ECE 298JA **AE #2 − Version 1.34 October 4, 2017** **Fall 2017**

Univ. of Illinois **Due Wed, Oct 11, 2017** **Prof. Allen**

Topic of this homework: Linear systems of equations; Gaussian elimination; Matrix permutations; Overspecified systems of equations; Analytic Geometry; Ohm's law; Two-port networks

Deliverable: Answers to problems

1 Nonlinear (quadratic) to linear equations

In the following problems we deal with algebraic equations in more than one variable, that are not linear equations. For example, the circle $x^2 + y^2 = 1$ is just such an equation. It may be solve for $y(x) = \pm\sqrt{1 - x^2}$.

If we let $z_+ = x + y\jmath = x + \jmath\sqrt{1 - x^2} = e^{\theta\jmath}$, we obtain the equation for half a circle ($y > 0$). The entire circle is described by the magnitude of z, as $|z|^2 = (x + y\jmath)(x - y\jmath) = 1$.

1. Given the curve defined by the equation:
$$x^2 + xy + y^2 = 1 \tag{1}$$
   (a) Find the function $y(x)$.
   (b) Using matlab/octave, plot $y(x)$, and describe the graph.
   (c) What is the name of this curve?
2. Find the solution (in x, p, and q) to the following equations:
$$\begin{aligned} x + y &= p \\ xy &= q \end{aligned}$$
3. Find an equation that is linear in y starting from equations that are quadratic (2^{nd} degree) in the two unknowns[1] x, y:
$$x^2 + xy + y^2 = 1 \tag{2}$$
$$4x^2 + 3xy + 2y^2 = 3 \tag{3}$$

[1] This problem is taken from Stillwell, Exercise 6.2.1 (p. 91).

2 Gaussian elimination

Definitions (Appendix A, Class Notes):

1. *Scalar*: A number, e.g. $\{a, b, c, \alpha, \beta, \cdots\} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{I}, \mathbb{R}, \mathbb{C}\}$

2. *Vector*: A quantity having direction as well as magnitude, often denoted by a bold-face letter with an arrow, $\vec{\mathbf{x}}$. In matrix notation, this is typically represented as a single row $[x_1, x_2, x_3, \ldots]$ or single column $[x_1, x_2, x_3 \ldots]^T$ (where T indicates the transpose). In this class we will typically use column vectors. The vector may also be written out using unit vector notation to indicate direction. For example: $\vec{\mathbf{x}}_{3\times 1} = x_1\hat{x} + x_2\hat{y} + x_3\hat{z} = [x_1, x_2, x_3]^T$, where $\hat{x}, \hat{y}, \hat{z}$ are unit vectors in the x, y, z cartesian directions (here the vector's subscript 3×1 indicates its dimensions). The type of notation used may depend on the engineering problem you are solving.

3. *Matrix:* $A = \left[\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \vec{\mathbf{a}}_3, \cdots, \vec{\mathbf{a}}_M\right]_{N\times M} = \{a_{n,m}\}_{N\times M}$, can be a non-square matrix if the number of elements in each of the vectors (N) is not equal to the number of vectors (M). When $M = N$, the matrix is square. It may be inverted if its determinant $|A| = \prod \lambda_k \neq 0$ (where λ_k are the eigenvalues).

   We shall only work with 2×2 and 3×3 square matrices throughout this course.

4. *Linear system of equations*: $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ where $\vec{\mathbf{x}}$ and $\vec{\mathbf{b}}$ are vectors and matrix A is a square.

   (a) *Inverse:* The solution of this system of equations may be found by finding the inverse $\vec{\mathbf{x}} = A^{-1}\vec{\mathbf{b}}$

   (b) *Equivalence:* If two systems of equations $A_0\vec{\mathbf{x}} = \vec{\mathbf{b}}_0$ and $A_1\vec{\mathbf{x}} = \vec{\mathbf{b}}_1$ have the same solution (i.e., $\vec{\mathbf{x}} = A_0^{-1}\vec{\mathbf{b}}_0 = A_1^{-1}\vec{\mathbf{b}}_1$), they are said to be equivalent.

   (c) *Augmented matrix:* The first type of augmented matrix is defined by combining the matrix with the right-hand-side. For example, given the linear system of equations $A\vec{\mathbf{x}} = \vec{\mathbf{y}}$

   $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

   then the augmented matrix is

   $$A|y = \left[\begin{array}{cc|c} a & b & y_1 \\ c & d & y_2 \end{array}\right]$$

   A second type of augmented matrix may be used for finding the inverse of a matrix (rather than solving a specific instance of linear equations $Ax = b$). In this case the augmented matrix is

   $$A|I = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

   Performing Gaussian elimination on this matrix, until the left side becomes the identity matrix, yields A^{-1}. This is because multiplying both sides by A^{-1} gives $A^{-1}A|A^{-1}I = I|A^{-1}$.

5. *Permutation matrix* (P): A matrix that is equivalent to the identity matrix, but with scrambled rows (or columns). Such a matrix has the properties $det(P) = \pm 1$ and $P^2 = I$. For the 2x2 case, there is only one permutation matrix:

   $$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A permutation matrix P swaps rows or columns of the matrix it operates on. For example, in the 2x2 case, pre-multiplication swaps the rows

$$PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ a & b \end{bmatrix},$$

whereas post-multiplication swaps the *columns*

$$AP = \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ \beta & \alpha \end{bmatrix}$$

For the 3x3 case there are $3 \cdot 2 = 6$ such matrices, including the original 3x3 identity matrix (swap a row with the other 2, then swap the remaining two rows).

6. *Gaussian elimination (GE) matrices* G_k: There are 3 types of elementary row operations, which may be performed without fundamentally altering a system of equations (e.g. the resulting system of equations is *equivalent*). These operations are (1) swap rows (e.g. using a permutation matrix), (2) scale rows, or (3) perform addition/subraction of two scaled rows. All such operations can be performed using matrices.

   For lack of a better term, we'll describe these as 'gaussian elimination' or 'GE' matrices.[2] We will categorize any matrix that performs only elementary row operations (but any number of them) as a 'GE' matrix. Therefore, cascade of GE matrices is also a GE matrix.

   Consider the GE matrix
$$G = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

   (a) Pre-multiplication scales and adds the *rows*

$$GA = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} a & b \\ a - \alpha & b - \beta \end{bmatrix}$$

   The result is a Gaussian elimination operation.

   (b) Post-multiplication adds and scales *columns.*

$$AG = \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} a - b & b \\ \alpha - \beta & \beta \end{bmatrix}$$

   Here the second column is subtracted from the first, and placed in the first. The second column is untouched. **This operation is *not* a Gaussian elimination**. Therefore, to put Gaussian elimination operations in matrix form, we form a cascade of **pre-multiply** matrices.

   Here $\det(G) = 1$, $G^2 = I$, which won't always be true if we scale by a number greater than 1. For instance, if $G = \begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix}$ (scale and add), then we have $\det(G) = 1$, $G^n = \begin{bmatrix} 1 & 0 \\ n \cdot m & 1 \end{bmatrix}$.

[2]The term 'elementary matrix' may also be used to refer to a matrix that performs an elementary row operation. Typically, each elementary matrix differs from the identity matrix by one single row operation. A cascade of elementary matrices could be used to perform Gaussian elimination.

2.1 Problems

Find the solution to the following 3x3 matrix equation $Ax = b$ by Gaussian elimination. Show your intermediate steps. You can check your work at each step using Matlab.

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 8 \end{bmatrix}.$$

1. Show (i.e., verify) that the first GE matrix G_1, which zeros out all entries in the first column, is given by

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

   Identify the elementary row operations that this matrix performs.

2. Find a second GE matrix, G_2, to put $G_1 A$ in upper triangular form. Identify the elementary row operations that this matrix performs.

3. Find a third GE matrix, G_3, which scales each row so that its leading term is 1. Identify the elementary row operations that this matrix performs.

4. Finally, find the last GE matrix, G_4, that subtracts a scaled version of row 3 from row 2, and scaled versions of rows 2 and 3 from row 1, such that you are left with the identity matrix ($G_4 G_3 G_2 G_1 A = I$).

5. Solve for $[x_1, x_2, x_3]^T$ using the augmented matrix format $G_4 G_3 G_2 G_1 [A|b]$ (where $[A|b]$ is the augmented matrix). Note that if you've performed the preceding steps correctly, $x = G_4 G_3 G_2 G_1 b$.

3 Two linear equations

In this exercise we transition from a general pair of equations

$$\begin{aligned} &\text{f(x,y)=0} \\ &\text{g(x,y)=0} \end{aligned}$$

to the important case of *two linear equations*

$$\begin{aligned} y &= ax + b \\ y &= \alpha x + \beta. \end{aligned}$$

Note that, to help keep track of the variables, Roman coefficients (a, b) are used for the first equation and Greek (α, β) for the second.

1. What does it mean, graphically, if these two linear equations have
   (a) a unique solution,
   (b) a non-unique solution, or
   (c) no solution?
2. Assuming the two equations have a unique solution, find the solution for x and y.
3. When will this solution fail to exist (for what conditions on a, b, α, and β)?
4. Write the equations as a 2x2 matrix equation of the form $A\vec{x} = \vec{b}$, where $\vec{x} = [x, y]^T$.
5. Finding the inverse of the 2x2 matrix, and solve the matrix equation for x and y.
6. Discuss the properties of the determinant of the matrix (Δ) in terms of the slopes of the two equations (a and α).
7. An application of *linear functional relationships* between *two variables*:

   2x2 matrices are used to describe 2-port networks, as will be discussed in Lec 16. Transmission lines are a great example, where both voltage and current must be tracked as they travel along the line. Figure 1 shows an example segment of a transmission line.

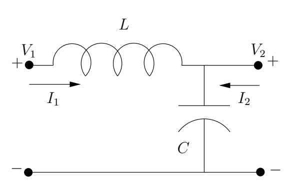


Figure 1: *This figure shows a cell from an LC transmission line. The index* 1 *is at the input on the left and 2 represents the output, on the right.*

   Suppose you are given the following pair of linear relationships between the input (source) variables V_1 and I_1, and the output (load) variables V_2 and I_2 of the transmission line.

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \jmath & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}.$$

   (a) Let the output (the load) be $V_2 = 1$ and $I_2 = 2$ (i.e., $V_2/I_2 = 1/2$ $[\Omega]$). Find the input voltage and current, V_1 and I_1.
   (b) Let the input (source) be $V_1 = 1$ and $I_1 = 2$. Find the output voltage and current V_2 and I_2.

4 Linear equations with three unknowns

This problem is similar to the previous problem, except we consider 3 dimensions. Consider two linear equations in unknowns x, y, z, representing planes:

$$a_1 x + b_1 y + z = c_1 \tag{1}$$
$$a_2 x + b_2 y + z = c_2 \tag{2}$$

1. In terms of the geometry (i.e., think graphically), under what conditions do these two linear equations have (a) a unique solution, (b) a non-unique solution, or (c) no solution?

2. Given 2 equations in 3 unknowns, the closest we can come to a 'unique' solution is an equation in (x, y), (y, z), or (x, z). Find a solution in terms of x and y by substituting one equation into the other.

3. Now consider the intersection of the planes at some arbitrary constant height, $z = z_0$. Write the modified plane equations as a 2x2 matrix equation in the form $A\vec{x} = \vec{b}$ where $\vec{x} = [x, y]^T$, and find the unique solution in x and y using matrix operations.

4. When will this solution fail to exist (for what conditions on a_1, a_2, b_1, b_2, etc.)?

5. Now, write the system of equations as a 3x3 matrix equation in x, y, z given the additional equation $z = z_0$ (e.g. put it in the form $A\vec{x} = \vec{b}$ where $\vec{x} = [x, y, z]^T$).

6. The determinant of a 3x3 matrix is given by

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

   For the 3x3 matrix equation you wrote in the previous part, find the determinant. How is the determinant related to the 2x2 case? Why?

7. Put the following systems of equations in matrix form, and use Matlab to find (i) the determinant of the matrix, (ii) the matrix inverse, and (iii) the solution (x, y, z). **If it is not possible to complete (i-iii), state why.**

   (a)

$$\begin{aligned} x + 3y + 2z &= 1 \\ x + 4y + z &= 1 \\ x + y &= 1 \end{aligned}$$

   (b)

$$\begin{aligned} x + 3y + 2z &= 1 \\ 2x + 6y + 4z &= 1 \\ x + y &= 1 \end{aligned}$$

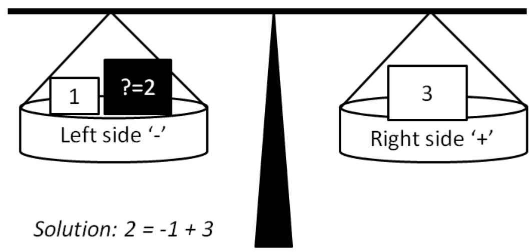
5 Integer equations: applications and solutions (20 pts)

Any equation for which we seek only integer solutions is called a *Diophantine* equation.

5.1 A practical example of using a Diophantine equation

"A merchant had a 40-pound weight that broke into 4 pieces. When the pieces were weighed, it was found that each piece was a whole number of pounds and that the four pieces could be used to weigh every integral weight between 1 and 40 pounds. What were the weights of the pieces?" - *Bachet de Bèziriac (1623 CE)*[3]

Here, weighing is performed using a balance scale having two pans, with weights being put on either pan. Thus, given weights of 1 and 3 pounds, one can weigh a 2-pound weight by putting the 1-pound weight in the same pan with the 2-pound weight, and the 3-pound weigh in the other pan. Then, the scale will be balanced. A solution to the four weights for Bachet's problem is $\mathbf{1 + 3 + 9 + 27 = 40}$ pounds.



Problem: Show how the combination of 1, 3, 9, & 27 pound weights may be used to weigh **1,2,3,...8, 28, and 40** pounds of milk (or something else, such as flour). Assuming that the milk is in the *left* pan, provide the position of the weights using a negative sign '-' to indicate the left pan and a positive sign '+' to indicate the right pan. For example, if the left pan has 1 pound of milk, then 1 pound of milk in the right pan, '+1' will balance the scales.

Hint: It is helpful to write the answer in matrix form. Set the vector of values to be weighed equal to a matrix indicating the pan assignments, multiplied by a vector of the weights $[1, 3, 9, 27]^T$. The pan assignments matrix should only contain the values -1 (left pan), +1 (right pan), and 0 (leave out). You can indicate these using '-', '+', and blank spaces.

[3]Taken from: Joseph Rotman, "A first course in abstract algebra," *Chapter 1, Number Theory* p. 50

6 Ohm's Law

In general, impedance is defined as the ratio of a force over a flow. For electrical circuits, the voltage is the 'force' and the current is the 'flow.' Ohm's law states that the voltage across and the current through a circuit element are related by the *impedance* of that element (which may be a function of frequency). For resistors, the voltage over the current is called the *resistance*, and is a constant (e.g. the simplest case, $V/I = R$). For inductors and capacitors, the voltage over the current is a frequency-dependent impedance (e.g. $V/I = Z(s)$, where s is the complex frequency $s \in \mathbb{C}$).

The impedance concept also holds in mechanics and acoustics. In mechanics, the 'force' is equal to the mechanical force on an element (e.g. a mass, dashpot, or spring), and the 'flow' is the velocity. In acoustics, the 'force' is pressure, and the 'flow' is the volume velocity or particle velocity of air molecules.

Case	Force	Flow	Impedance	units
Electrical	voltage (V)	current (I)	Z	Ohms [Ω]
Mechanics	force (F)	velocity (V)	Z	Mechanical Ohms [Ω]
Acoustics	pressure (P)	particle velocity (U)	Z	Acoustic Ohms [Ω]
Thermal	temperature (T)	heat-flux (J)	Z	Thermal Ohms [Ω]

1. The resistance of a lightbulb, measured cold, is about 100 ohms. As it lights up, the resistance of the metal filiment increases. Ohm's law says that the current

$$\frac{V}{I} = R(T).$$

   where T is the temperature. In the United States, the voltage is 120 volts (RMS) at 60 [Hz]. Find the current when the light is first switched on.

2. The power, in Watts [W], is the product of the force and the flow. What is the power of the light bulb of this example?

3. State the impedance $Z(s)$ of each of the following circuit elements:
   (a) A resistor with resistance R
   (b) An inductor with inductance L
   (c) A capacitor with capacitance C

7 2-port network analysis

Perform a simple analysis of electrical 2-port networks, shown in Figure 1. This can be a mechanical system if the capacitors are taken to be springs, and inductors taken as mass, as in the suspension of the wheels of a car. In an acoustical circuit, the low-pass filter could be a car muffler. While the physical representations will be different, the equations and the analysis are exactly the same.

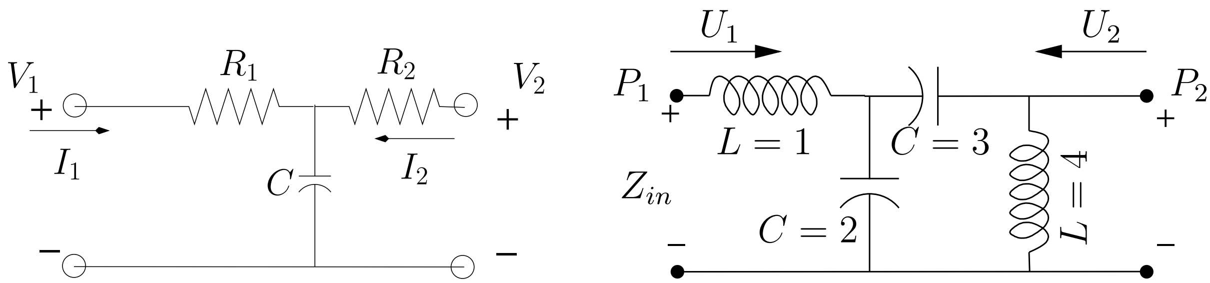


Figure 1: **Left: A lowpass RC electrical filter.** *The circuit elements R_1, R_2, and C are defined in the problems below.* **Right: A band-pass acoustic filter.** *Here, the pressure P is analogous to voltage, and the velocity U is analogous to current. The circuit elements are labeled with their L and C values as integers, to make the algebra simple.*

The definition of the ABCD *transmission matrix* (T) is

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \tag{1}$$

The *impedance matrix*, where the determinant $\Delta_T = AD - BC$, is given by

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{C} \begin{bmatrix} A & \Delta_T \\ 1 & D \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{2}$$

1. Derive the formula for the impedance matrix (Eq. 2) given the transmission matrix definition (Eq. 1). Show your work.

2. Consider a single circuit element with impedance $Z(s)$
   (a) What is the ABCD matrix for this element if it is in 'series'?
   (b) What is the ABCD matrix for this element if it is 'shunt'?

3. Find the ABCD matrix for each of the circuits of Figure 1. For each circuit, **(i)** show the *cascade* of transmission matrices in terms of the complex frequency $s \in \mathbb{C}$, then **(ii)** substitute $s = 1j$ and calculate the total transmission matrix at this single frequency.
   (a) Left circuit (let $R_1 = R_2 = 10$ kΩ 'kilo-ohms', and $C = 10$ nF 'nano-farads')
   (b) Right circuit (use L and C values given in the figure), where the pressure P is analogous to the voltage V, and the velocity U is analogous to the current I.

4. Convert both transmission (ABCD) matrices to impedance matrices using Equation 2. Do this for the specific frequency $s = 1j$, as in the previous part (feel free to use Matlab for your computation).