ECE 298JA **VC #1 – Version 3.03 November 14, 2017** **Fall 2017**

Univ. of Illinois **Due Mon, Dec 4, 2017** **Prof. Allen**

Topic of this homework: Vector algebra and fields in $\mathbb{R}^3$; Gradient and scalar Laplacian operator; Definitions of Divergence and Curl; Gauss's (divergence) & Stokes' (Curl) Law
Schwarz inequality; Quadratic forms; System postulates;

1 Vector algebra in $\mathbb{R}^3$.

Definitions of the dot, cross and triple product of vectors $\boldsymbol{A}\cdot\boldsymbol{B}$, $\boldsymbol{A}\times\boldsymbol{B}$ and $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})$ may be found in the class notes in Appendix A.2 (Vectors in $\mathbb{R}^3$). *Note:* $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) \neq (\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C}$. $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$.

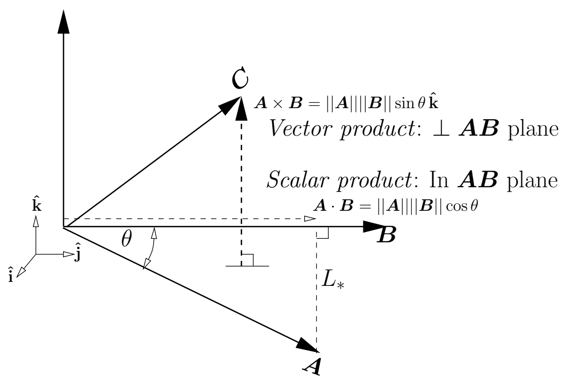


Figure 1: *Definitions of vectors $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$ (vectors in $\mathbb{R}^3$) used in the definition of $\boldsymbol{A}\cdot\boldsymbol{B}$, $\boldsymbol{A}\times\boldsymbol{B}$ and $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})$. There are two algebraic vector products, the* scalar (dot) product *$\boldsymbol{A}\cdot\boldsymbol{B}\in\mathbb{R}$ and the* vector (cross) product *$\boldsymbol{A}\times\boldsymbol{B}\in\mathbb{R}^3$. Note that the result of the dot product is a scalar, while the vector product yields a vector, which is $\perp$ to the plane containing $\boldsymbol{A},\boldsymbol{B}$. This is figure Fig. 1.13 from Lec. 14.*

To Do:

1. **Dot product $\mathbf{A}\cdot\mathbf{B}$**
   (a) If $\boldsymbol{A} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$ and $\boldsymbol{B} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}$, write out the definition of $\mathbf{A}\cdot\mathbf{B}$.
   (b) The dot product is often defined as $||\boldsymbol{A}||\,||\boldsymbol{B}||\cos(\theta)$, where $||\boldsymbol{A}|| = \sqrt{\boldsymbol{A}\cdot\boldsymbol{A}}$ and θ is the angle between $\boldsymbol{A},\boldsymbol{B}$. If $||\boldsymbol{A}|| = 1$, describe how the dot product relates to the vector $\mathbf{B}$.
2. **Cross product $\mathbf{A}\times\mathbf{B}$**
   (a) If $\boldsymbol{A} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$ and $\boldsymbol{B} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}$, write out the definition of $\mathbf{A}\times\mathbf{B}$.
   (b) Show that the cross product is equal to the area of the parallelogram formed by $\boldsymbol{A},\boldsymbol{B}$, namely $||\boldsymbol{A}||\,||\boldsymbol{B}||\sin(\theta)$, where $||\boldsymbol{A}|| = \sqrt{\boldsymbol{A}\cdot\boldsymbol{A}}$ and θ is the angle between $\boldsymbol{A},\boldsymbol{B}$.
3. **Triple product $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})$**
   Let $\boldsymbol{A} = [a_1,a_2,a_3]^T$, $\boldsymbol{B} = [b_1,b_2,b_3]^T$, $\boldsymbol{C} = [c_1,c_2,c_3]^T$ be three vectors in $\mathbb{R}^3$.
   (a) Starting from the definition of the dot and cross product, explain using a diagram and/or words, how one shows that: $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.
   (b) Describe why $|\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})|$ is the volume of parallelepiped generated by $\boldsymbol{A}$,$\boldsymbol{B}$ and $\boldsymbol{C}$.
   (c) Explain why three vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are in one plane if and only if the triple product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{0}$.
4. Given two vectors $\boldsymbol{A},\hat{\boldsymbol{B}}$ in the $\hat{\mathbf{i}},\hat{\mathbf{j}}$ plane (see Fig. 1), with $\boldsymbol{B} = \hat{\mathbf{j}}$ (i.e., $||\hat{\boldsymbol{B}}|| = 1$). Show that $\boldsymbol{A}$ may be split into two orthogonal parts, one in the direction of $\boldsymbol{B}$ and the other perpendicular ($\perp$) to $\boldsymbol{B}$.

$$\begin{aligned}\boldsymbol{A} &= (\boldsymbol{A}\cdot\hat{\boldsymbol{B}})\hat{\boldsymbol{B}} + \hat{\boldsymbol{B}}\times(\boldsymbol{A}\times\hat{\boldsymbol{B}}) \\ &= \boldsymbol{A}_{\parallel} + \boldsymbol{A}_{\perp}.\end{aligned}$$

2 Scalar fields and the ∇ operator

For a scalar field $\phi(x,y,z)$ in $\mathbb{R}^3$, the gradient ∇ and Laplacian $\nabla\cdot\nabla = \nabla^2$ operators are defined as

$$\nabla\phi(x,y,z) = \left[\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right]\phi(x,y,z) = \frac{\partial \phi}{\partial x}\hat{x} + \frac{\partial \phi}{\partial y}\hat{y} + \frac{\partial \phi}{\partial z}\hat{z} \qquad \text{(a } \textit{vector}\text{)}$$

$$(\nabla\cdot\nabla)\phi(x,y,z) = \nabla\cdot(\nabla\phi(x,y,z)) = \nabla^2\phi(x,y,z) = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\phi(x,y,z) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \qquad \text{(a } \textit{scalar}\text{)}$$

To Do:

1. Let $T(x,y) = x^2 + y$ be an analytic scalar temperature field in 2 dimensions (single-valued $\in \mathbb{R}^2$).
   (a) Find the gradient of $T(\mathbf{x})$ and make a sketch of T and the gradient..
   (b) Compute $\nabla^2 T(\mathbf{x})$, to determine if $T(\mathbf{x})$ satisfies Laplace's equation.
   (c) Sketch the iso-temperature contours at T = -10, 0, 10 degrees.
   (d) The *heat flux*[1] is defined as $\vec{J}(x,y) = -\kappa(x,y)\nabla T$ where $\kappa(x,y)$ is a constant denoting thermal conductivity at the point (x,y). Assuming $\kappa = 1$ everywhere (the medium is homogenous), plot the vector $\vec{J}(x,y) = -\nabla T$ at $x = 2$, $y = 1$. Be clear about the origin, direction and length of your result.
   (e) Find the vector $\perp$ to $\nabla T(x,y)$, namely tangent to the iso-temperature contours. Hint: Sketch it for one (x,y) point (e.g., 2,1) and then generalize.
   (f) The thermal resistance R_T is defined as the potential drop ΔT over the magnitude of the heat flux $|\vec{J}|$. At a single point the thermal resistance is

   $$R_T(x,y) = -\nabla T/\vec{J}$$

   How is $R_T(x,y)$ related to the thermal conductivity $\kappa(x,y)$?

2. **Acoustic wave equation:** *Note: In the following problem, we will work in the frequency domain.*
   The basic equations of acoustics in 1 dimension are

   $$-\frac{\partial}{\partial x}\mathcal{P} = \rho_o s\vec{\mathcal{V}} \quad \text{and} \quad -\frac{\partial}{\partial x}\vec{\mathcal{V}} = \frac{s}{\eta_o P_o}\mathcal{P}.$$

   Here $\mathcal{P}(x,\omega)$ is the pressure (in the frequency domain), $\mathcal{V}(x,\omega)$ is the volume velocity (integral of the velocity over the wave-front having area A), $s = \sigma + \omega\jmath$, $\rho_o = 1.2$ is the specific density of air, $\eta_o = 1.4$ and P_o is the atmospheric pressure (i.e., 10^5 [Pa]) (see the handout Appendix F.2 for details). **Note that the pressure field $\mathcal{P}$ is a scalar (pressure does not have direction), while the volume velocity field $\vec{\mathcal{V}}$ is a vector (velocity has direction).**

   We can generalize these equations to 3 dimensions using the ∇ operator

   $$-\nabla\mathcal{P} = \rho_o s\vec{\mathcal{V}} \quad \text{and} \quad -\nabla\cdot\vec{\mathcal{V}} = \frac{s}{\eta_o P_o}\mathcal{P}.$$

   (a) Starting from these two basic equations, derive the scalar wave equation in terms of the pressure $\mathcal{P}$,

   $$\nabla^2\mathcal{P} = \frac{s^2}{c_0^2}\mathcal{P}$$

   where c_0 is a constant representing the speed of sound.
   (b) What is c_0 in terms of η_0, ρ_0, and P_0?
   (c) Rewrite the pressure wave equation in the time domain, using the time derivative property of the Laplace transform (e.g. $dx/dt \leftrightarrow sX(s)$). For your notation, define the time-domain signal using a lowercase letter, $p(x,y,z,t) \leftrightarrow \mathcal{P}$.

[1] The heat flux is proportional to the change in temperature times the thermal conductivity κ of the medium `https://en.wikipedia.org/wiki/Heat_equation#Derivation_in_one_dimension`.

3 Vector fields and the ∇ operator

When ∇ operates on a vector there are two forms (as with vector products), the *scalar product*[2] (e.g., $\nabla \cdot \boldsymbol{A}$) and the *vector product*[3] (e.g., $\nabla \times \boldsymbol{A}$). These vector operations are defined in the class notes (Appendix A), and repeated here for convenience.

Defining $\boldsymbol{A} = [a_x a_y a_z]^T$ (i.e., $\boldsymbol{A} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$), then the dot product with ∇ is

$$\nabla \cdot \boldsymbol{A} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \qquad \text{"The } \textbf{divergence} \text{ of } \boldsymbol{A}\text{."}$$

The cross product with ∇ is defined as

$$\nabla \times \boldsymbol{A} = \det \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ a_x & a_y & a_z \end{bmatrix} \qquad \text{"The } \textbf{curl} \text{ of } \boldsymbol{A}\text{."}$$

Note the shorthand notation ∂_x, which represents $\frac{\partial}{\partial x}$.

The vector Laplacian $\boldsymbol{\nabla}^2$ defined by the repeated cross product

$$\nabla\times\nabla\times\boldsymbol{A} = \nabla\nabla\cdot\boldsymbol{A} - \boldsymbol{\nabla^2}\boldsymbol{A}.$$

It operates on vectors and is used with Maxwell's equations for defining the vector wave equation. We shall explore this in the final exercises VC-2.

To Do:

3.1 Vector Algebra

1. Let $\mathbf{R}(x, y, z) \equiv x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$:
   (a) If a, b, c are constants, what is $\mathbf{R}(x, y, z) \cdot \mathbf{R}(a, b, c)$?
   (b) If a, b, c are constants, what is $\frac{d}{dt}[\mathbf{R}(x, y, z) \cdot \mathbf{R}(a, b, c)]$?
2. Find the **divergence** and **curl** of the following vector fields:
   (a) $\vec{v} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$
   (b) $\vec{v}(x, y, z) = x\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$
   (c) $\mathbf{v}(x, y, z) = x\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + \log(z)\hat{\mathbf{k}}$
   (d) $\mathbf{v}(x, y, z) = \nabla(1/x + 1/y + 1/z)$

3.2 Vector & scalar field identities

1. Find the **divergence** and **curl** of the following vector fields:
   (a) $\mathbf{v} = \nabla\phi$, where $\phi(x, y) = xe^y$
   (b) $\mathbf{v} = \nabla \times \mathbf{A}$, where $\mathbf{A} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$
   (c) $\mathbf{v} = \nabla \times \mathbf{A}$, where $\mathbf{A} = y\hat{\mathbf{i}} + x^2\hat{\mathbf{j}} + z\hat{\mathbf{k}}$
2. For any differentiable vector field $\vec{V}$, write down two vector-calculus identities that are equal to zero.
3. What is the most general form of a vector field may be expressed in, in terms of scalar Φ and vector $\vec{A}$ potentials?
4. Perform the following calculations. If you can state the answer without doing the calculation, explain why.
   (a) Let $\mathbf{v} = \sin(\mathbf{x})\hat{\mathbf{i}} + \mathbf{y}\hat{\mathbf{j}} + \mathbf{z}\hat{\mathbf{k}}$. Find $\nabla \cdot (\nabla \times \vec{v})$ *Hint: Look at Lec 41 on page 83 of the notes, Eq. 1.58, 59.*
   (b) Let $\mathbf{v} = \sin(\mathbf{x})\hat{\mathbf{i}} + \mathbf{y}\hat{\mathbf{j}} + \mathbf{z}\hat{\mathbf{k}}$. Find $\nabla \times (\nabla\sqrt{\vec{v} \cdot \vec{v}})$
   (c) Let $\mathbf{v}(x, y, z) = \nabla[x + y^2 + \sin(\log(z)]$. Find $\nabla \times \mathbf{v}(x, y, z)$.

[2]Since dot product of two vectors is a scalar, it may be denoted the scalar product (of two vectors).
[3]Since a cross product of two vectors is a vector, it may be denoted the *vector product*.

3.3 Integral theorems

1. In a few words, identify the law, define what it means, and explain the following formula:

$$\int_{\mathcal{S}} \hat{\mathbf{n}} \cdot \mathbf{v}\, dA = \int_{\mathcal{V}} \nabla \cdot \mathbf{v}\, dV$$

2. What is the name of this formula?

$$\int_S (\nabla \times \vec{V}) \cdot d\vec{S} = \oint_C \vec{V} \cdot d\vec{R}$$

   Give one important application.

3. Describe a key application of the vector identity

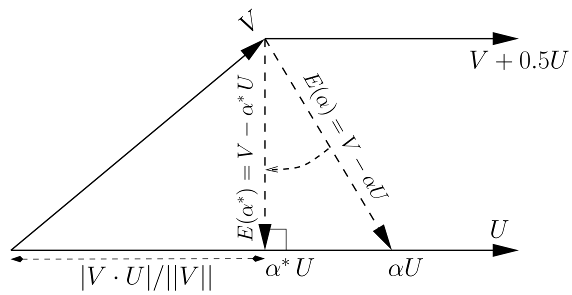
$$\nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \boldsymbol{\nabla}^{\mathbf{2}}\vec{V}$$

4 Schwarz inequality

Given two vectors in N-dimensional space $\boldsymbol{U}, \boldsymbol{V}$ and their weighted sum

$$\boldsymbol{E}(a) = \boldsymbol{V} + a\boldsymbol{U}$$

($a \in \mathbb{R}$). Below is a picture of these three vectors for an arbitrary value of a and a specific $a = a^*$.



1. Find the value of $a^* \in \mathbb{R}$ such that the length (norm) of $\boldsymbol{E}$ (i.e., $||\boldsymbol{E}|| \geq 0$) is minimum? Hint minimize

$$||\boldsymbol{E}||^2 = \boldsymbol{E} \cdot \boldsymbol{E} = (\boldsymbol{V} + a\boldsymbol{V}) \cdot (\boldsymbol{V} + a\boldsymbol{U}) \geq 0 \tag{1}$$

   with respect to a.

2. Find the formula for $||\boldsymbol{E}(a^*)||^2 \geq 0$. Hint: Substitute a^* into Eq. 1, and show that this results in the *Schwarz inequality*

$$|\boldsymbol{U} \cdot \boldsymbol{V}| \leq ||\boldsymbol{U}||||\boldsymbol{V}||.$$

3. What is the geometrical meaning of the dot product of two vectors?

4. Give the formula for the dot product between two vectors. Explain the meaning based on Fig. 4.

5. Write the formula for the "dot product" between two vectors: $\boldsymbol{U} \cdot \boldsymbol{V}$ in $\mathbb{R}^n$ in polar form (e.g., assume the angle between the vectors is equal to θ).

6. How is this related to the Pythagorean theorem?

7. Starting from $||\boldsymbol{U} + \boldsymbol{V}||$ derive the *triangle inequality*

$$||\boldsymbol{U} + \boldsymbol{V}|| \leq ||\boldsymbol{U}|| + ||\boldsymbol{V}||.$$

8. The *triangular inequality* $||\boldsymbol{U} + \boldsymbol{V}|| \leq ||\boldsymbol{U}|| + ||\boldsymbol{V}||$ is true for 2 and 3 dimensions: Does it hold for 5 dimensional vectors?

5 Quadratic forms

A matrix that has positive eigen-values is said to be *positive-definite.* The eigen-values are real if the matrix is symmetric, so this is a necessary condition for the matrix to be positive-definite. This condition is related to conservation of energy since the power is the voltage times the current. Given an impedance matrix,

$$\mathbf{V} = \mathbf{Z}\mathbf{I}$$

Then the power $\mathcal{P}$ is

$$\mathcal{P} = \mathbf{I} \cdot \mathbf{V} = \mathbf{I} \cdot \mathbf{Z}\mathbf{I},$$

which must be positive definite for the system to obey conservation of energy. For the following problems, consider the 2×2 $\mathbf{Z}$ matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}.$$

1. Solve for the power $\mathcal{P}(i_1, i_2)$ by multiplying out the matrix equation below (which is in *quadratic form*) ($\mathbf{I} \equiv \begin{bmatrix} i_1 & i_2 \end{bmatrix}^T$)
$$\mathcal{P}(i_1, i_2) = \mathbf{I}^T \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \mathbf{I}.$$
2. Is the impedance matrix *positive definite*? Show your work by finding the eigenvalues of the matrix $\mathbf{Z}$.
3. Should an impedance matrix always be positive definite? Explain.

6 System Classification

Provide a one sentence definition of the following properties:

L/NL : linear(L)/nonlinear(NL):

TI/TV : time-invariant(TI)/time varying(TV):

P/A : passive(P)/active(A):

C/NC : causal(C)/non-causal(NC):

Re/Clx : real(Re)/complex(Clx):

1. (8) Along the rows of the table, classify the following *systems*: In terms of a table having 5 columns, labeled with the abbreviations: L/NL, TI/TV, P/A, C/NC, Re/Clx:

					Category		
#	Case:	Definition	L/NL	TI/TV	P/A	C/NC	Re/Clx
1	Resistor	$v(t) = r_0\, i(t)$					
2	Inductor	$v(t) = L\frac{di}{dt}$					
3	Switch	$v(t) \equiv \begin{cases} 0 & t \le 0 \\ V_0 & t > 0. \end{cases}$					
5	Transistor	$I_{out} = g_m(V_{in})$					
7	"Resistor"	$v(t) = r_0\, i(t+3)$					
8	modulator	$f(t) = e^{i2\pi t} g(t)$					

2. (5) Using the same classification scheme, characterize the following equations:

#	Case:	L/NL	TI/TV	P/A	C/NC	Re/Clx
1	$A(x)\frac{d^2y(t)}{dt^2} + D(t)y(x,t) = 0$					
2	$\frac{dy(t)}{dt} + \sqrt{t}\; y(t) = \sin(t)$					
3	$y^2(t) + y(t) = \sin(t)$					
4	$\frac{\partial^2 y}{\partial t^2} + xy(t+1) + x^2 y = 0$					
5	$\frac{dy(t)}{dt} + (t-1)\; y^2(t) = ie^t$					