ECE 298JA | DE #3 − Version 2.02 November 3, 2017 | Fall 2017

Univ. of Illinois | Due Mon, Nov 10, 2017 | Prof. Allen

Topic of this homework: Laplace Transforms

1 Laplace transforms

Given a *Laplace transform* ($\mathcal{L}$) pair $f(t) \leftrightarrow F(s)$, the frequency domain will always be upper-case [e.g. $F(s)$] and the time domain lower case [$f(t)$] and causal (i.e., $f(t<0)=0$). The definition of the forward transform ($f(t) \rightarrow F(s)$) is

$$F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt,$$

where $s = \sigma + j\omega$ is the complex Laplace frequency in [radians] and t is time in [seconds].

The *inverse Laplace transform* ($\mathcal{L}^{-1}$), $F(s) \rightarrow f(t)$ is defined as

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s)e^{st}ds = \frac{1}{2\pi j}\oint_{\mathcal{C}} F(s)e^{st}ds$$

with $\sigma_0 > 0 \in \mathbb{R}$ is a positive constant.

As discussed in the lecture notes (Section 1.4.7, p. 72) we may use the Cauchy Residue Theorem (CRT), to evaluate the $\mathcal{L}^{-1}$, by requiring closure of the contour $\mathcal{C}$ at $\omega\jmath \rightarrow \pm j\infty$

$$\oint_{\mathcal{C}} = \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} + \int_{\subset_\infty},$$

where the path represented by '$\subset_\infty$' is a semicircle of infinite radius. For a causal, 'stable' (e.g. doesn't "blow up" in time) signal, all of the poles of $F(s)$ *must* be inside of the Laplace contour, in the full (closed) left-half s-plane ($\sigma \leq 0$).

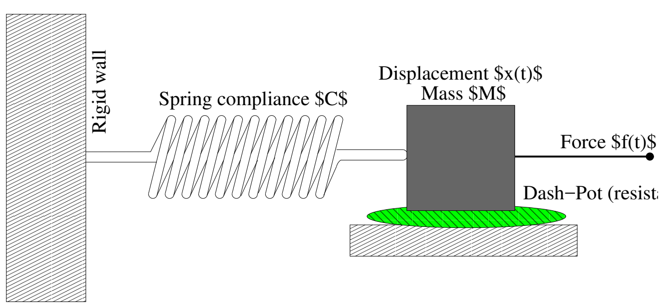

Figure 1: *This three element mechanical resonant circuit consisting of a spring, mass and dash-pot (e.g., viscous fluid).*

Hooke's Law for a spring states that the force $f(t)$ is proportional to the displacement $x(t)$, i.e., $f(t) = Kx(t)$. The formula for a dash-pot is $f(t) = Rv(t)$, and Newton's famous formula for mass is $f(t) = d[Mv(t)]/dt$, which for constant M is $f(t) = Mdv/dt$.

The equation of motion for the mechanical oscillator in Fig. 1 is given by Newton's second law; the sum of the forces must balance to zero

$$M\frac{d^2}{dt^2}x(t) + R\frac{d}{dt}x(t) + Kx(t) = f(t). \tag{1}$$

These three constants, the mass M, resistance R and stiffness K are all real and positive.. The dynamical variables are the driving force $f(t) \leftrightarrow F(s)$, the position of the mass $x(t) \leftrightarrow X(s)$ and its velocity $v(t) \leftrightarrow V(s)$, with $v(t) = dx(t)/dt \leftrightarrow V(s) = sX(s)$.

Newton's second law (c1650) is the mechanical equivalent of Kirchhoff's (c1850) voltage law (KCL), which states that the sum of the voltages around a loop must be zero. The gradient of the voltage results in a force on a charge (i.e., $F = qE$).

Equation 1 may be re-expressed in terms of impedances, the ratio of the force to velocity, once it is transformed into the Laplace frequency domain.

The key idea that every impedance must be complex analytic and ≥ 0 for $\sigma > 0$ was first proposed by Otto Brune in his PhD at MIT, supervised by a student of Arnold Sommerfeld, Ernst Guilliman, an MIT ECE professor, who played a major role in the development of circuit theory. Brune's primary (non-MIT) advisor was Cauer, who was also well trained in 19th century German mathematics.[1]

1.1 Brune Impedance

A Brune impedance is defined as the ratio of the force $F(s)$ over the flow $V(s)$, and may be expressed in residue form as

$$Z(s) = c_0 + \sum_{k=1}^{K} \frac{c_k}{s - s_k} = \frac{N(s)}{D(s)}. \tag{2}$$

It trivially follows that[2]

$$D(s) = \prod_{k=1}^{K} (s - s_k) \qquad \text{and} \qquad c_k = \lim_{s \to s_k} (s - s_k) D(s) = \prod_{n'=1}^{K-1} (s - s_n),$$

where the prime on index n' means that $n = k$ is not included in the product.

There are several important theorems here, best summarized as *Brune's Theorem* on positive-real functions. But it goes beyond this since the impedance matrix and the transmission matrix are a rearrangement of the same matrix equation(see the Lecture notes for the details; for example, 2-port transfer functions, and their input impedance, have the same poles).

1. Find the Laplace transform ($\mathcal{L}$) of the three force relations in terms of the force $F(s)$ and the velocity V(s), along with the electrical equivalent impedance:
   (a) Hooke's Law $f(t) = Kx(t)$.
   (b) Dash-pot resistance $f(t) = Rv(t)$.
   (c) Newton's Law for Mass $f(t) = Mdv(t)/dt$.
2. Take the Laplace transform ($\mathcal{L}$) of Eq. 1, and find the total impedance $Z(s)$ of the mechanical circuit.
3. What are $N(s)$ and $D(s)$ (e.g. Eq. 2)?
4. Assume that $M = R = K = 1$, find the residue form of the admittance $Y(s) = 1/Z(s)$ (e.g. Eq. 2) in terms of the roots $s_{\pm}$. You may check your answer with the Matlab's `residue` command.
5. By applying the CRT, find the inverse Laplace transform ($\mathcal{L}^{-1}$). Use the residue form of the expression that you derived in the previous exercise.

[1] It must be noted that Prof. 'Mac' Van Valkenburg from the University of IL., was arguably more influential in circuit theory, during the same period. Mac's book are certainly more accessible, but perhaps less widely cited.

[2] Is the ABCD $C(s)$ the same as the impedance denominator $D(s)$ here? I think it is.

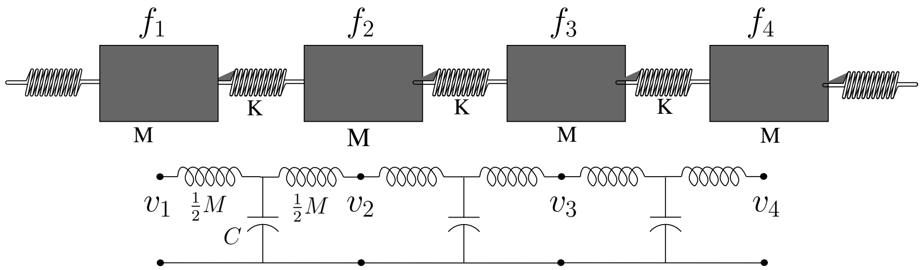

Figure 2: *Depiction of a train consisting of cars, treated as a mass M and linkages, treated as springs of stiffness K or compliance $C = 1/K$. Below it is the electrical equivalent circuit, for comparison. The mass is modeled as an inductor and the springs as capacitors to ground. The velocity is analogous to a current and the force $f_n(t)$ to the voltage $v_n(t)$.*

1.2 Transfer functions

In this problem, we will look at the transfer function of a two-port network, shown in Fig. 2. We wish to model the dynamics of a freight-train having N such cars. The model of the train consists of masses connected by springs.

The velocity *transfer function* for this system is defined as the ratio of the output to the input velocity. Consider the engine on the left pulling the train at velocity V_1 and each car responding with a velocity of V_n. Then

$$H(s) = \frac{V_N(s)}{V_1(s)}$$

is the frequency domain ratio of the last car having velocity V_N to V_1, the velocity of the engine, at the left most spring (i.e., coupler).

To do: Use the ABCD method to find the matrix representation of Fig. 2. Consistent with the figure, break the model into cells each consisting of three elements: a series inductor representing half the mass ($L = M/2$), a shunt capacitor representing the spring ($C = 1/K$), and another series inductor representing half the mass ($L = M/2$). Each cell is symmetric, making the model a cascade of identical cells.

At each node define the force $f_n(t) \leftrightarrow F_n(\omega)$ and the velocity $v_n(t) \leftrightarrow V_n(\omega)$ at junction n.

1. Write the ABCD matrix $\boldsymbol{T}$ for a single cell, composed of series mass $M/2$, shunt compliance C and series mass $M/2$, that relates the input node 1 to node 2 where

$$\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} F_2(\omega) \\ -V_2(\omega) \end{bmatrix}$$

   Note that here the mechanical force F is analogous to electrical voltage, and the mechanical velocity V is analogous to electrical current.

2. Assuming that $N = 2$ and that $F_2 = 0$ (two mass problem), find the transfer function $H(s) \equiv V_2/V_1$. From the results of the $\boldsymbol{T}$ matrix you determined above, find

$$H_{21}(s) = \left.\frac{V_2}{V_1}\right|_{F_2=0}$$

3. Find $h_{21}(t)$, the inverse Laplace transform of $H_{21}(s)$.

4. What is the input impedance $Z_2 = F_2/V_2$ if $F_3 = -r_0 V_3$?

5. Simplify the expression for Z_2 with $N \to \infty$ by assuming that:
   1) $F_3 = -r_0 V_3$ (i.e., V_3 cancels), 2) $s^2 MC << 1$: 3) $r_0 = \sqrt{M/C}$

6. State the ABCD matrix relationship between the first and Nth node, in terms of of the cell matrix.
7. Given a $\boldsymbol{T}$ (ABCD) transmission matrix, the eigenvalues are and vectors are given in Appendix C of the Notes (p. 143), repeated here.

Eigenvalues:

$$\begin{bmatrix}\lambda_+ \\ \lambda_-\end{bmatrix} = \frac{1}{2}\begin{bmatrix}(A+D) - \sqrt{(A-D)^2+4BC} \\ (A+D) + \sqrt{(A-D)^2+4BC}\end{bmatrix}$$

Due to symmetry, $A = D$, this simplifies to $\lambda_\pm = A \mp \sqrt{BC}$ so that the eigen matrix is

$$\Lambda = \begin{bmatrix} A - \sqrt{BC} & 0 \\ 0 & A + \sqrt{BC}\end{bmatrix}$$

Eigenvectors: The eigenvectors simplifying even more

$$\begin{bmatrix}\boldsymbol{E}_\pm\end{bmatrix} = \begin{bmatrix}\frac{1}{2C}\left[(A-D) \mp \sqrt{(A-D)^2+4BC}\right] \\ 1\end{bmatrix} = \begin{bmatrix}\mp\sqrt{\frac{B}{C}} \\ 1\end{bmatrix}$$

Eigen matrix:

$$\boldsymbol{E} = \begin{bmatrix}-\sqrt{\frac{B}{C}} & +\sqrt{\frac{B}{C}} \\ 1 & 1\end{bmatrix}, \qquad \boldsymbol{E}^{-1} = \frac{1}{2}\begin{bmatrix}-\sqrt{\frac{C}{B}} & 1 \\ +\sqrt{\frac{C}{B}} & 1\end{bmatrix}$$

To do: What is the velocity transfer function $H_{N1} = \frac{V_N}{V_1}$? Hint: Use an eigen matrix diagonalization, as we did for the Pell equation (Appendix C).

References

Greenberg, M. D. (1988). *Advanced Engineering Mathematics*. Prentice Hall, Upper Saddle River, NJ, 07458.