Chapter 2
Algebraic Equations

2.1 Problems AE-1

Topics of this homework: Fundamental theorem of algebra, polynomials, analytic functions and their inverse, convolution, Newton’s root finding method, Riemann zeta function. Deliverables: Answers to problems

Note: The term analytic is used in two different ways. (1) An analytic function is a function that may be expressed as a locally convergent power series; (2) analytic geometry refers to geometry using a coordinate system.

Polynomials and the fundamental theorem of algebra (FTA)

Problem # 1: A polynomial of degree $N$ is defined as

$$P_N(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N.$$  

– 1.1: How many coefficients $a_n$ does a polynomial of degree $N$ have?

Ans:

– 1.2: How many roots does $P_N(x)$ have?

Ans:

Problem # 2: The fundamental theorem of algebra (FTA)
2.1: State and then explain the FTA.

Ans:

2.2: Using the FTA, prove your answer to question 1.2. Hint: Apply the FTA to prove how many roots a polynomial $P_N(x)$ of order $N$ has.

Ans:

Problem # 3: Consider the polynomial function $P_2(x) = 1 + x^2$ of degree $N = 2$ and the related function $F(x) = 1/P_2(x)$. What are the roots (e.g., zeros) $x_\pm$ of $P_2(x)$? Hint: Complete the square on the polynomial $P_2(x) = 1 + x^2$ of degree 2, and find the roots.

Ans:

Problem # 4: $F(x)$ may be expressed as ($A, B, x_\pm \in \mathbb{C}$)

$$F(x) = \frac{A}{x - x_+} + \frac{B}{x - x_-},$$

where $x_\pm$ are the roots (zeros) of $P_2(x)$, which become the poles of $F(x)$; $A$ and $B$ are the residues. The expression for $F(x)$ is sometimes called a partial fraction expansion or residue expansion, and it appears in many engineering applications.

4.1: Find $A, B \in \mathbb{C}$ in terms of the roots $x_\pm$ of $P_2(x)$.

Ans:

4.2: Verify your answers for $A$ and $B$ by showing that this expression for $F(x)$ is indeed equal to $1/P_2(x)$.

Ans:
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– 4.3: Give the values of the poles and zeros of $P_2(x)$.

**Ans:**

– 4.4: Give the values of the poles and zeros of $F(x) = 1/P_2(x)$.

**Ans:**

2.1.1 Analytic functions

**Overview:** Analytic functions are defined by infinite (power) series. The function $f(x)$ is said to be analytic at any value of constant $x = x_o$, where there exists a convergent power series

$$P(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n$$

such that $P(x_o) = f(x_o)$. The point $x = x_o$ is called the expansion point. The region around $x_o$ such that $|x - x_o| < 1$ is called the radius of convergence, or region of convergence (RoC). The local power series for $f(x)$ about $x = x_o$ is defined by the Taylor series:

$$f(x) \approx f(x_o) + \frac{df}{dx} \bigg|_{x=x_o} (x - x_o) + \frac{1}{2!} \frac{d^2 f}{dx^2} \bigg|_{x=x_o} (x - x_o)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n f(x)}{dx^n} \bigg|_{x=x_o} (x - x_o)^n.$$

Two classic examples are the geometric series\(^1\) where $a_n = 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n,$$

and the exponential function where $a_n = 1/n!$, Eq. 3.2.11 (p. 68). The coefficients for both series may be derived from the Taylor formula.

**Problem # 5: The geometric series**

– 5.1: What is the region of convergence (RoC) for the power series Eq. AE-1.2 of $1/(1-x)$ given above—for example, where does the power series $P(x)$ converge to the function value $f(x)$? State your answer as a condition on $x$. Hint: What happens to the power series when $x > 1$?

**Ans:**

\(^1\)The geometric series is not defined as the function $1/(1-x)$, it is defined as the series $1 + x + x^2 + x^3 + \cdots$, such that the ratio of consecutive terms is $x$. 
– 5.2: In terms of the pole, what is the RoC for the geometric series in Eq. AE-1.2?

Ans:

– 5.3: How does the RoC relate to the location of the pole of $1/(1 - x)$?

Ans:

– 5.4: Where are the zeros, if any, in Eq. AE-1.2?

Ans:

– 5.5: Assuming $x$ is in the RoC, prove that the geometric series correctly represents $1/(1 - x)$ by multiplying both sides of Eq. AE-1.2 by $(1 - x)$.

Ans:

Problem # 6: Use the geometric series to study the degree N polynomial. It is very important to note that all the coefficients $c_n$ of this polynomial are 1.

$$P_N(x) = 1 + x + x^2 + \cdots + x^N = \sum_{n=0}^{N} x^n.$$  \hspace{1cm} (AE-1.3)

– 6.1: Prove that

$$P_N(x) = \frac{1 - x^{N+1}}{1 - x}.$$  \hspace{1cm} (AE-1.4)

Ans:
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- 6.2: What is the RoC for Eq. AE-1.3?
  **Ans:**

- 6.3: What is the RoC for Eq. AE-1.4?
  **Ans:**

- 6.4: How many poles does $P_N(x)$ (Eq. AE-1.3) have? Where are they?
  **Ans:**

- 6.5: How many zeros does $P_N(x)$ (Eq. AE-1.4) have? State where they in the complex plane.
  **Ans:**

- 6.6: Explain why Eqs. AE-1.3 and AE-1.4 have different numbers of poles and zeros.
  **Ans:**

- 6.7: Is the function $1/(1 - x)$ analytic outside of the RoC?
  **Ans:**
– 6.8: Extra credit. Evaluate $P_N(x)$ at $x = 0$ and $x = 0.9$ for the case of $N = 100$, and compare the result to that from Matlab.

% sum the geometric series and $P_{100}(0.9)$
clear all; close all; format long
N=100; x=0.9; S=0;
for n=0:N
    S=S+x^n
end
P100=(1-x^(N+1))/(1-x);
disp(sprintf('S= %g, P100= %g, error= %g', S, P100, S-P100))

Ans:

Problem # 7: The exponential series

– 7.1: What is the RoC for the exponential series Eq. 3.2.11?
Ans:

– 7.2: Let $x = j$ in Eq. 3.2.11, and write out the series expansion of $e^x$ in terms of its real and imaginary parts.
Ans:

– 7.3: Let $x = j\theta$ in Eq. 3.2.11, and write out the series expansion of $e^x$ in terms of its real and imaginary parts. How does your result relate to Euler’s identity ($e^{j\theta} = \cos(\theta) + j\sin(\theta)$)?
Ans:
2.1.2 Inverse analytic functions and composition

Overview: It may be surprising, but every analytic function has an inverse function. Starting from the function \((x, y \in \mathbb{C})\)

\[ y(x) = \frac{1}{1-x} \]

the inverse is

\[ x = \frac{y - 1}{y} = 1 - \frac{1}{y}. \]

Problem # 8: Consider the inverse function described above

- 8.1: Where are the poles and zeros of \(x(y)\)?
Answ:

- 8.2: Where (for what condition on \(y\)) is \(x(y)\) analytic?
Answ:

Problem # 9 Consider the exponential function \(z(x) = e^x (x, z \in \mathbb{C})\).

- 9.1: Find the inverse \(x(z)\).
Answ:

- 9.2: Where are the poles and zeros of \(x(z)\)?
Answ:
Problem # 10: Composition.

– 10.1: If \( y(s) = \frac{1}{1 - s} \) and \( z(s) = e^s \), compose these two functions to obtain \( (y \circ z)(s) \). Give the expression for \( (y \circ z)(s) = y(z(s)) \). Ans:

– 10.2: Where are the poles and zeros of \( (y \circ z)(s) \)? Ans:

– 10.3: Where (for what condition on \( x \)) is \( (y \circ z)(x) \) analytic? Ans:

Convolution

Multiplying two short or simple polynomials is not demanding. However, if the polynomials have many terms, it can become tedious. For example, multiplying two 10th-degree polynomials is not something one would want to do every day.

An alternative is a method called convolution, as described in Sec. 3.4 (p. 78).

Problem # 11: Convolution of sequences. Practice convolution (by hand!!) using a few simple examples. Show your work!!! Check your solution using Matlab.

– 11.1: Convolve the sequence \( \{0, 1, 1, 1, 1\} \) with itself. Ans:

– 11.2: Calculate \( \{1, 1\} \ast \{1, 1\} \ast \{1, 1\} \).
Problem # 12: Multiplying two polynomials is the same as convolving their coefficients.

\[ f(x) = x^3 + 3x^2 + 3x + 1 \]
\[ g(x) = x^3 + 2x^2 + x + 2. \]

– 12.1: In Octave/Matlab, compute \( h(x) = f(x) \cdot g(x) \) in two ways: (1) use the commands \texttt{roots} and \texttt{poly}, and (2) use the convolution command \texttt{conv}. Confirm that both methods give the same result.

\textbf{Ans:}

– 12.2: What is \( h(x) \)?

\textbf{Ans:}

Newton’s root-finding method

Problem # 13: Use Newton’s iteration to find the roots of the polynomial

\[ P_3(x) = 1 - x^3. \]

– 13.1: Draw a graph describing the first step of the iteration starting with \( x_0 = (1/2, 0) \).

\textbf{Ans:}
13.2: Calculate $x_1$ and $x_2$. What number is the algorithm approaching?

**Ans:**

13.3: Does Newton’s method work for $P_2(x) = 1 + x^2$? If so, why? Hint: What are the roots in this case?

**Ans:**

### Riemann zeta function $\zeta(s)$

**Definitions and preliminary analysis:** The zeta function $\zeta(s)$ is defined by the complex analytic power series

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots.$$  

This series converges, and thus is valid, only in the RoC given by $\Re s = \sigma > 1$, since there $|n^{-\sigma}| \leq 1$. To determine its formula in other regions of the $s$ plane, one must extend the series via analytic continuation (see p. 67).

**Euler product formula:** As Euler first published in 1737, one may recursively factor out the leading prime term, which results in Euler’s product formula.\(^2\) Multiplying $\zeta(s)$ by the factor $1/2^s$ and subtracting from $\zeta(s)$ remove all the terms $1/(2n)^s$ (e.g., $1/2^s + 1/4^s + 1/6^s + 1/8^s + \cdots$)

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \cdots - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \cdots\right),$$  

which results in

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \cdots.$$  

**Problem #14: Questions about the Riemann zeta function.**

14.1: What is the RoC for Eq. AE-1.6?

**Ans:**

\(^2\)This is known as Euler’s sieve, as distinguished from the Eratosthenes sieve.
– 14.2: Repeat the algebra of Eq. AE-1.5 using the lead factor of $1/3^s$.

**Ans:**

– 14.3: What is the RoC for Eq. AE-1.6?

**Ans:**

– 14.4: Repeat the algebra of Eq. AE-1.5 for all prime scale factors (i.e., $1/5^s$, $1/7^s$, $1/11^s$, ... $1/\pi_k^s$, ...) to show that

$$
\zeta(s) = \prod_{\pi_k \in \mathbb{P}} \frac{1}{1 - \pi_k^{-s}} = \prod_{\pi_k \in \mathbb{P}} \zeta_k(s),
$$

(AE-1.7)

where $\pi_p$ represents the $p$th prime.

**Ans:**

– 14.5: Given the product formula, identify the poles of $\zeta_p(s)$ ($p \in \mathbb{Z}$), which is important for defining the RoC of each factor. For example, the $p$th factor of Eq. AE-1.7, expressed as an exponential, is

$$
\zeta_p(s) \equiv \frac{1}{\prod_{\pi \neq p} (1 - \pi^{-s})} = \frac{1}{1 - e^{-sT_p}},
$$

(AE-1.8)

where $T_p \equiv \ln \pi_p$.

**Ans:**
- 14.6: Plot Eq. AE-1.8 using \texttt{zviz} for $p = 1$. Describe what you see.

\textbf{Ans:}