ECE 298JA **AE #1 – Version 1.29 September 29, 2017** **Fall 2017**

Univ. of Illinois **Due Mon, Oct 2, 2017** **Prof. Allen**

Topic of this homework: Fundamental theorem of algebra, polynomials, analytic functions, convolution, analytic geometry, composition, intersection.

Deliverable: Answers to problems

Note: The term 'analytic' is used in two different ways. (1) An analytic function is a function that may be expressed as a locally convergent power series; (2) analytic geometry refers to geometry using a coordinate system.

1 Polynomials and the fundamental theorem of algebra (FTA)

A polynomial of degree N is defined as

$$P_N(x) = a_0 + a_1 x + a_2 x^2 \cdots a_N x^N$$

1. How many coefficients a_n does a polynomial of degree N have?
2. How many roots does $P_N(x)$ have?
3. The *fundamental theorem of algebra* (FTA)
   (a) State the FTA.
   (b) Using the FTA, *prove* your answer to question (2) above.
4. Consider the polynomial function $P_2(x) = 1 + x^2$ of degree $N = 2$, and the related function $F(x) = 1/P_2(x)$.
   (a) What are the roots (e.g. 'zeros') $x_\pm$ of $P_2(x)$?
   (b) $F(x)$ may be expressed as $(A, B, x_\pm \in \mathbb{C})$

$$F(x) = \frac{A}{x - x_+} + \frac{B}{x - x_-}, \tag{1}$$

   where $x_\pm$ are the roots (zeros) of $P_2(x)$, which become the *poles* of $F(x)$, and A, B are the *residues*. The expression for $F(x)$ is sometimes called a 'partial fraction expansion' or 'residue expansion,' and it appears in many engineering applications.
      i. Find $A, B \in \mathbb{C}$ in terms of the roots $x_\pm$ of $P_2(x)$.
      ii. Verify your answers for A, B by showing that this expression for $F(x)$ is indeed equal to $1/P_2(x)$.
   (c) The *poles* of a function $G(x)$ are defined as values x_p where $G(x_p) \to \infty$; the *zeros* are defined as values x_z where $G(x_z) = 0$.
   Hint: Do not forget to consider $f(x)$ as $x \to \pm\infty$
      i. Give the values of the poles and zeros of $P_2(x)$.
      ii. Give the values of the poles and zeros of $F(x) = 1/P_2(x)$.

2 Analytic functions

Analytic functions are defined by infinite (power) series. The function $f(x)$ is analytic at any value of $x = x_0$ where there exists a convergent power series $P(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $P(x_0) = f(x_0)$. The local power series for $f(x)$ near $x = x_0$ is often obtained by finding the *Taylor series*:

$$
\begin{aligned}
f(x) &\approx f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0}(x - x_0) + \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=x_0}(x - x_0)^2 + \ldots \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n}{dx^n} f(x)\right|_{x=x_0} (x - x_0)^n
\end{aligned}
$$

The point $x = x_0$ is called the series expansion point.

When the expansion point is at $x_0 = 0$, the series is denoted a *MacLaurin series*. Two classic examples are the *geometric series*[1]

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n \tag{1}$$

and the exponential function

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{2}$$

These expressions may both be derived as the Taylor (MacLaurin) series about $x = 0$.

1. The geometric series

   (a) What is the region of convergence (ROC) for the power series of $1/(1-x)$ given above (e.g. where does the power series $P(x)$ converge to the function value $f(x)$)? State your answer as a condition on x. *Hint: What happens to the power series when $x > 1$?*

   (b) How does the ROC relate to the location of the pole of $1/(1-x)$?

   (c) Assuming x is in the ROC, prove that the geometric series correctly represents $1/(1-x)$ by multiplying both sides of Eq. 1 by $(1-x)$.

   (d) Use the geometric series to study the degree N polynomial (It is very important to note that all the coefficients of this polynomial are 1)

   $$P_N(x) = 1 + x + x^2 + \ldots + x^N = \sum_{n=0}^{N} x^n. \tag{3}$$

      i. Prove that

      $$P_N(x) = \frac{1 - x^{N+1}}{1 - x} \tag{4}$$

      ii. How many poles does $P_N(x)$ have? Where are they?

      iii. How many zeros does $P_N(x)$ have? Where are they?

      iv. Explain why Eq. 3 and 4 have different numbers of poles and zeros.

   (e) Is the function $1/(1-x)$ analytic outside of the ROC stated in part (a)? *Hint: Can it be represented by a different power series outside this ROC?*

[1] The geometric series is *not* defined as the function $1/(1-x)$, it is defined as the series $1 + r + r^2 + r^3 + \ldots$, such that the ratio of consecutive terms is r.

2. The exponential series

   (a) What is the region of convergence (ROC) for the exponential series given above (e.g. where does the power series $P(x)$ converge to the function value $f(x)$)?

   (b) Let $x = j$ in Eq. 2, and write out the series expansion of e^x in terms of its real and imaginary parts.

   (c) Let $x = j\theta$ in Eq. 2, and write out the series expansion of e^x in terms of its real and imaginary parts. How does your result relate to Euler's identity ($e^{j\theta} = \cos(\theta) + j\sin(\theta)$)?

3 Inverse analytic functions and composition

It may be surprising, but every analytic function has an inverse function. Starting from the function ($x, y \in \mathbb{C}$)

$$y(x) = \frac{1}{1-x}$$

the inverse is

$$x = \frac{y-1}{y} = 1 - \frac{1}{y}.$$

1. Considering the inverse function described above

   (a) Where are the poles and zeros of $x(y)$?

   (b) Where (for what condition on y) is $x(y)$ analytic?

2. Considering the exponential function $z(x) = e^x$ $(x, z \in \mathbb{C})$

   (a) Find the inverse $x(z)$.

   (b) Where are the poles and zeros of $x(z)$?

3. Compose these two functions $(y \circ z)(x)$

   (a) Give the expression for $(y \circ z)(x) = y(z(x))$.

   (b) Where are the poles and zeros of $(y \circ z)(x)$?

   (c) Where (for what condition on x) is $(y \circ z)(x)$ analytic?

4 Convolution

Multiplying two polynomials, when they are short or simple, is not demanding. However if they have many terms, it can become tedious. For example, multiplying two 10^{th} degree polynomials is not something one would want to do every day.

An alternative is a method called convolution, as described in Lecture 13.

1. Convolution of sequences

   Practice convolution (by hand!!) using a few simple examples. Show you work!!! You may check your solution using Matlab.

   (a) Convolve the sequence [0 1 1 1 1] with itself.

   (b) Convolve [1 1] with itself, then convolve the result with [1 1] again (e.g., calculate $[1, 1] \star [1, 1] \star [1, 1]$).

2. Multiplication of polynomials

   In class, it was shown that multiplying two polynomials is the same as convolving their coefficients. Consider

$$f(x) = x^3 + 3x^2 + 3x + 1$$
$$g(x) = x^3 + 2x^2 + x + 2$$

   In Matlab, compute $h(x) = f(x) \cdot g(x)$ two ways using (a) the commands `roots` and `poly`, and (b) the convolution command `conv`. Confirm that both methods give the same result. That is, compute the convolution $[1, 3, 3, 1] \star [1, 2, 1, 2]$.

   What is $h(x)$?

5 Intersection and analytic geometry

To find the Euclid's formula, it was necessary to study the intersection of a circle and a secant line. Consider the unit circle of radius 1, centered at $(x, y) = (0, 0)$

$$x^2 + y^2 = 1$$

and the line through $(-1, 0)$

$$y = t(x + 1).$$

If $0 < t < 1$, the line intersects the circle at a second point $(+x, +y)$ in the positive x, y quadrant.

1. Draw the circle and the line, given a positive slope $0 < t < 1$.
2. Substitute $y = t(x + 1)$ (the line equation) into the equation for the circle, and solve for $x(t)$. *Hint: Because the line intersects the circle at two points, you will get two solutions for x. One of these solutions is the trivial solution $x = -1$.*
3. Substitute the $x(t)$ you found back into the line equation, and solve for $y(t)$.
4. Let $t = q/p$ be a rational number, where p and q are integers. Find $x(p, q)$ and $y(p, q)$.
5. Substitute $x(p, q)$ and $y(p, q)$ into the equation for the circle, and show how Euclid's formula for the Pythagorean triples is generated.

6 Newton's root-finding method (Extra credit)

Newton used the iteration[2]

$$x_{n+1} = x_n - \frac{P_N(x_n)}{P'_N(x_n)}$$

to find roots of the polynomial $P_N(x_n)$. Here $P'_N(x) = dP_N(x)/dx$. This relation may be explored as a graph, which puts Newton's method in the realm of analytic geometry. The function $P'_N(x)$ is the slope of the polynomial $P_N(x)$ at x_n. The value of x_n is the estimate of the root after n iterations. x_0 is the initial guess.

Example: When the polynomial is $P_2 = 1 - x^2$, so $P'_2(x) = -2x$ Newton's iteration becomes

$$x_{n+1} = x_n + \frac{1 - x_n^2}{2x_n}.$$

[2] `https://en.wikipedia.org/wiki/Newton's_method`

To start the iteration ($n = 0$) we need an initial guess for x_0, which is a "best guess" of where the root will be. If we let $x_0 = 1/2$, then

$$x_1 = x_0 - \frac{1 - x_0^2}{2x_0} = x_0 + \frac{1}{2}\left(x_0 - 1/x_0\right).$$

1. Let $P_2(x) = 1 - x^2$, and $x_0 = 1/2$. Draw a graph describing the first step of the iteration.
2. Calculate x_1 and x_2. What number is the algorithm approaching? Is it a root of P_2?
3. Write a Matlab script to check your answer for part (a).
   (a) For $n = 4$, what is the absolute difference between the root and the estimate, $|x_r - x_4|$?
   (b) What happens if $x_0 = -1/2$?
4. Does Newton's method work for $P_2(x) = 1 + x^2$? Why?[3] Hint: What are the roots in this case?
5. What if you let $x_0 = (1 + \jmath)/2$ for the case of $P_2(x) = 1 + x^2$?

[3] https://en.wikipedia.org/wiki/Newton's_method#Complex_functions