ECE 298JA **DE #2 – Version 3.01 October 29, 2017** **Fall 2017**

Univ. of Illinois **Due Wed, Nov 6, 2017** **Prof. Allen**

Topic of this homework: Cauchy-Riemann conditions; Integration of complex functions; Cauchy's theorem, integral formula, residue theorem; power series; Riemann sheets and branch cuts; inverse Laplace transforms

1 FTCC and integration in the complex plane

Recall that, according to the Fundamental Theorem of Complex Calculus (FTCC)

$$F(z) = F(z_0) + \int_{z_0}^{z} f(\zeta)d\zeta, \tag{1}$$

where $z_0, z, \zeta, F \in \mathbb{C}$.

1. For a closed interval $[a, b]$, the FTCC can be stated as

$$\int_a^b f(z)dz = F(b) - F(a), \tag{2}$$

meaning that the result of the integral is independent of the path from $x = a$ to $x = b$. What condition(s) is (are) required of the integrand $f(z)$ such that the FTCC holds?

2. For the function $f(z) = c^z$, where $c \in \mathbb{C}$ is an arbitrary complex constant, use the Cauchy-Riemann (CR) equations to show that $f(z)$ is analytic for all $z \in \mathbb{C}$.

3. In the following problems, solve the integral

$$I = \int_{\mathcal{C}} f(z)dz$$

for the path $\mathcal{C}$. In some cases this is an indefinite integral.

$$I(z) = \int_{z_0}^{z} f(\zeta)d\zeta$$

The function $f(z) = c^z$, where $c \in \mathbb{C}$ is given for each problem below. *Hint: Can you apply the FTCC?*

   (a) $c = 1/e = 1/2.7183\ldots$ where $\mathcal{C}$ is $\zeta = 0 \to i \to z$

   (b) $c = 2$ where $\mathcal{C}$ is $\zeta = 0 \to (1 + i) \to z$

   (c) $c = i$ where the path $\mathcal{C}$ is an inward spiral described by $z(t) = 0.99^t e^{i2\pi t}$ for $t = 0 \to t_0 \to \infty$

   (d) $c = e^{t-\tau_0}$ where $\tau_0 > 0$ is a real number, and $\mathcal{C}$ is $z = (1 - i\infty) \to (1 + i\infty)$. *Hint: Do you recognize this integral? If you do not recognize the integral, please do not spend a lot of time trying to solve it via the 'brute force' method.*

2 Cauchy's theorems for integration in the complex plane

There are three basic definitions related to Cauchy's integral formula. They are all related, and can greatly simplify integration in the complex plane.

1. **Cauchy's (Integral) Theorem** (Stillwell, p. 319; Boas, p. 45)

$$\oint_{\mathcal{C}} f(z)dz = 0,$$

if and only if $f(z)$ is complex-analytic inside of $\mathcal{C}$.

This is related to the Fundamental Theorem of Complex Calculus (FTCC)

$$\int_a^b f(z)dz = F(b) - F(a)$$

where $F(z)$ is the *anti-derivative* of $f(z)$, namely $f(z) = dF/dz$. The FTCC requires $f(z)$ to be complex-analytic for all $z \in \mathbb{C}$. By closing the path (contour $\mathcal{C}$), Cauchy's theorem and the following theorems allow us to integrate functions that may not be complex-analytic for all $z \in \mathbb{C}$.

2. **Cauchy's Integral Formula** (Boas, p. 51; Stillwell, p. 220)

$$\frac{1}{2\pi j}\oint_{\mathcal{C}} \frac{w(z)}{z - z_0}dz = \begin{cases} w(z_0), z_0 \in \mathcal{C} \text{ (inside)} \\ 0, z_0 \notin \mathcal{C} \text{ (outside)} \end{cases}$$

Here $w(z)$ is required to be analytic everywhere within (and on) the contour $\mathcal{C}$. The result $w(z_0)$ is called the *residue* of the pole z_0 of the function $f(z) = w(z)/(z - z_0)$.

3. **(Cauchy's) Residue Theorem** (Boas, p. 72)

$$\oint_{\mathcal{C}} f(z)dz = 2\pi j \sum_{k=1}^{K} \text{Res}_k$$

Where Res_k are the *residues* of all poles of $f(z)$ enclosed by the contour $\mathcal{C}$.

How to calculate the residues: This can be rigorously defined as

$$\text{Res}_k = \lim_{z \to z_k} [(z - z_k)f(z)]$$

Consider the function $f(z) = w(z)/(z - z_k)$, where we have factored $f(z)$ to isolate the first-order pole at $z = z_k$. If the remaining factor $w(z)$ is analytic at z_k, then the residue of the pole at $z = z_k$ is $w(z_k)$.

To do:

1. In one or two brief sentences, describe the relationships between the three theorems:
   (a) (1) and (2)
   (b) (1) and (3)
   (c) (2) and (3)

2. Consider the function with poles at $z = \pm j$

$$f(z) = \frac{1}{1+z^2} = \frac{1}{(z-j)(z+j)}$$

Apply Cauchy's theorems to solve the following integrals. **Show your work and state which theorem(s) you used** to answer each question.

(a) $\oint_{\mathcal{C}} f(z)dz$ where $\mathcal{C}$ is a circle centered at $z = 0$ with a radius of $\frac{1}{2}$.

(b) $\oint_{\mathcal{C}} f(z)dz$ where $\mathcal{C}$ is a circle centered at $z = j$ with a radius of 1.

(c) $\oint_{\mathcal{C}} f(z)dz$ where $\mathcal{C}$ is a circle centered at $z = 0$ with a radius of 2.

3 Integration in the complex plane

In the following questions, you'll be asked to integrate $f(s) = u(\sigma,\omega) + iv(\sigma,\omega)$ around the contour $\mathcal{C}$ for complex $s = \sigma + i\omega$,

$$\oint_{\mathcal{C}} f(s)ds.$$

Follow the directions carefully for each question. When asked to state where the function is and is not analytic, you are not required to use the Cauchy-Riemann equations (but you should if you can't answer the question 'by inspection').

1. $f(s) = \frac{1}{s}$

   (a) State where the function is and is not analytic.

   (b) Explicitly evaluate the integral when $\mathcal{C}$ is the unit circle, defined as $s = e^{i\theta}$, $0 \le \theta \le 2\pi$.

   (c) Evaluate the same integral using Cauchy's theorem and/or the residue theorem.

2. $f(s) = \frac{1}{s^2}$

   (a) State where the function is and is not analytic.

   (b) Explicitly evaluate the integral when $\mathcal{C}$ is the unit circle, defined as $s = e^{i\theta}$, $0 \le \theta \le 2\pi$.

   (c) What does your result imply about the residue of the 2^{nd} order pole at $s = 0$?

3. $f(s) = e^{st}$

   (a) State where the function is and is not analytic.

   (b) Explicitly evaluate the integral when $\mathcal{C}$ is the square $(\sigma,\omega) = (1,1) \to (-1,1) \to (-1,-1) \to (1,-1) \to (1,1)$.

   (c) Evaluate the same integral using Cauchy's theorem and/or the residue theorem.

4. $f(s) = \frac{1}{s+2}$

   (a) State where the function is and is not analytic.

   (b) Let $\mathcal{C}$ be the unit circle, defined as $s = e^{i\theta}$, $0 \le \theta \le 2\pi$. Evaluate the integral using Cauchy's theorem and/or the residue theorem.

   (c) Let $\mathcal{C}$ be a circle of radius 3, defined as $s = 3e^{i\theta}$, $0 \le \theta \le 2\pi$. Evaluate the integral using Cauchy's theorem and/or the residue theorem.

5. $f(s) = \frac{1}{2\pi i}\left[\frac{e^{st}}{s+2}\right]$

(a) State where the function is and is not analytic.

(b) Let $\mathcal{C}$ be a circle of radius 3, defined as $s = 3e^{i\theta}$, $0 \le \theta \le 2\pi$. Evaluate the integral using Cauchy's theorem and/or the residue theorem.

(c) Let $\mathcal{C}$ contain the entire left-half s-plane. Evaluate the integral using Cauchy's theorem and/or the residue theorem. Do you recognize this integral?

6. $f(s) = \pm\frac{1}{\sqrt{s}}$ (e.g. $f^2 = \frac{1}{s}$)

(a) State where the function is and is not analytic.

(b) This function is multivalued. How many Riemann sheets do you need in the domain (s) and the range (f) to fully represent this function? Indicate (e.g. using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.

(c) Explicitly evaluate the integral

$$\int_{\mathcal{C}} \frac{1}{\sqrt{z}} dz$$

when $\mathcal{C}$ is the unit circle, defined as $s = e^{i\theta}$, $0 \le \theta \le 2\pi$. Is this contour 'closed'? State why or why not.

(d) Explicitly evaluate the integral

$$\int_{\mathcal{C}} \frac{1}{\sqrt{z}} dz$$

when $\mathcal{C}$ is *twice* around the unit circle, defined as $s = e^{i\theta}$, $0 \le \theta \le 4\pi$. Is this contour 'closed'? State why or why not. *Hint: Note that* $\sqrt{e^{i(\theta+2\pi)}} = \sqrt{e^{i2\pi}e^{i\theta}} = e^{i\pi}\sqrt{e^{i\theta}} = -1\sqrt{e^{i\theta}}$

(e) What does your result imply about the residue of the (twice-around $\frac{1}{2}$ order) pole at $s = 0$?

4 A two-port network application for the Laplace transform

The Laplace transform (LT) and inverse Laplace transform (ILT): Recall that the *Laplace transform* (LT) $f(t) \leftrightarrow F(s)$[1] of a causal function $f(t)$ is

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt.$$

where $s = \sigma + j\omega$ is complex frequency[2] in [radians] and t is time in [seconds]. Causal functions and the Laplace transform are particularly useful for describing *systems*, which have no response until a signal enters the system (e.g. at $t = 0$).

To define the *inverse Laplace transform* (ILT) we need to understand integration in the complex plane

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s)e^{st}ds = \frac{1}{2\pi j}\oint_{\mathcal{C}} F(s)e^{st}ds$$

The Laplace contour $\mathcal{C}$ actually includes two pieces

$$\oint_{\mathcal{C}} = \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} + \int_{\subset\infty}$$

[1]Many loosely adhere to the convention that the frequency domain uses upper-case [e.g. $F(s)$] while the time domain uses lower case [$f(t)$]

[2]While radians are useful units for calculations, when providing physical insight in discussions of problem solutions, it is easier to work with Hertz, since frequency in [Hz] and time in [s] are mentally more more natural units than radians. The same is true of degrees vs radians. Boas (p. 10) recommends the use degrees over radians. He gives the example of $3\pi/5$ [radians], which is more easily visualize as 108°.

where the path represented by '$\subset_\infty$' is a semicircle of infinite radius with $\sigma \to -\infty$. It is somewhat tricky to do, but it may be proved that the integral over the contour $\subset_\infty$ goes to zero. For a causal, 'stable' (e.g. doesn't blow up over time) signal, all of the poles of $F(s)$ *must* be inside of the Laplace contour, in the left-half s-plane.

Transfer functions Linear, time-invariant systems are described by an ordinary differential equations. For example, consider the first-order linear differential equation

$$a_1 \frac{d}{dt} y(t) = b_1 \frac{d}{dt} x(t) + b_0 x(t). \tag{3}$$

This equation describes the relationship between the input ($x(t)$) and output ($y(t)$) of the system. If we define Laplace transforms $y(t) \leftrightarrow Y(s)$ and $x(t) \leftrightarrow X(s)$, then this equation may be written in the frequency domain as

$$a_1 s Y(s) = b_1 s X(s) + b_0 X(s).$$

The *transfer function* for this system is defined as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_1 s + b_0}{a_1 s} = \frac{b_1}{a_1} + \frac{b_0}{a_1 s}$$

In this problem, we will look at the transfer function of a simple two-port network, shown in Figure 1. This network is an example of a RC low-pass filter, which acts as a *leaky integrator.*

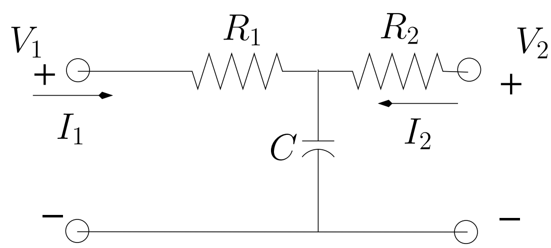

Figure 1: This three element electrical circuit is a system that acts to low-pass filter the signal voltage $V_1(\omega)$, to produce signal $V_2(\omega)$.

To do:

1. Use the ABCD method to find the matrix representation of Fig. 1.
2. Assuming that $I_2 = 0$, find the transfer function $H(s) \equiv V_2/V_1$. From the results of the ABCD matrix you determined above, show that
$$H(s) = \frac{1}{1 + R_1 C s}.$$
3. The transfer function $H(s)$ has one pole. Where is the pole? Find the *residue* of this pole.
4. Find $h(t)$, the inverse Laplace transform of $H(s)$.
5. Assuming that $V_2 = 0$ find $Y_{12}(s) \equiv I_2/V_1$.
6. Find the input impedance to the *right-hand side* of the system, $Z_{22}(s) \equiv V_2/I_2$ for two cases:

(a) $I_1 = 0$

(b) $V_1 = 0$

7. Compute the determinant of the ABCD matrix. *Hint: It is always 1.*

8. Compute the derivative of $H(s) = \left.\frac{V_2}{V_1}\right|_{I_2=0}$.

5 With the help of a computer...

In the following problems, we will look at some of the concepts from this homework using Matlab.

Note: We are using the `syms` *function which requires Matlab's symbolic math toolbox. Prof. Allen was able to get the* `symbolic` *toolbox to work with Octave, but the install was quite complicated. If it works for you, that's great, but if it doesn't, don't waste your time trying to install it. Use Matlab if you can, (via the EWS lab in person or one of their online services), or use an alternative symbolic-math tool, such as Wolfram Alpha (https://www.wolframalpha.com/).*

1. To find the Taylor series expansion about $s = 0$ of

$$F(s) = -\log(1-s),$$

first consider the derivative and its Taylor series (about $s = 0$)

$$F'(s) = \frac{1}{1-s} = \sum_{n=0}^{\infty} s^n.$$

Then, integrate this series term by term

$$F(s) = -\log(1-s) = \int^s F'(s)ds = \sum_{n=0}^{\infty} \frac{s^n}{n}.$$

Alternatively you may use Matlab to do this with the following Matlab commands

```
syms s
taylor(-log(1-s),7)
```

To do:

(a) Try these Matlab commands. Give the first 7 terms of the Taylor series (confirm that Matlab agrees with the formula derived above).

(b) What is the inverse Laplace transform of this series? Consider the series term by term.

2. The function $1/\sqrt{z}$ has a branch point at $z = 0$, thus it is singular there.

(a) Can you apply Cauchy's integral theorem when integrating around the unit circle?

(b) Here is a Matlab code that computes $\int_0^{4\pi} \frac{dz}{\sqrt{z}}$ using Matlab's symbolic analysis package:

```
 syms  z
 I=int(1/sqrt(z))
 J = int(1/sqrt(z),exp(-j*pi),exp(j*pi))
eval(J)
```

To do: Run this script. What answers do you get for I and J?

(c) Modify this code to integrate $f(z) = 1/z^2$ *once* around the unit circle. What answers do you get for I and J?

3. Bessel functions can describe waves in a cylindrical geometry (e.g. vibrations of a drum head). The Bessel function has a Laplace transform with a branch cut

$$J_0(t)u(t) \leftrightarrow \frac{1}{\sqrt{1+s^2}}$$

To do:

(a) Try the following Matlab commands, and then comment on your findings.

```
%Take the inverse LT of 1/sqrt(1+s^2)
syms s
I=ilaplace(1/(sqrt((1+s^2))));
disp(I)

%Find the Taylor series of the LT
T = taylor(1/sqrt(1+s^2),10);
disp(T);

%Verify this
syms t
J=laplace(besselj(0,t));
disp(J);

%now plot the Bessel function
t=0:0.1:10*pi;
b=besselj(0,t);
plot(t/pi,b);
grid on;
```

(b) When did Friedrich Bessel live? What did he use Bessel functions for?

4. Using `zviz`, for each of the following functions

i. Describe the plot generated by `zviz`

ii. Is this function a Brune impedance (i.e., does this function obey $\Re Z(\sigma > 0) \geq 0$)? *Hint: Consider the phase (color). Plot* `zviz Z` *for a reminder of the colormap.*

(a) `zviz 1./sqrt(1+S.^2)`

(b) `zviz 1./sqrt(1-S.^2)`

References

Greenberg, M. D. (1988). *Advanced Engineering Mathematics*. Prentice Hall, Upper Saddle River, NJ, 07458.

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