ECE 298JA **VC #2 – Version 3.0 December 5, 2017** **Fall 2017**

Univ. of Illinois **Due Wed, Dec 13, 2017** **Prof. Allen**

Topic of this homework: Maxwell's equations (ME) and variables ($\boldsymbol{E}, \boldsymbol{D}; \boldsymbol{B}, \boldsymbol{H}$); Compressible and rotational properties of vector fields; Fundamental Theorem of Vector Calculus (Helmholtz' Theorem); Riemann zeta function; Wave equation.

Notation: The following notation is used in this assignment:

1. $s = \sigma + j\omega$ is the *Laplace frequency*, as used in the Laplace transform.
2. A Laplace transform pair are indicated by the symbol $\leftrightarrow$: e.g., $f(t) \leftrightarrow F(s)$.
3. π_k is the k^{th} prime (i.e., $\pi_k \in \mathbb{P}$, e.g., $\pi_k = [2, 3, 5, 7, 11, 13 \cdots]$ for $k = 1..6$).

1 Partial differential equations (PDEs): Wave equation

1. Show that d'Alembert's solution, $\varrho(x,t) = f(t - x/c) + g(t + x/c)$, is a solution to the acoustic pressure wave equation, in 1-dimension:
$$\frac{\partial^2 \varrho(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \varrho(x,t)}{\partial t^2}$$
where $f(\xi)$ and $g(\xi)$ are arbitrary functions.

2. Solution to the wave equation in spherical coordinates (i.e, 3-dimensions):
   (a) Write out the wave equation in spherical coordinates $\varrho(\rho, \theta, \phi, t)$. Only consider the radial term ρ (i.e., dependence on angles θ, ϕ is assumed to be zero). *Hint: The form of the Laplacian as a function of the number of dimensions is given in the last appendix on Transmission lines and Acoustic Horns. Alternatively, look it up on the internet or in a calculus book.*

   (b) Show that the following is true:
$$\nabla^2_\rho R(\rho) \equiv \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\rho^2\frac{\partial}{\partial\rho}R(\rho) = \frac{1}{\rho}\frac{\partial^2}{\partial\rho^2}\rho R(\rho) \tag{1}$$
   Hint: Expand both sides of the equation.

   (c) Use the results from Eq. 1 to show that the solution to the spherical wave equation is
$$\nabla^2_\rho\, \varrho(\rho,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\,\varrho(\rho,t) \tag{2}$$
$$\varrho(\rho,t) = \frac{f(t-\rho/c)}{\rho} + \frac{g(t+\rho/c)}{\rho} \tag{3}$$

   (d) With $f(\xi) = \sin(\xi)$ and $g(\xi) = e^\xi u(\xi)$ [$u(\xi)$ is the step function] (Eq. 3) write down the solutions to the spherical wave equation.

(e) Sketch this last case for several times (e.g., 0, 1 2 seconds), and describe the behavior of the pressure $\varrho(\rho, t)$ as a function of time t and radius ρ.

(f) What happens when the inbound wave reaches the center at $\rho = 0$?

2 Helmholtz formula

Every differentiable vector field may be written as the sum of a *scalar potential* ϕ and *vector potential* $\boldsymbol{w}$. This relationship is best known as *The Fundamental theorem of vector calculus* (*Helmholtz' formula*).

$$\boldsymbol{v} = -\nabla\phi + \nabla\times\boldsymbol{w} \tag{4}$$

where ϕ is the *scalar potential* and $\boldsymbol{w}$ is the *vector potential.* This formula seems a natural extension of the algebraic $\boldsymbol{A}\cdot\boldsymbol{B} \perp \boldsymbol{A}\times\boldsymbol{B}$, since $\boldsymbol{A}\cdot\boldsymbol{B} \propto \|\boldsymbol{A}\|\|\boldsymbol{B}\|\cos(\theta)$ and $\boldsymbol{A}\times\boldsymbol{B} \propto \|\boldsymbol{A}\|\|\boldsymbol{B}\|\sin(\theta)$ as developed in the notes (Fig. A.1). Thus these orthogonal components have magnitude 1 when we take the norm, due to Euler's identity ($\cos^2(\theta) + \sin^2(\theta) = 1$).

Field type	Definition (most common)	Generator (form of potential)	Test (on $\boldsymbol{v}$)
Irrotational	$\nabla\times\boldsymbol{v} = 0$	$\boldsymbol{v} = -\nabla\phi$	$\nabla\times\boldsymbol{v} = 0$
Rotational	$\nabla\times\boldsymbol{v} \neq 0$	$\boldsymbol{v} = -\nabla\phi + \nabla\times\boldsymbol{w}$	$\nabla\times\boldsymbol{v} \neq 0$
Incompressible	$\nabla\cdot\boldsymbol{v} = 0$	$\boldsymbol{v} = \nabla\times\boldsymbol{w}$	$\nabla\cdot\boldsymbol{v} = 0$
Compressible	$\nabla\cdot\boldsymbol{v} \neq 0$	$\boldsymbol{v} = -\nabla\phi + \nabla\times\boldsymbol{w}$	$\nabla\cdot\boldsymbol{v} \neq 0$
Conservative	$\boldsymbol{v} = -\nabla\phi$	$\boldsymbol{v} = -\nabla\phi$	$\nabla\times\boldsymbol{v} = 0$
Solenoidal	$\nabla\cdot\boldsymbol{v} = 0$	$\boldsymbol{v} = \nabla\times\boldsymbol{w}$	$\nabla\cdot\boldsymbol{v} = 0$

Table 1: *Definitions of irrotational, rotational, incompressible and compressible. A solenoidal field is an alternative name for an incompressible field, and a conservative field is irrotational.*

Helmholtz' formula separates a vector field (i.e., $\boldsymbol{v}(\boldsymbol{x})$) into *compressible* and *rotational* parts:

1. The *rotational* (e.g. angular) part is defined by the vector potential $\boldsymbol{w}$, requiring $\nabla\times\nabla\times\boldsymbol{w} \neq 0$. A field is *irrotational* (conservative) when $\nabla\times\boldsymbol{v} = 0$, meaning that the field $\boldsymbol{v}$ can be generated using only [1] a scalar potential, $\boldsymbol{v} = \nabla\phi$ (note this is how a conservative field is usually defined, by saying there exists some ϕ such that $\boldsymbol{v} = \nabla\phi$).

2. The *compressible* (e.g. radial) part of a field is defined by the scalar potential ϕ, requiring $\nabla\cdot\nabla\phi = \nabla^2\phi \neq 0$. A field is *incompressible* (solenoidal) when $\nabla\cdot\boldsymbol{v} = 0$, meaning that the field $\boldsymbol{v}$ can be generated using only a vector potential, $\boldsymbol{v} = \nabla\times\boldsymbol{w}$.

[1] A note about the relationship between the generating function and the test: You might imagine special cases where $\nabla\times\boldsymbol{w} \neq 0$ but $\nabla\times\nabla\times\boldsymbol{w} = 0$ (or $\nabla\phi \neq 0$ but $\nabla^2\phi = 0$). In these cases, the vector (or scalar) potential can be recast as a scalar (or vector) potential.
Example: Consider a field $\boldsymbol{v} = \nabla\phi_0 + \boldsymbol{b}$ where $\boldsymbol{b} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$. Note that $\boldsymbol{b}$ can actually be generated by either a scalar potential ($\phi_1 = \frac{1}{2}[x^2 + y^2 + z^2]$, such that $\nabla\phi_1 = \boldsymbol{b}$) or a vector potential ($\boldsymbol{w}_0 = \frac{1}{2}[z^2\hat{\mathbf{i}} + x^2\hat{\mathbf{j}} + y^2\hat{\mathbf{k}}]$, such that $\nabla\times\boldsymbol{w}_0 = \boldsymbol{b}$). We find that $\nabla\times\boldsymbol{v} = 0$, therefore $\boldsymbol{v}$ must be irrotational. Therefore, we say this irrotational field is generated by $\nabla\phi = \nabla(\phi_0 + \phi_1)$.

The definitions and generating potential functions of irrotational (conservative) and incompressible (solenoidal) fields naturally follow from two key vector identities:

1. $\nabla \cdot (\nabla \times \boldsymbol{w}) = 0$
2. $\nabla \times (\nabla \phi) = 0$

Exercises:

1. Define the following:
   (a) A *conservative* vector field
   (b) A *irrotational* vector field
   (c) An *incompressible* vector field
   (d) A *solenoidal* vector field
2. When is a conservative field irrotational?
3. When is a incompressible field irrotational?
4. For each of the following, (i) compute $\nabla \cdot \boldsymbol{v}$, (ii) compute $\nabla \times v$, (iii) classify the vector field (e.g., conservative, irrotational, incompressible, etc.):
   (a) $\boldsymbol{v}(x, y, z) = -\nabla[3yx^3 + y \log(xy)]$
   (b) $\boldsymbol{v}(x, y, z) = xy\hat{\mathbf{i}} - z\hat{\mathbf{j}} + f(z)\hat{\mathbf{k}}$
   (c) $\boldsymbol{v}(x, y, z) = \nabla \times [x\hat{\mathbf{i}} - z\hat{\mathbf{j}}]$

3 Maxwell's Equations

The variables have the following names and defining equations:

Symbol	Equation	Name	Units
$\boldsymbol{E}$	$\nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$	Electric Field strength	[Volts/m]
$\boldsymbol{D} = \epsilon_o \boldsymbol{E}$	$\nabla \cdot \boldsymbol{D} = \rho$	Electric Displacement (flux density)	[Col/m^2]
$\boldsymbol{H}$	$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \dot{\boldsymbol{D}}$	Magnetic Field strength	[Amps/m]
$\boldsymbol{B} = \mu_o \boldsymbol{H}$	$\nabla \cdot \boldsymbol{B} = 0$	Magnetic Induction (flux density)	[Webers/m^2]

Note that $\boldsymbol{J} = \sigma \boldsymbol{E}$ is the *current density* (which has units of [Amps/m^2]). Furthermore the *speed of light in vacuo* is $c_o = 3 \times 10^8 = 1/\sqrt{\mu_0 \epsilon_0}$ [m/s], and the *characteristic resistance* of light $r_0 = 377 = \sqrt{\mu_0/\epsilon_0}$ [Ω (i.e., ohms)].

Exercises:

1. The *speed of light* in-vacuo is $c_o = 1/\sqrt{\mu_o \epsilon_o} \approx 3 \times 10^8$ [m/s]. The *characteristic resistance* in in-vacuo is $r_o = \sqrt{\mu_0/\epsilon_o} \approx 377\,[\Omega]$. Find a formula for the in-vacuo *permittivity* ϵ_o and *permeability* in terms of c_o and r_o. Based on your formula, what are the numeric values values of ϵ_o and μ_o?

2. The electric Maxwell equation is $\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$, where $\mathbf{E}$ the *Electric field strength* and $\dot{\mathbf{B}}$ is the time rate of change of the *magnetic induction field*, or simply the magnetic flux density. Consider this equation integrated over a two-dimensional surface S, where $\hat{n}$ is a unit vector normal to the surface (you may also find it useful to define the closed path C around the surface):

$$\iint_S [\nabla \times \mathbf{E}] \cdot \hat{\mathbf{n}} dS = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \hat{\mathbf{n}} dS$$

   (a) Apply Stokes' theorem to the left-hand side of the equation.

   (b) Consider the right-hand side of the equation. How is it related to the magnetic flux Ψ through the surface S?

   (c) Assume the right-hand side of the equation is zero. Can you relate your answer to part (a) to one of Kirchhoff's laws?

3. The magnetic Maxwell Equation is $\nabla \times \boldsymbol{H} = \boldsymbol{C} \equiv \boldsymbol{J} + \dot{\boldsymbol{D}}$, where $\boldsymbol{H}$ is the *magnetic field strength*, $\boldsymbol{J} = \sigma \mathbf{E}$ is the conductive (resistive) current density and the *displacement current* $\dot{\boldsymbol{D}}$ is the time rate of change of the *electric flux density* $\boldsymbol{D}$. Here we defined a new variable $\mathbf{C}$ as the total current density.

   (a) First consider the equation over a two dimensional surface S,

$$\iint_S [\nabla \times \mathbf{H}] \cdot \hat{\mathbf{n}} dS = \iint_S [\mathbf{J} + \dot{\mathbf{D}}] \cdot \hat{\mathbf{n}} dS = \iint_S \mathbf{C} \cdot \hat{\mathbf{n}} dS$$

   Apply Stokes' theorem to the left-hand side of this equation. In a sentence or two, explain the meaning of the resulting equation. *Hint: What is the right-hand side of the equation?*

   (b) Now consider this equation in three dimensions. Take the divergence of both sides, and integrate over a volume V (closed surface S).

$$\iiint_V \nabla \cdot [\nabla \times \mathbf{H}] dV = \iiint_V \nabla \cdot \mathbf{C} dV$$

      i. What happens to the left-hand side of this equation? *Hint: Can you apply a vector identity?*

      ii. Apply the divergence theorem (sometimes known as Gauss's theorem) to the right hand side of the equation, and interpret your result. *Hint: Can you relate your result to one of Kirchhoff's laws?*

4. When $\mathbf{V}(x, y, z) = \nabla(1/x + 1/y + 1/z)$ what is $\nabla \times \mathbf{V}(x, y, z)$?

5. When was Maxwell born (and die)? How long did he live (within ± 10 years)?

3.1 Capacitor analysis

1. Find the solution to the Laplace equation between two infinite [2] parallel plates, separated by a distance of d. Assume that the left plate, at $x = 0$, is at a voltage of $V(0) = 0$, and the right plate, at $x = d$, is at a voltage of $V_d \equiv V(d)$.

   (a) Write down Laplace's equation in one dimension for $V(x)$.

   (b) Write down the general solution to your differential equation for $V(x)$.

   (c) Apply the boundary conditions $V(0) = 0$ and $V(d) = V_d$ to solve for the constants in your equation from the previous part.

   (d) Find the charge density <u>per unit area</u> ($\sigma = Q/A$, where Q is charge and A is area) on the surface of each plate. *Hint:* $\mathbf{E} = -\nabla V$, *and Gauss's Law states that* $\iint_S \mathbf{D} \cdot \hat{\mathbf{n}} dS = Q_{enclosed}$.

   (e) Determine the per-unit-area capacitance C of the system.

4 Webster Horn Equation

Horns provide an important generalization of the solution of the 1D wave equation, in regions where the properties (i.e., area of the tube) vary along the axis of wave propagation. Classic applications of horns is vocal tract acoustics, loudspeaker design, cochlear mechanics, any case having wave propagation.

To do: Write out the formula for the Webster Horn equation, and explain the variables.

5 Riemann zeta function ($\zeta(s)$)

The zeta function $\zeta(s)$ is defined by the complex analytic power series

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots . \tag{5}$$

This series converges, and thus is valid, only in the region of convergence (ROC) given by $\Re s = \sigma > 1$ since there $|n^{-\sigma}| < 1$. To determine its formula in other regions of the s plane one must extend the series via analytic continuation.

5.1 Euler product formula

As was first published by Euler in 1737, one may recursively factor out the leading prime term, resulting in Euler's product formula.[3] Multiplying $\zeta(s)$ by the factor $1/2^s$, and subtracting from

[2] We study plates that are infinite because this means the electric field lines will be perpendicular to the plates, running directly from one plate to the other. However, we will solve for per-unit-area characteristics of the capacitor.

[3] This is known as *Euler's sieve*, as distinguish from the *Eratosthenes sieve*.

$\zeta(s)$, removes all the terms $1/(2n)^s$ (e.g., $1/2^s + 1/4^s + 1/6^s + 1/8^s + \cdots$)

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} \cdots - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \cdots\right), \tag{6}$$

which results in

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \cdots. \tag{7}$$

1. Repeat this with a lead factor $1/3^s$ applied to Eq. 7.
2. Repeat this process, with all prime scale factors, (i.e., $1/5^s, 1/7^s, \cdots, 1/\pi_k^s, \cdots$), and show that
$$\zeta(s) = \prod_{\pi_k \in \mathbb{P}} \frac{1}{1 - \pi_k^{-s}} = \prod_{\pi_k \in \mathbb{P}} \zeta_k(s). \tag{8}$$

5.1.1 Poles of $\zeta_p(s)$

Given the product formula we may identify the poles of $\zeta_p(s)$ ($p \in \mathbb{Z}$), which is important for defining the ROC of each factor. For example, the p^{th} factor of Eq. 8, expressed as an exponential, is

$$\zeta_p(s) \equiv \frac{1}{1 - \pi_p^{-s}} = \frac{1}{1 - e^{-sT_p}}, \tag{9}$$

where $T_p \equiv \ln \pi_p$ (π_p represents the p^{th} prime).

Plot $\zeta_p(s)$ using `zviz` for $p = 1$. Describe what you see.

5.2 Inverse Laplace transform

Take the inverse Laplace transform of $\zeta_p(s) \leftrightarrow z_p(t)$ (Eq. 9) and describe the result in words. *Hint: Consider the geometric series representation*

$$\zeta_p(s) = \frac{1}{1 - e^{-sT_p}} = \sum_{k=0}^{\infty} e^{-skT_p}, \tag{10}$$

for which you can easily look up (or may have memorized) the inverse Laplace transform of each term.

5.2.1 Inverse transform of Product of factors

The time domain version of Eq. 8 may be written as the convolution of all the $z_k(t)$ factors

$$z(t) \equiv z_2 \star z_3(t) \star z_5(t) \star z_7(t) \cdots \star z_p(t) \cdots, \tag{11}$$

where $\star$ represents time convolution.

Explain what this means in physical terms. Start with two terms (e.g., $z_1(t) \star z_2$).