ECE 298JA   **DE #1 − Version 3.05 November 7, 2017**   **Fall 2017**

Univ. of Illinois   **Due Mon, Oct 30**   **Prof. Allen**

Topic of this homework: Complex numbers and functions (ordering and algebra); Complex power series; Fundamental theorem of calculus (real and complex); Cauchy-Riemann conditions; Multivalued functions (branch cuts and Riemann sheets)

1 Two fundamental theorems of calculus

Fundamental Theorem of Calculus (Leibniz): According to the Fundamental Theorem of (Real) Calculus (FTC)

$$F(x) = F(a) + \int_a^x f(\xi)d\xi, \tag{1}$$

where $x, a, \xi, F \in \mathbb{R}$. This is an *indefinite integral* (since the upper limit is unspecified). It follows that

$$\frac{dF(x)}{dx} = \frac{d}{dx}\int_a^x f(x)dx = f(x).$$

This justifies also calling the indefinite integral the *anti-derivative*.

For a closed interval $[a, b]$, the FTC is

$$\int_a^b f(x)dx = F(b) - F(a), \tag{2}$$

thus the integral is independent of the path from $x = a$ to $x = b$.

Fundamental Theorem of *Complex* Calculus: According to the Fundamental Theorem of Complex Calculus (FTCC)

$$f(z) = f(z_0) + \int_{z_0}^z F(\zeta)d\zeta, \tag{3}$$

where $z_0, z, \zeta, F \in \mathbb{C}$. It follows that

$$\frac{df(z)}{dz} = \frac{d}{dz}\int_{z_0}^z F(\zeta)d\zeta = F(z).$$

To do:

1. Consider Equation 1. What is the condition on $f(x)$ for which this formula is true?

2. Consider Equation 3. What is the condition on $f(z)$ for which this formula is true?

3. Perform the following integrals ($z = x + iy \in \mathbb{C}$)

   (a) $I = \int_0^{1+j} zdz$

   (b) $I = \int_0^{1+j} zdz$, but this time make the path explicit: from 0 to 1, with y=0, and then to y=1, with x=1.

   (c) Do your results agree with Equation 2?

4. Perform the following integrals on the closed path C, which we define to be the unit circle. You should substitute $z = e^{i\theta}$ and $dz = ie^{i\theta}d\theta$, and integrate from $[-\pi, \pi]$ to go once around the unit circle.

   (a) $\int_C z\,dz$

   (b) $\int_C \frac{1}{z}\,dz$

   (c) Do your results agree with Equation 2? If not, do you know why not?

2 Cauchy-Riemann Equations

For the following problem: $i = \sqrt{-1}$, $s = \sigma + i\omega$, and $F(s) = u(\sigma, \omega) + iv(\sigma, \omega)$.

In class I showed that the integration of a complex analytic function is independent of the path, formally known as the *Fundamental theorem of complex calculus*. The derivative of $F(s)$ is defined as

$$\frac{df}{ds} = \frac{d}{ds}\left[u(\sigma, \omega) + jv(\sigma, \omega)\right]. \tag{4}$$

If the integral is independent of the path, then the derivative must also be independent of direction

$$\frac{df}{ds} = \frac{\partial f}{\partial \sigma} = \frac{\partial f}{\partial j\omega}. \tag{5}$$

1. The Cauchy-Riemann (CR) conditions

$$\frac{\partial u(\sigma, \omega)}{\partial \sigma} = \frac{\partial v(\sigma, \omega)}{\partial \omega} \quad \text{and} \quad \frac{\partial u(\sigma, \omega)}{\partial \omega} = -\frac{\partial v(\sigma, \omega)}{\partial \sigma}$$

   may be used to show where Equation 5 holds.

   To do:

   (a) Assuming Equation 5 is true, use it to derive the CR equations.

   (b) Merge the CR equations to show that u and v obey Laplace's equation ($\nabla^2 u(\sigma, \omega) = 0$ and $\nabla^2 v(\sigma, \omega) = 0$). One may conclude that the real and imaginary parts of complex analytic functions must obey these conditions.

2. Apply the CR equations to the following functions. State for which values of $s = \sigma + i\omega$ the CR conditions do or do not hold (e.g. where the function $F(s)$ is or is not analytic). *Hint: Recall your answers to problem 1.2 of this assignment.*

   (a) $F(s) = e^s$

   (b) $F(s) = 1/s$

3 Complex Power Series

1. In each case derive (e.g. using Taylor's formula) the power series of $w(s)$ about $s = 0$ and state the ROC of your series. If the power series doesn't exist, state why! *Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at $s = 0$.*

To do:

   (a) $1/(1-s)$

   (b) $1/(1-s^2)$.

   (c) $1/(1-s)^2$

   (d) $1/(1+s^2)$. *Hint: This series will be <u>very ugly</u> to derive if you try to take the derivatives $\frac{d^n}{ds^n}[1/(1+s^2)]$. Using the results of our previous homework, you should represent this function as $w(s) = -0.5i/(s-i) + 0.5i/(s+i)$.*

   (e) $1/s$

   (f) $1/(1-|s|^2)$

2. Consider the function $w(s) = 1/s$

   (a) Expand this function as a power series about $s = 1$. What is the ROC?

   (b) Expand $w(s)$ as a power series in $s^{-1} = 1/s$ about $s^{-1} = 1$ (same as $s = 1$). What is the ROC? *Hint: Let $z = s^{-1}$.*

3. Consider the function $w(s) = 1/(2-s)$

   (a) Expand $w(s)$ as a power series in $s^{-1} = 1/s$. State the ROC as a condition on $|s^{-1}|$. *Hint: Let $z = s^{-1}$.*

   (b) Find the inverse function $s(w)$. Where are the poles and zeros of $s(w)$, and where is it analytic?

4   Branch cuts and Riemann sheets

1. Consider the function $[w(z)]^2 = z$. This function can also be written as $w(z) = \sqrt{z_\pm}$. Define $z = re^{\phi j}$ and $w(z) = \rho e^{\theta j} = \sqrt{r}e^{\phi j/2}$.
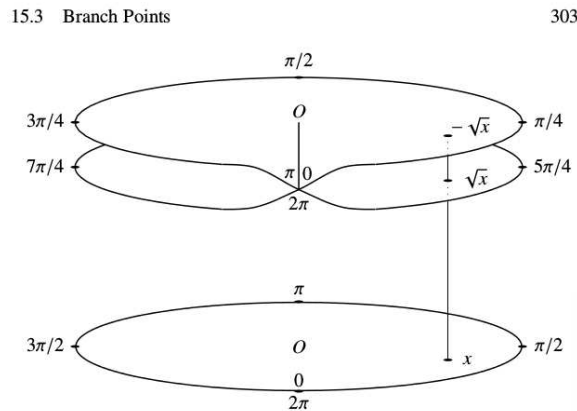

Figure 15.6: Branch point for the square root

Figure 1: *Here we show the mapping for the square root function $w(z) = \sqrt{z_\pm} = \sqrt{r}e^{j\phi/2}$ which has two single-valued sheets in z, corresponding to the two signs of the square root. The lower sheet is z_+, and the upper sheet is z_+. The location of the branch cut may be moved by rotating the z coordinate system. For example, $w(z) = \sqrt{\pm jz}$ and $w(z) = \pm\sqrt{-z}$ have a different branch cuts, as may be easily verified using the Matlab commands `zviz(-j*z)` and `zviz(-z)`. A function is analytic on the branch cut, since it is analytic everywhere. If a Taylor series is formed on the branch cut, it will describe the function on the two different sheets. Thus the Taylor series is agnostic to the location of the cut, as it only describes the function uniquely (as a single valued function), valid in its local region of convergence. Figure taken from Stillwell p. 303.*

(a) How many Riemann sheets do you need in the *domain* (z) and the *range* (w) to fully represent this function as single valued?

Indicate (e.g. using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.

(b) Use `zviz.m` to plot the positive and negative square roots $+\sqrt{z}$ and $-\sqrt{z}$. Describe what you see.

(c) Where does `zviz.m` place the branch cut for this function?

(d) Must it necessarily be in this location?

2. Consider the function $w(z) = \log(z)$. As before define $z = re^{\phi\jmath}$ and $w(z) = \rho e^{\theta\jmath}$.

   (a) Describe with a sketch, and then discuss the *branch cut* for $f(z)$.

   (b) What is the inverse of this function, $z(f)$? Does this function have a branch cut (if so, where is it)?

3. Using `zviz.m`, show that
$$\tan^{-1}(z) = -\frac{\jmath}{2} \log \frac{\jmath - z}{\jmath + z}. \tag{6}$$

That is, plot both these function and verify they are the same function.

4. Algebraically justify Eq. 6. Hint: Let $w(z) = \tan^{-1}(z)$, $z(w) = \tan w = \sin w / \cos w$, then solve for e^{wj}.