ECE 298JA **AE #3 – Version 1.23 October 12, 2017** **Fall 2017**

Univ. of Illinois **Due Mon, Oct 16, 2017** **Prof. Allen**

Topic of this homework: Visualizing complex functions; Bilinear/Möbius transform; Riemann sphere.

1 Algebra with complex variables

1. One can always say that 3<4, namely that real numbers have *order*. One way to view this is to take the difference, and compare to zero, as in $4 - 3 > 0$. Here, we will explore how complex variables may be ordered. Define the complex variable $z = x + iy \in \mathbb{C}$.

   (a) Explain the meaning of $|z_1| > |z_2|$.

   (b) If $x_1, x_2 \in \mathbb{R}$ (are *real* numbers), define the meaning of $x_1 > x_2$. *Hint: Take the difference.*

   (c) Explain the meaning of $z_1 > z_2$.

   (d) (*not graded*) If time were complex how might the world be different?

2. It is sometimes necessary to consider a function $w(z) = u + iv$ in terms of the real functions, $u(x, y)$ and $v(x, y)$ (e.g. separate the real and imaginary parts). Similarly, we can consider the inverse $z(w) = x + iy$ where $x(u, v)$ and $y(u, v)$ are real functions.

   (a) Find $u(x, y)$ and $v(x, y)$ for $w(z) = 1/z$.

   (b) Find $u(x, y)$ and $v(x, y)$ for $w(z) = c^z$ with complex constant $c \in \mathbb{C}$ for the following cases

      i. $c = e$

      ii. $c = 1$ (recall that $1 = e^{i2\pi k}$ for $k = 0, 1, 2, \ldots$)

   (c) Find $u(x, y)$ for $w(z) = \sqrt{z}$. *Hint: Begin with the inverse function* $z = w^2$.

2 Visualizing complex functions

The mapping from $z = x + iy$ to $w(z) = u(x, y) + iv(x, y)$ is a $2 \cdot 2 = 4$ dimensional graph. This is difficult to visualize, because for each point in the domain z, we would like to represent both the magnitude and phase (or real and imaginary parts) of $w(z)$. A good way to visualize these mappings is to use color (hue) to represent the phase and intensity (dark to light) to represent the magnitude.[1] The Matlab program `zviz.m` has been provided to do this (see Lecture 17 on the class website):
http://jontalle.web.engr.illinois.edu/uploads/298.17/zviz.zip

To use the program in Matlab/Octave, use the syntax `zviz <function of z>` (for example, type `zviz z.^2`). Note the period between z and `^2`. This will render a 'domain coloring' (aka colorized) version of the function. Examples you can render with `zviz` are given in the comments at the top of the `zviz.m` program. A good example for testing is `zviz z-sqrt(j)`, which should show a dark spot (a zero) at $(1 + 1\jmath)/\sqrt{2} = 0.707(1 + 1\jmath)$.

Example: Figure 1 shows a colorized plot of $w(z) = \sin(\pi(z - i)/2)$ due to the Matlab/Octave command `zviz sin(pi*(z-i)/2)`. The abscissa (horizontal axis) is the real x axis and the ordinate (vertical axis) is the complex iy axis. The graph is offset along the ordinate axis by $1i$, since the argument $z - i$ causes a shift of the sine function by 1 in the positive imaginary direction. The visable

[1] This is also called 'domain coloring': `https://en.wikipedia.org/wiki/Domain_coloring`

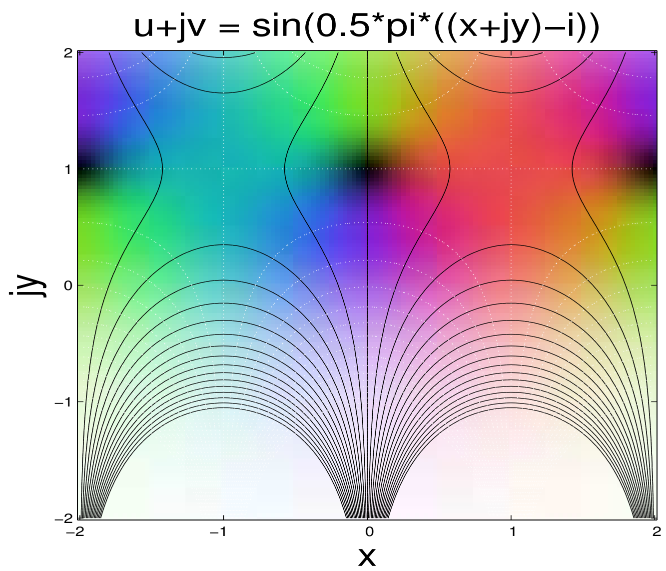

Figure 1: *Plot of* $\sin(0.5\pi(z-i))$.

zeros of $w(z)$ appear as dark regions at $(-2,1),(0,1),(2,1)$. As a function of x, $w(x+1\jmath)$ oscillates between red (phase is zero degrees), meaning the function is positive and real, and sea-green (phase is $180°$), meaning the function is negative and real.

Along the vertical axis, the function is either a $\cosh(y)$ or $\sinh(y)$, depending on x. The intensity becomes lighter as $|w|$ increases, and white as $w \to \infty$. the intensity becomes darker as $|w|$ decreases, and black as $w \to 0$.

To do: For the following functions, explain what you see (i.e., understand) about the `zviz` plot of each function. Do <u>not</u> include printouts of the plots with your homework. Write down your observations, in prose.

1. $w(z) = z$. Summarize how the color (hue) relate to the phase of z and how the color intensity relate to the magnitude of z?)
2. z^2
3. e^z
4. $\cos(\pi z/2)$
5. $\cosh(\pi z)$

3 Fundamental theorem of algebra (FTA)

1. State the fundamental theorem of algebra (FTA).

4 Möbius transforms and infinity

The bilinear transform: The *bilinear z transform* (a specific case of the Möbius transformation) is used in signal processing to design a digital (discrete-time) filter $H(z)$ given an analog (continuous time) filter $H(s)$. The goal of the transform is to take a function of analog frequency ω_a, where $\omega_a \in (-\infty, \infty)$, and map it to a finite digital frequency range, $\omega_d \in [-\pi, \pi]$. You will learn more about this if you take ECE 310.

The bilinear z transform is expressed in terms of the complex Laplace frequency $s \equiv \sigma_a + j\omega_a$, where ω_a is the analog frequency in radians/second, and $z \equiv \rho e^{j\omega_d}$, where $\rho = |z|$ and ω_d is the digital frequency (it is an angle, in radians). The bilinear transform is given by

$$s = \alpha \frac{1 - z^{-1}}{1 + z^{-1}}, \tag{1}$$

where α is a real constant.

1. Suppose you are given the analogue low-pass filter $h(t) = e^{-t}u(t)$, which has a frequency response given by

   $$H(s) = \frac{1}{s+1} = \int_{0^-}^{\infty} h(t)e^{-st}dt,$$

   where $s = \sigma + \omega \jmath$.

   Use the bilinear z transform (Eq. 1) to find the discrete time filter $H(z)$. *Hint: Look at matlab/Octave command* `help bilinear`. *Your answer should be a composition of $H(s)$ and Eq. 1.*

2. Substitute $s = j\omega_a$ and $z = e^{j\omega_d}$ ($\sigma_a, \sigma_d = 0$) into the Eq. 1 to determine the relationship between ω_a, ω_d. Express your final result using a tangent function. *Hint: Try to form sine and cosine terms!* Recall that $\sin(\omega) = (e^{j\omega} - e^{-j\omega})/2j$ and $\cos(\omega) = (e^{j\omega} + e^{-j\omega})/2$.

3. By hand, draw a graph of the relationship you found the previous part, $\omega_a = f(\omega_d)$. Make sure to specify the behavior of ω_a at $\omega_d = 0, \pm\pi/2, \pm\pi$.

4. Explain how this relationship maps the analog frequency $\omega_a \to \pm\infty$ to a the digital frequency ω_d.

5. Draw the s and z planes, showing the real parts on the horizontal axes and the imaginary parts on the vertical axes. Mark (e.g. using thick lines) which sets of values are considered when $\sigma_a, \sigma_d = 0$.

6. Geometrically, what is the effect of this Möbius transform? Consider your drawing in the previous part.