Concepts in Engineering Mathematics

Mathematics and its History by John Stillwell

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Abstract

It is widely acknowledged that *interdisciplinary science* is the backbone of modern scientific research. However such a curriculum is not taught, in part because there are few people to teach it, and due to its inherent complexity and breadth. Mathematics, Engineering and Physics (MEP) are at the core of such studies. To create such an interdisciplinary program, a unified MEP curriculum is needed. This unification could take place based on a core mathematical training from a historical perspective, starting with Euclid or before (i.e., Chinese mathematics), up to modern information theory and logic. As a bare minimum, the *fundamental theorems of mathematics* (arithmetic, algebra, calculus, vector calculus, etc.) need to be appreciated by every MEP student.

At the core of this teaching are 1) partial differential equations (e.g., Maxwell's Eqs), 2) linear algebra of (several) complex variables, and 3) complex vector calculus (e.g., Laplace transforms).

If MEP were taught a common mathematical language, based on a solid training in mathematical history [**?**], students would be equipped to 1) teach and exercise interdisciplinary science and 2) easily communicate with other M, E, and P scientists.

The idea is to teach the history of the development of these core topics, so that the student can fully appreciate the underlying principles. Understanding these topics based on their history (e.g., the people who created them, what they were attempting to do, and their basic mind-set), makes the subject uniformly understandable to every student. The present method, using abstract proofs, with no (or few) figures or physical principles, lacks the intuition and motivation of the original creators of these theories. Such a sterile approach is not functional for many students, resulting in their poor intuition.

In the beginning . . .

The very first documented mathematics:

- Chinese 5000 BC (aka BCE)
- Babylonians (Mesopotamia/Iraq) 1800 BCE
- Archimedes 300 BCE
  - Geometric series
  - Vol of sphere, Area of Parabola
  - Hydrostatics
- Euclid active in Alexandria during the reign of Ptolemy I ≈323 BCE
  - Euclid's *Elements* is (was?) the most influential works of mathematics.
  - Geometry every student is assumed to learn in High School
- Burning of the Library of Alexandria results in the destruction of 'All recorded knowledge' 391 BCE

Acoustics & Music

BCE Pythagoras; Aristotle

16^{th} Mersenne, Marin 1588-1647; *Harmonie Universelle 1636*, *Father of acoustics*;
Galilei, Galileo, 1564-1642; *Frequency Equivalence 1638*

17^{th} Newton, Sir Issac 1686; Hooke, Robert; Boyle, Robert 1627-1691;

18^{th} Bernoulli, Daniel (#3); Euler; Lagrange; d'Alembert;

19^{th} Gauss; Laplace; Fourier; Helmholtz; Heaviside; Bell, AG;
Rayleigh, Lord (aka: Strutt, William)

20^{th} Campbell, George; Hilbert, David; Noether, Emmy; Fletcher, Harvey;
Nyquist, Harry; Bode, Henrik; Dudley, Homer; Shannon, Claude;

Chronological history, by century

500^{th} BCE Chinese (quadratic equation)

180^{th} BCE Babylonia (Mesopotamia/Iraq) (quadratic equation)

6^{th} BCE Diophantus; Pythagoras (Thales) and "tribe"

4^{th} BCE Archimedes; Euclid (quadratic equation)

7^{th} CE Brahmagupta (negative numbers; quadratic equation)

15^{th} Copernicus 1473-1543 Renaissance mathematician & astronomer

16^{th} Tartaglia (cubic eqs); Bombelli (complex numbers); Galileo

17^{th} Newton 1642-1727 *Principia* 1687; Mersenne; Huygen; Pascal; Fermat, Descartes (analytic geometry); Bernoullis Jakob, Johann & son Daniel

18^{th} Euler 1748 *Student of Johann Bernoulli*; d'Alembert 1717-1783; Kirchhoff; Lagrange; Laplace; Gauss 1777-1855

19^{th} Möbius, Riemann 1826-1866, Galois, Hamilton, Cauchy 1789-1857, Maxwell, Heaviside, Cayley

20^{th} Hilbert; Einstein; ...

Stream 1: Number systems

The development of representation proceeded at a deadly-slow pace:

- Natural numbers (positive Integers) 5000 CE: Bees can count
- Rational numbers: Egyptians c1000 CE; Pythagoras 500 CE
- Prime numbers (Fundamental Thm. Arithmetic); Euclid algorithm: Greatest Common prime Divisor (Ex: 15=3*5, 30=2*3*5: gcd=3)
- Negative integers: 628 AD Brahmagupt used negative numbers to represent debt.
- Zero: By the ninth century zero had entered the Arabic numeral system
- Real numbers: Pythagoras knew of irrational numbers ($\sqrt{2}$)
- Complex numbers: 1572 *"Bombelli is regarded as the inventor of complex numbers . . . "* http://www-history.mcs.st-andrews.ac.uk/Biographies/Bombelli.html http://en.wikipedia.org/wiki/Rafael_Bombelli & p. 258
- Power Series: Gregory-Newton interpolation formula c1670, p. 175
- Point at infinity and the Riemann sphere 1850
- Analytic functions p. 267 c1800; Analytic Impedance $Z(s)$ 1893

Outcomes: Roots of systems of polynomials

- Fundamental theorem of:
    - Arithmetic
    - Calculus
    - Algebra, ...
- Other key outcomes:
    - Complex analytic functions (complex roots are finally accepted!)
    - Complex Taylor Series of complex numbers (ROC)
    - Riemann mapping theorem
    - Cauchy Integral Theorem (Residue integration (i.e., Green's Thm))
    - Laplace Transform and its inverse
    - Complex Impedance (Ohm's Law) 1893: A. Kennelly 1861-1939:

14.6 The Fundamental Theorem of Algebra

Fundamental theorem of algebra: *Every polynomial equation $p(z) = 0$ has a solution in the complex numbers. As Descartes observed, a solution $z = a$ implies that $p(z)$ has a factor $z - a$. The quotient $q(z) = p(z)/(z - a)$ is then a polynomial of lower degree.*

. . . We can go on to factorize $p(z)$ into n linear factors.

. . . d'Alembert (1746) observed that for polynomials $p(z)$ with real coefficients, if $z = u + iv$ is a solution of $p(z) = 0$, then so is its conjugate $z^* = u - iv$. Thus the imaginary linear factors of a real $p(z)$ can always be combined in pairs with real coefficients.

p. 285

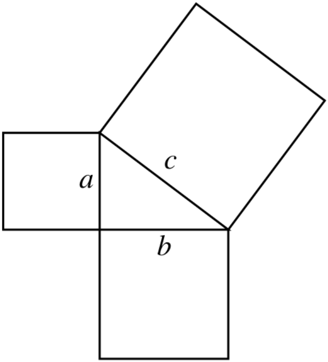
Pythagoras's end

- The Pythagoras Theorem (PT) is the oldest "math cornerstone"
- PT is the entrypoint to three main streams in mathematics
  - Numbers $\mathbb{Z}_+, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathcal{C}, \mathcal{F}$ (i.e., Int, Rats, Reals, Complex, Functions)
  - Geometry (e.g., lines, circles, spheres, toroids, ...)
  - Infinity (irrationals, limits, ∞)
- The Pathagoreans were destroyed by fear:

  > *Whether the complete rule of number (integers) is wise remains to be seen. It is said that when the Pythagoreans tried to extend their influence into politics they met with popular resistance. Pythagoras fled, but he was murdered in nearby Metapontum in 497 BCE.* *p. 16*

Pythagorean Theorem: $c^2 = a^2 + b^2$

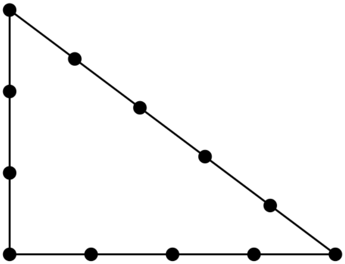
- Pythagoras assumed that $[a, b, c] \in \mathbb{Z}$ (i.e., are integers)
- This relationship has a deep meaning and utility
- Note both discrete lengths and areas are intertwined



Proof of Pythagoras's Theorem

Unit-length jointed sides (e.g., bricks)

Integer property of Pythagoras's Theorem:



- Pythagoras required that the sides are $\in \mathbb{Z}$ (e.g.: [3,4,5])
- Note that $3 = \sqrt{4+5}$ (i.e., $a = \sqrt{b+c}$) Why?

Pythagorean triplets: $b = \sqrt{c^2 - a^2}$

Applications in architecture and scheduling (quantized units)

Exercises

The integer pairs (a, c) in Plimpton 322 are

a	c
119	169
3367	4825
4601	6649
12709	18541
65	97
319	481
2291	3541
799	1249
481	769
4961	8161
45	75
1679	2929
161	289
1771	3229
56	106

Figure 1.3: Pairs in Plimpton 322

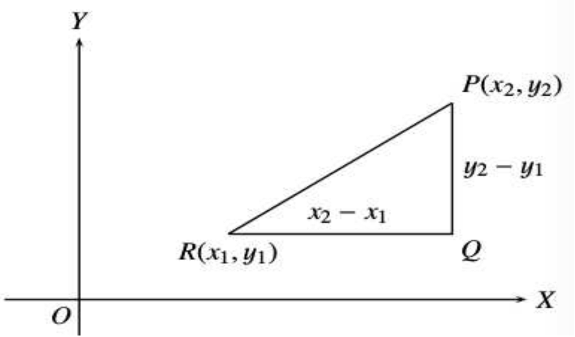
1.2.1 For each pair (a, c) in the table, compute $c^2 - a^2$, and confirm that it is a perfect square, b^2. (Computer assistance is recommended.)

You should notice that in most cases b is a "rounder" number than a or c.

1.2.2 Show that most of the numbers b are divisible by 60, and that the rest are divisible by 30 or 12.

Pythagorean formula defines "Euclidean lengths"

- Distance is related to length first defined as the "Euclidean length"



- Extended definitions of length requre:
  - *line integrals* (i.e., calculus: $\int_a^b f(x) \cdot dx$) c1650
  - *Complex vector dot products* $||x||^2 = \sum_k x_k^2$, $||x - y||$
  - N-dimensional complex "Hilbert space," (i.e., "normed" vector spaces)

Generalization of Pythagorean Formula

Generalization onto a circle (i.e., geometry):

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1 \tag{1}$$

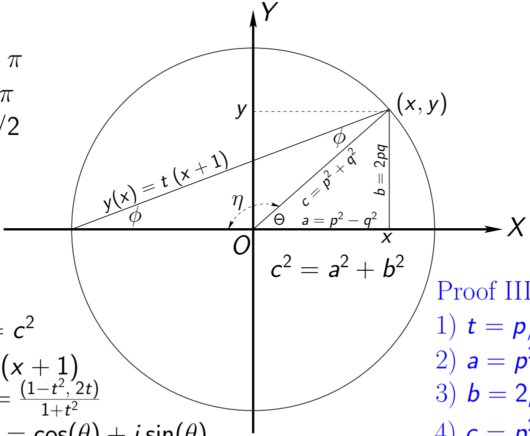
This leads to:

1. *Homogeneous Coordinates* (Möbius, 1827)
2. *Riemann sphere & surface*, *Branch cuts* and *Topology* 1851
3. The Riemann sphere *closes the plane*: thus ∞ is *complex analytic*.
4. Non-euclidean Geometry was soon discovered.
5. Newton's concept of topological *Genus* was starting to be understood
   - Genus 0 is related to the Padé approximation (ratio of polynomials), reminiscent of rational numbers (Potential theory)

Euclid's Formula

Proof I:

1) $2\phi + \eta = \pi$
2) $\eta + \Theta = \pi$
3) $\therefore \phi = \Theta/2$



Proof II:

1) $x^2 + y^2 = c^2$
2) $y(x) = t\,(x + 1)$
3) $\therefore (x, y) = \frac{(1-t^2,\, 2t)}{1+t^2}$
4) $e^{i\theta} = \frac{1+it}{1-it} = \cos(\theta) + i\sin(\theta)$

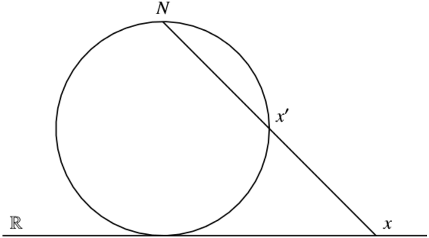
Proof III:

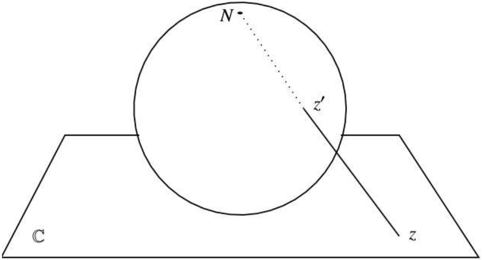
1) $t = p/q$
2) $a = p^2 - q^2$
3) $b = 2pq$
4) $c = p^2 + q^2$

Choose $p, q, N \in \mathbb{Z}_+$, with $p = q + N$, then $c^2 = a^2 + b^2$

Ex: $N = 1, q = 1, p = 2$: $a = 2^2 - 1^2 = 3$, $b = 2 \cdot 2 \cdot 1 = 4$, $c = 2^2 + 1^2 = 5$

1→2 and 2→3 dimensions by projection (Ch. 15)





Prime numbers π_k

- The primes are integers not “divisable” (no remainder) by any other integer other than 1 and itself
- $\pi_k|_{k=1}^{\infty} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots\}$
- $2^n - 1$ is typically prime (e.g., $2^n - 1$ is not prime for $n = 4, 6, 8, 9, 10$)
- Are most primes odd? All primes but 2 are odd
- How many primes are divisible by 3? Only 3
- Fundamental theorem of arithmetic: *Every integer N may be written as a product of primes π_k, of multiplicity m_k*

$$N = \Pi_k \pi_k^{m_k}$$

- Examples:
  - $14 = \pi_1 \cdot \pi_4^2$
  - $28 = 7 \cdot 2 \cdot 2 = \pi_4 \cdot \pi_1^2$
  - $3881196 = 2^2 \cdot 11^3 \cdot 27^2 = \pi_1^2 \cdot \pi_2^6 \cdot \pi_5^3$

Irrational numbers $\mathbb{J} \equiv \not{\mathbb{Q}}$

- The relations between the integers were assumed to be a reflection of the physical world
- This followed from the musical scale and other observations
- However this observation broke with the diagonal of a unit-square, where

$$d = \sqrt{1^2 + 1^2} = \sqrt{2} \tag{2}$$

- They soon "proved" $\sqrt{2}$ was not *rational*, thus
- It was termed *irrational* $\not{\mathbb{Q}} \equiv \mathbb{J}$ (*not* the ratio of two integers)

Real numbers $\mathbb{R}$

- Once irrational numbers were appreciated, it became clear that *real* numbers $\mathbb{R}$ must also exist
- This appreciation came slowly
- Integers are a subset of reals
- Prime numbers are a special subset of integers

Pythagorean Motto: *All is number*

- Integers were linked to Physics: i.e., Music and Planetary orbits
- The identification of irrational numbers spoiled this concept
- Yet today:

> *With the digital computer, digital audio, and digital video coding everything, at least approximately, [is transformed] into sequences of whole numbers, [thus] we are closer than ever to a world in which "all is number."* *p. 16*

Mapping the multi-valued square root of $w = \pm\sqrt{x+iy}$

- This provides a deep (essential) analytic insight.

15.3 Branch Points 303

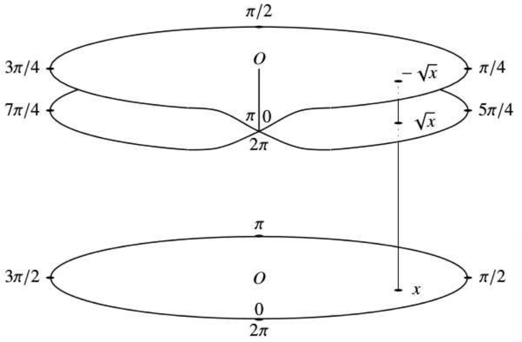


Figure 15.6: Branch point for the square root

- The Riemann Surface of the cubic $y^2 = x(x-a)(x-b)$ has Genis 1 (torus) (p. 307). *Elliptic functions* naturally follow.

Jakob Bernoulli #1 (1654-1705)

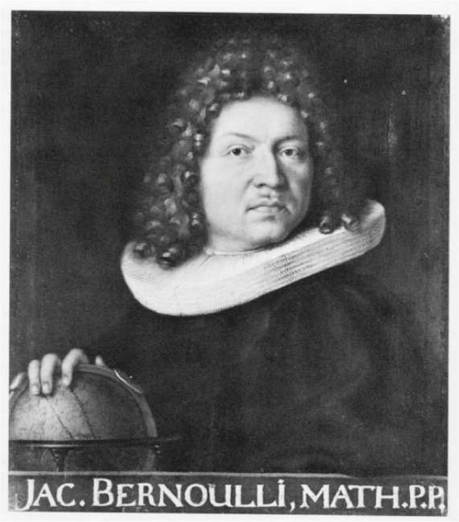


Figure 13.10: Portrait of Jakob Bernoulli by Nicholas Bernoulli

Johann Bernoulli (#2) 10^{th} child; Euler's advisor

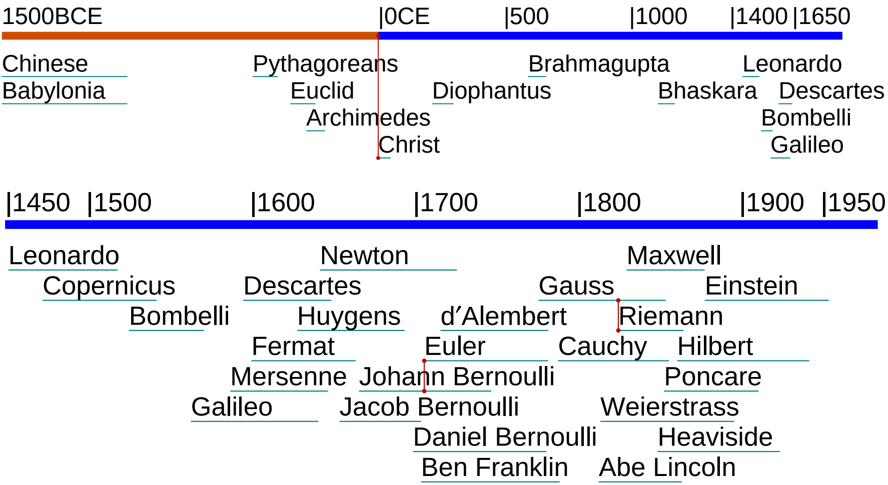
Figure 13.11: Johann Bernoulli

Leonhard Euler, most prolific of all mathematicians

Figure 10.4: Leonhard Euler

d'Alembert, creative but a bit "sloppy" (ideas were stolen)

Time Line 16-21 CE



Mathematics and its History (MH)

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