's (curl) theorems

1550 L 40 J.C. Maxwell unifies Electricity and Magnetism with the formula for the speed of light

1551 Basic definitions of  $E, H, B, D$

1552 O. Heaviside's (1884) vector form of Maxwell's EM equations and the *vector wave equation*

1553 How a loud-speaker works

1554 L 41 *The Fundamental Thm of vector calculus*

1555 *Incompressible and Irrotational fluids and the two defining vector identities*

1556

### 1557 1.5.5 Lec 39 $\nabla, \nabla \cdot, \nabla \times$ & Vector operators

1558 There are three key vector differential operators that we need to understand Maxwell's equations. The  
1559 *gradient* transforms a potential, such as a voltage  $V(x, y, z)$  into a vector, such as the electric field  
1560 vector  $\mathbf{E}$ . The *divergence*  $\nabla \cdot \mathbf{E}(x, y, z)$  transforms a vector field into a scalar field. Finally the curl  
1561  $\nabla \times \mathbf{A}(x, y, z)$  transforms a vector into a vector.

1562 To define these three operations we first need to define scalar and vector fields. These are concepts  
1563 that you already understand. It is the terminology that needs to be mastered, not a new concept.

1564 Think of a voltage field in space, say between two finite sized capacitor plates. In such a case, the  
1565 voltage is given by a *scalar field*  $V(x, y, z)$ . A scalar field is also called a *potential*. Somewhat confusing  
1566 is that one may also define *vector potentials* which is three scalar potentials turned into a vector. So  
1567 this term is more than one use. It is therefore important to recognize the intended use of the field.

1568 This can be gleaned from the units. Volts is a scalar field.

The simplest example of a scalar potential is the voltage between two very large (think  $\infty$ ) conducting parallel planes, or plates (large so that we can ignore the edge effects). In this case the voltage varies linearly between the two plates. For example

$$V(x, y, z) = V_0(1 - x)$$

1569 is a scalar potential, thus it is scalar field (i.e., potential). At  $x = 0$  the voltage is  $V_0$  and at  $x = 1$  the  
1570 voltage is zero. Between 0 and 1 the voltage varies linearly. Thus  $V(x, y, z)$  defines a *scalar field*.

1571 If the same setup were used but the two plates were  $1 \times 1$  [cm<sup>2</sup>], with a 1 [mm] air gap, there will  
1572 be a small “fringe” effect at the edges that would slightly modify the ideal fields. The hope is that  
1573 this effect can be made small so that it does not ruin the capacitor composed of the two plates. If we  
1574 are given a set of three scalar fields, we define a *vector field*. If the three elements of the vector are  
1575 potentials, then we have a vector potential.

### 1576 Gradient operator $\nabla$

The gradient operator takes a scalar fields and outputs a vector field. This is exactly what the gradient does. Given any scalar field  $V(x, y, z)$  it outputs a vector field<sup>51</sup>

$$\mathbf{E}(x, y, z) = [E_x(x, y, z), E_y(x, y, z), E_z(x, y, z)]^T = -\nabla V(x, y, z).$$

1577 To understand these three operations we therefore need to define the domain and range of their  
1578 operation, as specified in Table 5.1.

### 1579 1.5.6 Lec 40: Definitions of E, H, B, D and Maxwell’s equations

#### 1580 Maxwell’s Equations

1581 Once you have mastered the three basic vector operations, the gradient, divergence and curl, you are  
1582 able to understand Maxwell’s equations. Like the vector operations, these equations may be written  
1583 in integral or vector form. The notation is basically the same since the concept is the same. The only  
1584 difference is that with Maxwell’s equations we are dealing with well defined physical quantities. The  
1585 scalar and vector fields take on meaning, and units. Thus to understand these important equations,  
1586 one must master the units, and equally important, the names of the four fields that are manipulated  
1587 by these equations.

1588 We may now restate everything defined above in terms of two types of vector fields that decompose  
1589 every vector field. Thus another name for the Fundamental Theorem of Vector Calculus is the *Helmholtz*  
1590 *decomposition*. An *irrotational field* is define as one that is “curl free,” namely the vector potential is a  
1591 constant. An *incompressible field* is one that is “diverge free,” namely the scalar potential is a constant.  
1592 Just to confuse matters, the incompressible field is also called a *solenoidal field*. I recommend that you  
1593 know this term (as it is widely used), but never use it. Rather use incompressible as a more meaningful  
1594 and physical term.

### 1595 1.5.7 Lec 41 Fundamental Theorem of Vector calculus (Helmholtz theorem)

#### 1596 The Fundamental Theorem of Vector Calculus

1597 Here we define the basic vector operations based on the  $\nabla$  “operator,” the *gradient*, *divergence* and  
1598 the *curl*. These operations may be defined in terms of integral operations on a surface in 1, 2 or 3  
1599 dimensions, and then taking the limit as that surface goes to zero. These operators are required to  
1600 understand Maxwell’s Equations, the crown jewel of mathematical physics.

---

<sup>51</sup>As before vectors are columns, which take up space on the page, thus we write them as rows and take the transpose to properly format them.

1601 **Incompressible and Irrotational vector fields**

1602 One of the most important fundamental theorems is that of vector calculus. This is also known as  
 1603 *Helmholtz theorem*. This theorem is very easily stated but less easily to appreciate. But a physical  
 1604 description of what is going on will help.

1605 A vector field may be split into two parts, that are independent. Think of linear and angular  
 1606 momentum. They are also independent in that they represent different ways to store energy. An  
 1607 object with mass can be moving along a path and rotating at the same time. The two modes of motion  
 1608 define two different types of kinetic energy, translational and rotational. In some real sense, Helmholtz  
 1609 theorem quantifies this independence.

**The Fundamental Theorem of Vector Calculus:** This theorem is also known as Helmholtz' theorem. It states that every differentiable vector field may be written as the sum of two terms, a scalar part and a vector part expressed in terms of a scalar potential  $\phi(x, y, z)$  (think voltage) and a vector potential (think magnetic vector potential). Specifically

$$\mathbf{E} = -\nabla\phi + \nabla \times \mathbf{A}. \quad (1.40)$$

To show that this relationship splits the vector field  $\mathbf{E}$  into two parts we need to add to the mix two key vector identities, that are always true (assuming they exist, i.e, that the fields are differentiable):

$$\nabla \times \nabla\phi(x, y, z) = 0, \quad (1.41)$$

or in words, the *curl of the divergence* = 0, and

$$\nabla \cdot \nabla \times \mathbf{A} = 0 \quad (1.42)$$

1610 or the *divergence of the gradient* = 0. These identities are easily verified by working out a few examples  
 1611 based on the definitions of the three operators, for example in terms of their integral definitions. They  
 1612 also have an important physical meaning, as indicated above, that every vector field may be split into  
 1613 its translational and rotational parts, as with our example of momentum.

1614 If we apply these two identities to Helmholtz's theorem  
 1615 (Eq. 1.40), we can appreciate the significance of the theo-  
 1616 rem. It is a form of proof actually, once you have satisfied  
 1617 yourself that the vector identities are true. In fact one can  
 1618 work backward using a physical argument, that rotational  
 1619 momentum and thus energy is independent from transla-  
 1620 tional momentum, thus energy. Again this all goes back to  
 1621 the definitions of rotation and translational forces, hidden  
 1622 in the vector operations. Once these forces are made clear,  
 1623 the meaning of the vector operations all take on a very well  
 1624 defined meaning, and the mathematical constructions, centered  
 1625 around Helmholtz's theorem, begins to provide some  
 1626 common-sense meaning.

Specifically if we take the divergence of Eq. 1.40, and use the divergence vector identity

$$\nabla \cdot \mathbf{E} = \nabla \cdot \{-\nabla\phi + \nabla \times \mathbf{A}\} = -\nabla \cdot \nabla\phi = -\nabla^2\phi.$$

1627 since the divergence vector identity removes the vector po-  
 1628 tential  $\mathbf{A}(x, y, z)$ .

Likewise if we take the curl of Eq. 1.40, and use the curl vector identity

$$\nabla \times \mathbf{E} = \nabla \times \{-\nabla\phi + \nabla \times \mathbf{A}\} = \nabla \times \nabla \times \mathbf{A},$$



H. v. Helmholtz.

Figure 1.16: von Helmholtz portrait taken from the English translation of his 1858 paper "On integrals of the hydrodynamic equations that correspond to Vortex motions" (in German) (von Helmholtz, 1978).



1629 since using the curl vector identity, removes the scalar field  $\phi(x, y, z)$ .

There is a third vector identity that needs to be mentioned for later use

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.$$

The best way to think of this relationship is that it *defines* the vector Laplacian  $\nabla^2 \mathbf{A}$ . In other words, think of this identity the definition of the left hand side of

$$\nabla^2 \mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}).$$

## 1630 WEEK 15

40.15.0

1631

1632 L 42 Quasi-static approximation and applications:

1633 The Kirchoff's Laws and the *Telegraph wave equation*, starting from Maxwell's equations The  
1634 telegraph wave equation starting from Maxwell's equations

1635 Quantum Mechanics

1636 L 43 Last day of class: Review of Fund Thms of Mathematics:

1637 Closure on Numbers, Algebra, Differential Equations and Vector Calculus,

1638 The Fundamental Thms of Mathematics & their applications:

1639 Theorems of Mathematics; Fundamental Thms of Mathematics (Ch. 9); Normal modes vs. eigen-  
1640 states, delay and quasi-statics;

1641 – Reading Day

1642 VC-1 Due

### 1643 1.5.8 Lec 42: Kirchoff's Laws and the quasistatic approximation

1644 - The term *quasistatics* indicates a type of approximation that is widely used when reducing a problem  
1645 based on partial differential equations to one of a scalar differential equation. It is important to  
1646 understand the nature of this approximation so that it is not miss-applied. Quasistatics is a way of  
1647 reducing a three dimensional problem to a 1 dimensional problem. This approximation is at the heart  
1648 of transmission line theory. Lets begin with an example: The acoustic wave equation describes how  
1649 the scalar pressure  $p(x, y, z, t)$  propagates in three dimensions. If the wave propagation is restricted  
1650 to a pipe, such as an organ pipe, or to a string, as in a guitar string, we do not need to worry about  
1651 the transverse directions. What needs to be modeled by the equations is the wave propagation along  
1652 the pipe or string. Thus we replace the three-dimensional wave with a one-dimensional wave, without  
1653 further thought.

1654 However if we wish to be more precise about this reduction in geometry, we need to consider  
1655 the quasistatic approximation, as it makes some assumptions about what is happening in the other  
1656 directions, and quantifies their effects. Taking the case of wave propagation in a tube, say the ear  
1657 canal, there is the main wave direction, down the tube. But there is also wave propagation in the  
1658 transverse direction, perpendicular to the direction of propagation. As shown in Table 3.1 (p. 108),  
1659 the key statement of the quasistatic approximation is that the wavelength in the transverse direction  
1660 is much larger than the radius of the pipe. This is equivalent to saying that the radial wave reaches the  
1661 walls and is reflected back, in a time that is small compared to the distance propagated down the pipe.  
1662 Clearly the speed of sound down the pipe and in the transverse direction is the same if the medium is  
1663 homogeneous (i.e., air or water). Thus the sound reaches the walls and is returned to the center line  
1664 in a time that the axial wave traveled about 1 diameter along the pipe. So if the distance traveled is

several diameters, the radial parts of the wave have time to come to equilibrium. So the question one must ask is, what are the conditions of such an equilibrium. The most satisfying answer to this is to look at the internal forces on the air, due to the gradients in the pressure.

The pressure  $p(x, y, z, t)$  is a potential, thus its gradient is a force density  $\mathbf{f}(x, y, z, t) = -\nabla p(x, y, z, t)$ . What this equation tells us is that as the pressure wave approaches that of a plane wave, the radial (transverse) forces go to zero. If the tube has a curvature, or a change in area, then there will be local forces that create radial flow. But after traveling a few diameters, these forces will come to equilibrium and the wave will return to a plane wave. The internal stress caused by a change in area must settle out very quickly. There is a very important caveat however: it is only at low frequencies that the plane wave can dominate. At frequencies such that the wavelength is very small compared to the diameter, the distance traveled between reflections is much greater than a few diameters. Fortunately the frequencies where this happens are so high that they play no role in frequencies that we care about. This effect is referred to as *cross-modes* which imply some sort of radial standing waves. In fact such modes exist in the ear canal, but on the eardrum where the speed of sound is much slower than that of air. Because of the slower speed, the ear drum has cross-modes, and these may be seen in the ear canal pressure. Yet they seem to have a negligible effect on our ability to hear sound with good fidelity. The point here is that the cross modes are present, but we call upon the quasistatic approximation as a justification for ignoring them, to get closer to the first-order physics.

**Breakdown of the quasistatic approximation at high frequencies:** If we wonder, for the sake of wonderment, what happens at high frequencies where the quasistatic approximation begins to break down, we need to consider other significant physics of the system. In acoustics there are two basic effects that have been ignored by assuming that wave propagation is dictated by the wave equation, viscosity and thermal effects. In fact, it turns out that these two loss mechanisms are related, but to understand why is quite difficult. However Helmholtz, with some help from Kirchhoff, figured this out and published it between 1863 (Helmholtz, 1863b) and 1868 (Kirchhoff, 1868). Their theory was summarized by Lord Rayleigh (Rayleigh, 1896) and then experimentally verified to be correct by Warren P. Mason (Mason, 1928). The nature of the correction is that the wave number is extended to be of form

$$\kappa(s) = \frac{s + \beta_0\sqrt{s}}{c_0}, \quad (1.43)$$

where the forwarded  $P_-$  and backward  $P_+$  pressure waves propagate as

$$P_{\pm}(s, x) = e^{-\kappa(s)x}, e^{-\bar{\kappa}(s)x} \quad (1.44)$$

and  $\bar{\kappa}(s)$  is the complex conjugate of  $\kappa(s)$ , and  $\Re\kappa(s) > 0$ . The term  $\beta_0\sqrt{s}$  affects both the real and imaginary parts of  $\kappa(s)$ . The real part is a frequency dependent loss and the imaginary part introduces a frequency dependent velocity (Mason, 1928).

The frequency where the loss-less part equals the lossy part is an important parameter of the system. This frequency is  $s_0 + \beta_0\sqrt{s_0} = 0$ , or  $\sqrt{s_0} = \beta_0$  or  $f_0 = \beta_0^2/2\pi$ .

Assuming air at 23.5° [C],  $c_0 = \sqrt{\eta_0 P_0/\rho_0} \approx 344$  [m/s] is the speed of sound,  $\eta_0 = c_p/c_v = 1.4$  is the ratio of specific heats,  $\mu = 18.5 \times 10^{-6}$  [Pa-s] is the viscosity,  $\rho_0 \approx 1.2$  [kgm/m<sup>3</sup>] is the density,  $P_0 = 10^5$  [Pa] (1 atm).

The constant  $\beta_0 = P\eta'/2S\sqrt{\rho_0}$

$$\eta' = \sqrt{\mu} \left[ 1 + \sqrt{5/2} \left( \eta^{1/2} - \eta^{-1/2} \right) \right]$$

is a thermodynamic constant, P is the perimeter of the tube and S the area (Mason, 1928).

For a cylindrical tube having radius  $R = 2S/P$ ,  $\beta_0 = \eta'_0/R\sqrt{\rho}$ . To get a feeling for the magnitude of  $\beta_0$  consider a 7.5 [mm] tube (i.e., the average diameter of the adult ear canal). Then  $\eta' = 6.6180 \times 10^{-3}$  and  $\beta_0 = 1.6110$ . Using these conditions the wave-number cutoff frequency is  $1.611^2/2\pi = 0.4131$  [Hz].

1695 At 1 kHz the ratio of the loss over the propagation is  $\beta_0/\sqrt{|s|} = 1.6011/\sqrt{2\pi 10^3} \approx 2\%$ . At 100 [Hz]  
 1696 this is a 6.4% effect.<sup>52</sup>

1697 Mason shows that the wave speed drops from 344 [m/s] at 2.6 [kHz] to 339 [m/s] at 0.4 [kHz], which  
 1698 is a 1.5% reduction in the wave speed. In terms of the losses, this is much larger effect. At 1 [kHz]  
 1699 the loss is 1 [dB/m] for a 7.5 [mm] tube. Note that the loss and the speed of sound vary inversely with  
 1700 the radius. As the radius approaches the boundary layer thickness (the radial distance such that the  
 1701 loss is  $e^{-1}$ ), the effect of loss dominates.

1702 In Section 5.4.1 we shall look at some simple problems where we use the quasistatic effect and  
 1703 derive the Kirchhoff voltage and current equations, starting from Maxwell's equations.

### 1704 1.5.9 Lec 43: Final Review for Final Exam

#### 1705 Summary

1706 Mathematics began as a simple way of keeping track of how many things there were. But eventually  
 1707 Physics and Mathematics evolved together as tools to help us navigate our environment, not just phys-  
 1708 ically around the globe, but how to solve daily problems such as food, water and waste management,  
 1709 understand the solar system and the stars, defend ourselves, use tools of war, etc.

1710 Based on the historical record of the abacus, one can infer that people precisely understood the  
 1711 concept of counting, addition, subtraction and multiplication (recursive addition).

1712 There is some evidence that the abacus, a simple counting tool, formalizing the addition of very  
 1713 large numbers, was introduced to the Chinese by the Romans, where it was used for trade.

1714 However this working knowledge of arithmetic did not to show up in written number systems. The  
 1715 Roman numerals were not useful for doing calculations done on the abacus. The final answer would  
 1716 then be expressed in terms of the Roman number system.

1717 According to the known written record, the number zero (null) had no written symbol until the  
 1718 time of Brahmagupta (628 CE). One should not assume the concept of zero was not understood simply  
 1719 because there was no symbol for it in the Roman Numeral system. Negative numbers and zero would  
 1720 be obvious when using the abacus. Numbers between the integers would be represented as *rational*  
 1721 *numbers*  $\mathbb{Q}$  since any number may be approximated with arbitrary accuracy with rations numbers.

1722 Mathematics is the science of formalizing a repetitive method into a set of rules, and then general-  
 1723 izing it as much as possible. Generalizing the multiplication and division algorithm, to different types  
 1724 of numbers, becomes increasingly more complex as we move from integers to rational numbers, irra-  
 1725 tional numbers, real and complex numbers and ultimately, vectors and matrices. How do you multiply  
 1726 two vectors, or multiply and divide one matrix by another? Is it subtraction as in the case of two  
 1727 numbers? Multiplying and dividing polynomials (by long division) generalizes these operations even  
 1728 further. Linear algebra is further important generalization, fallout from the Fundamental Theorem of  
 1729 Algebra, and essential for solving the generalizations of the number systems.

1730 Many of the concepts about numbers naturally evolved from music, where the length of a string  
 1731 (along with its tension) determined the pitch (Stillwell, 2010, pp. 11, 16, 153, 261). Cutting the string's  
 1732 length by half increased the frequency by a factor of 2. One forth of the length increases the frequency  
 1733 by a factor of 4. One octave is a factor of 2 and two octaves a factor of 4 while a half octave is  $\sqrt{2}$ . The  
 1734 musical scale was soon factored into rational parts. This scale almost worked, but did not generalize  
 1735 (sometimes known as *Pythagoreas' comma*<sup>53</sup>), resulting in today's *well tempered scale*, which is based  
 1736 on 12 equal geometric steps along one octave, or 1/12 octave ( $\sqrt[12]{2} \approx 1.05946 \approx 18/17 = 1 + 1/17$ ).

1737 But the concept of a *factor* was clear. Every number may be written as either a sum, or a product  
 1738 (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is  
 1739 based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of a second  
 1740 degree polynomial was understood, which lead to a generalization of factoring, since the polynomial, a  
 1741 sum of terms, may be written in factored form. If you think about this a bit, it is sort of an amazing

<sup>52</sup>/home/jba/Mimosa/2C-FindLengths.16/doc.2-c\_calib.14/m/MasonKappa.m

<sup>53</sup>[https://en.wikipedia.org/wiki/Pythagorean\\_comma](https://en.wikipedia.org/wiki/Pythagorean_comma)

1742 idea, that needed to be discovered (Stillwell, 2010, p. ). This concept lead to an important string of  
 1743 theorems on factoring polynomials, and how to numerically describe physical quantities. Newton was  
 1744 one of the first to master these tools with his proof that the orbits of the planets are ellipses, not circles.  
 1745 This lead him to expanding functions in terms of their derivatives and power series. Could these sums  
 1746 be factored? The solution to this problem led to calculus.

1747 So mathematics, a product of the human mind, is a highly successful attempt to explain the physical  
 1748 world. All aspects of our lives were impacted by these tools. Mathematical knowledge is power. It  
 1749 allows one to think about complex problems in increasingly sophisticated ways. An equation is a  
 1750 mathematical sentence, expressing deep knowledge. Witnessed  $E = mc^2$  and  $\nabla^2\psi = \ddot{\psi}$ .

1751 **Reading List:** The above concepts come straight from mathematical physics, as developed in the  
 1752 17<sup>th</sup>–19<sup>th</sup> centuries. Much of this was first developed in acoustics by Helmholtz, Stokes and Rayleigh,  
 1753 following in Green’s footsteps, as described by Lord Rayleigh (1896). When it comes to fully appreciat-  
 1754 ing Green’s theorem and reciprocity, I have found Rayleigh (1896) to be a key reference. If you wish to  
 1755 repeat my reading experience, start with Brillouin (1953), followed by Sommerfeld (1952); Pipes (1958).  
 1756 Second tier reading contains many items Morse (1948); Sommerfeld (1949); Morse and Feshbach (1953);  
 1757 Ramo et al. (1965); Feynman (1970); Boas (1987). A third tier might include Helmholtz (1863a); Fry  
 1758 (1928); Lamb (1932); Bode (1945); Montgomery et al. (1948); Beranek (1954); Fagen (1975); Lighthill  
 1759 (1978); Hunt (1952); Olson (1947). It would be a mistake to ignore other massive physics writings by  
 1760 stalwart authors, J.C. Slater<sup>54</sup> and Landau and Lifshitz,<sup>55</sup> and their impressive series of Mathematical  
 1761 Physics books.

1762 You must enter at a level that allows you to understand. Successful reading of these books critically  
 1763 depends on what you already know. A rudimentary (high school) level of math comprehension must  
 1764 be mastered first. Read in the order that helps you best understand the material.

1765 Without a proper math vocabulary, mastery is hopeless. I suspect that one semester of college  
 1766 math can bring you up to speed. This book is my attempt to present this level of understanding.

---

<sup>54</sup>[https://en.wikipedia.org/wiki/John\\_C.\\_Slater](https://en.wikipedia.org/wiki/John_C._Slater)

<sup>55</sup><https://www.amazon.com/Mechanics-Third-Course-Theoretical-Physics/dp/0750628960>

1767

## Chapter 2

1768

# Number Systems: Stream 1

1769 This chapter is devoted to *Number Systems* (Stream 1), starting with the counting numbers  $\mathbb{N}$ . In this  
1770 chapter we delve more deeply into the details of the topics of Lectures 4-9.

1771

## 2.1 Week 2

1772 In Section 2.1.1 we explore in more detail the two fundamental theorems of prime numbers, working  
1773 out a sieve example, and explore the logarithmic integral  $Li(N)$  which approximates the density of  
1774 primes  $\rho_k(N)$  up to prime  $N$ .

1775 The topics of Section 1.2.4 consider the practical details of computing the *greatest common divisor*  
1776 (GCD) of two integers  $m, n$  (Matlab's routine `1=gcd(m,n)`), with detailed examples and comparing  
1777 the algebraic and matrix methods. Homework assignments will deal with these two methods. Finally  
1778 we discuss the relationship between coprimes and the GCD. In Section 1.2.5 we defined the Continued  
1779 Fraction algorithm (CFA), a method for finding rational approximations to irrational numbers. **The**  
1780 **CFA and GCD are closely related, but the relation needs to be properly explained.** In Section 1.2.7 we  
1781 derive *Euclid's formula*, the solution for the Pythagorean triplets (PT), based on Diophantus's *chord/-*  
1782 *tangent* method. This method is used many times throughout the course notes, first for computing  
1783 Euclid's formula for the PTs, then for finding a related formula in Section 1.2.8 for the solutions to  
1784 Pell's equation, and finally for finding the mapping from the complex plane to the extended complex  
1785 plane (the Riemann sphere).

1786 Finally in Section 1.2.9 the general properties of the *Fibonacci sequence* is discussed. This equation  
1787 is a special case of the second order digital resonator (well known in digital signal processing), so it  
1788 has both historical and practical application for engineering. The general solution of the Fibonacci  
1789 is found by taking the Z-transform and finding the roots, resulting in an eigenvalue expansion (Appendix  
1790 D).

1791

### 2.1.1 Lec 4 Two theorems on primes

1792

#### Theorem 1: *Fundamental Theorem of Arithmetic*

**Factoring integers:** Every integer  $n \in \mathbb{N}$  has a unique factorization (Stillwell, 2010, p. 43)

$$n = \prod_{k=1}^K \pi_k^{\beta_k}, \quad (2.1)$$

1793 where  $k = 1, \dots, K$  indexes the integer's  $K$  prime factors  $\pi_k$  and their *multiplicity*  $\beta_k$ .

1794 **Examples:**  $2312 = 2^3 \cdot 17^2 = \pi_1^3 \pi_7^2$  (i.e.,  $\pi_1 = 2, \beta_1 = 3; \pi_7 = 17, \beta_7 = 2$ )  
1795  $2313 = 3^2 \cdot 257 = \pi_3^2 \pi_{55}$  (i.e.,  $\pi_2 = 3, \beta_3 = 2; \pi_{55} = 257, \beta_{55} = 1$ )

Integers 2312 and 2313 are said to be *coprime*, since they have no common factors. *Coprimes* may be identified via the *greatest common divisor*:

$$\gcd(a, b) = 1$$

1796 using the *Euclidean algorithm* (Stillwell, 2010, p. 41).

### 1797 **Theorem 2: Prime Number Theorem**

The primes are a random field since there is no way to predict when the next prime will occur. Thus one needs to use statistical methods to characterize their density. Based on a sample of approximately 3 million primes, Gauss showed empirically that the average total number of primes less than  $N$  is

$$\sum_{n=1}^N \delta_n \sim \frac{N}{\ln N}.$$

1798 These primes were obtained by manual calculations “as a pastime” in 1792-3 (Goldstein, 1973).

1799 Define  $\delta_n = 1$  if  $n$  is a prime, and zero otherwise.<sup>1</sup>.

It follows that the *average density of primes* is  $\rho_\pi(N) \sim 1/\ln N$ , thus

$$\rho_\pi(N) \equiv \frac{1}{N} \sum_{n=1}^N \delta_n \approx \frac{1}{N} Li(N) \equiv \frac{1}{N} \int_2^N \frac{d\xi}{\ln \xi},$$

1800 where  $Li(N)$  is the *offset logarithmic integral* (Stillwell, 2010, p. 585). The primes are distributed  
1801 as  $1/\ln(n)$  since the average total number of primes is proportional to the *logarithmic integral*  $Li(n)$   
1802 (Goldstein, 1973; Fine, 2007).

1803 Here is a Matlab code that tests this formula:

```
1804 %Computes density of primes from average density
1805
1806 NP=1e6; % 10^6 primes
1807 p=primes(NP); %compute primes
1808 delta=zeros(1,NP); delta(p)=1; %put 1 at each prime
1809 PI=cumsum(delta); %Number of primes vs N
1810 rho=PI./(1:NP); %estimate of the density of primes= PI(N)./N
1811 %
1812 figure(1)
1813 semilogy(rho); %plot of density vs number of primes
1814 title('Density of primes vs. N'); ylabel('\rho(N)'); xlabel('N'); grid on
```

1815 From the Prime Number Theorem it is clear that the density of primes is large (they are not scarce).  
1816 As best I know there are no methods to find primes other than by the sieve method (Section ??, p. ??).  
1817 If there is any good news it is that they only need to be computed once, and saved. In practical  
1818 applications this doesn't help much given their large number. In theory, given primes  $\pi_n$  up to  $n = N$ ,  
1819 the density  $\rho_\pi(N)$  could help one search for a particular prime of known size  $N$ , by estimating how  
1820 many primes there are in the neighborhood of  $N$ .

1821 Not surprisingly, playing with primes has been a popular pastime of mathematicians. Perhaps this  
1822 is because those who have made inroads, providing improved understanding, have become famous.

<sup>1</sup>You may view  $\delta_n$  for the first 100 numbers with the one-line Matlab/Octave command `stem(isprime(1:100))`

### 1823 2.1.2 Lec 5 Greatest common divisor (GCD)

1824 Multiplying two numbers together, or dividing one by the other, is very inexpensive on today's computer  
 1825 hardware. However, factoring a large integer (i.e.,  $10^3$  digits) into its primes, is very expensive. When  
 1826 the integers are large, it is so costly that it cannot be done in a lifetime, even with the fastest computers.

1827 The obvious question is: "Can we find the largest common factor  $k = \text{gcd}(m, n)$  without factoring  
 1828  $(m, n)$ ?" The answer is "yes," with the *Euclidean algorithm* (EA). While the EA falls short of factoring,  
 1829 it is fast and easily implemented.

1830 If the two integer are in factored form, the answer is trivial. For example  $5 = \text{gcd}(5 \cdot 13, 5 \cdot 17)$ , and  
 1831  $17 = \text{gcd}(17 \cdot 53, 17 \cdot 3 \cdot 31)$ . But what about  $\text{gcd}(901, 1581)$ ? So the problem that computing the GCD  
 1832 solves is when the factors are not known. Since  $901 = 53 \cdot 17$  and  $1581 = 3 \cdot 17 \cdot 31$ ,  $\text{gcd}(901, 1581) = 17$ ,  
 1833 which is not obvious.

1834 In Matlab the GCD may be computed using `k=gcd(m,n)`, which only allows integers as arguments  
 1835 (and removes the sign).

**Matrix method:** The GCD may be written a matrix recursion, based on Eq. 2.1.2. The two starting numbers are given by the vector  $(m_0, n_0)$ . The recursion is then

$$\begin{bmatrix} m_{k+1} \\ n_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_k \\ n_k \end{bmatrix}$$

1836 This recursion continues until  $m_{k+1} < n_{k+1}$ , at which point  $m$  and  $n$  must be swapped. Because the  
 1837 output depends on the input, this is a nonlinear recursion (Postulate P1 (Linear/nonlinear) of Section  
 1838 3.5.1, p. 103).

1839 The direct method is inefficient because in recursively subtract  $n$  many times until the resulting  
 1840  $m$  is less than  $n$ , as shown in Fig. 2.1. It also must test for  $m < n$  at each iteration, and then swap  
 1841  $m$  and  $n$  once that condition is met. This recursion is repeated until  $m_{k+1} = 0$ . At that stage the  
 1842 GCD is  $n_{k+1}$ . Figure 2.1, along with the above matrix relation, give the best insight into the Euclidean  
 1843 Algorithm, but at the cost of low efficiency.

1844 Below is a Matlab code to find the gcd based on the strict definition of the EA:

```
1845 n = gcd(m,n)
1846 while m ~=0
1847     A=m; B=n;
1848     m=max(A,B); n=min(A,B); %m>n
1849     m=m-n;
1850 end
```

1851 This program keeps looping until  $m = 0$ . It first finds the min and max of the inputs, sets  $m$  as the  
 1852 max and  $n$  as the minimum. The next line  $m = m - n$  removes the smaller number from the larger  
 1853 one. It then loops back and repeats the cycle. Thus the EA is a two step recursive method.

**Division with rounding method:** This method implements Eq. 2.2. It is not necessary to test that  $m_{k+1} < n_{k+1}$ . After computing the new value of  $n$ , using the floor function, the old value of  $m$  is saved as the new value of  $n$  (thus  $m_{k+1} > n_{k+1}$ )

$$\begin{bmatrix} m_{k+1} \\ n_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor \frac{m}{n} \rfloor \end{bmatrix} \begin{bmatrix} m_k \\ n_k \end{bmatrix}. \quad (2.2)$$

1854 where  $\lfloor x \rfloor$  finds the integer part of  $x$  ( $\lfloor x \rfloor$  rounds toward  $-\infty$ ). The method terminates when  $m_{k+1} = 0$ .  
 1855 The previous values  $m_k, n_k$  are the solutions to Bézout's identity ( $\text{gcd}(n, m) = 1$ , namely  $m_k m_0 + n_k n_0 =$   
 1856  $1$ ), since the terminal state and the GCD of  $a, b$  is  $m - n \lfloor m/n \rfloor = 0$ , for which  $n = \text{gcd}(a, b)$ .

1857 Below is 1-line vectorized code that is much more efficient than the direct matrix method:

```

1858 k = gcd(m,n) %entry point: input m,n; output k=gcd(m,n)
1859 M=[abs(m),abs(n)]; %init M
1860 while M(2) ~=0 % < n*eps to ‘almost work’ with irrational inputs
1861     M = [M(2) - M(1)*floor(M(2)/M(1)); M(1)]; %M = [M(1); M(2)] with M(1)<M(2)
1862 end

```

1863 With a minor extension in the test for “end,” this code can be made to work with irrational inputs:  
 1864 e.g.:  $(n\pi, m\pi)$ .

1865 This method calculates the number of times  $n < m$  must subtract from  $m$  using the floor function.  
 1866 Note that the new value of  $m$  ( $M(1)$ ) is always less than  $n$  ( $M(2)$ ), and must remain greater or equal  
 1867 to zero. This one-line vector operation is then repeated until the remainder ( $M(1)$ ) is 0. The gcd is  
 1868 then  $n$  ( $M(2)$ ). When using irrational numbers, this still works except the error is never exactly zero,  
 1869 due to IEEE 754 rounding. Thus the criterion must be that the error is within some small factor times  
 1870 the smallest number (which in Matlab is the number  $\text{eps} = 2.220446049250313 \times 10^{-16}$ ).

1871 Thus without factoring the two numbers, the Euclidean algorithm recursively finds the gcd simply  
 1872 by ordering the two numbers and then updating them. Perhaps this is best seen with some examples.

1873 The GCD is an important and venerable method, useful in engineering and mathematics, but, as  
 1874 best I know, is not typically taught in the traditional engineering curriculum.

1875 **Graphical meaning of the GCD:** The Euclidean algorithm is actually very simple when viewed  
 1876 graphically. In Fig. 2.1 we show what is happening as one approaches the threshold. After reaching  
 1877 the threshold, the two number must be swapped, which is addressed by upper row of Eq. 2.2.

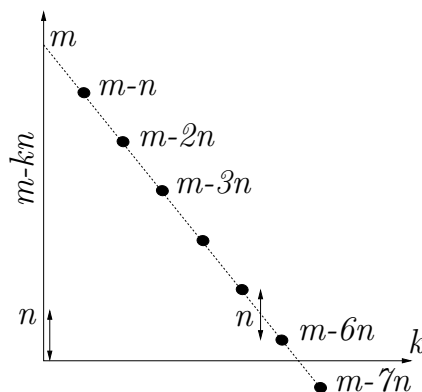


Figure 2.1: The Euclidean Algorithm recursively subtracts  $n$  from  $m$  until the remainder  $m - kn$  is either less than  $n$  or zero. For the case depicted here the value of  $k$  that renders the remainder less than  $n$  is  $k = 6$ . If one more step were taken ( $k = 7$ ) the remainder becomes negative. By linear interpolation we can find that  $m - an = 0$  when  $a = m/n$ , which for this example is close to  $a=6.5$ . In this example  $6 = \text{floor}(m/n) < n$ .

Multiplication is simply recursive addition, and finding the gcd takes advantage of this fact. Lets take a trivial example,  $(9,6)$ . Taking the difference of the larger from the smaller, and writing multiplication as sums, helps one see what is going on. Since  $6=3*2$ , this difference may be written two different ways

$$9 - 6 = (3 + 3 + 3) - (3 + 3) = 0 + 0 + 3 = 3,$$

or

$$9 - 6 = (3 + 3 + 3) - (2 + 2 + 2) = 1 + 1 + 1 = 3.$$

1878 Written out the first way, it is 3, because subtracting  $(3+3)$  from  $(3+3+3)$  leaves 3. Written out  
 1879 in the second way, each 3 is matched with a -2, leaving 3 ones, which add to 3. Of course the two  
 1880 decompositions must yield the same result because  $2 \cdot 3 = 3 \cdot 2$ . Thus finding the remainder of the  
 1881 larger number minus the smaller yields the gcd of the two numbers.



1882 **Coprimes:** When the gcd of two integers is 1, the only common factor is 1. This is of key importance  
 1883 when trying to find common factors between the two integers. When  $1 = \text{gcd}(m, n)$  they are said to be  
 1884 *coprime*, which can be written as  $m \perp n$ . By definition, the largest common factor of coprimes is 1.  
 1885 But since 1 is not a prime ( $\pi_1 = 2$ ), they have no common primes.

1886 **Generalizations of GCD:** The GCD may be generalized in several significant ways. For example  
 1887 what is the GCD of two polynomials? To answer this question one must factor the two polynomials to  
 1888 identify common roots. This will be discussed in more detail in Section 3.2.2.

### 1889 2.1.3 Lec 6 Continued Fraction Expansion (CFA)

1890 **Continued Fractions and circuit theory:** One of the most powerful generalizations of the CFA  
 1891 seems to be the expansion of a function of a complex variable, such as the expansion of an impedance  
 1892  $Z(s)$ , as a function of complex frequency  $s$ . This idea is described in Fig. 2.2 and Eq. 2.3. This  
 1893 is especially interesting in that it leads to a physical interpretation of the impedance in terms of a  
 1894 transmission line (horn), a structure well known in acoustics having a variable area  $A(x)$  as function of  
 1895 the range variable  $x$ .

1896 The CFA expansion is of great importance in circuit theory, where it is equivalent to an infinitely  
 1897 long segment of transmission line, composed of series and shunt impedance elements. Thus such a  
 1898 cascade network composed of 1 ohm resistors, has an input impedance of  $(1 + \sqrt{5})/2 \approx 1.6180$  [ohms].

The CFA may be extended to monomials in  $s$ . For example consider the input impedance of a cascade L-C transmission line as shown in Fig. 2.2. The input impedance of this transmission line is given by a continued fraction expansion of the form

$$Z_{in} = sL + \frac{1}{sC + \frac{1}{sL + \frac{1}{sC + \frac{1}{\dots}}}} \quad =: [sL; sC, sL, sC, \dots]. \quad (2.3)$$

1899 where we have again used the bracket notation to describe the CFA coefficients.

In some ways, Eq. 2.3 is reminiscent of the Taylor series expansion about  $s = 0$ , yet very different. In the limit, as the frequency goes to zero ( $s \rightarrow 0$ ), the impedance of the inductors go to zero, and that of the capacitors go to  $\infty$ . In physical terms, the inductors become short circuits, while the capacitors become open circuits. The precise relation may be quantified by the use of composition, described in Fig. 1.11 of Section 2.1.3 (p. 52). Specifically

$$\begin{bmatrix} P_1 \\ U_1 \end{bmatrix} = \begin{bmatrix} 1 & sL \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC & 1 \end{bmatrix} \dots \begin{bmatrix} 1 & sL \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC & 1 \end{bmatrix} \begin{bmatrix} 1 & sL \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC & 1 \end{bmatrix} \begin{bmatrix} P_2 \\ -U_2 \end{bmatrix}. \quad (2.4)$$

1900 It seems possible that this is the CFA generalization of the Taylor series expansion, built on composition.  
 1901 If we were to do the algebra we would find that  $A(s), B(s), C(s), D(s)$  (i.e., Sections 1.3.6, 3.3.2)  
 1902 are ratios of polynomials having rational expansions as Taylor series. This seems like an important  
 1903 observation, that should have support beyond that of the engineering literature (Campbell, 1903;  
 1904 Brillouin, 1953; Ramo et al., 1965). Its interesting that (Brillouin, 1953) credits (Campbell, 1903).

In terms of the TL, it is a long piece of wire, with a delay determined by the velocity and the length, in units of cells each of length  $\Delta$ . There are two basic parameters that characterize a transmission line, the characteristic resistance  $r_0 = \sqrt{Z/Y}$  and the wave number

$$\kappa = 1/\sqrt{ZY} = s/\sqrt{LC} = s/c,$$

which gives  $c = \sqrt{LC}$ . Each of these is a constant as  $\Delta \rightarrow 0$ , and in that limit the waves travel as

$$f(t - x/c) = e^{-\kappa x} e^{-st},$$

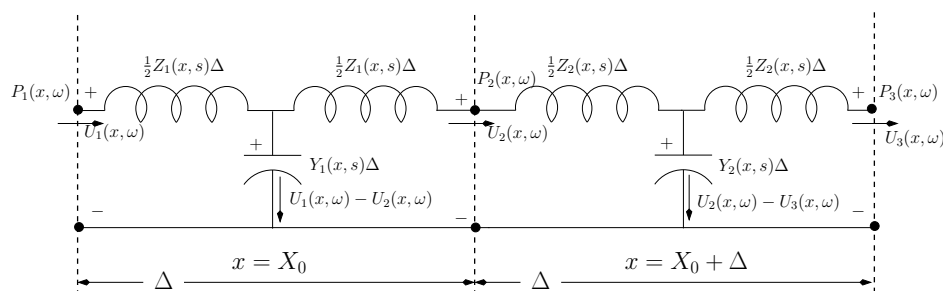


Figure 2.2: This transmission line (TL) is known as a low-pass filter wave-filter (Campbell, 1922), composed of series inductors having an impedance  $Z_k = sL$  and shunt admittance  $Y_k = sC$ . The characteristic resistance of the TL is  $r_0 = \sqrt{Z_k/Y_k}$  and the wave velocity is  $\kappa = \sqrt{Z_k Y_k} = s/\sqrt{LC} = s/c$ . Thus the wave velocity is  $c = 1/\sqrt{LC}$ . The length  $\Delta$  [m] defines the physical size of each cell. For long wavelengths the wave-filter acts as a pure delay. But when the frequency increases above a cutoff frequency  $f_c > c/\Delta$ , the wavelength  $\lambda = c/f$  is less than the size of a section  $\Delta$ , and the system response becomes very high-order low-pass filter.

1905 with a wave resistance ( $r_0 = \sqrt{L/C}$ ). The total delay  $T = L/c$  where  $\Delta$  is the TL cell length in meters  
 1906 and  $c$  is the velocity  $c = 1/\sqrt{LC}$  in meters/second.

1907 Since the CFA has a physical representation as a transmission line, as shown in Fig. 2.2, it can be  
 1908 of high utility for the engineer.<sup>2</sup> The theory behind this will be discussed in greater detail in Chapter  
 1909 5. If you're ready to jump ahead, read the interesting book by Brillouin (1953) and the collected works  
 1910 of Campbell (1937).

## 1911 2.2 Week 3

### 1912 2.2.1 Lec 7 Pythagorean triplets (PTs) and Euclid's formula ,

1913 Pythagorean triplets (PTs) have many applications in architecture and scheduling, which explains why  
 1914 they are important and heavily studied. For example, if one wished to construct a triangle with a  
 1915 perfect  $90^\circ$  angle, then the materials need to be squared off as shown in Fig. 1.8. The lengths of the  
 1916 sides need to satisfy PTs.

1917 **Derivation of Euclid's formula:** The problem is to find integer solutions to the Pythagorean  
 1918 theorem (Eq. 1.1, p. 15). The solution method, said to be due to Diophantus, is call a *chord/tangent*  
 1919 method (Stillwell, 2010, p. 48). The method composes (Section 3.2.3) a line and a circle, where the  
 1920 line defines a chord within the circle (its not clear where the tangent line might go). The slope of the  
 1921 line is then taken to be rational, allowing one to determine integer solutions of the intersections points.  
 1922 This solution for *Pythagorean triplets*  $[a, b, c]$  is known as *Euclid's formula* (Eq. 1.3, p. 1.3 (Stillwell,  
 1923 2010, p. 4–9, 222).

1924 The derivation methods of Diophantus have been lost, but Fermat and Newton figured out what  
 1925 Diophantus must have done (Stillwell, 2010, p. 49). Since Diophantus worked before algebra was  
 1926 invented, he described all the equations in prose (Stillwell, 2010, p. 93).

1927 **Derivation of Euclid's formula:** The derivation is outlined in Fig. 2.3. Starting from two integers  
 1928  $[p > q > 0] \in \mathbb{N}$ , composing a line having a rational slope  $t = p/q$ , with a circle (Stillwell, 2010, p. 6),  
 1929 reveals the formula for the Pythagorean triplets.

1930 The construction starts with a circle and a line, which is terminated at the point  $(-1, 0)$ . The slope  
 1931 of the line is the free parameter  $t$ . By composing the circle and the line (i.e., solving for the intersection  
 1932 of the circle and line), the formula for the intersection point  $(a, b)$  may be determined in terms of  $t$ ,  
 1933 which will then be taken as the rational slope  $t = p/q \in \mathbb{Q}$ .

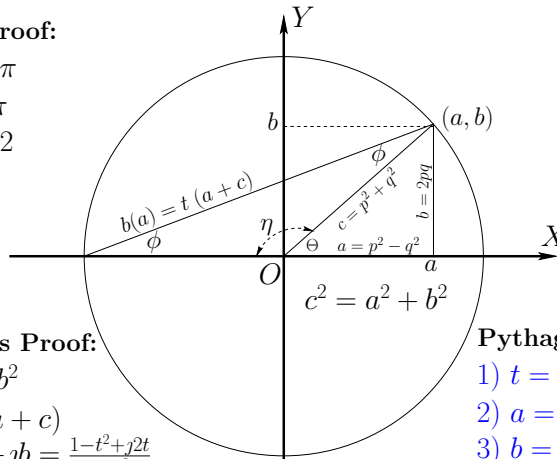
<sup>2</sup>Continued fraction expansions of functions are know in the circuit theory literature as a *Cauer* synthesis (Van Valkenburg, 1964b).

**Euclid's formula for Pythagorean triplets  $[a, b, c]$**

5.2.4

**Euclidean Proof:**

- 1)  $2\phi + \eta = \pi$
- 2)  $\eta + \Theta = \pi$
- 3)  $\therefore \phi = \Theta/2$



**Diophantus's Proof:**

- 1)  $c^2 = a^2 + b^2$
- 2)  $b(a) = t(a + c)$
- 3)  $\zeta(t) \equiv a + jb = \frac{1-t^2+j2t}{1+t^2}$
- 4)  $\zeta = |c|e^{i\theta} = |c|\frac{1+it}{1-it} = |c|(\cos(\theta) + i \sin(\theta))$

**Pythagorean triplets:**

- 1)  $t = p/q \in \mathbb{Q}$
- 2)  $a = p^2 - q^2$
- 3)  $b = 2pq$
- 4)  $c = p^2 + q^2$

Figure 2.3: Derivation of Euclid's formula for the Pythagorean triplets  $[a, b, c]$ , based on a composition of a line, having a rational slope  $t = p/q \in \mathbb{Q}$ , and a circle  $c^2 = a^2 + b^2, [a, b, c] \in \mathbb{N}$ . This analysis is attributed to Diophantus (Di-o-phan-tus) (250 CE), and today such equations are called Diophantine (Di-o-phan'tine) equations. PTs have applications in architecture and scheduling, and many other practical problems.

1934 In Fig. 2.3 there are three panels, two labeled "Proofs." The *Euclidean Proof* shows the angle  
 1935 relationships of two triangles, the first an isosceles triangle formed by the chord, having slope  $t$  and  
 1936 two equal sides formed from the radius of the circle, and a second right triangle having its hypotenuse  
 1937 as the radius of the circle and its right angle vertex at  $(a, 0)$ . As shown, it is this smaller right triangle  
 1938 that must satisfy Eq. 1.1. The inner right triangle has its hypotenuse  $c$  between the origin of the circle  
 1939 (O) to the point  $(a, b)$ . Side  $a$  forms the  $x$  axis and side  $b$  forms the  $y$  ordinate. Thus by construction  
 1940 Eq. 1.1 must be obeyed.

1941 The *Diophantus Proof* is the heart of Diophantus' (250 CE) derivation, obtained by composing a  
 1942 line and a circle, as shown in Fig. 2.3. Diophantus's approach was to fix the line at  $x = -c$  having a  
 1943 rational slope  $t = p/q \in \mathbb{Q}$ . He then solved for the intersection of the line and the circle, at  $(a, b)$ .

The formula for the line is  $b(a) = t(a + c)$ , which goes through the points  $(-c, 0)$  and  $(a, b)$ . The circle is given by  $a^2 + b^2 = c^2$ . Composing the line with the circle gives

$$\begin{aligned}
 a^2 + (t(a + c))^2 &= c^2 \\
 a^2 + t^2(a^2 + 2ac + c^2) &= c^2 \\
 (1 + t^2)a^2 + 2ct^2a + c^2(t^2 - 1) &= 0
 \end{aligned}$$

This last equation is a quadratic equation in  $a$ . In some sense it is not really a quadratic equation, since we know that  $a = -c$  is a root. Dividing by  $1 + t^2$

$$a^2 + \frac{2ct^2}{1 + t^2}a + \frac{c^2(t^2 - 1)}{1 + t^2} = 0,$$

makes it easier to complete the square, delivering the roots:

$$\begin{aligned} \left(a + \frac{ct^2}{1+t^2}\right)^2 - \left(\frac{ct^2}{1+t^2}\right)^2 + \frac{c^2(t^2-1)}{1+t^2} &= 0 \\ \left(a + \frac{ct^2}{1+t^2}\right)^2 - \frac{c^2t^4}{(1+t^2)^2} + \frac{c^2(t^2-1)(t^2+1)}{(1+t^2)^2} &= 0 \\ \left(a + \frac{ct^2}{1+t^2}\right)^2 - \frac{c^2t^4 + c^2(t^4-1)}{(1+t^2)^2} &= 0 \\ \left(a + \frac{ct^2}{1+t^2}\right)^2 &= \left(\frac{c}{1+t^2}\right)^2 \end{aligned}$$

1944 The second to last equation simplifies (magic happens) because the known root  $a = -c$  is embedded  
1945 in the result.

Taking the square root gives the two roots

$$\begin{aligned} a_{\pm} + \frac{ct^2}{1+t^2} &= \pm \frac{c}{1+t^2} \\ (1+t^2)a_{\pm} &= -ct^2 \pm c = -c(t^2 \mp 1) \\ a_{\pm} &= -c \frac{t^2 \mp 1}{1+t^2}. \end{aligned}$$

1946 The known root is  $a_+ = -c$ , because when the sign is +, the numerator and denominator terms cancel.  
The root we have been looking for is  $a_-$

$$a_- = c \frac{1-t^2}{1+t^2},$$

which allows us to solve for  $b_-$

$$\begin{aligned} b_- &= \pm \sqrt{c^2 - a_-^2} \\ &= \pm c \sqrt{1 - \left(\frac{1-t^2}{1+t^2}\right)^2} \\ &= \pm c \sqrt{\frac{(1+t^2)^2 - (1-t^2)^2}{(1+t^2)^2}} \\ &= \pm \frac{2ct}{t^2+1}. \end{aligned}$$

Therefore the coordinates  $(a, b)$ , the intersection point of the line and circle, are

$$(a(t), b(t)) = c \frac{[1-t^2, 2t]}{1+t^2}.$$

1947  
1948 To obtain the Pythagorean triplets, as given in Fig. 2.3 and Eq. 1.3 of Section 1.2.7 (p. 37), set  
1949  $t = p/q$ , assuming  $p > q \in \mathbb{Z}$ , and simplify.

**Complex roots:** Defining the root as a complex number  $\zeta(\Theta) \equiv a + bj$  forces  $a \perp b$  (i.e., forces the right triangle) and gives us polar coordinates, as defined by the figure as the Euclidean Proof

$$\zeta(\Theta) = |c|e^{\Theta j} = |c|(\cos(\Theta) + j \sin(\Theta)).$$

This naturally follows since

$$\zeta = |c|e^{j\Theta(t)} = |c|\frac{1-t^2+2tj}{1+t^2} = |c|\frac{(1+jt)(1+jt)}{(1+tj)(1-tj)} = (q+pj)\sqrt{\frac{q+jp}{q-pj}}.$$

1950 Examples of PTs include  $a = 2^2 - 1^2 = 3$ ,  $b = 2 \cdot 2 \cdot 1 = 4$ , and  $c = 2^2 + 1^2 = 5$ ,  $3^2 + 4^3 = 5^2$ .

1951 Defining  $p = q + N$  ( $N \in \mathbb{N}$ ) gives slightly better parametric representation of the answers, as the  
 1952 pair  $(q, N)$  are a more systematic representation than  $(p, q)$ , because the condition  $p > q$  is accounted  
 1953 for, so the general properties of the solutions are expressed more naturally. Note that  $b+c$  must always  
 1954 be a perfect square since  $b+c = (p+q)^2 = (2q+N)^2$ , as first summarized by Fermat Stillwell (2010,  
 1955 p. 212).

1956 s

### 1957 2.2.2 Lec 8 Pell's Equation

1958 **Eigenvalue solution to Pell's equation:** To provide a full understanding of what was known to  
 1959 the Pythagoreans, it is helpful to provide the full solution to this recursive matrix equation, based on  
 1960 what we know today.

1961 As shown in Fig. 1.10,  $(x_n, y_n)$  may be written as a power series of the 2x2 matrix  $A$ . To find the  
 1962 powers of a matrix, the well know modern approach is to *diagonalize* the matrix. For the 2x2 matrix  
 1963 case, this is relatively simple. The final result written out in detail for the general solution  $(x_n, y_n)$ , as  
 1964 detailed in Appendix D (p. 139):

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = j^n \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = E \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix} E^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{2.5}$$

The eigen-values are  $\lambda_{\pm} = j(1 \pm \sqrt{2})$  while the eigen-matrix and its inverse are

$$E = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.8165 & 0.8165 \\ 0.5774 & -0.5774 \end{bmatrix}, \quad E^{-1} = \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & \sqrt{2} \\ 1 & -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0.6124 & 0.866 \\ 0.6124 & -0.866 \end{bmatrix}$$

The relative “weights” on the two eigen-solutions are equal, as determined by

$$E^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & \sqrt{2} \\ 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We still need to prove that

$$\frac{x_n}{y_n} \xrightarrow{\infty} \sqrt{N},$$

1965 which follows intuitively from Pell's equation, since as  $(x_n, y_n) \rightarrow \infty$ , the difference between  $x^2$  and  
 1966  $2y^2$ , the  $(\pm 1)$  becomes negligible.

## 1967 WEEK 4

1968

### 1969 2.3 Week 4

#### 1970 2.3.1 Lec 9 Fibonacci Numbers

1971 The Fibonacci sequence is famous in number theory. It is said that the sequence commonly appears in  
 1972 physical systems. Fibonacci numbers are related to the “golden ratio”  $(1 + \sqrt{5})/2$ , which could explain  
 1973 why these numbers appear in nature.

But from a mathematical point of view, the Fibonacci sequence does not seem special. It is generated by a linear recursion relationship, where the next number is the sum of the previous two (Eq. 1.6, p. 40)

$$x_{n+1} = x_n + x_{n-1}. \quad (2.6)$$

1974 The term *linear* means that the principle of superposition holds (P1 (linear/nonlinear) of Section  
1975 3.5.1). To understand the meaning of this we need to explore the z-transform, the discrete-time version  
1976 of the Laplace transform. We will return to this in Chapter 4.

A related linear recurrence relation is that the next output be the average of the previous two

$$x_{n+1} = \frac{x_n + x_{n-1}}{2}.$$

1977 In some ways this relationship is more useful than the Fibonacci recursion, since it perfectly removes  
1978 oscillations of the form  $-1^n$  (it is a 2-sample *moving average*, a trivial form of low-pass filter). And it  
1979 is stable, unlike the Fibonacci sequence, with stable real eigenvalues (digital-poles) at  $\lambda_{\pm} = (1, -0.5)$ .  
1980 Perhaps biology prefers unstable poles (to propagate growth?).

The most general 2d order recurrence relationships (i.e., digital filter) is

$$x_{n+1} = -bx_n - cx_{n-1},$$

1981 with filter constants  $b, c \in \mathbb{R}$  and poles at (completing the square),  $\lambda_{\pm} = -b/2 \pm \sqrt{c - b^2/2}$ .

Equation 2.6 may be written as a 2x2 matrix relationship. If we define  $y_{n+1} = x_n$  then Eq. 2.6 is equivalent to (Eq. 1.7, p. 40)

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}. \quad (2.7)$$

1982 The first equation is  $x_{n+1} = x_n + y_n$  while the second is  $y_{n+1} = x_n$ , which is the same as  $y_n = x_{n-1}$ .  
1983 Note that the Pell 2x2 recursion is similar in form to the Fibonacci recursion. This removes mystique  
1984 from both equations.

## General properties of the Fibonacci numbers<sup>a</sup>

$$x_n = x_{n-1} + x_{n-2}$$

- This is a 2-sample *moving average* difference equation with an unstable pole
- $x_n = [0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots]$ , assuming  $x_0 = 0, x_1 = 1$ :
- Analytic solution (Stillwell, 2010, p. 194):  $\sqrt{5} x_n \equiv \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \rightarrow \left(\frac{1+\sqrt{5}}{2}\right)^\infty$ 
  - $\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = \frac{1+\sqrt{5}}{2}$
  - Example:  $34/21 = 1.6190 \approx \frac{1+\sqrt{5}}{2} = 1.6180$  **0.10% error**
- Matlab's `rat(1 + sqrt(5)) = 3 + 1/(4 + 1/(4 + 1/(4 + 1/(4)))) =: [3; 4, 4, 4, ...]`

<sup>a</sup>[https://en.wikipedia.org/wiki/Fibonacci\\_number](https://en.wikipedia.org/wiki/Fibonacci_number)

Figure 2.4: Properties of the Fibonacci numbers (Stillwell, 2010, p. 28).

1985 In the matrix diagonalization of the Pell equation we found that the eigenvalues were  $\lambda_{\pm} = 1 \mp \sqrt{N}$ ,  
1986 and the two solutions turned out to be powers of the eigenvalues. The solution to the Fibonacci recursion  
1987 may similarly be expressed in terms of a matrix. These two cases may thus be reduced by the same  
1988 2x2 eigenvalue solution method.

The eigenvalues of the Fibonacci matrix are

$$\det \begin{bmatrix} 1 - \lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \lambda - 1 = (\lambda - 1/2)^2 - (1/2)^2 - 1 = (\lambda - 1/2)^2 - 5/4 = 0$$

1989 thus  $\lambda_{\pm} = \frac{1 \pm \sqrt{5}}{2} = [1.618, -0.618]$ .







## 1991 Chapter 3

# 1992 Algebraic Equations: Stream 2

### 1993 3.1 Week 4

#### 1994 3.1.1 Lec 11 Algebra and geometry as physics

Before Newton could work out his basic theories, algebra needed to be merged with Euclid's early quantification of geometry. The key to putting geometry and algebra together is the Pythagorean theorem (Eq. 1.1), which is both geometry and algebra. To make the identification with geometry the sides of the triangle needed to be viewed as a length. This is done by recognizing that the area of a square is the square of a length. Thus a geometric proof requires one to show that the area of the square  $A = a^2$  plus the area of square  $B = b^2$  must equal the area of square  $C = c^2$ . There are many such constructions that show  $A + B = C$  for the right triangle. It follows that in terms of coordinates of each vertex, the length of  $c$  is given by

$$c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \quad (3.1)$$

1995 with  $a = x_2 - x_1$  and  $b = y_2 - y_1$ . Thus Eq. 1.1 is both an algebraic and a geometrical statement. This  
1996 is not immediately obvious.

1997 Analytic geometry is based on coordinates of points, with the length given by Eq. 3.1. Geometry  
1998 treats lines as lengths without specifying the coordinates (Eq. 1.1). Algebra gave a totally new view  
1999 to the quantification of geometrical lengths by introducing a coordinate system. This resulted in an  
2000 entire new way to work with conic sections, which were now explained in terms of equations having  
2001 coordinate systems. When viewed through the lens of algebra, Eq. 1.1 is a circle having radius  $c$ .  
2002 Complex numbers provide an equivalent representations, since if  $z = x + yj$ , the unit circle is  $z = e^{j\theta}$   
2003 and  $|z|^2 = x^2 + y^2$ . Here we explore the relationships between points, represented as coordinates,  
2004 describing geometrical objects. We shall do this with simple examples from analytic geometry.

2005 For example, in terms of the geometry, the intersection of two circles can occur at two points, and  
2006 the intersection of two spheres gives a circle. These ideas may be verified using algebra, but in a very  
2007 different, since the line can traverse through the circle, like a piece of thread going through the eye of  
2008 a needle. In such cases the intersections are complex intersections.

2009 For each of these problems, the lines and circles may intersect, or not, depending on how they are  
2010 drawn. Yet we now know that even when they do not intersect on the sheet of paper, they still have  
2011 an intersection, but the solution is  $\in \mathbb{C}$ . Finding such solutions require the use of algebra rather than  
2012 geometry. These ideas were in the process of being understood, first by Fermat and Descartes, then by  
2013 Newton, followed by the Bernoulli family and Euler.

2014 **Complex analytic functions:** A very delicate point, that seems to have been ignored for centuries,  
2015 is that the roots of  $P_n(x)$  are, in general, complex, namely  $x_k \in \mathbb{C}$ . It seems a mystery that complex  
2016 numbers were not accepted once the quadratic equation was discovered, but they were not. Newton  
2017 called complex roots *imaginary*, presumably in a pejorative sense. The algebra of complex numbers

is first attributed to Bombelli in 1575, more than 100 years before Newton. One can only begin to imagine what Bombelli learned from Diophantus, following his discovery of Diophantus' Arithmetic, that he discovered in the Vatican library (Stillwell, 2010, p. 51).

It is interesting that Newton was using power series with fractional degree, thus requiring multi-valued solutions, much later to be known as *branch cuts* (c1851). These topics will be explored in Section 3.1.1.

When the argument is complex, analytic functions takes on an entirely new character. For example Euler's identity (1748) with  $z = x + yj \in \mathbb{C}$  results in  $e^z \in \mathbb{C}$  (Stillwell, 2010, p. 315)

$$e^z = e^x(\cos(y) + j \sin(y)).$$

It should be clear that the complex analytic functions results in a new category of algebra, with no further assumptions beyond allowing the argument to be complex.

Prior to 1851 most of the analysis *assumed* that the roots of polynomials were real ( $x_k \in \mathbb{R}$ ), even though there was massive evidence that they were complex ( $x_n \in \mathbb{C}$ ). This is clearly evident in Newton's work (c1687): When he found a non-real root, he ignore it (Stillwell, 2010, pp. 115-7). Euler (c1748) first derived the Zeta function as a function of real arguments  $\zeta(x)$  with  $\zeta, x \in \mathbb{R}$ . Cauchy (c1814) broke this staid thinking with his analysis of complex analytic functions, but it was Riemann thesis (c1851), when working with Gauss (1777-1855), which had a several landmark breakthroughs. In this work Riemann introduced the *extended complex plane*, which explained the *point* at infinity. He also introduced *Riemann sheets* and *Branch cuts*, which finally allowed mathematics to better describe the physical world (Section 1.4.2).

Once the argument of an analytic function is complex, for example an impedance  $Z(s)$ , or the Riemann Zeta function  $\zeta(s)$ , The development of complex analytic functions led to many new fundamental theorems. Complex analytic functions have poles and zeros, branch cuts, Riemann sheets and can be analytic at the point at infinity. Many of these properties were first worked out by Augustin-Louis Cauchy (1789-1857), who drew heavily on the much earlier work of Euler, expanding Euler's ideas into the complex plane (Chapter 4).

## Systems of equations

We don't need to restrict ourselves to polynomials in one variable. In some sense  $y(x) = ax + b$  is already an equation in two variables. Moving beyond such a simple example, we can work with the equation for a circle, having radius  $r$

$$y^2 + x^2 = r^2,$$

which is quadratic in two variables. Solving for roots  $y(x_r) = 0$  ( $y^2(x_r) = r^2 - x_r^2 = 0$ ) gives  $(r - x_r)(r + x_r)$ , which simply says that when the circle crosses the  $y = 0$  line at  $x_r = \pm r$ .

This equation may also be factored as

$$(y - xj)(y + xj) = r^2,$$

as is easily demonstrated by multiplying out the two monomials. This does not mean that a circle has complex roots. A root is defined by either  $y(x_r) = 0$  or  $x(y_r) = 0$ .

Writing the conic as a 2d degree polynomial gives

$$P_2(x) = ax^2 + bx + c,$$

with  $y^2(x) = -P_2(x)$ . Setting this equal to zero and completing the square (Eq. 1.12, p. 42), gives the equation for the roots

$$\left(x_{\pm} + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0,$$

or

$$x_{\pm} = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}.$$

The polynomial in factored form is

$$y^2 = -\left(x - \frac{b}{2a}\right)^2 + \left(\frac{b}{2a}\right)^2 - \frac{c}{a},$$

is a conic section, and becomes a circle with  $a = 1$ ,  $b = 0$  and  $c = -r^2$ .

## 3.2 Week 5

### 3.2.1 Lec 12 The physics behind complex analytic expressions: linear vs. nonlinear

The question we address here is “When do multi-variable complex analytic expressions appear in physics?” The most common example comes from the solution of the wave equation (Eq. 1.8) in three dimensions. Such cases arise in wave-guide problems, semiconductors, plasma waves, or for acoustic wave propagation in crystals (Brillouin, 1960) and the earth’s mantle (e.g., seismic waves, earthquakes, etc.). The solutions to these problems are based on the *eigenfunction* for the *vector wave equation* (see Chapter 5),

$$P(s, \mathbf{x}) = e^{st} e^{-\boldsymbol{\kappa} \cdot \mathbf{x}}, \quad (3.2)$$

where  $\mathbf{x} = [x\hat{x} + y\hat{y} + z\hat{z}]$  is a vector pointing in the direction of the wave,  $[\hat{x}, \hat{y}, \hat{z}]$  are unit vectors in the three dimensions and  $s = \sigma + \omega j$  [rad] is the Laplace frequency. The function  $\boldsymbol{\kappa}(s)$  is the complex vector *wave number*, which describes the propagation of a plane wave of radian frequency  $\omega$ , in the  $\mathbf{x}$  direction.

Just as the frequency  $s = \sigma + \omega j$  must be complex, it is important to allow the *wave number* function<sup>1</sup> to be complex,<sup>2</sup> because in general it will have a real part, to account for losses as the wave propagates. While it is common to assume there are no losses, in reality this assumption cannot be correct. In many cases it is an excellent approximation (e.g., even the losses of light in-vacuo are not zero) that gives realistic answers. But it is important to start with a notation that accounts for the most general situation, so that when losses must be accounted for, the notation need not change. With this in mind, we take the vector wave number to be complex

$$\boldsymbol{\kappa} = \mathbf{k}_r + \mathbf{k}_j,$$

where vector expression for the *lattice vector* is the imaginary part of  $\boldsymbol{\kappa}$

$$\Im \boldsymbol{\kappa} = \mathbf{k} = \frac{2\pi}{\lambda_x} \hat{x} + \frac{2\pi}{\lambda_y} \hat{y} + \frac{2\pi}{\lambda_z} \hat{z}, \quad (3.3)$$

is the vector wave number for three dimensional lossless plane-wave solutions ( $\mathbf{k}_r = 0$ ).

Equation Eq. 3.2 is linear in  $\mathbf{x}$ . If one takes the derivative with respect to either time or space,

$$\frac{\partial}{\partial t} e^{st} e^{-\boldsymbol{\kappa} \cdot \mathbf{x}} = s e^{st} e^{-\boldsymbol{\kappa} \cdot \mathbf{x}}, \quad \frac{\partial}{\partial x} e^{st} e^{-\boldsymbol{\kappa} \cdot \mathbf{x}} = \frac{2\pi}{\lambda_x} e^{st} e^{-\boldsymbol{\kappa} \cdot \mathbf{x}},$$

we find the eigenvalue of that derivative.

The units of  $\boldsymbol{\kappa}$  are reciprocal length [ $\text{m}^{-1}$ ] since  $\boldsymbol{\kappa} \cdot \mathbf{x}$  has units of radians. When there are losses  $\kappa_r(s) = \Re \boldsymbol{\kappa}(s)$  must be a function of frequency, due to the physics behind these losses. In many important cases, such as loss-less wave propagation in semiconductors,  $\boldsymbol{\kappa}(\mathbf{x})$  is a function of direction and position (Brillouin, 1960).

When the eigenfunction Eq. 3.2 is applied to the wave equation, a quadratic (degree 2) algebraic expression results, known as the *dispersion relation*. The three dimensional dispersion relation

$$\left(\frac{s}{c}\right)^2 = \boldsymbol{\kappa} \cdot \boldsymbol{\kappa} = \left(\frac{2\pi}{\lambda_x}\right)^2 + \left(\frac{2\pi}{\lambda_y}\right)^2 + \left(\frac{2\pi}{\lambda_z}\right)^2 = k_x^2 + k_y^2 + k_z^2 \quad (3.4)$$

<sup>1</sup>This function has many names in the literature, all of which are confusing. It is called the *wave number*, *propagation constant* and the Brillouin zone *dispersion function* (Brillouin, 1953, Ch. 1). However, its neither a number nor constant.

<sup>2</sup>In fact  $\boldsymbol{\kappa}(s)$  is a complex analytic function of the Laplace frequency  $s$ .

2059 is a complex analytic algebraic relationship in four variables, frequency  $s$  and the three complex *lattice*  
 2060 *wave numbers*. This represents a three-dimensional generalization of the well know relation between  
 2061 wavelength and frequency, i.e.,  $f\lambda = c$ . For lossless plane waves propagating in free space,  $|\kappa(s)| =$   
 2062  $\pm|s/c|$ , where the sign accounts for the direction of the plane wave.

2063 This scalar relation ( $f\lambda = c$ ) was first deduced by Galileo in the 16th century and was then explored  
 2064 further by Mersenne a few years later. This relationship would have been important to Newton when  
 2065 formulating the wave equation, which he needed to estimate the speed of sound. We shall return to  
 2066 this in Chapters 4 and 5.

2067 **Inner product space:** Another important example of algebraic expressions in mathematics is Hilbert's  
 2068 generalization of Eq. 1.1, known as the Schwarz inequality, shown in Fig. 3.1. What is special about  
 2069 this generalization is that it proves that when the vertex is  $90^\circ$ , the length of the leg is minimum.

Vectors may be generalize to have  $\infty$  dimensions:  $\vec{U}, \vec{V} = [v_1, v_2, \dots, v_\infty]$ . The inner product (i.e., dot product) between two such vectors generalizes the finite dimensional case

$$\vec{U} \cdot \vec{V} = \sum_{k=1}^{\infty} u_k v_k.$$

2070 As with the finite case, the *norm*  $\|\vec{U}\| = \sqrt{\vec{U} \cdot \vec{U}} = \sqrt{\sum u_k^2}$  is the dot product of the vector with itself,  
 2071 defining the *length* of the infinite component vector. Obviously there is an issue of convergence, if the  
 2072 norm for the vector to have a finite length.

It is a somewhat arbitrary requirement that  $a, b, c \in \mathbb{R}$  for the Pythagorean theorem (Eq. 1.1). This seems natural enough since the sides are lengths. But, what if they are taken from the complex numbers, as for the lossy vector wave equation, or the lengths of vectors in  $\mathbb{C}^n$ ? Then the equation generalizes to

$$\mathbf{c} \cdot \mathbf{c} = \|\mathbf{c}\|^2 = \sum_{k=1}^n |c_k|^2,$$

2073 where  $\|\mathbf{c}\|^2 = (\mathbf{c}, \mathbf{c})$  is the inner (dot) product of a vector  $\mathbf{c}$  with itself, where  $|c_k|$  is the magnitude the  
 2074 complex  $c_k$ . As before,  $\|\mathbf{c}\| = \sqrt{\|\mathbf{c}\|^2}$  is the *norm* of vector  $\mathbf{c}$ , akin to a length.

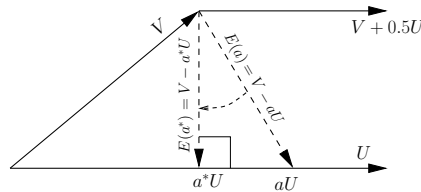


Figure 3.1: The Schwarz inequality is related to the shortest distance (length of a line) between the ends of the two vectors.  $\|U\| = \sqrt{(U \cdot U)}$  as the dot product of that vector with itself.

2075 **Schwarz inequality** The *Schwarz inequality* says that the magnitude of the inner product of two  
 2076 vectors is less than or equal to the lengths of the product of the two vectors. This follows from Fig. 3.1  
 2077 in terms of the geometry.

A proof of the Schwarz inequality is as follows: From these definitions we may define the minimum difference between the two vectors as the perpendicular from the end of one to the intersection of the second. As shown in Fig. 3.1,  $U \perp V$  may be found by minimizing the length of the vector difference:

$$\begin{aligned} \min_{\alpha} \|V - \alpha U\|^2 &= \|V\|^2 + 2\alpha V \cdot U + \alpha^2 \|U\|^2 > 0 \\ 0 &= \partial_{\alpha} (V - \alpha U) \cdot (V - \alpha U) \\ &= V \cdot U - \alpha^* \|U\|^2 \\ \therefore \alpha^* &= V \cdot U / \|U\|^2. \end{aligned}$$

The *Schwarz inequality* follows:

$$I_{\min} = \|V - \alpha^* U\|^2 = \|V\|^2 - \frac{|U \cdot V|^2}{\|U\|^2} > 0$$

$$0 \leq |U \cdot V| \leq \|U\| \|V\|$$

One may define the direction cosine between the two vectors as

$$\cos(\theta) = \frac{U \cdot V}{\|U\| \|V\|}$$

2078 which is a measure of the projection of one of the vectors onto the other, as shown by the figure. The  
2079  $\cos(\theta)$  does not represent a complex analytic function. Namely  $\theta \in \mathbb{R}$ , not  $\in \mathbb{C}$  (as best I know).

An important example of such a vector space includes the definition of the *Fourier Transform*, where we may set

$$U(\omega) = e^{-\omega_0 t} \quad V(\omega) = e^{\omega j t} \quad U \cdot V = \int_{\omega} e^{j\omega t} e^{-j\omega_0 t} \frac{d\omega}{2\pi} = \delta(\omega - \omega_0).$$

2080 We shall return to this topic in Lecture 1.3.9.

### 2081 Power vs. power series, linear vs. nonlinear

Another place where equations of second degree appear in physical applications is in energy and power calculations. The electrical power is given by the product of the voltage  $v(t)$  and current  $i(t)$  (or in mechanics as the force times the velocity). For example if we define  $\mathcal{P} = v(t)i(t)$  to be the power  $\mathcal{P}$  [Watts], then the *total energy* [Joules] at time  $t$  is (Van Valkenburg, 1964a, Chapter 14)

$$\mathcal{E}(t) = \int_0^t v(t)i(t)dt.$$

From this observe that the power is the rate of change of the total energy

$$\mathcal{P}(t) = \frac{d}{dt} \mathcal{E}(t).$$

**Ohm's Law and impedance:** The ratio of voltage over the current is call the *impedance* which has units of [Ohms]. For example given a resistor of  $R = 10$  [ohms],

$$v(t) = R i(t).$$

Namely 1 amp flowing through the resistor would give 10 volts across it. Merging the linear relation due to Ohm's law with the definition of power, shows that the instantaneous power in a resistor is quadratic in voltage and current

$$\mathcal{P} = v(t)^2/R = i(t)^2 R.$$

2082 Note that Ohm's law is linear in its relation between voltage and current whereas the power and energy  
2083 are *nonlinear*.

2084 Ohm's Law generalizes in a very important way, allowing the impedance (e.g., resistance) to be  
2085 a linear complex analytic function of complex frequency  $s = \sigma + \omega j$  (Kennelly, 1893; Brune, 1931a).  
2086 Impedance is a fundamental concept in many fields of engineering. For example:<sup>3</sup> Newton's second law  
2087  $F = ma$  obeys Ohm's law, with mechanical impedance  $Z(s) = sm$ . Hooke's Law  $F = kx$  for a spring  
2088 is described by a mechanical impedance  $Z(s) = k/s$ . In mechanics a "resistor" is called a *dashpot* and  
2089 its impedance is a positive and real constant.<sup>4</sup>

<sup>3</sup>In acoustics the pressure is a potential, like voltage. The force per unit area is given by  $f = -\nabla p$  thus  $F = -\int \nabla p dS$ . Velocity is analogous to a current. In terms of the velocity potential, the velocity per unit area is  $v = -\nabla \phi$ .

<sup>4</sup>[https://en.wikipedia.org/wiki/Impedance\\_analogy](https://en.wikipedia.org/wiki/Impedance_analogy)

2090 **Kirchhoff's Laws KCL, KVL:** The laws of electricity and mechanics may be written down using  
 2091 Kirchoff's Laws current and voltage laws, (KCL, KVL), which lead to linear systems of equations in  
 2092 the currents and voltages (velocities and forces) of the system under study, with complex coefficients  
 2093 having positive real parts.

2094 Points of major confusion are a number of terms that are misused, and overused, in the fields of  
 2095 mathematics, physics and engineering. Some of the most obviously abused terms are *linear/nonlinear*,  
 2096 *energy, power, power series*. These have multiple meanings, which can, and are, fundamentally in  
 2097 conflict.

**Transfer functions (Transfer matrix):** The only method that seems to work, to sort this out, is to cite the relevant physical application, in specific contexts. The most common touch point is a physical system that has an input  $x(t)$  and an output  $y(t)$ . If the system is linear, then it may be represented by its *impulse response*  $h(t)$ . In such cases the system equation is

$$y(t) = h(t) \star x(t) \leftrightarrow Y(\omega) = H(s)|_{\sigma=0} X(\omega),$$

2098 namely the convolution of the input with the impulse response gives the output. From Fourier analysis  
 2099 this relation may be written in the real frequency domain as a product of the Laplace transform of the  
 2100 impulse response, evaluated on the  $\omega j$  axis and the Fourier transform of the input  $X(\omega) \leftrightarrow x(t)$  and  
 2101 output  $Y(\omega) \leftrightarrow y(t)$ .

2102 **Mention ABCD Transfer matrix**

2103 If the system is nonlinear, then the output is not given by a convolution, and the Fourier and  
 2104 Laplace transforms have no obvious meaning.

2105 The question that must be addressed is why is the power said to be nonlinear whereas a power series  
 2106 of  $H(s)$  said to be linear. Both have powers of the underlying variables. This is massively confusing,  
 2107 and must be addressed. The question will be further addressed in Section 3.5.1 in terms of the system  
 2108 postulates of physical systems.

2109 **Whats going on? The domain variables must be separated from the codomain variables. In our**  
 2110 **example, the voltage and current are multiplied together, resulting in a nonlinear output, the power.**  
 2111 **If the frequency is squared, this is describing the degree of a polynomial. This is not nonlinear because**  
 2112 **it does not impact the signal output, it characterizes the Laplace transform of the system response.**

### 2113 3.2.2 Lec 13 Root classification of polynomials

2114 **Root classification for polynomials of Degree \* = 1-4 (p.102);**

2115 **Quintic (\* = 5) cannot be solved: Why?**

2116 **Fundamental Thm of Algebra (d'Alembert, ≈1760)**

2117

2118 **Add intro & merge convolution discussions.**

### 2119 Convolution

2120 As we discussed in Chapter 1, given the roots, the construction of higher degree polynomials, is greatly  
 2121 assisted by the convolution method. This has physical meaning, and gives insight into the problem of  
 2122 factoring higher order polynomials. By this method we can obtain explicit relations for the coefficients  
 2123 of any polynomial in terms of its roots.

Extending the example of Section 1.3.3, let's find the relations for the cubic. For simplicity, assume that the polynomial has been normalized so that the lead  $x^3$  term has coefficient 1. Then the cubic in terms of its roots  $[a, b, c]$  is a convolution of three terms

$$[1, a] \star [1, b] \star [1, c] = [1, a + b, ab] \star [1, c] = [1, a + b + c, ab + c(a + b), abc].$$

Working out the coefficients for a *quartic* gives

$$[1, a+b+c, ab+c(a+b), abc] \star [1, d] = [1, a+b+c+d, d(a+b+c)+c(a+b)+ab, d(ab+ac+bc)+abc, abcd].$$

2124 It is clear what is going on here. The coefficient on  $x^4$  is 1 (by construction). The coefficient for  $x^3$  is  
 2125 the sum over the roots. The  $x^2$  term is the sum over all possible products of pairs of roots, The linear  
 2126 term  $x$  is the sum over all triple products of the four roots, and finally the last term (a constant) is the  
 2127 product of the four roots.

2128 In fact this is a well known, a frequently quoted result from the mathematical literature, and trivial  
 2129 to show given an understand of convolution. If one wants the coefficients for the quintic, it is not even  
 2130 necessary to use convolution, as the pattern (rule) for all the coefficients is now clear.

2131 You can experiment with this numerically using Matlab's convolution routine `conv(a,b)`. Once  
 2132 we start studying Laplace and Fourier transforms, convolution becomes critically important because  
 2133 multiplying an input signal in the frequency domain by a transfer function, also a function of frequency,  
 2134 is the same a convolution of the time domain signal with the inverse Laplace transform of the transfer  
 2135 function. So you didn't need to learn how to take a Laplace transform, and then learn convolution.  
 2136 We have learned convolution first independent of the Fourier and Laplace transforms.

2137 When the coefficients are real, the roots must appear as conjugate pairs. This is an important  
 2138 symmetry.

For the case of the quadratic we have the relations between the coefficients and the roots, found by  
 completing the square. This required isolating  $x$  to a single term, and solving for it. We then proceeded  
 to find the coefficients for the cubic and quartic case, after a few lines of calculation. For the quartic

$$\begin{aligned} a_4 &= 1 \\ a_3 &= a + b + c + d \\ a_2 &= d(a + b + c) + c(a + b) + ba \\ a_1 &= d(ab + ac + bd) + abc \\ a_0 &= abcd \end{aligned}$$

2139 These relationships are algebraically nonlinear in the roots. From the work of Galois, for  $N \geq 5$ , this  
 2140 system of equations is impossible to invert. Namely, given  $a_k$ , one may not determine the four roots  
 2141  $[a, b, c, d]$  analytically. One must use numeric methods.

To gain some insight, let us look at the problem for  $N = 2$ , which has a closed form solution:

$$\begin{aligned} a_2 &= 1 \\ a_1 &= a + b \\ a_0 &= ab \end{aligned}$$

2142 We must solve for  $[a, b]$  given twice the mean,  $2(a+b)/2$ , and the square of the geometric mean  $(\sqrt{ab})^2$ .  
 2143 Since we already know the answer (i.e, the quadratic formula). The solution was first worked out by the  
 2144 Babylonians (2000 BCE) Stillwell (2010, p. 92). It is important to recognize that for physical systems,  
 2145 the coefficients  $a_k$  are real. This requires that the roots come in conjugate pairs ( $b = a^*$ ), thus  $ab = |a|^2$   
 2146 and  $a + b = 2\Re a$ , which makes the problem somewhat more difficult, due to the greater symmetry.

2147 Once you have solved this problem, feel free to attempt the cubic case. Again, the answer is known,  
 2148 after thousands of years of searching. The solution to the cubic is given in (Stillwell, 2010, pp. 97-9),  
 2149 as discovered by Cardano in 1545. According to Stillwell "The solution of the cubic was the first  
 2150 clear advance in mathematics since the time of the Greeks." The ability to solve this problem required  
 2151 algebra, and the solutions were complex numbers. The denial of complex numbers was, in my view, the  
 2152 main stumbling block in the progress of these solutions. For example, how can two parallel lines have  
 2153 a solution? Equally mystifying, how can a circle and a line, that do not intersect, have intersections?  
 2154 From the algebra we know that they do. This was a basic problem that needed to be overcome. This

2155 story is still alive,<sup>5</sup> because the cubic solution is so difficult.<sup>6</sup> One can only begin to imagine how much  
 2156 more difficult the quartic is, solved by Cardano's student Ferrair, and published by Cardano in 1545.  
 2157 The impossibility of the quintic was finally resolved in 1826 by Able (Stillwell, 2010, p. 102).

2158 Finally with these challenges behind them, Analytic Geometry, relating of algebra and geometry,  
 2159 via coordinate systems, was born.

### 2160 3.2.3 Lec 14: Analytic Geometry

2161 Lec 14: Early Analytic Geom (Merging Euclid and Descartes): Composition of degrees  $n, m$  gives  
 2162 degree  $m \cdot n$

2163 Composition, Intersection and Gaussian elimination

2164 The first "algebra" (al-jabr) is credited to al-Khwarizmi (830 CE). Its invention advanced the theory  
 2165 of polynomial equations in one variable, Taylor series, and composition versus intersections of curves.  
 2166 The solution of the quadratic equation had been worked out thousands of year earlier, but with algebra  
 2167 a general solution could be defined. The Chinese had found the way to solve several equations in  
 2168 several unknowns, for example, finding the values of the intersection of two circles. With the invention  
 2169 of algebra by al-Khwarizmi, a powerful tool became available to solve the difficult problems.

2170 **Composition, Elimination and Intersection** In algebra there are two contrasting operations on  
 2171 functions: *composition* and *Elimination*.

2172 **Composition:** Composition is the merging of functions, by feeding one into the other. If the two  
 2173 functions are  $f, g$  then their composition is indicated by  $f \circ g$ , meaning the function  $y = f(x)$  is  
 2174 substituted into the function  $z = g(y)$ , giving  $z = g(f(x))$ .

2175 Composition is not limited to linear equations, even though that is where it is most frequently  
 2176 applied. To compose two functions, one must substitute one equation into the other. That requires  
 2177 solving for that substitution variable, which is not always possible in the case of nonlinear equations.  
 2178 However many tricks are available that may work around this restrictions. For example if one equation  
 2179 is in  $x^2$  and the other in  $x^3$  or  $\sqrt{x}$ , it may be possible to multiply the first by  $x$  or square the second.  
 2180 The point is that one of the variables must be isolated so that when it is substituted into the other  
 2181 equations, the variable is removed from the mix.

**Examples:** Let  $y = f(x) =: x^2 - 2$  and  $z = g(y) =: y + 1$ . Then

$$g \circ f = g(f(x)) = (x^2 - 2) + 1 = x^2 - 1. \quad (3.5)$$

In general composition does not commute (i.e.,  $f \circ g \neq g \circ f$ ), as is easily demonstrated. Swapping the order of composition for our example gives

$$f \circ g = f(g(y)) = z^2 - 2 = (y + 1)^2 - 2 = y^2 + 2y - 1. \quad (3.6)$$

2182 **Intersection:** Complimentary to composition is *intersection* (i.e., decomposition) (Stillwell, 2010,  
 2183 pp. 119,149). For example, the intersection of two lines is defined as the point where they meet. This  
 2184 is not to be confused with finding roots. A polynomial of degree  $N$  has  $N$  roots, but the points where  
 2185 two polynomials intersect has nothing to do with the roots of the polynomials. The intersection is a  
 2186 function (equation) of lower degree, implemented with Gaussian elimination.

<sup>5</sup>M. Kac, *How I became a mathematician.* American Scientist (72), 498–499.

<sup>6</sup><https://www.google.com/search?client=ubuntu&channel=fs&q=Kac+%22how+I+became+a%22+1984+pdf&ie=utf-8&oe=utf-8>



**Intersection of two lines** Unless they are parallel, two lines meet at a point. In terms of linear algebra this may be written as 2 linear equations (left) along with the intersection point  $[x_1, x_2]^T$ , given by the inverse of the 2x2 set of equations (right).<sup>7</sup>

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad (3.7)$$

2187 where  $\Delta = ab - cd$  is called the *determinant*. By substituting the right expression into the left, and  
 2188 taking the inverse we obtain the intersection point. If  $\Delta = 0$  there can be no solution, in which case  
 2189 the two lines are parallel (they meet at infinity.)

2190 Algebra will give the solution when geometry cannot. When the two curves fail to intersect on the  
 2191 real plane, the solution still exists, but is complex valued. In such cases, geometry, which only considers  
 2192 the real solutions, fails. For example, when the coefficients  $[a, b, c, d]$  are complex, the solution exists,  
 2193 but the determinant can be complex. Thus algebra is much more general than geometric. Geometry  
 2194 fails when the solution has a complex intersection.

## 2195 3.3 Week 6

### 2196 3.3.1 Lec 15 Gaussian Elimination

2197 **Example problems using Gaussian Elimination:** Gaussian Elimination is valid for nonlinear  
 2198 systems of equations. Till now we have emphasized the reduction of linear systems of equations.

**Problem 1:** Two lines *in a plane* either intersect or are parallel, in which case they are said to meet at  $\infty$ . Does this make sense? The two equations that describe this may be written in matrix form as  $Ax = b$ , which written out as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad (3.8)$$

The intersection point  $x_0, y_0$  is given by the solution two these two equations

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad (3.9)$$

2199 where  $\Delta = a_{11}a_{22} - a_{12}a_{21}$  is the determinant of matrix  $A$  (Matlab's `det(A)` function).

It is useful to give an interpretation of these two equations. Each row of the 2x2 matrix defines a line in the  $(x, y)$  plane. The top row is

$$a_{11}x + a_{12}y = b_1.$$

Normally we would write this equation as  $y(x) = \alpha x + \beta$ , where  $\alpha$  is the slope and  $\beta$  is the *intercept* (i.e.,  $y(0) = \beta$ ). In terms of the elements of matrix  $A$ , the slope of the first equation is  $\alpha = -a_{11}/a_{12}$  while the slope of the second is  $\alpha = -a_{21}/a_{22}$ . The two slopes are equal (the lines are parallel) when  $-a_{11}/a_{12} = -a_{21}/a_{22}$ , or written out

$$\Delta = a_{11}a_{22} - a_{12}a_{21} = 0.$$

2200 Thus when the determinate is zero, the two lines are parallel and there is no solution to the equations.

<sup>7</sup>It is very important to note when writing the equations in matrix format, the unknowns are  $x_1, x_2$  whereas in the original equations they were  $y, x$ . The starting equations are  $ay_1 + bx_1 = c$  and  $cy_1 + dx_2 = d$  but in matrix format the names are changed. The first equation is  $ax_1 + bx_2 = y_1$  and  $cx_1 + dx_2 = y_2$ . The first time you meet this scrambling of terminology it can be very confusing. In matrix equations the coordinates of the graph are  $(x_1, x_2)$  rather than the usual  $x, y$ .

This 2x2 matrix equation is equivalent to a  $2^d$  degree polynomial. If we seek an eigenvector solution  $[e_1, e_2]^T$  such that

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \lambda \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad (3.10)$$

the 2x2 equation becomes singular, and  $\lambda$  is one of the roots of the polynomial. One may proceed by merging the two terms to give

$$\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (3.11)$$

Clearly this new matrix has no solution, since if it did,  $[e_1, e_2]^T$  would be zero, which is nonsense. If it has no solution, then the determinant of the matrix must be zero. Forming this determinate gives

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$

thus we obtain the following quadratic equation for the roots  $\lambda_{\pm}$  (eigenvalues)

$$\lambda_{\pm}^2 - (a_{11} + a_{22})\lambda_{\pm} + \Delta = 0.$$

2201 When  $\Delta = 0$ , one eigenvalue is zero while the other is  $a_{11} + a_{22}$ , which is known as the *trace* of the  
2202 matrix.

2203 **In summary:** Given a “linear” equation for the point of intersection of two lines, we see that there  
2204 must be two points of intersection, as there are always two roots of the quadratic *characteristic poly-*  
2205 *nomial*. However the two lines only intersect at one point. Whats going on? What is the meaning of  
2206 *this second root?*

Needs work. 2207 Some simple examples will help. The eigenvalues depend on the relative slopes of the lines, which  
2208 in general can become complex. The intercepts are dependent on  $\mathbf{b}$ . Thus when the RHS is zero, the  
2209 eigenvalues are irrelevant. This covers the very simple examples. When one eigenvalue is real and the  
2210 other is imaginary, more interesting things are happening since the slope of one line is real and the  
2211 slope of the other is pure imaginary. The lines can intersect in the real plane, and again in the complex  
2212 plane.

Lets try an example of two lines, slopes of 1 and 2:  $y_1 = x + a$  and  $y_2 = 2x + b$ . In matrix form Let

$$\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad (3.12)$$

The determinate is  $\Delta = -1$ , thus the solution is

$$\begin{bmatrix} y_0 \\ x_0 \end{bmatrix} = -1 \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a - b \\ a - b \end{bmatrix}. \quad (3.13)$$

2213 Thus the two real lines having slopes of 1 and 2 having intercepts of  $a$  and  $b$ , meet  $(x_0, y_0) = (2a - b, a - b)$ .  
2214 We may verify by substituting  $x = a - b$  into the starting equations  $y_1 = (a - b) + a = 2a - b$  and  
2215  $y_2 = 2(a - b) + b = 2a - b$ , which each  $2a - b$ .

While there is a unique solution, there are two eigenvalues, given by the roots of

$$(1 - \lambda_{\pm})(-2 - \lambda_{\pm}) + 1 = 0.$$

If we transfer the sign from one monomial to the other

$$(-1 + \lambda_{\pm})(2 + \lambda_{\pm}) + 1 = 0$$

and reorder for simplicity

$$(\lambda_{\pm} - 1)(\lambda_{\pm} + 2) + 1 = 0$$

we obtain the quadratic for the roots

$$\lambda_{\pm}^2 + \lambda_{\pm} - 1 = 0.$$

Completing the square gives

$$(\lambda_{\pm} + 1/2)^2 = 3/4.$$

or

$$\lambda_{\pm} = -1/2 \pm \sqrt{3}/2.$$

2216 The question is, what is the relationship between the eigenvalues and the final solution, if any? Maybe  
 2217 none. The solution  $(x_0, y_0)$  is reasonable, and its not clear that the eigenvalues play any useful role  
 2218 here, other than to predict there is a second solution. I'm confused.

**Two lines in 3-space:** In three dimensions

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (3.14)$$

2219 Each row of the matrix describes a plane, which is said to be linear in the unknowns  $(x, y, z)$ . Thus the  
 2220 system of linear equations represents three planes, which must intersect at one point. If two planes are  
 2221 parallel, there is no real solution. In this case the intersection by the third plane generates two parallel  
 2222 lines.

2223 As in the 2x2 case, one may convert this linear equation into a cubic polynomial by setting the  
 2224 determinant of the matrix, with  $-\lambda$  subtracted from the diagonal, equal to zero. That is,  $\det(A - \lambda I) =$   
 2225 0. Here  $I$  is the matrix with 1 on the diagonal and zero off the diagonal.

**Simple example:** As a simple example, let the first plane be  $z = 0$  (independent of  $x, y$ ), the second  
 parallel plane be  $z = 1$  (independent of  $(x, y)$ ) and the third plane be  $x = 0$  (independent of  $y, z$ ). This  
 results in the system of equations

$$\begin{bmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad (3.15)$$

2226 Writing out the three equations we find  $a_{13}z = 0$ ,  $a_{23}z = 1$ , and  $a_{31}x = 0$ . Note that  $\det(A) = 0$  (we  
 2227 need to learn how to compute the 3x3 determinant). This means the three planes never intersect at  
 2228 one point. Use Matlab to find the eigenvalues.

### 2229 3.3.2 Lec 16 Matrix composition: Bilinear and ABCD transformations

#### 2230 The Transmission matrix

A transmission matrix is a 2x2 matrix that characterizes a 2-port circuit, one having an input and  
 output voltage and current, as shown in Fig. 1.11. The input is the voltage and current  $V_1, I_1$  and the  
 output is the voltage and current  $V_2, -I_2$ , with the current always defined to flow into the port. For  
 any such a linear network, the input-output relations may be written in a totally general way as

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}.$$

2231 In Section 1.3.6 we showed that a cascade of such matrices is composition. We shall show below that  
 2232 the justification of this relationship is based on the composition of bilinear transformations.

Expanding Eq. 1.27 into its individual equations demonstrates the linear form of the relations

$$V_1 = A(s)V_2 - B(s)I_2 \quad I_1 = C(s)V_2 - D(s)I_2,$$

2233 quantifying the relationship between the input voltage and current to its output voltage and current.

2234 Define  $H(s) = V_2/V_1$  as the *transfer function*, as the ratio of the output voltage  $V_2$  over the input  
2235 voltage  $V_1$ , under the constraint that the output current  $I_2 = 0$ . From this definition  $H(s) = 1/A(s)$ .

In a similar fashion we may define the meaning of all four functions as

$$A(s) \equiv \left. \frac{V_1}{V_2} \right|_{I_2=0} \qquad B(s) \equiv -\left. \frac{V_1}{I_2} \right|_{V_2=0} \qquad (3.16)$$

$$C(s) \equiv \left. \frac{I_1}{V_2} \right|_{I_2=0} \qquad D(s) \equiv -\left. \frac{I_1}{I_2} \right|_{V_2=0} \qquad (3.17)$$

2236 From Eq. 1.27 one may compute any desired quantity, specifically those quantities defined in  
2237 Eq. 3.17, the open circuit voltage transfer function ( $1/A(s)$ ), the short-circuit transfer current ( $1/D(s)$ )  
2238 and the two transfer impedances  $B(s)$  and  $1/C(s)$ .

2239 In the engineering fields this matrix composition is called the *Transmission matrix*, also known as  
2240 the ABCD method. It is a powerful method that is easy to learn and use, that gives important insights  
2241 into transmission lines, and thus even the 1 dimensional wave equation.

#### 2242 Derivation of ABCD matrix for example of Fig. 1.11 (p. 52).

2243 The derivation is straight forward by the application of Ohm's Law, as shown in Section 1.3.6.

The convenience of the ABCD matrix method is that the output of one is identically the input of  
the next. Cascading (composing) the results for the series inductor with the shunt compliance leads to  
the 2x2 matrix form that precisely corresponds to the transmission line CFA shown in Fig. 2.2,

$$\begin{bmatrix} V_n(s) \\ I_n(s) \end{bmatrix} = \begin{bmatrix} 1 & sL_n \\ 1 & 0 \end{bmatrix} \begin{bmatrix} V_{n+1}(s) \\ -I_{n+1}(s) \end{bmatrix}. \qquad (3.18)$$

This matrix relation characterizes the series mass term  $sL_n$ . A second equation maybe be used for the  
shunt capacitance term  $sY_n(s)$

$$\begin{bmatrix} V_n(s) \\ I_n(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ sC_n & 0 \end{bmatrix} \begin{bmatrix} V_{n+1}(s) \\ -I_{n+1}(s) \end{bmatrix}. \qquad (3.19)$$

2244 The positive constants  $L_n, C_n \in \mathbb{R}$  represent the series mass (inductance) and the shunt compliance  
2245 (capacitance) of the mechanical (electrical) network. The integer  $n$  indexes the series and shunt sections,  
2246 that are composed one following the next.

2247 **Matrix composition and the bilinear transform:** Now that we have defined the composition of  
2248 two functions, we will use it to define the *Möbius* or *bilinear* transformation. Once you understand how  
2249 this works, hopefully you will understand why it is the unifying element in many important engineering  
2250 problems.

The bilinear transformation is given by

$$w = \frac{a + bz}{c + dz}$$

This takes one complex number  $z = x + iy$  and transforms it into another complex number  $w = u + iv$ .  
This transformation is *bilinear* in the sense that its linear in both the input and output side of the  
equation. This may be seen when written as

$$(c + dz)w = a + bz,$$

since this relation is linear in the coefficients  $[a, b, c, d]$ . An important example is the transformation  
between impedance  $Z(s)$  and reflectance  $\Gamma(s)$ ,

$$\Gamma(s) = \frac{Z(s) - r_0}{Z(s) + r_0},$$

2251 which is widely used in transmission line problems. In this example  $w = \Gamma, z = Z(s), a = -r_0, b =$   
 2252  $1, c = r_0, d = 1.$

If we define a second bilinear transformation (this could be the transformation from reflectance back to impedance)

$$r = \frac{\alpha + \beta w}{\gamma + \delta w},$$

and then compose the two **something astray wrt arguments**

$$w \circ r = \frac{a + b r}{c + d r} = \frac{a(\gamma + \delta w) + b(\alpha + \beta w)}{c(\gamma + \delta w) + d(\alpha + \beta w)} = \frac{a\gamma + b\alpha + (a\delta + b\beta)w}{c\gamma + d\alpha + (c\delta + d\beta)w},$$

something surprising happens. The composition  $w \circ r$  may be written in matrix form, as the product of two matrices that represents each bilinear transform. This may be seen as true by inspecting the coefficients of the composition  $w \circ r$  (shown above) and the product of the two matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} (a\gamma + b\alpha) & (a\delta + b\beta) \\ (c\gamma + d\alpha) & (c\delta + d\beta) \end{bmatrix}.$$

2253 The the power of this composition property of the bilinear transform may be put to work solving  
 2254 important engineering problems, using *transmission matrices*.

2255 **3.3.3 Lec 17 Introduction to the Branch cut and Riemann sheets**

2256 Branch cuts are required to preserve the single-valued nature of complex analytic functions. When an  
 2257 analytic function is multi-valued, some method needs to be devised to allow the multi-valued complex  
 2258 analytic function to be expanded as a Taylor series, which is necessarily single-valued. It follows that  
 2259 each single-valued sheet must have a different expansion, valid out to the nearest pole (or singularity).  
 2260 We shall explain these ideas with the simplest case, the double-valued square root function  $w(z) = \pm\sqrt{z}$ ,  
 2261 as shown in Fig. 1.15 (p. 65).

2262 **3.4 Week 7**

2263 **3.4.1 Lec 18 Complex analytic mappings (domain coloring)**

2264 When one uses complex analytic functions it is helpful to understand their properties in the complex  
 2265 plane. In this sections we explore several well-known functions using domain coloring, discussed in  
 2266 some detail in Section 1.3.8, p. 55. For the following figures the coordinate systems are defined by  
 2267  $s = \sigma + \omega j = \sqrt{x^2 + y^2}e^{j\theta}$  and  $w = u + vj = \sqrt{u^2 + v^2}e^{j\psi}$ .

2268 For the first example (Fig. 3.2)  
 2269  $w(s) = s^2$  and its inverse  $s(w) = \sqrt{w}$   
 2270 are shown. On the left the red region,  
 2271 corresponding to  $0^\circ$  [degrees], appears  
 2272 at both 0 and 180 ( $u = \pm 1$ ) in the  $w$   
 2273 plane. This is because in polar coordinates  $s^2 = |s|^2 e^{2j\theta}$  where  $\theta$  is the angle  
 2274 of  $s = |s|e^{j\theta}$ . The square causes  
 2275 the phase to rotate twice around for  
 2276 once around the  $s$  plane. Namely the  
 2277 angle is doubled, and the magnitude  
 2278 squared. Due to the faster changing  
 2279 phase in  $w$ , there are two red regions,  
 2280 one when  $\theta = 0$  and the second at  
 2281

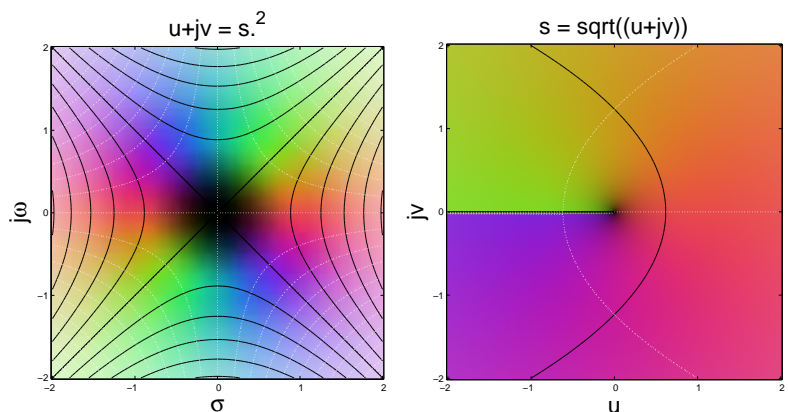


Figure 3.2: Here the Cartesian coordinate map between  $s = \sigma + \omega j$  and  $w = u + vj$ . The left shows the mapping  $w(s) = s^2$ . The right shows the lower branch of the inverse  $s(w) = \sqrt{w}$ , as shown in Fig. 1.15.

2282  $\theta = \pi$  ( $\angle w(s) = 2\theta$ ). The black spot  
 2283 is dilated due to the squaring of the radius (expanding it).

2284 On the right the  $\sqrt{w} = \sqrt{|w|}e^{\phi/2}$ . Because the angle of  $w$  is divided by two, it takes twice as much  
 2285 phase (in  $w$ ) to cover the same angle. Thus the red region ( $0^\circ$ ) is expanded. We barely see the violet  
 2286  $90^\circ$  and yellow  $-90^\circ$  angles. There is a *branch cut* running from  $w = 0$  to  $w = \infty$ . As the branch cut  
 2287 is crossed, the function switches *Riemann sheets*, going from the top sheet (shown here) to the bottom  
 2288 sheet (not shown). Figure 1.15 in Section 3.3.3 depicts what is going on with these two sheets, and  
 2289 show the branch cut from the origin (point O) to  $\infty$ . In this depiction the first sheet ( $+\sqrt{z}$ ) is on the  
 2290 bottom, while the second sheet ( $\sqrt{z}$ ) is on top. For every value of  $z$  there are two possible outcomes,  
 2291  $\pm\sqrt{z}$ , represented by the two sheets.

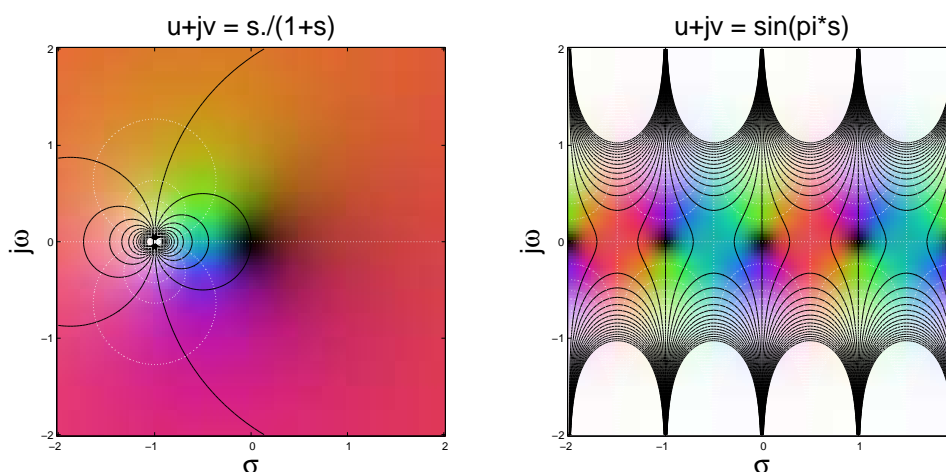


Figure 3.3: On the left is the function  $w(s) = s/(1+s)$ , having a zero at  $s = 0$  and a pole at  $s = -1$ . On the right is  $w(s) = \sin(\pi s)$ . See the discussion in the text for an interpretation of these complex maps.

2292 Two more examples are given in Fig. 3.3 to interpret the two complex mappings  $w = s/(1+s)$   
 2293 (left) and  $w = \sin(\pi s)$  (right). On the left is  $s/(1+s)$  which is a zero at  $s = 0$ , resulting in a dark  
 2294 spot, and a pole at  $s = -1$  resulting in a white spot. Note how the white and black contour lines are  
 2295 always perpendicular where they cross, just as in the calibration plots for the x and y axes, shown in  
 2296 Fig. 1.13 in Section 1.3.8 (p. 55).

2297 The right-hand plot of  $w(s) = \sin(\pi s)$  is equally interesting. Along the  $\sigma$  axis (real part of  $s$ ) the  
 2298 function is the periodic  $\sin(\sigma)$  function. The dark spots are at  $\sigma = k\pi$ , with  $k \in \mathbb{Z}$ . This is the normal  
 2299  $\sin(\pi\sigma)$  function with zeros at  $0, \pm\pi, 2 \pm \pi, \dots$ . When we stray off the  $\omega_j = 0$  axis, the function either  
 2300 goes to zero (black) or  $\infty$  (white). This behavior carries the same  $2\pi$  periodicity as it has along the  
 2301  $\omega = 0$  line. These figure are worthy of careful study to develop an intuition for complex functions of  
 2302 complex variables. In Section 1.3.8 we shall explore more complex mappings, and in greater detail.

2303 In the third example (Fig. 3.4) we show  $w = \tan(z)$  and its inverse  $z = \tan^{-1}(w)$ . The tangent  
 2304 function has zeros where  $\sin(z)$  has zeros (e.g., at  $z = 0$ ) and poles where  $\cos(z)$  is zero (e.g., at  $\pm\pi/2$ ).  
 2305 The inverse function  $s = \text{atan}(w)$  has a zero at  $w = 0$  and branch cuts eliminating from  $z = \pm\pi$ .

2306 It becomes most interesting to study polynomials of degree 5 and 4, with one zero removed, to  
 2307 demonstrate the Fundamental Theorem of Algebra. Recall that degree 5 is not analytically tractable,  
 2308 and must be investigated numerically.

### 2309 3.4.2 Lec 19 Signals and Systems: Fourier vs. Laplace Transforms

2310 Signals and Systems: Fourier vs. Laplace Transforms AE-3

### 2311 3.4.3 Lec 20 Role of Causality and the Laplace Transform

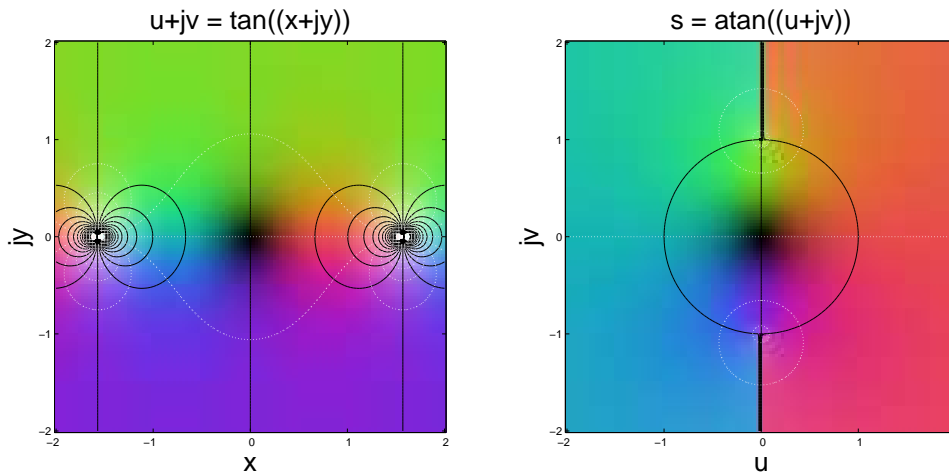


Figure 3.4: On the left is the function  $w(z) = \tan(z)$ . On the right is the inverse  $s = \tan^{-1}(w)$ . Of special interest is  $z \mapsto \text{atan}(i \cdot z) = (i/2) \cdot \log((1+z)/(1-z))$ .

2312 Role of Causality and the Laplace Transform:

2313  $u(t) \leftrightarrow 1/s$  (LT)

2314  $2\tilde{u}(t) \equiv 1 + \text{sgn}(t) \leftrightarrow 2\pi\delta(\omega) + 2/j\omega$  (FT)

### 2315 3.5 Week 8

#### 2316 3.5.1 Lec 21 The 9 postulates of System of algebraic Networks

2317 Physical system obey very important rules, that follow from the physics. It is helpful to summarize  
 2318 these physical restrictions in terms of postulates, presented in terms of a taxonomy, or categorization  
 2319 method, of the fundamental properties of physical systems. Nine of these are listed below. These nine  
 2320 come from a recently published paper (Kim and Allen, 2013). It is possible that given time, others  
 2321 could be added.

2322 A taxonomy of physical systems comes from a systematic summary of the laws of physics, which  
 2323 includes at least the nine basic network postulates, described in Section 1.3.11. To describe each of  
 2324 the network postulates it is helpful to begin with the 2-port transmission (aka ABCD, chain) matrix  
 2325 representation, discussed in Section 3.3.2 (p. 99).

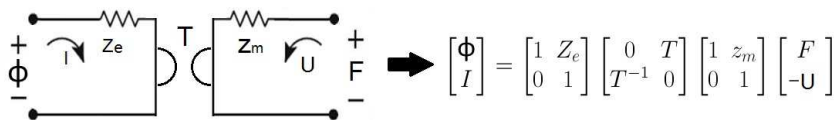


Figure 3.5: A schematic representation of a 2-port ABCD electro-mechanic system using Hunt parameters  $Z_e(s)$ ,  $z_m(s)$ , and  $T(s)$ : electrical impedance, mechanical impedances, and transduction coefficient (Hunt, 1952; Kim and Allen, 2013). Also  $V(f)$ ,  $I(f)$ ,  $F(f)$ , and  $U(f)$  are the frequency domain voltage, current, force, and velocity respectively. Notice how the matrix method ‘factors’ the 2-port model into three  $2 \times 2$  matrices. This allows one to separate the physical modeling from the algebra. It is a standard impedance convention that the flows  $I(f)$ ,  $U(f)$  are always defined into the port. Thus it is necessary to apply a negative sign on the velocity  $-U(f)$  so that it has an outward flow, to feed the next cell with an inward flow. **Replace  $\Phi$  with  $V$ .**

As a specific example we show the 2-port *transmission matrix* for an acoustic transducer (loud-speaker), shown in Fig. 3.5, characterized by the equation

$$\begin{bmatrix} \Phi_i \\ I_i \end{bmatrix} = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} \begin{bmatrix} F_l \\ -U_l \end{bmatrix} = \frac{1}{T} \begin{bmatrix} z_m(s) & z_e(s)z_m(s) + T^2 \\ 1 & z_e(s) \end{bmatrix} \begin{bmatrix} F_l \\ -U_l \end{bmatrix}. \quad (3.20)$$

2326 This equation comes from the product of the three  $2 \times 2$  matrices representing each of the three elements  
 2327 of the figure.

Add figures as s notebook (p. 35), these postulates.

2328 This figure represents the motor of the loudspeaker (not the box that it comes in). The system con-  
 2329 sists of three elements, the electrical input impedance  $Z_e(s)$ , a gyrator, which is similar to a transformer,  
 2330 but relates current to force, and an output mechanical impedance  $Z_m(s)$ . This circuit describes what  
 2331 is needed to fully characterize its operation, from electrical input to mechanical (acoustical) output.

2332 The input is electrical (voltage and current)  $[\Phi_i, I_i]$  and the output (load) are the mechanical (force  
 2333 and velocity)  $[F_l, U_l]$ . The first matrix is the general case, expressed in terms of four unspecified  
 2334 functions  $A(s), B(s), C(s), D(s)$ , while the second matrix is for the specific example of Fig. 3.5. The  
 2335 four entries are the electrical driving point impedance  $Z_e(s)$ , the mechanical impedance  $z_m(s)$  and the  
 2336 transduction  $T = B_0 l$  where  $B_0$  is the magnetic flux strength and  $l$  is the length of the wire crossing the  
 2337 flux. Since the transmission matrix is anti-reciprocal, its determinate  $\Delta_T = -1$ , as is easily verified.

2338 Other common transduction examples of cross-modality transduction include current–thermal (ther-  
 2339 moelectric effect) and force–voltage (piezoelectric effect). These systems are all reciprocal, thus the  
 2340 transduction has the same sign.

### 2341 Impedance matrix

These nine postulates describe the properties of a system having an input and an output. For the  
 case of an electromagnetic transducer (Loudspeaker) the system is described by the 2-port, as shown in  
 Fig. 3.5. The electrical input impedance of a loudspeaker is  $Z_e(s)$ , defined by

$$Z_e(s) = \left. \frac{V(\omega)}{I(\omega)} \right|_{U=0}.$$

2342 Note that this *driving-point impedance* must be causal, thus it has a Laplace transform and therefore is  
 2343 a function of the complex frequency  $s = \sigma + j\omega$ , whereas the Fourier transforms of the voltage  $V(\omega)$  and  
 2344 current  $I(\omega)$  are functions of the real radian frequency  $\omega$ , since the time-domain voltage  $v(t) \leftrightarrow V(\omega)$   
 2345 and the current  $i(t) \leftrightarrow I(\omega)$  are signals that may start and stop at any time (they are not typically  
 2346 causal).

The corresponding 2-port *impedance matrix* for Fig. 3.5 is

$$\begin{bmatrix} \Phi_i \\ F_l \end{bmatrix} = \begin{bmatrix} z_{11}(s) & z_{12}(s) \\ z_{21}(s) & z_{22}(s) \end{bmatrix} \begin{bmatrix} I_i \\ U_l \end{bmatrix} = \begin{bmatrix} Z_e(s) & -T(s) \\ T(s) & z_m(s) \end{bmatrix} \begin{bmatrix} I_i \\ U_l \end{bmatrix}. \quad (3.21)$$

2347 Such a description allows one to define *Thèvenin parameters*, a very useful concept used widely in  
 2348 circuit analysis and other network models from other modalities.

2349 The impedance matrix is an alternative description of the system, but with generalized forces  $[\Phi_i, F_l]$   
 2350 on the left and generalized flows  $[I_i, U_l]$  on the right. A rearrangement of the equations allows one to  
 2351 go from the ABCD to impedance set of parameters (Van Valkenburg, 1964b). The electromagnetic  
 2352 transducer is anti-reciprocal (P6),  $z_{12} = -z_{21} = T = B_0 l$ .

### 2353 Additional or modified postulates

2354 The postulates must go beyond postulates P1-P6 defined by Carlin and Giordano (Section 1.3.11,  
 2355 p. 61), when there are interaction of waves and a structured medium, along with other properties not  
 2356 covered by classic network theory. Assuming QS, the wavelength must be large relative to the medium's  
 2357 lattice constants. Thus the QS property must be extended to three dimensions, and possibly to the  
 2358 cases of an-isotropic and random media.

2359 **Causality: P1** As stated above, due to causality the negative properties (e.g., negative refractive  
 2360 index) must be limited in bandwidth, as a result of the Cauchy-Riemann conditions. However even  
 2361 causality needs to be extended to include the delay, as quantified by the d'Alembert solution to the  
 2362 wave equation, which means that the delay is proportional to the distance. Thus we generalize P1  
 2363 to include the space dependent delay. When we wish to discuss this property we denote it *Einstein*  
 2364 *causality*, which says that the delay must be proportional to the distance  $x$ , with impulse response  
 2365  $\delta(t - x/c)$ .



2366 **Linearity: P2** The wave properties of may be non-linear (P2). This is not restrictive as most  
 2367 physical systems are naturally nonlinear. For example, a capacitor is inherently nonlinear: as the  
 2368 charge builds up on the plates of the capacitor, a stress is applied to the intermediate dielectric due to  
 2369 the electrostatic force  $F = qE$ . In a similar manner, an inductor is nonlinear. Two wires carrying a  
 2370 current are attracted or repelled, due to the force created by the flux. The net force is the product of  
 2371 the two fluxes due to each current.

2372 In summary, most physical systems are naturally nonlinear, it's simply a matter of degree. An  
 2373 important counter example is a amplifier with negative feedback, with very large open-loop gain.  
 2374 There are, therefore, many types of non-linear, instantaneous and those with memory (e.g., hysteresis).  
 2375 Given the nature of P1, even an instantaneous non-linearity may be ruled out. The linear model is so  
 2376 critical for our analysis, providing fundamental understanding that we frequently take this postulate  
 2377 for granted.

2378 **Real time response: P3** The impulse response of every physical system is real, vs. complex. This  
 2379 requires that the Laplace Transform have conjugate-symmetric symmetry  $H(s) = H^*(s^*)$ , where the \*  
 2380 indicates conjugation (e.g.,  $R(\sigma, \omega) + X(\sigma, \omega) = R(\sigma, \omega) - X(\sigma, -\omega)$ ).

2381 **Passive/Active: P4** We greatly extend P4 by building in the physics behind conservation of energy:  
 2382 Otto Brune's *positive Real* (PR aka physically realizable) condition. Following up on the earlier work  
 2383 of his primary PhD thesis advisor Wilhelm Cauer (1900-1945), and working with Norbert Weiner and  
 2384 Vannevar Bush at MIT, Otto Brune mathematically characterized the properties of every PR 1-port  
 2385 driving point impedance.

When the input resistance of the impedance is real, the system is said to be *passive*, which means the system obeys conservation of energy. The real part of  $Z(s)$  is positive if and only if the corresponding reflectance is less than 1 in magnitude. The definition of the reflectance of  $Z(s)$  is defined as a bilinear transformation of the *impedance*, normalized by its *surge resistance*  $r_0$  (Campbell, 1903)

$$\Gamma(s) = \frac{Z(s) - r_0}{Z(s) + r_0}.$$

The surge resistance is defined in terms of the inverse Laplace transform of  $Z(s) \leftrightarrow z(t)$ , which must have the form

$$z(t) = r_0\delta(t) + \zeta(t),$$

2386 where  $\zeta(t) = 0$  for  $t < 0$ . It naturally follows that  $\gamma(t) \leftrightarrow \Gamma(s)$  is zero for negative and zero time,  
 2387 namely  $\gamma(0) = 0, t \leq 0$ . at

Given any linear PR impedance  $Z(s) = R(\sigma, \omega) + jX(\sigma, \omega)$ , having real part  $R(\sigma, \omega)$  and imaginary part  $X(\sigma, \omega)$ , the impedance is defined as being PR (Brune, 1931b) if and only if

$$R(\sigma \geq 0, \omega) \geq 0. \tag{3.22}$$

2388 That is, the real part of any PR impedance is non-negative everywhere in the right half  $s$  plane ( $\sigma \geq 0$ ).  
 2389 This is a very strong condition on the complex analytic function  $Z(s)$  of a complex variable  $s$ . This  
 2390 condition is equivalent to any of the following statements: 1) There are no poles or zeros in the right  
 2391 half plane ( $Z(s)$  may have poles and zeros on the  $\sigma = 0$  axis). 2) If  $Z(s)$  is PR then its reciprocal  
 2392  $Y(s) = 1/Z(s)$  is PR (the poles and zeros swap). 3) If the impedance may be written as the ratio  
 2393 of two polynomials (a limited case, related to the quasistatics approximation, P9) having degrees  $N$   
 2394 and  $L$ , then  $|N - L| \leq 1$ . 4) The angle of the impedance  $\theta \equiv \angle Z$  lies between  $[-\pi \leq \theta \leq \pi]$ . 5)  
 2395 The impedance *and its reciprocal* are *complex analytic* in the right half plane, thus they each obey the  
 2396 Cauchy Riemann conditions there.

The PR (positive real or Physically realizable) condition assures that every impedance is *positive-definite* (PD), thus guaranteeing conservation of energy is obeyed (Schwinger and Saxon, 1968, p.17).

This means that the total energy absorbed by any PR impedance must remain positive for all time, namely

$$\mathcal{E}(t) = \int_{-\infty}^t v(t)i(t) dt = \int_{-\infty}^t i(t) \star z(t) i(t) dt > 0,$$

2397 where  $i(t)$  is *any* current,  $v(t) = z(t) \star i(t)$  is the corresponding voltage and  $z(t)$  is the real causal  
2398 impulse response of the impedance, e.g.,  $z(t) \leftrightarrow Z(s)$  are a Laplace Transform pair. In summary, if  
2399  $Z(s)$  is PR,  $\mathcal{E}(t)$  is PD.

As discussed in detail by Van Valkenburg, any rational PR impedance can be represented as a *rational polynomial fraction expansion* (residue expansion), which can be expanded into first-order poles as

$$Z(s) = K \frac{\prod_{i=1}^L (s - n_i)}{\prod_{k=1}^N (s - d_k)} = \sum_n \frac{\rho_n}{s - s_n} e^{j(\theta_n - \theta_d)}, \quad (3.23)$$

2400 where  $\rho_n$  is a complex scale factor (residue). Every pole in a PR function has only simple poles *and*  
2401 zeros, requiring that  $|L - N| \leq 1$  (Van Valkenburg, 1964b).

2402 Whereas the PD property clearly follows P3 (conservation of energy), the physics is not so clear.  
2403 Specifically what is the physical meaning of the specific constraints on  $Z(s)$ ? In many ways, the  
2404 impedance concept is highly artificial, as expressed by P1-P7.

2405 When the impedance is not rational, special care must be taken. An example of this is the semi-  
2406 inductor  $M\sqrt{s}$  and semi-capacitor  $K/\sqrt{s}$  due, for example, to the *skin effect* in EM theory and viscous  
2407 and thermal losses in acoustics, both of which are frequency dependent boundary-layer diffusion losses.  
2408 They remain positive-real but have a branch cut, thus are double valued in frequency.

2409 By building in the physics behind conservation of energy: Otto Brune's *positive-real* (PR) condition.  
2410 Following up on the earlier work of his primary PhD thesis advisor Wilhelm Cauer (1900-1945), and  
2411 working with Norbert Wiener and Vannevar Bush at MIT, Otto Brune mathematically characterized  
2412 the properties of every PR 1-port driving point impedance (Brune, 1931b).

Given any linear PR impedance  $Z(s) = R(\sigma, \omega) + jX(\sigma, \omega)$ , having real part (resistance)  $R(\sigma, \omega)$  and imaginary part (reactance)  $X(\sigma, \omega)$ , the impedance is defined as being PR (Brune, 1931a) if and only if

$$R(\sigma \geq 0, \omega) \geq 0. \quad (3.24)$$

2413 That is, the real part of any PR impedance is non-negative everywhere in the right half  $s$  plane ( $\sigma \geq 0$ ).  
2414 This is a very strong condition on the complex analytic function  $Z(s)$  of a complex variable  $s$ . This  
2415 condition is equivalent to any of the following statements: 1) There are no poles or zeros in the right  
2416 half plane ( $Z(s)$  may have poles and zeros on the  $\sigma = 0$  axis). 2) If  $Z(s)$  is PR then its reciprocal  
2417  $Y(s) = 1/Z(s)$  is PR (the poles and zeros swap). 3) If the impedance may be written as the ratio of two  
2418 polynomials (a limited case) having degrees  $N$  and  $L$ , then  $|N - L| \leq 1$ . 4) The angle of the impedance  
2419  $\theta \equiv \angle Z$  lies between  $[-\pi \leq \theta \leq \pi]$ . 5) The impedance *and its reciprocal* are *complex analytic* in the  
2420 right half plane, thus they each obey the Cauchy Riemann conditions.

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$$\mathcal{E}(t) = \int_{-\infty}^t v(t)i(t) dt = \int_{-\infty}^t i(t) \star z(t) i(t) dt > 0,$$

2421 where  $i(t)$  is *any* current,  $v(t) = z(t) \star i(t)$  is the corresponding voltage and  $z(t)$  is the real causal  
2422 impulse response of the impedance, e.g.,  $z(t) \leftrightarrow Z(s)$  are a Laplace Transform pair. In summary, if  
2423  $Z(s)$  is PR,  $\mathcal{E}(t)$  is PD.

As discussed in detail by Van Valkenburg, any rational PR impedance can be represented as a *rational polynomial fraction expansion* (residue expansion), which can be expanded into first-order poles as

$$Z(s) = K \frac{\prod_{i=1}^L (s - n_i)}{\prod_{k=1}^N (s - d_k)} = \sum_n \frac{\rho_n}{s - s_n} e^{j(\theta_n - \theta_d)}, \quad (3.25)$$

2424 where  $\rho_n$  is a complex scale factor (residue). Every pole in a PR function has only simple poles *and*  
 2425 zeros, requiring that  $|L - N| \leq 1$  (Van Valkenburg, 1964a).

2426 Whereas the PD property clearly follows P3 (conservation of energy), the physics is not so clear.  
 2427 Specifically what is the physical meaning of the specific constraints on  $Z(s)$ ? In many ways, the  
 2428 impedance concept is highly artificial, as expressed by P1-P7.

2429 When the impedance is not rational, special care must be taken. An example of this is the semi-  
 2430 inductor  $M\sqrt{s}$  and semi-capacitor  $K/\sqrt{s}$  due, for example, to the *skin effect* in EM theory and viscous  
 2431 and thermal losses in acoustics, both of which are frequency dependent boundary-layer diffusion losses.  
 2432 They remain positive-real but have a branch cut, thus are double valued in frequency.

2433 **Time invariant: P5** The meaning of *time-invariant* requires that the impulse response of a system  
 2434 does not change over time. This requires that the system coefficients of the differential equation  
 2435 describing the system are constant (independent of time).

**Rayleigh Reciprocity: P6** Reciprocity is defined in terms of the unloaded output voltage that  
 results from an input current. Specifically

$$\begin{bmatrix} z_{11}(s) & z_{12}(s) \\ z_{21}(s) & z_{22}(s) \end{bmatrix} = \frac{1}{C(s)} \begin{bmatrix} A(s) & \Delta_T \\ 1 & D(s) \end{bmatrix}, \quad (3.26)$$

where  $\Delta_T = A(s)D(s) - B(s)C(s) = \pm 1$  for the reciprocal and anti-reciprocal systems respectively.  
 This is best understood in term of Eq. 3.21. The off-diagonal coefficients  $z_{12}(s)$  and  $z_{21}(s)$  are defined  
 as

$$z_{12}(s) = \left. \frac{\Phi_i}{U_l} \right|_{I_i=0} \quad z_{21}(s) = \left. \frac{F_l}{I_i} \right|_{U_l=0}$$

2436 The these off-diagonal elements are equal [ $z_{12}(s) = z_{21}(s)$ ] the system is said to obey *Rayleigh reci-*  
 2437 *procity*. If they are opposite in sign [ $z_{12}(s) = -z_{21}(s)$ ], the system is said to be *anti-reciprocal*. If  
 2438 a network has neither of the reciprocal or anti-reciprocal characteristics, then we denote it as *non-*  
 2439 *reciprocal* (McMillan, 1946). The most comprehensive discussion of reciprocity, even to this day, is that  
 2440 of Rayleigh (1896, Vol. I). The reciprocal case may be modeled as an ideal transformer (Van Valkenburg,  
 2441 1964a) while for the anti-reciprocal case the generalized force and flow are swapped across the 2-port.  
 2442 An electromagnetic transducer (e.g., a moving coil loudspeaker or electrical motor) is anti-reciprocal  
 2443 (Kim and Allen, 2013; Beranek and Mellow, 2012), requiring a gyrator rather than a transformer, as  
 2444 shown in Fig. 3.5.

2445 **Reversibility: P7** A second 2-port property is the *reversible/non-reversible* postulate. A reversible  
 2446 system is invariant to the input and output impedances being swapped. This property is defined by  
 2447 the input and output impedances being equal.

2448 Referring to Eq. 3.26, when the system is *reversible*  $z_{11}(s) = z_{22}(s)$  or in terms of the transmission  
 2449 matrix variables  $\frac{A(s)}{C(s)} = \frac{D(s)}{C(s)}$  or simply  $A(s) = D(s)$  assuming  $C(s) \neq 0$ .

2450 An example of a non-reversible system is a transformer where the turns ratio is not one. Also an  
 2451 ideal operational amplifier (when the power is turned on) is non-reversible due to the large impedance  
 2452 difference between the input and output. Furthermore it is *active* (it has a power gain, due to the  
 2453 current gain at constant voltage) (Van Valkenburg, 1964b).

2454 Generalizations of this lead to group theory, and *Noether's theorem*. These generalizations apply  
 2455 to systems with many modes whereas quasistatics holds when operate below a cutoff frequency (Table  
 2456 3.1), meaning that like the case of the transmission line, there are no propagating transverse modes.  
 2457 While this assumption is never exact, it leads to highly accurate results because the non-propagating  
 2458 evanescent transverse modes are attenuated over a short distance, and thus, in practice, may be ignored  
 2459 (Montgomery et al., 1948; Schwinger and Saxon, 1968, Chap. 9-11).

2460 We extend the Carlin and Giordano postulate set to include (P7) Reversibility, which was refined by  
 2461 Van Valkenburg (1964a). To satisfy the reversibility condition, the diagonal components in a system's

2462 impedance matrix must be equal. In other words, the input force and the flow are proportional to the  
 2463 output force and flow, respectively (i.e.,  $Z_e = z_m$ ).

2464 **Spatial invariant: P8** The characteristic impedance and wave number  $\kappa(s, x)$  may be strongly  
 2465 frequency and/or spatially dependent, or even be negative over some limited frequency ranges. Due to  
 2466 *causality*, the concept of a negative group velocity must be restricted to a limited bandwidth (Brillouin,  
 2467 1960). As is made clear by Einstein's theory of relativity, all materials must be strictly causal (P1),  
 2468 a view that must therefore apply to acoustics, but at a very different time scale. We first discuss  
 2469 generalized postulates, expanding on those of Carlin and Giordano.

2470 **The QS constraint: P9** When a system is described by the wave equation, delay is introduced  
 2471 between two points in space, which depends on the wave speed. When the wavelength is large compared  
 2472 to the delay, one may successfully apply the *quasistatic approximation*. This method has wide-spread  
 2473 application, and is frequency used without mention of the assumption. This can lead to confusion,  
 2474 since the limitations of the approximation may not be appreciated. An example is the use of QS in  
 2475 Quantum Mechanics. The QS approximation has wide spread use when the signals may be accurately  
 2476 approximated by a band-limited signal. Examples include KCL, KVL, wave guides, transmission lines,  
 2477 and most importantly, impedance. The QS property is not mentioned in the six postulates of Carlin  
 2478 and Giordano (1964), thus they need to be extended in some fundamental ways.

2479 When the dimensions of a cellular structure in the material are much less than the wavelength, can  
 2480 the QS approximation be valid. This effect can be viewed as a *mode filter* that suppresses unwanted (or  
 2481 conversely enhances the desired) modes (Ramo et al., 1965). Qs may be applied to a 3 dimensional  
 2482 specification, as in a semiconductor lattice. But such applications fall outside the scope of this text  
 2483 (Schwinger and Saxon, 1968).

2484 Although I have never seen the point discussed in the literature, the QS approximation is applied  
 2485 when defining Green's theorem. For example, Gauss's Law is not true when the volume of the container  
 2486 violates QS, since changes in the distribution of the charge have not reached the boundary, when doing  
 2487 the integral. Thus such integral relationships assume that the system is in quasi steady-state (i.e., that  
 2488 QS holds).

Table 3.1: There are several ways of indicating the quasi-static (QS) approximation. For network theory there is only one lattice constant  $a$ , which must be much less than the wavelength (wavelength constraint). These three constraints are not equivalent when the object may be a larger structured medium, spanning many wavelengths, but with a cell structure size much less than the wavelength. For example, each cell could be a Helmholtz resonator, or an electromagnetic transducer (i.e., an earphone).

Measure	Domain
$ka < 1$	Wavenumber constraint
$\lambda > 2\pi a$	Wavelength constraint
$f_c < c/2\pi a$	Bandwidth constraint

2489 Formally, QS is defined as  $ka < 1$  where  $k = 2\pi/\lambda = \omega/c$  and  $a$  is the cellular dimension or the size  
 2490 of the object ( $k$  and  $a$  can be vectors). Schelkunoff may have been the first to formalize this concept  
 2491 (Schelkunoff, 1943) (but not the first to use it, as exemplified by the Helmholtz resonator). George  
 2492 Ashley Campbell was the first to use the concept in the important application of a wave-filter, some  
 2493 30 years before Schelkunoff (Campbell, 1903). These two men were 40 years apart, and both worked  
 2494 for the telephone company (after 1929, called AT&T Bell Labs) (Fagen, 1975).

2495 There are alternative definitions of the QS approximation, depending on the geometrical cell struc-  
 2496 ture. The alternatives are outlined in Table 3.1.

2497 **Summary**

2498 A transducer converts between modalities. We propose the general definition of the nine system  
2499 postulates, that include all transduction modalities, such as electrical, mechanical, and acoustical. It  
2500 is necessary to generalize the concept of the QS approximation (P9) to allow for guided waves.

2501 Given the combination of the important QS approximation, along with these space-time, linearity,  
2502 and reciprocity properties, a rigorous definition and characterization a system can thus be established.  
2503 It is based on a taxonomy of such materials, formulated in terms of material and physical properties  
2504 and in terms of extended network postulates.

2505 **3.5.2 Lec 22 Exam II (Evening)**



2506 Chapter 4

2507 Ordinary Differential Equations:  
2508 Stream 3a

2509 WEEK 8

23.9.0

2510

2511 Week 8 Friday *Stream 3*

2512 L 23 The amazing Bernoulli family; Fluid mechanics; airplane wings; natural logarithms  
2513 The transition from geometry → algebra → algebreic geometry → real analytic → complex  
2514 analytic  
2515 From Bernoulli to Euler to Cauchy and Riemann

2516 4.1 Week 8

2517 4.1.1 Lec 23 Newton and early calculus & the Bernoulli Family

2518 Newton and Calculus

2519 Bernoulli family

2520 Euler standard periodic (circular) function package

2521 The period of analytic discovery:

2522 Coming out of the dark ages, from algebra, to analytic geometry, to calculus.

2523 Starting with real analytic functions by Euler, we move to complex analytic functions with Cauchy.

2524 Integration in the complex plane is finally conquered.

**Lect DE 25.9 Stream 3: ∞ and Sets** 25.9.1

The development of real representations proceeded at a deadly-slow pace:

- *Real numbers*  $\mathbb{R}$ : Pythagoras knew of irrational numbers ( $\sqrt{2}$ )
- *Complex numbers*  $\mathbb{C}$ : 1572 “Bombelli is regarded as the inventor of complex numbers . . . ” <http://www-history.mcs.st-andrews.ac.uk/Biographies/Bombelli.html> [http://en.wikipedia.org/wiki/Rafael\\_Bombelli](http://en.wikipedia.org/wiki/Rafael_Bombelli) & p. 258
- *Power Series*: Gregory-Newton interpolation formula c1670, p. 175
- Point at infinity and the Riemann sphere 1851
- Analytic functions p. 267 c1800; Impedance  $Z(s)$  1893

2525

### Stream 3 Infinity

- Infinity  $\infty$  was not “understood” until 19<sup>th</sup> CE
- $\infty$  is best defined in terms of a limit
- Limits are critical when defining calculus
- Set theory is the key to understanding Limits
- Open vs close sets determine when a limit exists (or not)
- Thus, to fully understand limits, one needs to understand set theory
- Related is the convergence of a series
- Every convergent series has a *Region of Convergence* (ROC)
- When the ROC is Complex:
  - Example of  $\frac{1}{1-x}$  vs.  $\frac{1}{i-x}$ : The ROC is 1 for both cases
  - Why?
  - The case of the Heaviside step function  $u(t)$  & the Fourier Transform

2526

### Irrational numbers and limits (Ch. 4)

- How are irrational numbers interleaved with the integers?
- Between  $n$  and  $2n$  there is always an irrational number:

Chebyshev said, and I say it again. There is always a prime between  $n$  and  $2n$ . -p. 585<sub>2</sub>

2527

- *Prime number theorem*: The number of of primes is approximately( the density of primes is  $\rho_{\pi}(n) \propto 1/\ln(n)$ ).
- The number of primes less than  $n$  is  $n$  times the density, or

$$N(n) = n/\ln(n).$$

- The formula for *entropy* is  $\mathcal{H} = -\sum_n p_n \log p_n$ .  
Could there be some hidden relationship lurking here?

### Stream 3: $\infty$ and Sets

25.9.2

- Understanding  $\infty$  has been a primary goals since Euclid
- The Riemann sphere solves this fundamental problem
- The point at  $\infty$  simply “another point” on the Riemann sphere

2528



2529 **Open vs. closed sets****Influence of open vs. closed set****7.3.6**

2530

- Important example: LT vs. FT step function: Dirac step vs Fourier step:
- $u(t) \leftrightarrow \frac{1}{s}$  vs.  $\tilde{u}(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$

2531

**WEEK 9****23.9.0**

2532

2533

Week 9 Monday

2534

L 24 Power series and integration of functions (ROC)

2535

Fundamental Theorem of calculus (Leibniz theorem of integration)

2536

 $1/(1-x) = \sum_{k=0}^{\infty} x^k$  with  $x \in \mathbb{R}$ 

2537

L 25 Integration in the complex plane: Three theorems

2538

Integration of  $1/s$  on the unit circle, and on a unit circle centered about  $s = 1 + i$ .

2539

2540

L 26 Cauchy-Riemann conditions

2541

Real and imaginary parts of analytic functions obey Laplace's equation.

2542

Infinite power Series and analytic function theory; ROC

2543

2544

**4.2 Week 9**

2545

**4.2.1 Lec 24 Power series and complex analytic functions**

2546

L 24: Power series and complex analytic function

2547

**4.2.2 Lec 25 Integration in the complex plane**

2548

L 25: Integration in the complex plane; Infinite power Series and analytic function theory; ROC

2549

Real and imaginary parts of analytic functions obey Laplace's equation.

2550

Colorized plots of analytic functions. How to read the plots and what they tell us?

2551

**4.2.3 Lec 26 Cauchy Riemann conditions: Complex-analytic functions**

2552

L 26: Cauchy Riemann conditions: Complex-analytic functions

2553

**WEEK 10****26.10.0**

2554

2555

L 27 Differentiation in the complex plane: Fundamental Thm of complex calculus (FTCC);

2556

Complex Analytic functions; ROC in the complex plane

2557

 $Z(s) = R(s) + jX(s)$ : real and imag parts obey Laplace's Equation

2558

Basic equations of mathematical Physics: Wave equation, Diffusion equation, Laplace's Equation

2559

Motivation: Dispersion relation for the wave equation  $\kappa \cdot \kappa = s^2/c_0^2$

2560 L 28 Three Fundamental theorems of complex integral calculus  
 2561  $\int_0^z = F(\zeta)d\zeta = F(z) - F(0)$ :  $dZ(s)/ds$  independent of direction  
 2562 Integration in the complex plane; Integrals independent of limits  
 2563 Cauchy-Riemann conditions

2564 L 29 Inverse Laplace transform  
 2565 Inverse Laplace transform: Poles and Residue expansions;  
 2566 Application of the Fundamental Thm of Complex Calculus  
 2567 The Inverse Laplace Transform (ILT); poles and the Residue expansion: The case for causality  
 2568 ROC as a function of the sign of time in  $e^{st}$  (How does causality come into play?)  
 2569 Examples.

## 2570 4.3 Integration and differentiation in the complex plane

### 2571 4.3.1 Lec 27 Differentiation in the complex plane

2572 L 27: Differentiation in the complex plane: CR conditions?  
 2573 Motivation: Inverse Laplace transform  
 2574 ROC in the complex plane  
 2575 Basic equations of mathematical Physics: Wave equation, Diffusion equation, Laplace's Equation  
 2576 Motivation: Dispersion relation for the wave equation  $\kappa \cdot \kappa = s^2/c_0^2$

### 2577 4.3.2 Lec 28 Three complex Integral Theorems

2578 L 28: Integration in the complex plane: Basic definitions of Three theorems  
 2579 Integration of  $1/s$  on the unit circle, and on a unit circle centered about  $s = 1 + i$ .  
 2580  
 2581 Moved from Lec 3 (page 31)

2582 **Set Theory:** Set theory is a topic that can be inadequately addressed in the undergraduate Engi-  
 2583 neering and Physics curriculum, and is relatively young to mathematics. The set that a number is  
 2584 drawn from is crucially important when taking limits.

### 2585 4.3.3 Lec 29 Inverse Laplace Transform

2586 L 29: Inverse Laplace transform: Poles and Residue expansions;  
 2587 Application of the Fundamental Thm of Complex Calculus  
 2588 Examples.

## Stream 3: Infinity and irrational numbers Ch 4

2.1.6

- Limit points, open vs. closed sets are fundamental to modern mathematics
- These ideas first appeared with the discovery of  $\sqrt{2}$ , and  $\sqrt{n}$  [https://en.wikipedia.org/wiki/Spiral\\_of\\_Theodorus](https://en.wikipedia.org/wiki/Spiral_of_Theodorus) and related constructions (factoring the square, Pell's Eq. p. 44)

2589 Infinity and irrational  $\mathbb{Q}$  numbers

**The fundamental theorem of calculus**

2.1.7

Let  $A(x)$  be the area under  $f(x)$ . Then

$$\begin{aligned}\frac{d}{dx}A(x) &= \frac{d}{dx} \int f(\eta)d\eta \\ &= \lim_{\delta \rightarrow 0} \frac{A(x + \delta) - A(x)}{\delta}\end{aligned}$$

and/or

$$A(b) - A(a) = \int_a^b f(\eta)d\eta$$

- Stream 3 is about limits
- Integration and differentiation (Calculus) depend on limits
- Limits are built on open vs. closed sets

**WEEK 11**

30.11.0

2590

2591

2592 L 30 Inverse Laplace transform &amp; Cauchy Residue Theorem

2593 L 31 Case for causality Closing the contour as  $s \rightarrow \infty$ ; Role of  $\Re st$ 2594 **DE-3**

2595 L 32 Properties of the LT:

2596 1) Modulation, 2) Translation, 3) convolution, 4) periodic functions

2597 Tables of common LTs

2598 **4.4 Integration in the complex plane**2599 **4.4.1 Lec 30** Inverse Laplace Transform & Cauchy residue theorem

2600 L30: The Inverse Laplace Transform (ILT); poles and the Residue expansion: The case for causality

2601 ROC as a function of the sign of time in  $e^{st}$  (How does causality come into play?)2602 **4.4.2 Lec 31** The case for causality2603 L31: Closing the contour as  $s \rightarrow \infty$ ; Role of  $\Re st$ 

2604

2605 **4.4.3 Lec 32** Laplace transform properties: Modulation, time translation, etc.

2606 L32: Detailed examples of the Inverse LT:

2607 1) Modulation, 2) Translation, 3) convolution, 4) periodic functions

2608 Tables of common LTs

**WEEK 12**

33.12.0

2609

2610

- 2611 L 33 Multi-valued functions in the complex plane; Branch cuts  
 2612 The extended complex plane (regularization at  $\infty$ ) (Stillwell, 2010, p. 280)  
 2613 Complex analytic functions of Genus 1 (Stillwell, 2010, p. 343)
- 2614 L 34 Euler's vs. Riemann's Zeta function  $\zeta(s)$ : Poles at the primes  
 2615 colored plot of  $\zeta(s)$   
 2616 ??Sterling's formula??
- 2617 L 35 Exam III

## 2618 4.5 Complex plane concepts

### 2619 4.5.1 Lec 33 Multi-valued complex functions, Branch Cuts, Extended plane

- 2620 L33: Multi-valued functions in the complex plane; Branch cuts  
 2621 The extended complex plane (regularization at  $\infty$ ) (Stillwell, 2010, p. 280)  
 2622 Complex analytic functions of Genus 1 (Stillwell, 2010, p. 343)

### 2623 4.5.2 Lec 34 The Riemann Zeta function $\zeta(s)$

- 2624 L34: Euler's vs. Riemann's Zeta function  $\zeta(s)$ : Poles at the primes  
 2625 domain coloring of  $\zeta(s)$   
 2626 ??Sterling's formula??

**Table 4.1: Physical meaning of each factor of  $\zeta(s)$**

**4.2.7**

- Series expansion

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad \text{ROC: } |x| < 1$$

- If time  $T$  is a positive delay, then from the Laplace transform

$$\delta(t-T) \leftrightarrow \int_0^\infty \delta(t-T)e^{st} dt = e^{-sT}$$

- Each factor of  $\zeta(s)$  is an  $\infty$  sum of delays
- For example for  $\pi_1 = 2$ , ( $T = \ln(2)$ , thus  $2^{-2} = e^{-s \ln 2}$ )

$$\sum_n \delta(t-nT) \leftrightarrow \frac{1}{1-2^{-s}} = 1 + e^{-sT} + e^{-s2T} + \dots$$

*Table 4.1: Each prime number defines a delay  $T_k = \ln(\pi_k)$ , which in turn defines a pole in the complex  $s$  plane. The series expansion of this pole is a train of delta functions that are one-sided periodic in the delta  $T$ . Thus each factor in the  $\zeta(s)$  function defines a pole, having an incommensurate delay, since each pole is defined by a unique prime. Following this simple logic, we may interpret  $\zeta(s)$  as being the Laplace transform of  $Zeta(t)$ , the cascade of quasi-periodic impulse responses, each with a recursive delay, determined by a prime. Note that 48100 = 10 · (2 · 5 · 13 · 37) is the sampling frequency [Hz] of modern CD players. This corresponds to the 20<sup>th</sup> harmonic of the US line frequency (60 [Hz]).<sup>b</sup>*

<sup>a</sup>since  $\text{gcd}(48100, 60) = 20$  and  $\text{gcd}(48100, 50) = 50$ .

<sup>b</sup>since  $\text{gcd}(48100, 60) = 20$  and  $\text{gcd}(48100, 50) = 50$ .

## 2627 Riemann Zeta Function $\zeta(s)$

This very important analytic function is the credible argument for true deeper understanding of the power to the analytic function. Just like the Pythagorean theorem is important to all mathematics,

the zeta function is important to analysis, with many streams of analysis emanating from this form. For example the analytic Gamma function  $\Gamma(s)$  is a generalization of the factorial by the relationship

$$n! = \Gamma(s - 1).$$

Another important relationship is

$$\sum_{k=n}^{\infty} k = nu_n = u_n \star u_n$$

where the  $\star$  represents convolution. If this is treated in the frequency domain then we obtain z-transforms of a very simple second-order pole<sup>1</sup>

$$nu_n \leftrightarrow \frac{2}{(z-1)^2}.$$

This follows from the geometric series

$$\frac{1}{1-z} = \sum_n z^n$$

2628 with  $z = e^s$ , and the definition of convolution.

The Laplace transform does not require that the series converge, rather that the series have a region of convergence that is properly specified. Thus the non-convergent series  $nu_n$  is perfectly well defined, just like

$$tu(t) = u(t) \star u(t) \leftrightarrow \frac{1!}{s^2}$$

is well defined, in the Laplace transform sense. More generally

$$t^n u(t) \leftrightarrow \frac{n!}{s^{n+1}}.$$

From this easily understood relationship we can begin to understand  $\Gamma(s)$ , as the analytic extension of the factorial. Its definition is simply related to the inverse Laplace transform, which is an integral. But to go there we must be able to think in the complex frequency domain. In fact we have the very simple definition for  $\Gamma(p)$  with  $p \in \mathbb{C}$

$$t^{p-1} u(t) \leftrightarrow \frac{\Gamma(p)}{s^p}$$

2629 which totally explains  $\Gamma(p)$ . Thinking in the time domain is crucial for my understanding.

2630 **An example is a digital filter, which is linear. Such a system is shown in Fig. 4.3, where the two**  
 2631 **functions are second order digital filters. The input signal  $x[n]$  enters from the left, is filtered by the**  
 2632 **first filter, producing output  $y[n]$ . This is then filtered again by the filter in the second box to produce**  
 2633 **signal  $z[n]$ . For this simple case of two linear filters the operation *commute*.**

### 2634 4.5.3 Lec 35 Exam III

2635 L 35: Exam III

2636 Thanksgiving Holiday 11/19–11/27 2016

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<sup>1</sup>Need to verify the exact form of these relationships, not work from memory

### Riemann Zeta Function $\zeta(s)$

4.2.5

- Integers appear as the “roots” (aka eigenmodes) of  $\zeta(s)$
- Basic properties ( $s = \sigma + i\omega$ )

$$\zeta(s) \equiv \sum_1^{\infty} \frac{1}{n^s} \quad \sigma = \Re(s) > 0$$

– What is the region of convergence (ROC)?

- The amazing Euler-Riemann Product formula (Stillwell, 2010, Sect. 10.7:)

$$\begin{aligned} \zeta(s) &= \prod_k \frac{1}{1 - \pi_k^{-s}} = \prod_k \frac{1}{1 - \left(\frac{1}{\pi_k}\right)^s} = \prod_k \frac{1}{1 - \frac{1}{\pi_k^s}} \\ &= \frac{1}{1 - 2^{-s}} \cdot \frac{1}{1 - 3^{-s}} \cdot \frac{1}{1 - 5^{-s}} \cdot \frac{1}{1 - 7^{-s}} \cdots \frac{1}{1 - \pi_n^{-s}} \cdots \end{aligned}$$

- Euler c1750 assumed  $s \subset \mathbb{R}$ . Riemann c1850 extended  $s \subset \mathbb{C}$

Figure 4.1: The zeta function arguably the most important of the special functions of analysis because it connects the primes to analytic function theory in a fundamental way.

### Plot of $\angle\zeta(s)$

4.2.6

Angle of Riemann Zeta function  $\angle\zeta(z)$  as a function of complex  $z$

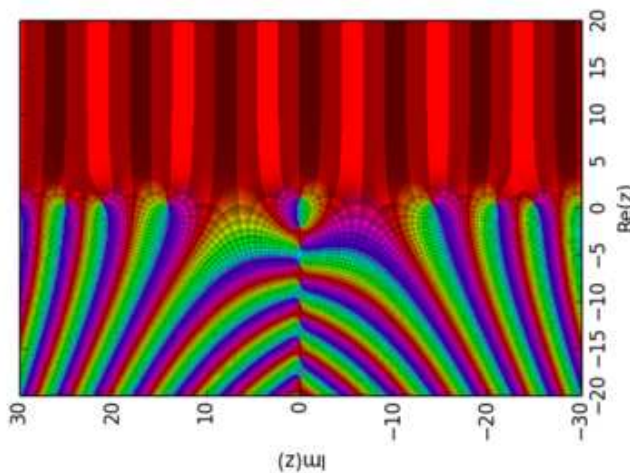


Figure 4.2:  $\angle\zeta(z)$ : Red  $\Rightarrow \angle\zeta(z) < \pm\pi/2$

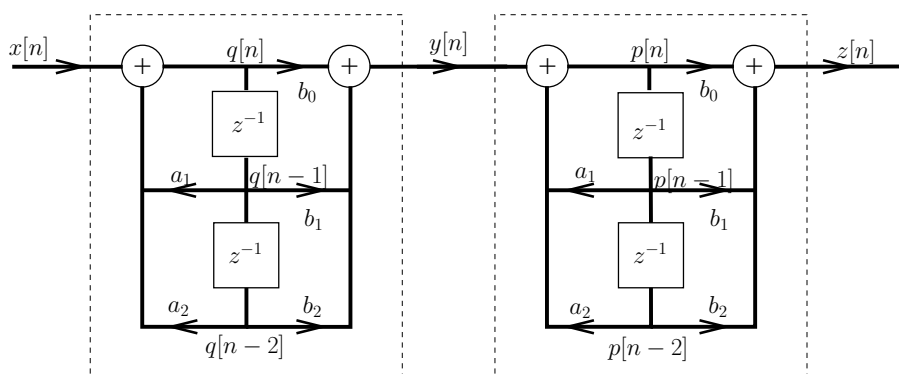


Figure 4.3: Example of a signal flow diagram for the composition of signals  $z = g \circ f(x)$  with  $y = f(x)$  and  $z = g(y)$ .

2637 **Chapter 5**

2638 **Vector Calculus: Stream 3b**

2639 **WEEK 13**

36.13.0

2640

2641 L 36 Scaler wave equations and the Webster Horn equation; WKB method  
2642 A real-world example of large delay, where the branch-cut placement is critical

2643

2644 L 37 Partial differential equations of Physics  
2645 Scaler wave equation and its solution in 1 and 3 Dimensions  
2646 **VC-1**

2647 L 38 Vector dot and cross products  $A \cdot B, A \times B$   
2648 Gradient, divergence and curl

2649 – Thanksgiving Holiday 11/19–11/27 2016

2650 **5.1 Stream 3b**

2651 **5.1.1 Lec 36** Scalar Wave equation

2652 **5.1.2 Lec 37** Partial differential equations of physics

2653 Scalar wave equations and the Webster Horn equation; WKB method  
2654 Example of a large delay, where a branch-cut placement is critical (i.e., phase unwrapping)

2655 L 37: Partial differential equations of Physics  
2656 Scalar wave equation and its solution in 1 and 3 Dimensions

2657 **5.1.3 Lec 38** Gradient, divergence and curl vector operators

2658 L 38: Vector dot and cross products  $A \cdot B, A \times B$   
2659 Gradient, divergence and curl vector operators

2660 **WEEK 14**

37.14.0

2661

- 2662 L 39 Gradient, divergence and curl: Gauss's (divergence) and Stokes's (curl) theorems
- 2663 L 40 J.C. Maxwell unifies Electricity and Magnetism with the formula for the speed of light
- 2664 Basic definitions of  $E, H, B, D$
- 2665 O. Heaviside's (1884) vector form of Maxwell's EM equations and the *vector wave equation*
- 2666 How a loud-speaker works
- 2667 L 41 *The Fundamental Thm of vector calculus*
- 2668 *Incompressible and Irrotational fluids and the two defining vector identities*
- 2669

## 2670 5.2 Thanksgiving Holiday 11/19–11/27 2016

2671 Thanksgiving Vacation: 1 week of rest

## 2672 5.3 Vector Calculus

### 2673 5.3.1 Lec 39 Geometry of Gradient, divergence and curl vector operators

2674 Geometry of Gradient, divergence and curl vector operators

#### Lec 39: Review of vector field calculus

39.14.2

- Review of last few lectures: Basic definitions
  - *Field*: i.e., Scalar & vector fields are functions of more than one variable
  - “Del:”  $\nabla \equiv [\partial_x, \partial_y, \partial_z]^T$
  - Gradient:  $\nabla\phi(x, y, z) \equiv [\partial_x\phi, \partial_y\phi, \partial_z\phi]^T$
- *Helmholtz Theorem*:  
 2675 Every vector field  $\mathbf{V}(x, y, z)$  may be uniquely decomposed into *compressible* & *rotational* parts
 
$$\mathbf{V}(x, y, z) = -\nabla\phi(x, y, z) + \nabla \times \mathbf{A}(x, y, z)$$
- Scalar part  $\nabla\phi$  is *compressible* ( $\nabla\phi = 0$  is *incompressible*)
- Vector part  $\nabla \times \mathbf{A}$  is *rotational* ( $\nabla \times \mathbf{A} = 0$  is *irrotational*)
- Key vector identities:  $\nabla \times \nabla\phi = 0$ ;  $\nabla \cdot \nabla \times \mathbf{A} = 0$
- Definitions of Divergence, Curl & Maxwell's Eqs;
- Closure: Fundamental Theorems of Integral Calculus

Name	Input	Output	Operator
Gradient	Scalar	<b>Vector</b>	$-\nabla()$
Divergence	<b>Vector</b>	Scalar	$\nabla \cdot ()$
Curl	<b>Vector</b>	<b>Vector</b>	$\nabla \times ()$

Table 5.1: *The three vector operators manipulate scalar and vector fields, as indicated here. The gradient converts scalar fields into vector fields. The divergence eats vector fields and outputs scalar fields. Finally the curl takes vector fields into vector fields.*



2676 **Gradient****Gradient:**  $\mathbf{E} = \nabla\phi(x, y, z)$ **39.14.3**

- Definition:  $\mathbb{R}^1 \xrightarrow{\nabla} \mathbb{R}^3$

$$\mathbf{E}(x, y, z) = [\partial_x, \partial_y, \partial_z]^T \phi(x, y, z) = \left[ \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right]_{x,y,z}^T$$

2677

- $\mathbf{E} \perp$  plane tangent at  $\phi(x, y, z) = \phi(x_0, y_0, z_0)$
- Unit vector in direction of  $\mathbf{E}$  is  $\hat{\mathbf{n}} = \frac{\mathbf{E}}{\|\mathbf{E}\|}$ , along *isocline*
- Basic definition

$$\nabla\phi(x, y, z) \equiv \lim_{|\mathcal{S}| \rightarrow 0} \left\{ \frac{\iiint_{\mathcal{S}} \phi(x, y, z) \hat{\mathbf{n}} dA}{|\mathcal{S}|} \right\}$$

$\hat{\mathbf{n}}$  is a unit vector in the direction of the gradient  
 $\mathcal{S}$  is the surface area centered at  $(x, y, z)$

2678 **Divergence****Divergence:**  $\nabla \cdot \mathbf{D} = \rho$ **39.14.4a**

- Definition:  $\mathbb{R}^3 \xrightarrow{\nabla \cdot} \mathbb{R}^1$

$$\nabla \cdot \mathbf{D} \equiv [\partial_x, \partial_y, \partial_z] \cdot \mathbf{D} = \left[ \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right] = \rho(x, y, z)$$

- Examples:

2679

- Voltage about a point charge  $Q$  [SI Units of Coulombs]

$$\phi(x, y, z) = \frac{Q}{\epsilon_0 \sqrt{x^2 + y^2 + z^2}} = \frac{Q}{\epsilon_0 R}$$

$\phi$  [Volts];  $Q$  [C]; Free space  $\epsilon_0$  *permittivity* ( $\mu_0$  *permeability*)

- *Electric Displacement* (flux density) around a point charge ( $\mathbf{D} = \epsilon_0 \mathbf{E}$ )

$$\mathbf{D} \equiv -\nabla\phi(R) = -Q\nabla \left\{ \frac{1}{R} \right\} = -Q\delta(R)$$

**Divergence: The integral definition****39.14.4b**

- Surface integral definition of *incompressible* vector field

2680

$$\nabla \cdot \mathbf{D} \equiv \lim_{|\mathcal{V}| \rightarrow 0} \left\{ \frac{\iint_{\mathcal{S}} \mathbf{D} \cdot \hat{\mathbf{n}} dA}{|\mathcal{V}|} \right\} = \rho(x, y, z)$$

$\mathcal{S}$  must be a closed surface

$\hat{\mathbf{n}}$  is the unit vector in the direction of the gradient

- $\hat{\mathbf{n}} \cdot \mathbf{D} \perp$  surface differential  $dA$

## Divergence: Gauss' Law

39.14.4c

- General case of a *Compressible* vector field
- Volume integral over charge density  $\rho(x, y, z)$  is total charge enclosed  $Q_{enc}$

2681

$$\iiint_{\mathcal{V}} \nabla \cdot \mathbf{D} dV = \iint_{\mathcal{S}} \mathbf{D} \cdot \hat{\mathbf{n}} dA = Q_{enc}$$

- Examples
  - When the vector field is *incompressible*
    - \*  $\rho(x, y, z) = 0$  [C/m<sup>3</sup>] over enclosed volume
    - \* Surface integral is zero ( $Q_{enc} = 0$ )
  - Unit point charge:  $D = \delta(R)$  [C/m<sup>2</sup>]

2682 **Curl**

$$\text{Curl: } \nabla \times \mathbf{H} = \mathbf{I} \text{ [amps/m}^2\text{]}$$

39.14.5a

- Definition:  $\mathbb{R}^3 \xrightarrow{\nabla \times} \mathbb{R}^3$

2683

$$\nabla \times \mathbf{H} \equiv \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ H_x & H_y & H_z \end{vmatrix} = \mathbf{I}$$

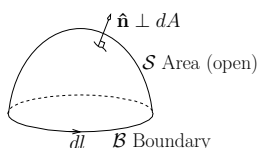
- Examples:
  - Maxwell's equations:  $\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$ ,  $\nabla \times \mathbf{H} = \sigma \mathbf{E} + \dot{\mathbf{D}}$ ,
  - $\mathbf{H} = -y\hat{x} + x\hat{y}$  then  $\nabla \times \mathbf{H} = 2\hat{z}$  constant *irrotational*
  - $\mathbf{H} = 0\hat{x} + 0\hat{y} + z^2\hat{z}$  then  $\nabla \times \mathbf{H} = \mathbf{0}$  is *irrotational*

2684 **Stokes' Law**

$$\text{Curl: Stokes Law}$$

39.14.5b

- Surface integral definition of  $\nabla \times \mathbf{H} = \mathbf{I}$  ( $\mathbf{I} \perp$  rotation plane of  $\mathbf{H}$ )



2685

$$\nabla \times \mathbf{H} \equiv \lim_{|\mathcal{S}| \rightarrow 0} \left\{ \frac{\iint_{\mathcal{S}} \hat{\mathbf{n}} \times \mathbf{H} dA}{|\mathcal{S}|} \right\} \quad (5.1)$$

$$\mathcal{I}_{enc} = \iint (\nabla \times \mathbf{H}) \cdot \hat{\mathbf{n}} dA = \oint_{\mathcal{B}} \mathbf{H} \cdot d\mathbf{l} \quad (5.2)$$

- Eq. (1):  $\mathcal{S}$  must be an *open surface* with closed boundary  $\mathcal{B}$   
 $\hat{\mathbf{n}}$  is the unit vector  $\perp$  to  $dA$   
 $\mathbf{H} \times \hat{\mathbf{n}} \in$  Tangent plane of  $A$  (i.e.,  $\perp \hat{\mathbf{n}}$ )
- Eq. (2): Stokes Law: Line integral of  $\mathbf{H}$  along  $\mathcal{B}$  is total current  $\mathcal{I}_{enc}$

2686 **5.3.2 Lec: 40 Introduction to Maxwell's Equation**

2687 L 40: J.C. Maxwell unifies Electricity and Magnetism with the formula for the speed of light  
 2688 Basic definitions of  $E, H, B, D$   
 2689 O. Heaviside's (1884) vector form of Maxwell's EM equations and the *vector wave equation*  
 2690 How a loud-speaker works.

2691 **5.3.3 Lec: 41 The Fundamental theorem of Vector Calculus**

2692 L 41: *The Fundamental Thm of vector calculus*  
 2693 *Incompressible and Irrotational fluids and the two defining vector identities*

2694 **WEEK 15**

40.15.0

2695

2696 L 42 Quasi-static approximation and applications:  
 2697 The Kirchoff's Laws and the *Telegraph wave equation*, starting from Maxwell's equations The  
 2698 telegraph wave equation starting from Maxwell's equations  
 2699 Quantum Mechanics

2700 L 43 Last day of class: Review of Fund Thms of Mathematics:  
 2701 Closure on Numbers, Algebra, Differential Equations and Vector Calculus,  
 2702 The Fundamental Thms of Mathematics & their applications:  
 2703 Theorems of Mathematics; Fundamental Thms of Mathematics (Ch. 9); Normal modes vs. eigen-  
 2704 states, delay and quasi-statics;

2705 – Reading Day  
 2706 VC-1 Due

2707 **5.4 Kirchhoff's Laws**2708 **5.4.1 Lec 42: The Quasi-static approximation and applications**

2709 L 42: The Kirchhoff's Laws and the *Telegraph wave equation*, starting from Maxwell's equations  
 2710 Quantum Mechanics

2711 **5.4.2 Lec 43: Last day of class: Review of Fund Thms of Mathematics**

2712 L 43: Closure on Numbers, Algebra, Differential Equations and Vector Calculus,  
 2713 The Fundamental Thms of Mathematics & their applications:  
 2714 Theorems of Mathematics; Fundamental Thms of Mathematics (Ch. 9)  
 2715 Normal modes vs. eigen-states, delay and quasi-statics;  
 2716 Reading Day

2717 **Properties****Closure: Properties of fields of Maxwell's Equations**

39.14.6

The variables have the following names and defining equations:

Symbol	Equation	Name	Units
$\mathbf{E}$	$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$	Electric Field strength	[Volts/m]
$\mathbf{D}$	$\nabla \cdot \mathbf{D} = \rho$	Electric Displacement (flux density)	[Col/m <sup>2</sup> ]
$\mathbf{H}$	$\nabla \times \mathbf{H} = \dot{\mathbf{D}}$	Magnetic Field strength	[Amps/m]
$\mathbf{B}$	$\nabla \cdot \mathbf{B} = 0$	Magnetic Induction (flux density)	[Weber/m <sup>2</sup> ]

*In vacuo*  $\mathbf{B} = \mu_0 \mathbf{H}$ ,  $\mathbf{D} = \epsilon_0 \mathbf{E}$ ,  $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$  [m/s],  $r_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377$  [ $\Omega$ ].

2719 **Vector field properties****Closure: Summary of vector field properties**

39.14.7

- Notation:  $\mathbf{v}(x, y, z) = -\nabla\phi(x, y, z) + \nabla \times \mathbf{w}(x, y, z)$
- Vector identities:  $\nabla \times \nabla\phi = 0$ ;  $\nabla \cdot \nabla \times \mathbf{w} = 0$

Field type	Generator:	Test (on $\mathbf{v}$ ):
Irrotational	$\mathbf{v} = \nabla\phi$	$\nabla \times \mathbf{v} = 0$
Rotational	$\mathbf{v} = \nabla \times \mathbf{w}$	$\nabla \times \mathbf{v} = \mathbf{J}$
Incompressible	$\mathbf{v} = \nabla \times \mathbf{w}$	$\nabla \cdot \mathbf{v} = 0$
Compressible	$\mathbf{v} = \nabla\phi$	$\nabla \cdot \mathbf{v} = \rho$

- Source density terms: Current:  $\mathbf{J}(x, y, z)$ , Charge:  $\rho(x, y, z)$ 
  - Examples:  $\nabla \times \mathbf{H} = \dot{\mathbf{D}}(x, y, z)$ ,  $\nabla \cdot \mathbf{D} = \rho(x, y, z)$

2721 **Fundamental Theorem of integral Calculus****Closure: Fundamental Theorems of integral calculus**

39.14.8

1.  $f(x) \in \mathbb{R}$  (Leibniz Integral Rule):  $F(x) = F(a) + \int_a^x f(x)dx$
2.  $f(s) \in \mathbb{C}$  (Cauchy's formula):  $F(s) = F(a) + \int_a^s f(\zeta)d\zeta$   
 –When integral is independent of path,  $F(s) \in \mathbb{C}$  obeys CR conditions  
 –Contour integration inverts causal Laplace transforms
3.  $\mathbf{F} \in \mathbb{R}^3$  (Helmholtz Formula):  $\mathbf{F}(x, y, z) = -\nabla\phi(x, y, z) + \nabla \times \mathbf{A}(x, y, z)$   
 –Decompose  $\mathbf{F}(x, y, z)$  as *compressible* and *rotational*
4. Gauss' Law (Divergence Theorem):  $Q_{enc} = \iiint \nabla \cdot \mathbf{D} dV = \iint_S \mathbf{D} \cdot \hat{\mathbf{n}} dA$   
 –Surface integral describes enclosed compressible sources
5. Stokes' Law (Curl Theorem):  $\mathcal{I}_{enc} = \iint (\nabla \times \mathbf{H}) \cdot \hat{\mathbf{n}} dA = \oint_{\mathcal{B}} \mathbf{H} \cdot d\mathbf{l}$   
 –Boundary vector line integral describes enclosed rotational sources
6. Green's Theorem ... Two-port boundary conditions  
 –*Reciprocity* property (*Theory of Sound*, Rayleigh, J.W.S., 1896)

**Closure: Quasi-static (QS) approximation**

39.14.9

- Definition:  $ka \ll 1$  where  $a$  is the size of object,  $\lambda = c/f$  wavelength
- This is equivalent to  $a \ll \lambda$  or
- $\omega \ll c/a$  which is a low-frequency approximation
- The QS approximation is widely used, but infrequently identified.
- All *lumped parameter models* (inductors, capacitors) are based on QS approximation as the lead term in a Taylor series approximation.



# Appendix A

## Notation

### A.1 Number systems

The notation used in this book is defined in this appendix so that it may be quickly accessed.<sup>1</sup> Where the definition is sketchy, Page numbers are provided where these concepts are fully explained, along with many other important and useful definitions. For example  $\mathbb{N}$  may be found on page 22.

#### A.1.1 Greek letters

The Greek letters used in this text include (at least)  $\alpha, \beta, \gamma, \delta, \epsilon, \kappa, \rho, \xi, \omega, \sigma, \phi, \psi, \zeta$ , and upper-case  $\Gamma, \Xi, \Phi, \Psi, \Delta, \Omega$ . Many of these imply a physical meaning. For example,  $\omega$  is the radian frequency,  $\rho$  is typically a density.  $\Phi, \phi, \Psi, \psi$  are commonly used to indicate angles of a triangle. For those of you who know this, its obvious. But at some time in your life, you had to learn these conventions. They were not implanted at birth.

Likely you do not know all of these Greek letters. But they are so commonly used in mathematics, you need to recognize them. Some of them are pronounced in a way that you may not know. The symbol  $\xi$  is pronounced “c,”  $\zeta$  is “zeta” and  $\chi$  is “kye.” I must assume you know how to pronounce the others, which are more phonetic in English. One advantage of learning L<sup>A</sup>T<sub>E</sub>X is that all of these math symbols are built in, and thus easily learned, by using this powerful open-source math-oriented word-processing system.

#### A.1.2 Double-Bold notation

Table A.1 indicates the symbol followed by a page number and the name of the number type. For example  $\mathbb{N}$  stands for the infinite set of *counting numbers*  $\{1, 2, 3, \dots\}$ . From any counting number you may get the next one by adding 1.

**Summary of various number types:** Counting number ( $\mathbb{N}$ ) are also know as the Cardinal numbers. The prime numbers ( $\mathbb{P}$ ) cannot be further factored. The counter example of  $-5 = -1 \cdot 5$  is questionable, as it could be included as a prime by a slight change in the definition. One may say that a real ( $\mathbb{R}$ ) is a complex number ( $\mathbb{C}$ ) with a zero imaginary part, thus real numbers are complex ( $\mathbb{R} \subset \mathbb{C}$ ).

Note that  $\mathbb{R} : \mathbb{I} \cup \mathbb{Q}, \mathbb{I} \perp \mathbb{Q}, \mathbb{Q} : \mathbb{Z} \cup \mathbb{F}$ .

We say that a number is in the set with the notation  $3 \in \mathbb{N} \in \mathbb{R}$ , which is read as “3 is in the set of counting numbers, which in turn in the set of real numbers,” or in vernacular language “3 is a real counting number.”

The *cardinality* of a set is denoted by taking the absolute value (e.g.,  $|\mathbb{N}|$ ).

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<sup>1</sup>[https://en.wikipedia.org/wiki/List\\_of\\_mathematical\\_symbols\\_by\\_subject#Definition\\_symbols](https://en.wikipedia.org/wiki/List_of_mathematical_symbols_by_subject#Definition_symbols)

Table A.1: Double-bold notation for the types of numbers. (#) is a page number.

Symbol (p. #)	Genus	Examples	Counter Examples
<b>N</b> (22)	Counting	1,2,17,3, $10^{20}$	0, -10, 5j
<b>P</b> (22)	Prime	2,17,3, $10^{20}$	0, 1, 4, $3^2$ , 12, -5
<b>Z</b> (22)	Integer	-1, 0, 17, 5j, $-10^{20}$	$1/2, \pi, \sqrt{5}$
<b>Q</b> (22)	Rational	2/1, 3/2, 1.5, 1.14	$\sqrt{2}, 3^{-1/3}, \pi$
<b>F</b> (22)	Fractional	1/2, 7/22	2/1, $1/\sqrt{2}$
<b>I</b> (23)	Irrational	$\sqrt{2}, 3^{-1/3}, \pi$	Vectors
<b>R</b> (23)	Reals	$\sqrt{2}, 3^{-1/3}, \pi$	$2\pi j$
<b>C</b> (117)	Complex	1, $\sqrt{2}j, 3^{-j/3}, \pi^j$	Vectors

## A.2 Periodic functions

Any periodic function may be indicated using double-parentheses notation. This is sometimes known as modular arithmetic. For example function

$$f((t))_T = f(t) = f(t \pm kT),$$

is periodic on  $t, T \in \mathbb{R}$  with a period of  $T$  and  $k \in \mathbb{Z}$ . This notation is useful when dealing with Fourier series of periodic functions.

When a discrete valued (e.g., time) sequence is periodic we use square brackets

$$f[[n]]_N = f[n] = f[n \pm kN],$$

with  $n, k, N \in \mathbb{Z}$  and period  $N$ . This notation will be used with discrete-time signals that are periodic, such as the case of the DFT.

## A.3 Vectors

Vectors are ordered sets of scalars. When we write them out, we use row notation, with the *transpose* symbol

$$[a, b, c]^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

*Vectors* are always columns. Row vectors are weights not vectors. A vector dot product is normally defined between weights and vectors, resulting in a real scalar. This is said to be a 3 *dimensional* vector. for example

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 + 2 + 3 = 6.$$

When the elements are complex, the transpose also takes the complex conjugate.

## A.4 Matrices

Unfortunately when working with matrices, the role of the weights and vectors can change, depending on the context. A useful way to view a matrix is as a set of column vectors, weighted by the elements



of the column-vector of weights multiplied from the right. For example

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1M} \\ a_{21} & a_{22} & a_{32} & \cdots & a_{3M} \\ & & \ddots & & \\ a_{N1} & a_{N2} & a_{N3} & \cdots & a_{NM} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_M \end{bmatrix} = w_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{21} \\ \dots \\ a_{N1} \end{bmatrix} + w_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \dots \\ a_{N2} \end{bmatrix} \dots w_M \begin{bmatrix} a_{1M} \\ a_{2M} \\ a_{3M} \\ \dots \\ a_{NM} \end{bmatrix},$$

where the weights are  $[w_1, w_2, \dots, w_M]^T$

Another way to view the matrix is a set of row vectors of weights, each of which re applied to the vector  $[w_1, w_2, \dots, w_M]^T$ .

The determinant of a matrix is denoted either as  $\det \mathbf{A}$  or  $|\mathbf{A}|$ , as in the absolute value. The inverse of a square matrix is  $\mathbf{A}^{-1}$  or  $\text{inv}\mathbf{A}$ . If  $|\mathbf{A}| = 0$ , the inverse does not exist.  $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A}$ .

Matlab’s notional convention for a row-vector is  $[a, b, c]$  and a column-vector is  $[a; b; c]$ . A prime on a vector takes the complex conjugate transpose. To suppress the conjugation, place a period before the prime. The `:` argument converts the array into a column vector, without conjugation. A tacit notation in Matlab is that *vectors* are columns and the index to a vector is a row vector. Matlab defines the notation `1:4` as the “row-vector”  $[1, 2, 3, 4]$ , which is unfortunate as it leads users to assume that the default vector is a row. This can lead to serious confusion later, as Matlab’s default vector is a column. I have not found the above convention explicitly stated, and it took me years to figure this out for myself.

Units are SI; Angles in degrees `[deg]` unless otherwise noted. The units for  $\pi$  are always in radians `[rad]`. Ex:  $\sin(\pi)$ ,  $e^{j90^\circ}$ ,  $e^{j\pi/2}$ .

when writing a complex number we shall try to use  $1j$  to indicate  $\sqrt{-1}$ . Matlab prefers this as well, as its explicit.

## A.5 Differential equations vs. Polynomials

A polynomial has *degree*  $N$  defined by the largest power. A quadratic equation is degree 2, and a cubic has degree 3. We shall indicate a polynomial by the notation

$$P_N(z) = z^N + a_{N-1}z^{N-1} \dots a_0.$$

Always normalize the polynomial so that  $a_N = 1$ . This will not change the roots, defined by Eq. 1.10 (p. 42). The coefficient on  $z^{N-1}$  is always the sum of the roots  $z_n$  ( $a_{N-1} = \sum_n^N z_n$ ), and  $a_0$  is always their product ( $a_0 = \prod_n^N z_n$ ).

Differential equations have *order* (polynomials have degree). If a second order differential equation is Laplace transformed (Lec. 1.3.10, p. 59), one is left with a degree 2 polynomial. For example:

$$\frac{d^2}{dt^2}y(t) + b\frac{d}{dt}y(t) + cy(t) = \alpha \left( \frac{d}{dt}x(t) + \beta x(t) \right) \leftrightarrow \tag{A.1}$$

$$(s^2 + bs + c)Y(s) = \alpha(s + \beta)X(s). \tag{A.2}$$

$$\frac{Y(s)}{X(s)} = \alpha \frac{s + \beta}{s^2 + bs + c} \equiv H(s) \leftrightarrow h(t). \tag{A.3}$$

Using the same argument as for polynomials, the lead coefficient must always be 1. The coefficient  $\alpha \in \mathbb{R}$  is called the *gain*. The complex variable  $s$  is the *Laplace frequency*.

The ratio of the output  $Y(s)$  over the input  $X(s)$  is called the system *transfer function*  $H(s)$ . When  $H(s)$  is the ratio of two polynomials in  $s$ , the transfer function is said to be *bilinear*, since it is linear in both the input and output. The roots of the numerator are called the *zeros* and those of the denominator, the *poles*. The inverse Laplace transform of the transfer function is called the system *impulse response*, which describes the system’s output signal  $y(t)$  for any given input signal  $x(t)$ , via convolution (i.e.,  $y(t) = h(t) \star x(t)$ ).

## A.6 Residue expansions and the ROC

With the new tool of analytic functions came the concept of the *region of convergence* (ROC) that defines the regions in the complex plane where the infinite series is valid. In other words, the function  $Z(s)$  and its analytic power series  $\sum_0^\infty c_n s^n$ , are equivalent over a region of  $s$  that lies within the ROC. When the series fails to converge, it no longer represents  $Z(s)$ . A helpful example is the series

$$\frac{1}{1+x^2} = \frac{1}{(1-xj)(1+xj)} = \frac{A}{1-xj} + \frac{B}{1+xj} = \frac{1}{2} \sum_{n=0}^{\infty} (+xj)^n + \frac{1}{2} \sum_{n=0}^{\infty} (-xj)^n,$$

which is valid for  $|x| < 1$ . At face value this function seems fine at  $x = 1$ , where it is equal to  $1/2$ . In fact the series fails to converge at precisely this value (the ROC is 1 for this example). Until one views  $x$  as complex, this behavior is not obvious.

A trivial analysis shows that  $A = 1/2$  and  $B = A$  since

$$1 = A(1+xj) + B(1-xj) = (A+Axj) + (B-Bxj) = A+B + (A-B)xj.$$

# Appendix B

## Linear algebra on 2x2 matrices

### B.1 Gaussian elimination

Definitions:

1. *Scalar*: A number, e.g.  $\{a, b, c, \alpha, \beta, \dots\} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{I}, \mathbb{R}, \mathbb{C}\}$
2. *Vector*: A quantity having direction as well as magnitude, often denoted by a bold-face letter with an arrow,  $\vec{x}$ . In matrix notation, this is typically represented as a single row  $[x_1, x_2, x_3, \dots]$  or single column  $[x_1, x_2, x_3 \dots]^T$  (where  $T$  indicates the transpose). In this class we will typically use column vectors. The vector may also be written out using unit vector notation to indicate direction. For example:  $\vec{x}_{3 \times 1} = x_1\hat{x} + x_2\hat{y} + x_3\hat{z} = [x_1, x_2, x_3]^T$ , where  $\hat{x}, \hat{y}, \hat{z}$  are unit vectors in the  $x, y, z$  cartesian directions (here the vector's subscript  $3 \times 1$  indicates its dimensions). The type of notation used may depend on the engineering problem you are solving.
3. *Matrix*:  $A = [\vec{a}_1, \vec{a}_2, \vec{a}_3, \dots, \vec{a}_M]_{N \times M} = \{a_{n,m}\}_{N \times M}$ , can be a non-square matrix if the number of elements in each of the vectors ( $N$ ) is not equal to the number of vectors ( $M$ ). When  $M = N$ , the matrix is square. It may be inverted if its determinant  $|A| = \prod \lambda_k \neq 0$  (where  $\lambda_k$  are the eigenvalues).

We shall only work with  $2 \times 2$  and  $3 \times 3$  square matrices throughout this course.

4. *Linear system of equations*:  $A\vec{x} = \vec{b}$  where  $\vec{x}$  and  $\vec{b}$  are vectors and matrix  $A$  is a square.
  - (a) *Inverse*: The solution of this system of equations may be found by finding the inverse  $\vec{x} = A^{-1}\vec{b}$
  - (b) *Equivalence*: If two systems of equations  $A_0\vec{x} = \vec{b}_0$  and  $A_1\vec{x} = \vec{b}_1$  have the same solution (i.e.,  $\vec{x} = A_0^{-1}\vec{b}_0 = A_1^{-1}\vec{b}_1$ ), they are said to be equivalent.
  - (c) *Augmented matrix*: The first type of augmented matrix is defined by combining the matrix with the right-hand-side. For example, given the linear system of equations  $A\vec{x} = \vec{y}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

then the augmented matrix is

$$A|y = \left[ \begin{array}{cc|c} a & b & y_1 \\ c & d & y_2 \end{array} \right]$$

A second type of augmented matrix may be used for finding the inverse of a matrix (rather than solving a specific instance of linear equations  $Ax = b$ ). In this case the augmented matrix is

$$A|I = \left[ \begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array} \right]$$

Performing Gaussian elimination on this matrix, until the left side becomes the identity matrix, yields  $A^{-1}$ . This is because multiplying both sides by  $A^{-1}$  gives  $A^{-1}A|A^{-1}I = I|A^{-1}$ .

5. *Permutation matrix (P)*: A matrix that is equivalent to the identity matrix, but with scrambled rows (or columns). Such a matrix has the properties  $\det(P) = \pm 1$  and  $P^2 = I$ . For the 2x2 case, there is only one permutation matrix:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A permutation matrix  $P$  swaps rows or columns of the matrix it operates on. For example, in the 2x2 case, pre-multiplication swaps the rows

$$PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ a & b \end{bmatrix},$$

whereas post-multiplication swaps the *columns*

$$AP = \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ \beta & \alpha \end{bmatrix}$$

For the 3x3 case there are  $3 \cdot 2 = 6$  such matrices, including the original 3x3 identity matrix (swap a row with the other 2, then swap the remaining two rows).

6. *Gaussian elimination (GE) matrices  $G_k$* : There are 3 types of elementary row operations, which may be performed without fundamentally altering a system of equations (e.g. the resulting system of equations is *equivalent*). These operations are (1) swap rows (e.g. using a permutation matrix), (2) scale rows, or (3) perform addition/subtraction of two scaled rows. All such operations can be performed using matrices.

For lack of a better term, we'll describe these as 'gaussian elimination' or 'GE' matrices.<sup>1</sup> We will categorize any matrix that performs only elementary row operations (but any number of them) as a 'GE' matrix. Therefore, cascade of GE matrices is also a GE matrix.

Consider the GE matrix

$$G = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

- (a) Pre-multiplication scales and adds the *rows*

$$GA = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} a & b \\ a - \alpha & b - \beta \end{bmatrix}$$

The result is a Gaussian elimination operation.

- (b) Post-multiplication adds and scales *columns*.

$$AG = \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} a - b & b \\ \alpha - \beta & \beta \end{bmatrix}$$

Here the second column is subtracted from the first, and placed in the first. The second column is untouched. **This operation is *not* a Gaussian elimination.** Therefore, to

---

<sup>1</sup>The term 'elementary matrix' may also be used to refer to a matrix that performs an elementary row operation. Typically, each elementary matrix differs from the identity matrix by one single row operation. A cascade of elementary matrices could be used to perform Gaussian elimination.

put Gaussian elimination operations in matrix form, we form a cascade of **pre-multiply** matrices.

Here  $\det(G) = 1$ ,  $G^2 = I$ , which won't always be true if we scale by a number greater than

1. For instance, if  $G = \begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix}$  (scale and add), then we have  $\det(G) = 1$ ,  $G^n = \begin{bmatrix} 1 & 0 \\ n \cdot m & 1 \end{bmatrix}$ .

### B.1.1 Problems

Find the solution to the following 3x3 matrix equation  $Ax = b$  by Gaussian elimination. Show your intermediate steps. You can check your work at each step using Matlab.

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 8 \end{bmatrix}.$$

1. Show (i.e., verify) that the first GE matrix  $G_1$ , which zeros out all entries in the first column, is given by

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Identify the elementary row operations that this matrix performs. [See Lab manual for solutions.](#)

2. Find a second GE matrix,  $G_2$ , to put  $G_1A$  in upper triangular form. Identify the elementary row operations that this matrix performs. [See Lab manual for solutions.](#)
3. Find a third GE matrix,  $G_3$ , which scales each row so that its leading term is 1. Identify the elementary row operations that this matrix performs. [See Lab manual for solutions.](#)
4. Finally, find the last GE matrix,  $G_4$ , that subtracts a scaled version of row 3 from row 2, and scaled versions of rows 2 and 3 from row 1, such that you are left with the identity matrix ( $G_4G_3G_2G_1A = I$ ). [See Lab manual for solutions.](#)
5. Solve for  $[x_1, x_2, x_3]^T$  using the augmented matrix format  $G_4G_3G_2G_1[A|b]$  (where  $[A|b]$  is the augmented matrix). Note that if you've performed the preceding steps correctly,  $x = G_4G_3G_2G_1b$ . [See Lab manual for solutions.](#)

## B.2 Gaussian Elimination of a 2x2 matrix

We shall now apply Gaussian elimination to find the solution  $[x_1, x_2]$  for the 2x2 matrix equation  $Ax = y$  (Eq. 3.7, left). We assume to know  $[a, b, c, d]$  and  $[y_1, y_2]$ . We wish to show that the intersection (solution) is given by the equation on the right.

Here we wish to prove that the left equation has an inverse given by the right equation:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

How to take inverse:

- 1) Swap the diagonal, 2) change the signs of the off-diagonal, and 3) divide by  $\Delta$ .

### B.2.1 Derivation of the inverse of a 2x2 matrix

1. Step 1: normalize the first column to 1.
2. Step 2: subtract the top equation from the lower.
3. Step 3: express result in terms of the determinate  $\Delta = ad - bc$ .

$$\begin{bmatrix} 1 & \frac{b}{a} \\ 1 & \frac{d}{c} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{a}y_1 \\ \frac{1}{c}y_2 \end{bmatrix} \quad \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{d}{c} - \frac{b}{a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ -\frac{1}{a} & \frac{1}{c} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

4. Step 4: These steps give the solution for  $x_2$ :  $x_2 = -\frac{c}{\Delta}y_1 + \frac{a}{\Delta}y_2$ .
5. Step 5: Finally the top equation is solved for  $x_1$ :  $x_1 = \frac{1}{a}y_1 - \frac{b}{a}x_2 = x_1 = \frac{1}{a}y_1 - \frac{b}{a}\left[-\frac{c}{\Delta}y_1 + \frac{a}{\Delta}y_2\right]$ .

In matrix format, in terms of the determinate  $\Delta = ab - cd$  becomes:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{a} - \frac{bc}{a\Delta} & \frac{b}{\Delta} \\ -\frac{c}{\Delta} & \frac{a}{\Delta} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \frac{\Delta-bc}{a} & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

**Summary:** This is a lot of algebra, that is why it is essential you memorize the formula for the inverse.

# Appendix C

## Eigenvector analysis

Here we show how to compute the eigenvalues and eigenvectors for the 2x2 Pell matrix

$$\mathbf{A} = \begin{bmatrix} 1 & N \\ 1 & 1 \end{bmatrix}.$$

The analysis applies to any matrix, but since we are concentrated on Pell's equation, we shall use the Pell matrix, for  $N = 2$ . By using a specific matrix we can check all the equations below with Matlab, which I advise you to do.

The Matlab command `[E,D]=eig(A)` returns the eigenvector matrix  $\mathbf{E}$

$$\mathbf{E} = [\mathbf{e}_+, \mathbf{e}_-] = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.8165 & -0.8165 \\ 0.5774 & 0.5774 \end{bmatrix}$$

and the eigenvalue matrix  $\Lambda$  (Matlab's D)

$$\Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} = \begin{bmatrix} 1 + \sqrt{2} & 0 \\ 0 & 1 - \sqrt{2} \end{bmatrix} = \begin{bmatrix} 2.4142 & 0 \\ 0 & -0.4142 \end{bmatrix}.$$

The factor  $\sqrt{3}$  on  $\mathbf{E}$  normalizes each eigenvector to 1 (i.e., The Matlab command `norm([\sqrt{2}, 1])` gives  $\sqrt{3}$ ).

In the following discussion we show how to determine  $\mathbf{E}$  and  $D$  (i.e,  $\Lambda$ ), given  $\mathbf{A}$ .

**Calculating the eigenvalue matrix ( $\Lambda$ ):** The matrix equation for  $\mathbf{E}$  is

$$\mathbf{A}\mathbf{E} = \mathbf{E}\Lambda. \tag{C.1}$$

Pre-multiplying by  $\mathbf{E}^{-1}$  diagonalizes  $\mathbf{A}$ , given the *eigenvalue matrix* ( $D$  in Matlab)

$$\Lambda = \mathbf{E}^{-1}\mathbf{A}\mathbf{E}. \tag{C.2}$$

Post-multiplying by  $\mathbf{E}^{-1}$  recovers  $\mathbf{A}$

$$\mathbf{A} = \mathbf{E}\Lambda\mathbf{E}^{-1}. \tag{C.3}$$

**Matrix power formula:** This last relation is the entire point of the eigenvector analysis, since it shows that any power of  $\mathbf{A}$  may be computed from powers of the eigen values. Specifically

$$\mathbf{A}^n = \mathbf{E}\Lambda^n\mathbf{E}^{-1}. \tag{C.4}$$

For example,  $\mathbf{A}^2 = \mathbf{A}\mathbf{A} = \mathbf{E}\Lambda(\mathbf{E}^{-1}\mathbf{E})\Lambda\mathbf{E}^{-1} = \mathbf{E}\Lambda^2\mathbf{E}^{-1}$ .

Equations C.1, C.2 and C.3 are the key to eigenvector analysis, and you need to memorize them. You will use them repeatedly throughout this course, and for a long time after it is over.

**Showing that  $\mathbf{A} - \lambda_{\pm}\mathbf{I}_2$  is singular:** If we restrict Eq. C.1 to a single eigenvector (one of  $\mathbf{e}_{\pm}$ ), along with the corresponding eigenvalue  $\lambda_{\pm}$ , we obtain a matrix equations

$$\mathbf{A}\mathbf{e}_{\pm} = \mathbf{e}_{\pm}\lambda_{\pm} = \lambda_{\pm}\mathbf{e}_{\pm}$$

Note the important swap in the order of  $\mathbf{e}_{\pm}$  and  $\lambda_{\pm}$ . Since  $\lambda_{\pm}$  is a scalar, this is legal (and critically important), since this allows us to remove (factored out)  $\mathbf{e}_{\pm}$

$$(\mathbf{A} - \lambda_{\pm}\mathbf{I}_2)\mathbf{e}_{\pm} = 0. \quad (\text{C.5})$$

This means that the matrix  $\mathbf{A} - \lambda_{\pm}\mathbf{I}_2$  must be singular, since when it operates on  $\mathbf{e}_{\pm}$ , which is not zero, it gives zero. It immediately follows that its determinant is zero (i.e.,  $|(\mathbf{A} - \lambda_{\pm}\mathbf{I}_2)| = 0$ ). This equation is used to uniquely determine the eigenvalues  $\lambda_{\pm}$ . Note the important difference between  $\lambda_{\pm}\mathbf{I}_2$  and  $\Lambda$  (i.e.,  $|(A - \Lambda)| \neq 0$ ).

**Calculating the eigenvalues  $\lambda_{\pm}$ :** The eigenvalues  $\lambda_{\pm}$  of  $\mathbf{A}$  may be determined from  $|(\mathbf{A} - \lambda_{\pm}\mathbf{I}_2)| = 0$

$$\begin{vmatrix} 1 - \lambda_{\pm} & N \\ 1 & 1 - \lambda_{\pm} \end{vmatrix} = (1 - \lambda_{\pm})^2 - N^2 = 0.$$

For our case of  $N = 2$ ,  $\lambda_{\pm} = (1 \pm \sqrt{2})$ .<sup>1</sup>

**Calculating the eigenvectors  $\mathbf{e}_{\pm}$ :** Once the eigenvalues have been determined, they are substitute them into Eq. C.5, which determines the eigenvectors  $\mathbf{E} = [\mathbf{e}_+, \mathbf{e}_-]$ , by solving

$$(\mathbf{A} - \lambda_{\pm})\mathbf{e}_{\pm} = \begin{bmatrix} 1 - \lambda_{\pm} & 2 \\ 1 & 1 - \lambda_{\pm} \end{bmatrix} \mathbf{e}_{\pm} = 0$$

where  $1 - \lambda_{\pm} = 1 - (1 \pm \sqrt{2}) = \mp\sqrt{2}$ .

Recall that Eq. C.5 is singular, because we are using an eigenvalue, and each eigenvector is pointing in a unique direction (This is why it is singular). You might respectively suggest that this equation has no solution. In some sense you would be correct. When we solve for  $\mathbf{e}_{\pm}$ , the two equations defined by Eq. C.5 *co-linear* (the two equations describe parallel lines). This follows from the fact that there is only one eigenvector for each eigenvalue.

Expecting trouble, yet proceeding to solve for  $\mathbf{e}_+ = [e_1^+, e_2^+]^T$ ,

$$\begin{bmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} e_1^+ \\ e_2^+ \end{bmatrix} = 0$$

This gives two identical equations  $-\sqrt{2}e_1^+ + 2e_2^+ = 0$  and  $e_1^+ - \sqrt{2}e_2^+ = 0$ . This is the price of an over-specified equation (the singular matrix is degenerate). The most we can determine is  $\mathbf{e}_+ = c[-\sqrt{2}, 1]^T$ , where  $c$  is a constant. We can determine eigenvector direction, but not its magnitude.

Following *exactly* the same procedure for  $\lambda_-$ , the equation for  $\mathbf{e}_-$  is

$$\begin{bmatrix} \sqrt{2} & 2 \\ 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} e_1^- \\ e_2^- \end{bmatrix} = 0$$

In this case the relation becomes  $e_1^- + \sqrt{2}e_2^- = 0$ , thus  $\mathbf{e}_- = c[\sqrt{2}, 1]^T$  where  $c$  is a constant.

**Normalization of the eigenvectors:** The two constants may be determined by normalizing the eigenvectors to have unit length. Since we cannot determine the length, we set it to 1. In some sense the degeneracy is resolved by this normalization. Thus  $c = 1/\sqrt{3}$ , since

$$c^2 \left( (\pm\sqrt{2})^2 + 1^2 \right) = 3c^2 = 1.$$

<sup>1</sup>It is a convention to order the eigenvalues from largest to smallest.



**Summary:** Thus far we have shown

$$\mathbf{E} = [\mathbf{e}_+, \mathbf{e}_-] = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} = \begin{bmatrix} 1 + \sqrt{2} & 0 \\ 0 & 1 - \sqrt{2} \end{bmatrix}.$$

**Verify that  $\Lambda = \mathbf{E}^{-1}\mathbf{A}\mathbf{E}$ :** To find the inverse of  $\mathbf{E}$ , 1) swap the diagonal values, 2) change the sign of the off diagonals, and 3) divide by the determinant  $\Delta = 2\sqrt{2}/\sqrt{3}$  (see Appendix B)

$$\mathbf{E}^{-1} = \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & \sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 0.6124 & 0.866 \\ -0.6124 & 0.866 \end{bmatrix}.$$

By definition for any matrix  $\mathbf{E}^{-1}\mathbf{E} = \mathbf{E}\mathbf{E}^{-1} = \mathbf{I}_2$ . Taking the product gives

$$\mathbf{E}^{-1}\mathbf{E} = \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & \sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2.$$

We wish to show that  $\Lambda = \mathbf{E}^{-1}\mathbf{A}\mathbf{E}$

$$\begin{bmatrix} 1 + \sqrt{2} & 0 \\ 0 & 1 - \sqrt{2} \end{bmatrix} = \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & \sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix},$$

which is best verified with Matlab.

**Verify that  $\mathbf{A} = \mathbf{E}\Lambda\mathbf{E}^{-1}$ :** We wish to show that

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 + \sqrt{2} & 0 \\ 0 & 1 - \sqrt{2} \end{bmatrix} \cdot \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & \sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix},$$

which is best verified with Matlab (or Octave).

I suggest that you verify  $\mathbf{E}\Lambda \neq \Lambda\mathbf{E}$  and  $\mathbf{A}\mathbf{E} = \mathbf{E}\Lambda$  with Matlab. Here is the Matlab program which does this:

```
A = [1 2; 1 1]; %define the matrix
[E,D] = eig(A); %compute the eigenvector and eigenvalue matrices
A*E-E*D %This should be $\approx 0$, within numerical error.
E*D-D*E %This is not zero
```

All the equations have been verified both with Matlab and Octave.



# Appendix D

## Solution to Pell's Equation (N=2)

Section 2.2.2 (p. 85) showed that the solution  $[x_n, y_n]^T$  to Pell's equation, for  $N = 2$ , is given by powers of Eq. 1.4. To find an explicit formula for  $[x_n, y_n]^T$ , one must compute powers of

$$\mathbf{A} = 1j \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}. \quad (\text{D.1})$$

We wish to find the solution to Pell's equation (Eq. 1.4), based on the recursive solution, Eq. 1.5 (p. 38). Thus we need is powers of  $A$ , that is  $A^n$ , which gives the a closed form expression for  $[x_n, y_n]^T$ . By the diagonalization of  $A$ , its powers are simply the powers of its eigenvalues. This diagonalization is called an *eigenvalue analysis*, a very general method rooted in linear algebra. This type of analysis allows us to find the solution to most of the linear the equations we encounter.

From Matlab with  $N = 2$  the eigenvalues of Eq. D.1 are  $\lambda_{\pm} \approx [2.4142j, -0.4142j]$  (i.e.,  $\lambda_{\pm} = 1j(1 \pm \sqrt{2})$ ). The final solution to Eq. D.1 is given in Eq. 2.5 (p. 85). The solution for  $N = 3$  is provided in Appendix D.1 (p. 139).

Once the matrix has been diagonalized, one may compute powers of that matrix as powers of the eigenvalues. This results in the general solution given by

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = 1j^n \mathbf{A}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1j^n \mathbf{E} \mathbf{\Lambda}^n \mathbf{E}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The eigenvalue matrix  $D$  is diagonal with the eigenvalues sorted, largest first. The Matlab command `[E,D]=eig(A)` is helpful to find  $D$  and  $E$  given any  $A$ . As we saw above,

$$\mathbf{\Lambda} = 1j \begin{bmatrix} 1 + \sqrt{2} & 0 \\ 0 & 1 - \sqrt{2} \end{bmatrix} \approx \begin{bmatrix} 2.414j & 0 \\ 0 & -0.414j \end{bmatrix}.$$

### D.1 Pell equation for N=3

This summarizes the solution of Pell's equation due to the Pythagoreans using matrix recursion, for the case of N=3. The integer solutions are shown in on the right. Note that  $x_n/y_n \rightarrow \sqrt{3}$ , in agreement with the Euclidean algorithm.<sup>1</sup> It seem likely that  $\beta_0$  could be absorbed in the starting solution, and then be removed from the generating function, other than as the known factor  $\beta_0^n$

Case of  $N = 3$ :  $[x_0, y_0]^T = [1, 0]^T$ ,  $\beta_0 = j/\sqrt{2}$ ; Pell-3:  $x_n^2 - 3y_n^2 = 1$ ;  $x_n/y_n \xrightarrow{\infty} \sqrt{3}$

Try other trivial solutions such as  $[-1, 0]^T$  and  $[\pm j, 0]^T$ . Perhaps this can provide a clue to the proper value of  $\beta_0$ .

---

<sup>1</sup>The matlab program for generating this solution is `PellSol3.m`.

$$\begin{array}{l}
\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \beta_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \beta_0 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \beta_0^2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \beta_0^2 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \beta_0^3 \begin{bmatrix} 10 \\ 6 \end{bmatrix} = \beta_0^3 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \\
\begin{bmatrix} x_4 \\ y_4 \end{bmatrix} = \beta_0^4 \begin{bmatrix} 28 \\ 16 \end{bmatrix} = \beta_0^4 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 6 \end{bmatrix} \\
\begin{bmatrix} x_5 \\ y_5 \end{bmatrix} = \beta_0^5 \begin{bmatrix} 76 \\ 44 \end{bmatrix} = \beta_0^5 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 28 \\ 16 \end{bmatrix}
\end{array}
\qquad
\begin{array}{l}
(1\beta_0)^2 - 3(1\beta_0)^2 = 1 \\
(4\beta_0^2)^2 - 3(2\beta_0^2)^2 = 1 \\
(10\beta_0^3)^2 - 3(6\beta_0^3)^2 = 1 \\
(28\beta_0^4)^2 - 3(16\beta_0^4)^2 = 1 \\
(76\beta_0^5)^2 - 3(44\beta_0^5)^2 = 1
\end{array}$$

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