ECE 298JA **NS #3 – Version 1.26 September 19, 2017** **Fall 2017**

Univ. of Illinois **Due Monday, Sept 18, 2017** **Prof. Allen**

Topic of this homework: Pythagorean triples, Pell's equation, Fibonacci sequence
Deliverable: Answers to problems

1 Pythagorean triples

Euclid's formula for the Pythagorean triples a, b, c is: $a = p^2 - q^2$, $b = 2pq$, and $c = p^2 + q^2$.

1. What condition(s) must hold for p and q such that a, b, and c are always positive and nonzero?
2. Solve for p and q in terms of a, b and c.
3. The ancient Babylonians (c2000BEC) cryptically recorded (a,c) pairs of numbers on a clay tablet, archeologically denoted *Plimpton-322*.

To Do: Find p and q for the first five pairs of a and c from the tablet entries:

Table 1: First five (a,c) pairs of Plimpton-322.

a	c
119	169
3367	4825
4601	6649
12709	18541
65	97

4. Based on Euclid's formula, show that $c > (a, b)$.
5. What happens when $c = a$?
6. Is $b + c$ a perfect square? Discuss.

2 Pell's equation

Pell's equation is one of the most historic (i.e., important) equations of Greek number theory, because it was used to show that $\sqrt{2} \in \mathbb{I}$. We seek integer solutions

$$x^2 - Ny^2 = 1.$$

As shown in Lec 8 of the lecture notes, the solutions x_n, y_n for the case of $N = 2$ are given by the 2x2 matrix recursion

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = 1\jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

with $[x_0, y_0]^T = [1, 0]^T$ and $1\jmath = \sqrt{-1} = e^{\jmath\pi/2}$. It follows that the general solution to Pell's equation for $N = 2$ is

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = (e^{\jmath\pi/2})^n \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

To calculate solutions to Pell's equation using the matrix equation above, we must calculate

$$A^n = e^{\jmath\pi n/2}\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}^n = e^{\jmath\pi n/2}\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}\cdots\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}$$

which becomes tedious for $n > 2$, since it requires $n \times 2 \times 2$ matrix multiplications.

Diagonalization of a matrix ("eigenvalue/eigenvector decomposition"): As derived in Appendix C of the lecture notes, the most efficient way to compute A^n is to *diagonalize* the matrix A, by finding its *eigenvalues* and *eigenvectors*.

The eigenvalues λ_k and eigenvectors $\vec{e}_k$ of a square matrix A are related by

$$A\vec{e}_k = \lambda_k \vec{e}_k, \tag{1}$$

such that multiplying an eigenvector $\vec{e}_k$ of A by the matrix A is the same as multiplying by a scalar, $\lambda_k \in \mathbb{C}$ (the corresponding eigenvalue). The complete eigenvalue problem may be written as

$$AE = E\Lambda.$$

If A is a 2×2 matrix,[1] the matrices E and Λ (of eigenvectors and eigenvalues, respectively) are

$$E = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} \qquad \Lambda = \begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix}$$

Thus, the matrix equation $AE = \begin{bmatrix}A\vec{e}_1 & A\vec{e}_2\end{bmatrix} = \begin{bmatrix}\lambda_1\vec{e}_1 & \lambda_2\vec{e}_2\end{bmatrix} = E\Lambda$ contains Eq. 1 for each eigenvalue-eigenvector pair.

The *diagonalization* of the matrix A refers to the fact that the matrix of eigenvalues, Λ, has non-zero elements only on the diagonal. The key result is found by post-multiplication of the eigen value matrix by E^{-1}, giving

$$AEE^{-1} = A = E\Lambda E^{-1}. \tag{2}$$

If we now take powers of A, the n^{th} power of A is

$$\begin{aligned}A^n &= (E\Lambda E^{-1})^n\\ &= E\Lambda E^{-1}E\Lambda E^{-1}\cdots E\Lambda E^{-1}\\ &= E\Lambda^n E^{-1}.\end{aligned} \tag{3}$$

This is a very powerful result, because the n^{th} power of a diagonal matrix is extremely easy to calculate:

$$\Lambda^n = \begin{bmatrix}\lambda_1^n & 0\\0 & \lambda_2^n\end{bmatrix}$$

Thus, from Eq. 3 we can calculate A^n using only two matrix multiplications

$$A^n = E\Lambda^n E^{-1}.$$

[1]These concepts may be easily extended to higher dimensions.

Finding the eigenvalues: The *eigenvalues* λ_k are determined by Eq. 1, by factoring out $\vec{e}_k$

$$A\vec{e}_k = \lambda_k \vec{e}$$
$$(A - \lambda_k I)\vec{e}_k = \vec{0}.$$

Matrix $I = [1, 0; 0, 1]^T$ is the *identity matrix*, having the dimensions of A, with elements δ_{ij} (i.e., diagonal elements $\delta_{11,22} = 1$ and off-diagonal elements $\delta_{12,21} = 0$).

The vector $\vec{e}_k$ is not zero, yet when operated on by $A - \lambda_k I$, the result must be zero. The only way this can happen is if the operator is degenerate (has no solution), that is

$$\det(A - \lambda I) = \det \begin{bmatrix} (a_{11} - \lambda) & a_{12} \\ a_{21} & (a_{22} - \lambda) \end{bmatrix} = 0. \tag{4}$$

This means that the two equations have the same slope (the equation is degenerate).

This determinant equation results in a second degree polynomial in λ

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0,$$

the roots of which are the eigenvalues of the matrix A.

Finding the eigenvectors: An *eigenvector* $\vec{e}_k$ can be found for each eigenvalue λ_k from Eq. 1,

$$(A - \lambda_k I)\vec{e}_k = \vec{0}.$$

The left side of the above equation becomes a column vector, where each element is an equation in the elements of $\vec{e}_k$, set equal to 0 on the right side. These equations are always *degenerate*, since the determinant is zero. Thus the two equatons have the same slope.

Solving for the eigenvectors is often confusing, because they have arbitrary magnitudes, $||\vec{e}_k|| = \sqrt{\vec{e}_k \cdot \vec{e}_k} = \sqrt{e_{k,1}^2 + e_{k,2}^2} = d$. From Eq. 1, you can only determine the *relative* magnitudes and signs of the elements of $\vec{e}_k$, so you will have to choose a magnitude d. It is common practice to *normalize* each eigenvector to have unit magnitude ($d = 1$).

To do:

Hint: Use Matlab's function `[E,Lambda] = eig(A)` *to check your results!*

1. Solutions to Pell's equation were used by the Pythagoreans to explore the value of $\sqrt{2}$. Explain why Pell's equation is relevant to $\sqrt{2}$.
2. Find the first 3 values of $[x_n, y_n]^T$ by hand and show that they satisfy Pell's equation for N=2.
3. By hand, find the eigenvalues $\lambda_\pm$ of the 2×2 Pell's equation matrix
$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$
4. By hand, show that the matrix of eigenvectors, E, is
$$E = \begin{bmatrix} \vec{e}_+ & \vec{e}_- \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}$$
5. Using the eigenvalues and eigenvectors you found for A, verify that
$$E^{-1}AE = \Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$$
6. Now that you have diagonalized A (Equation 3), use your results for E and Λ to solve for the $n = 10$ solution $[x_{10}, y_{10}]^T$ to Pell's equation with $N = 2$.

3 The Fibonacci sequence

The Fibonacci sequence is famous in mathematics, and has been observed to play a role in the mathematics of genetics. Let x_n represent the Fibonacci sequence,

$$x_n = x_{n-1} + x_{n-2}, \tag{5}$$

where the current output sample, x_n, is equal to the sum of the previous two inputs. This is a 'discrete time' *recurrence relation.* To solve for x_n, we require some *initial conditions.* In this exercise, let us define $x_0 = 1$ and $x_{n<0} = 0$. This leads to the Fibonacci sequence $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$ for $n = 0, 1, 2, 3, \ldots$.

Here we seek the general formula for x_n. Like the Pell's equation, Eq. 5 has a recursive, eigen decomposition solution. To find it we must recast x_n as a 2x2 matrix relation, and then proceed as we did for the Pell case.

1. Show that Eq. 5 is equivalent to the 2×2 matrix equation

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}. \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{6}$$

   and that the Fibonacci sequence x_n as described above may be generated by

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \qquad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

   What is the relationship between y_n and x_n?

2. Write a Matlab program to compute x_n using the matrix equation above (you don't need to turn in your code). Test your code using the first few values of the sequence. Using your program, what is x_{40}?
   Note: to make your program run faster, consider using the eigen decomposition of A, described by Eq. 3 from the Pell's equation problem.

3. Using the eigen decomposition of the matrix A (and a lot of algebra), it is possible to obtain the general formula for the Fibonacci sequence,

$$x_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right]. \tag{7}$$

   What are the eigenvalues $\lambda_\pm$ of the matrix A? How is the formula for x_n related to these eigenvalues?

4. Consider Eq. 7 in the limit as $n \to \infty$...
   (a) What happens to each of the two terms $[(1 \pm \sqrt{5})/2]^{n+1}$?
   (b) What happens to the ratio x_{n+1}/x_n?

5. Prove that[2]

$$\sum_1^N f_n^2 = f_N f_{N+1}.$$

[2] I found this problem on a workseet for Math 213 midterm (213practice.pdf).

6. Replace the Fibonacci sequence with

$$x_n = \frac{x_{n-1} + x_{n-2}}{2},$$

such that the value x_n is the average of the previous two values in the sequence.

   (a) What matrix A is used to calculate this sequence?

   (b) Modify your computer program to calculate the new sequence x_n. What happens as $n \to \infty$?

   (c) What are the eigenvalues of your new A? How do they relate to the behavior of x_n as $n \to \infty$? *Hint: you can expect the closed-form expression for x_n to be similar to Eq. 7.*

7. Now consider

$$x_n = \frac{x_{n-1} + 1.01x_{n-2}}{2}.$$

   (a) What matrix A is used to calculate this sequence?

   (b) Modify your computer program to calculate the new sequence x_n. What happens as $n \to \infty$?

   (c) What are the eigenvalues of your new A? How do they relate to the behavior of x_n as $n \to \infty$? *Hint: you can expect the closed-form expression for x_n to be similar to Eq. 7.*