Chapter 2

Algebraic Equations

2.1 Problems AE-1

Topics of this homework: Fundamental theorem of algebra, polynomials, analytic functions and their inverse, convolution, Newton's root finding method, Riemann zeta function. Deliverables: Answers to problems

Note: The term analytic is used in two different ways. (1) An analytic function is a function that may be expressed as a locally convergent power series; (2) analytic geometry refers to geometry using a coordinate system.

Polynomials and the fundamental theorem of algebra (FTA)

Problem # *1: A polynomial of degree N is defined as*

$$P_N(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N.$$

– 1.1: How many coefficients a_n does a polynomial of degree N have?

Ans:

– 1.2: How many roots does $P_N(x)$ have?

Ans:

Problem # *2: The* fundamental theorem of algebra *(FTA)*

– 2.1: State and then explain the FTA.

Ans:

– 2.2: Using the FTA, prove *your answer to question 1.2.* Hint: Apply the FTA to *prove* how many roots a polynomial $P_N(x)$ of order N has.
Ans:

Problem # *3: Consider the polynomial function $P_2(x) = 1 + x^2$ of degree $N = 2$ and the related function $F(x) = 1/P_2(x)$. What are the roots (e.g., zeros) $x_\pm$ of $P_2(x)$?* Hint: Complete the square on the polynomial $P_2(x) = 1 + x^2$ of degree 2, and find the roots.
Ans:

Problem # *4: $F(x)$ may be expressed as $(A, B, x_\pm \in \mathbb{C})$*

$$F(x) = \frac{A}{x - x_+} + \frac{B}{x - x_-}, \tag{AE-1.1}$$

where $x_\pm$ are the roots (zeros) of $P_2(x)$, which become the *poles* of $F(x)$; A and B are the *residues*. The expression for $F(x)$ is sometimes called a *partial fraction expansion* or *residue expansion*, and it appears in many engineering applications.

– 4.1:Find $A, B \in \mathbb{C}$ in terms of the roots $x_\pm$ of $P_2(x)$.
Ans:

– 4.2: Verify your answers for A and B by showing that this expression for $F(x)$ is indeed equal to $1/P_2(x)$.
Ans:

– *4.3: Give the values of the poles and zeros of $P_2(x)$.*
Ans:

– *4.4: Give the values of the poles and zeros of $F(x) = 1/P_2(x)$.*
Ans:

2.1.1 Analytic functions

Overview: Analytic functions are defined by infinite (power) series. The function $f(x)$ is said to be *analytic* at any value of constant $x = x_o$, where there exists a convergent power series

$$P(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n$$

such that $P(x_o) = f(x_o)$. The point $x = x_o$ is called the *expansion point*. The region around x_o such that $|x - x_o| < 1$ is called the *radius of convergence*, or region of convergence (RoC). The local power series for $f(x)$ about $x = x_o$ is defined by the Taylor series:

$$\begin{aligned} f(x) &\approx f(x_o) + \left.\frac{df}{dx}\right|_{x=x_o} (x - x_o) + \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=x_o} (x - x_o)^2 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n}{dx^n} f(x)\right|_{x=x_o} (x - x_o)^n. \end{aligned}$$

Two classic examples are the geometric series[1] where $a_n = 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \tag{AE-1.2}$$

and the exponential function where $a_n = 1/n!$, Eq. 3.2.11 (p. 68). The coefficients for both series may be derived from the Taylor formula.

Problem # 5: *The geometric series*

– *5.1: What is the* region of convergence *(RoC) for the power series Eq. AE-1.2 of $1/(1-x)$ given above—for example, where does the power series $P(x)$ converge to the function value $f(x)$? State your answer as a condition on x.* Hint: What happens to the power series when $x > 1$?
Ans:

[1]The geometric series is *not* defined as the function $1/(1-x)$, it is defined as the series $1 + x + x^2 + x^3 + \cdots$, such that the ratio of consecutive terms is x.

– 5.2: In terms of the pole, what is the RoC for the geometric series in Eq. AE-1.2?
Ans:

– 5.3: How does the RoC relate to the location of the pole of $1/(1-x)$*?*
Ans:

– 5.4: Where are the zeros, if any, in Eq. AE-1.2?

Ans:

– 5.5: Assuming x *is in the RoC, prove that the geometric series correctly represents* $1/(1-x)$ *by multiplying both sides of Eq. AE-1.2 by* $(1-x)$.
Ans:

Problem # *6: Use the geometric series to study the degree* N *polynomial. It is very important to note that all the coefficients* c_n *of this polynomial are* 1.

$$P_N(x) = 1 + x + x^2 + \cdots + x^N = \sum_{n=0}^{N} x^n. \tag{AE-1.3}$$

– 6.1: Prove that

$$P_N(x) = \frac{1 - x^{N+1}}{1 - x}. \tag{AE-1.4}$$

Ans:

– *6.2: What is the RoC for Eq. AE-1.3?*
Ans:

– *6.3: What is the RoC for Eq. AE-1.4?*
Ans:

– *6.4: How many poles does $P_N(x)$ (Eq. AE-1.3) have? Where are they?*
Ans:

– *6.5: How many zeros does $P_N(x)$ (Eq. AE-1.4) have? State where are they in the complex plane.*
Ans:

– *6.6: Explain why Eqs. AE-1.3 and AE-1.4 have different numbers of poles and zeros.*
Ans:

– *6.7: Is the function $1/(1-x)$ analytic outside of the RoC?*
Ans:

– *6.8: Extra credit. Evaluate $P_N(x)$ at $x = 0$ and $x = 0.9$ for the case of $N = 100$, and compare the result to that from Matlab.*

```
%sum the geometric series and P_100(0.9)
clear all;close all;format long
N=100; x=0.9; S=0;
for n=0:N
S=S+x^n
end
P100=(1-x^(N+1))/(1-x);
disp(sprintf('S= %g, P100= %g, error= %g',S,P100, S-P100))
```

Ans:

Problem # *7: The exponential series*

– *7.1: What is the RoC for the exponential series Eq. 3.2.11?*
Ans:

– *7.2: Let $x = \jmath$ in Eq. 3.2.11, and write out the series expansion of e^x in terms of its real and imaginary parts.*
Ans:

– *7.3: Let $x = \jmath\theta$ in Eq. 3.2.11, and write out the series expansion of e^x in terms of its real and imaginary parts. How does your result relate to Euler's identity ($e^{\jmath\theta} = \cos(\theta) + \jmath\sin(\theta)$)?*
Ans:

2.1.2 Inverse analytic functions and composition

Overview: It may be surprising, but every analytic function has an inverse function. Starting from the function ($x, y \in \mathbb{C}$)

$$y(x) = \frac{1}{1-x}$$

the inverse is

$$x = \frac{y-1}{y} = 1 - \frac{1}{y}.$$

Problem # *8: Consider the inverse function described above*

– *8.1: Where are the poles and zeros of $x(y)$?*
Ans:

– *8.2: Where (for what condition on y) is $x(y)$ analytic?*
Ans:

Problem # *9 Consider the exponential function $z(x) = e^x$ ($x, z \in \mathbb{C}$).*

– *9.1: Find the inverse $x(z)$.*
Ans:

– *9.2: Where are the poles and zeros of $x(z)$?*
Ans:

Problem # *10: Composition.*

– *10.1: If $y(s) = 1/(1-s)$ and $z(s) = e^s$, compose these two functions to obtain $(y \circ z)(s)$.* Give the expression for $(y \circ z)(s) = y(z(s))$. **Ans:**

– *10.2: Where are the poles and zeros of $(y \circ z)(s)$?*
Ans:

– *10.3: Where (for what condition on x) is $(y \circ z)(x)$ analytic?*
Ans:

Convolution

Multiplying two short or simple polynomials is not demanding. However, if the polynomials have many terms, it can become tedious. For example, multiplying two 10th-degree polynomials is not something one would want to do every day.

An alternative is a method called *convolution*, as described in Sec. 3.4 (p. 78).

Problem # *11: Convolution of sequences. Practice convolution (by hand!!) using a few simple examples. Show your work!!! Check your solution using Matlab.*

– *11.1: Convolve the sequence $\{0\ 1\ 1\ 1\ 1\}$ with itself.*
Ans:

– *11.2: Calculate $\{1, 1\} \star \{1, 1\} \star \{1, 1\}$.*

Ans:

Problem # *12: Multiplying two polynomials is the same as convolving their coefficients.*

$$f(x) = x^3 + 3x^2 + 3x + 1$$
$$g(x) = x^3 + 2x^2 + x + 2.$$

– 12.1: In Octave/Matlab, compute $h(x) = f(x) \cdot g(x)$ in two ways: (1) use the commands `roots` *and* `poly`*, and (2) use the convolution command* `conv`*. Confirm that both methods give the same result.*

Ans:

– 12.2: What is $h(x)$?

Ans:

Newton's root-finding method

Problem # *13: Use Newton's iteration to find the roots of the polynomial*

$$P_3(x) = 1 - x^3.$$

– 13.1: Draw a graph describing the first step of the iteration starting with $x_0 = (1/2, 0)$.
Ans:

– *13.2: Calculate x_1 and x_2. What number is the algorithm approaching?*
Ans:

– *13.3: Does Newton's method work for $P_2(x) = 1 + x^2$? If so, why?* Hint: What are the roots in this case?
Ans:

Riemann zeta function $\zeta(s)$

Definitions and preliminary analysis: The zeta function $\zeta(s)$ is defined by the complex analytic power series

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots .$$

This series converges, and thus is valid, only in the RoC given by $\Re s = \sigma > 1$, since there $|n^{-\sigma}| \le 1$. To determine its formula in other regions of the s plane, one must extend the series via analytic continuation (see p. 67).

Euler product formula: As Euler first published in 1737, one may recursively factor out the leading prime term, which results in Euler's product formula.[2] Multiplying $\zeta(s)$ by the factor $1/2^s$ and subtracting from $\zeta(s)$ remove all the terms $1/(2n)^s$ (e.g., $1/2^s + 1/4^s + 1/6^s + 1/8^s + \cdots$)

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \cdots\right), \tag{AE-1.5}$$

which results in

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \cdots . \tag{AE-1.6}$$

Problem # *14: Questions about the Riemann zeta function.*

– *14.1: What is the RoC for Eq. AE-1.6?*

Ans:

[2] This is known as *Euler's sieve*, as distinguished from the Eratosthenes sieve.

– 14.2: Repeat the algebra of Eq. AE-1.5 using the lead factor of $1/3^s$*.*

Ans:

– 14.3: What is the RoC for Eq. AE-1.6?

Ans:

– 14.4: Repeat the algebra of Eq. AE-1.5 for all prime scale factors (i.e., $1/5^s$*,* $1/7^s$*, . . .,* $1/\pi_k^s$*, . . .) to show that*

$$\zeta(s) = \prod_{\pi_k \in \mathbb{P}} \frac{1}{1-\pi_k^{-s}} = \prod_{\pi_k \in \mathbb{P}} \zeta_k(s), \tag{AE-1.7}$$

where π_p represents the pth prime.

Ans:

– 14.5: Given the product formula, identify the poles of $\zeta_p(s)$ *(*$p \in \mathbb{Z}$*), which is important for defining the RoC of each factor. For example, the* p*th factor of Eq. AE-1.7, expressed as an exponential, is*

$$\zeta_p(s) \equiv \frac{1}{1-\pi_p^{-s}} = \frac{1}{1-e^{-sT_p}}, \tag{AE-1.8}$$

where $T_p \equiv \ln \pi_p$.

Ans:

– *14.6: Plot Eq. AE-1.8 using* `zviz` *for* $p = 1$. *Describe what you see.*

Ans: