2.3 Problems AE-3

Topics of this homework:
Visualizing complex functions, bilinear/Möbius transformation, Riemann sphere.
Deliverables: Answers to problems

Two-port network analysis

Problem #1: Perform an analysis of electrical two-port networks, shown in Fig. ?? (page ??). This can be a mechanical system if the capacitors are taken to be springs and inductors taken as mass, as in the suspension of the wheels of a car. In an acoustical circuit, the low-pass filter could be a car muffler. While the physical representations will be different, the equations and the analysis are exactly the same.

The definition of the ABCD transmission matrix (\(T\)) is

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} =
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix}.
\]

(AE-3.1)

The impedance matrix, where the determinant \(\Delta_T = AD - BC\), is given by

\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} = \frac{1}{C}
\begin{bmatrix}
A & \Delta_T \\
\Delta_T & D
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}.
\]

(AE-3.2)

– 1.1: Derive the formula for the impedance matrix (Eq. AE-3.2) given the transmission matrix definition (Eq. AE-3.1). Show your work.

Sol: The formula may be easily derived by re-arranging the equations from the matrix (Eq. AE-3.2). Begin with

\[
V_1 = AV_2 - BI_2 \\
I_1 = CV_2 - DI_2
\]

From the second equation, we get

\[
V_2 = \frac{1}{C}I_1 + \frac{D}{C}I_2
\]

which gives (upon substitution)

\[
V_1 = \frac{A}{C}I_1 + \frac{AD}{C}I_2 - BI_2 = \frac{A}{C}I_1 + \left(\frac{AD}{C} - B\right)I_2
\]

which yields the matrix equation

\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} = \begin{bmatrix}
\frac{A}{C} & \left(\frac{AD}{C} - B\right) \\
1/C & D/C
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix} = \frac{1}{C}
\begin{bmatrix}
A & \Delta_T \\
\Delta_T & D
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}.
\]

(AE-3.3)

Problem #2: Consider a single circuit element with impedance \(Z(s)\).

– 2.1: What is the ABCD matrix for this element if it is in series?

Sol:

\[
\begin{bmatrix}
1 & Z(s) \\
0 & 1
\end{bmatrix}
\]
2.2: What is the ABCD matrix for this element if it is in shunt?

\[
\begin{bmatrix}
1 & 0 \\
1/Z(s) & 1
\end{bmatrix}
\]

\[\text{Problem # 3: Find the ABCD matrix for each of the circuits of Fig. ??}.
\]

For each circuit, (i) show the cascade of transmission matrices in terms of the complex frequency \(s \in \mathbb{C}\), then (ii) substitute \(s = j\) and calculate the total transmission matrix at this single frequency.

- **3.1**: Left circuit (let \(R_1 = R_2 = 10\) kilo-ohms and \(C = 10\) nano-farads)

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} = \begin{bmatrix}
1 & Z_1 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
Y_{C1} & 1
\end{bmatrix} \begin{bmatrix}
1 & Z_3 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix} = \begin{bmatrix}
1 & Z_1 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
sC_1 & 1
\end{bmatrix} \begin{bmatrix}
1 & Z_3 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix}
\]

Now we substitute the given values:

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} = \begin{bmatrix}
1 & 10^4 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
10^{-8} & 1
\end{bmatrix} \begin{bmatrix}
1 & 10^4 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 + j10^{-4} \\
-j10^{-8}
\end{bmatrix} \begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix}
\]

- **3.2**: Right circuit (use \(L\) and \(C\) values given in the figure), where the pressure \(P\) is analogous to the voltage \(V\), and the velocity \(U\) is analogous to the current \(I\).

\[
\begin{bmatrix}
P_1 \\
U_1
\end{bmatrix} = \begin{bmatrix}
1 & j \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
2j & 1
\end{bmatrix} \begin{bmatrix}
1 & \frac{1}{\tau_{C1}} \\
0 & 1
\end{bmatrix} \begin{bmatrix}
P_2 \\
-U_2
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
10^{-8} & j
\end{bmatrix} \begin{bmatrix}
P_1 \\
U_1
\end{bmatrix}
\]

I used Matlab/Octave to evaluate this script:

```
a=[1 j;0 1];b=[1 0;2j 1];c=[1 1/3j; 0 1];d=[1 0;1/4j 1]; T=a*b*c*d.
```

Finally I found \(T(2,1)\) to be \(19/12\) using the Matlab/Octave command: `rats(1.5833, 6)`

- **3.3**: Convert both transmission (ABCD) matrices to impedance matrices using Eq. AE-3.2. Do this for the specific frequency \(s = j\) as in the previous part (feel free to use Matlab/Octave for your computation).

\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} = \frac{1}{j10^{-8}} \begin{bmatrix}
1 + j10^{-4} & 1 \\
1 & 1 + j10^{-4}
\end{bmatrix} \begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}
\]

- **3.4**: Right circuit: Repeat the analysis as in question 3.3.

\[
\begin{bmatrix}
P_1 \\
P_2
\end{bmatrix} = \frac{1}{1.5833j} \begin{bmatrix}
-\frac{2}{3} & \frac{1}{3} \\
1 & 0
\end{bmatrix} \begin{bmatrix}
U_1 \\
U_2
\end{bmatrix}
\]
Algebra

Problem # 4: Fundamental theorem of algebra (FTA).

– 4.1: State the fundamental theorem of algebra (FTA).
Sol: There are multiple definitions of the FTA, which of course must be equivalent.
Here are three (equivalent) answers from Wikipedia

1. The fundamental theorem of algebra states that every non-constant single-variable polynomial with complex coefficients has at least one complex root. This may then be applied recursively till the degree is zero.
2. Every degree \( n \) polynomial with complex coefficients has, counted with multiplicity, exactly \( n \) roots. The equivalence of the two statements can be proven through the use of successive polynomial division.
3. The field of complex numbers is algebraically closed. Note: this one requires an understanding of the term algebraically closed.

Wikipedia warns:
In spite of its name, there is no purely algebraic proof of the theorem, since any proof must use the completeness of the reals (or some other equivalent formulation of completeness), which is not an algebraic concept.

Algebra with complex variables

Problem # 5: Order and complex numbers:
One can always say that \( 3 < 4 \)—namely, that real numbers have order. One way to view this is to take the difference and compare it to zero, as in \( 4 - 3 > 0 \). Here we will explore how complex variables may be ordered.

– 5.1: Explain the meaning of \( |z_1| > |z_2| \).
Sol: \( |z| = \sqrt{x^2 + y^2} \) is the length of \( z \), so the above expression says that a disk of radius \( |z_1| \) contains a second disk of radius \( |z_2| \).

– 5.2: If \( x_1, x_2 \in \mathbb{R} \) (are real numbers), define the meaning of \( x_1 > x_2 \). Hint: Take the difference.
Sol: This conditions is the same as \( x_1 - x_2 > 0 \). Order is meaningful on the real line, as a length.

– 5.3: Explain the meaning of \( z_1 > z_2 \).
Sol: It makes no sense to order complex numbers. A complex number has both a length and an angle (it is the same as a vector). The concept of an angle extends the sign of a real number, making order impossible. To show this, place to points on a plane, and ask which is larger than the other. The order of the \( x \) and \( y \) components, each have order. Thus order cannot be defined.

– 5.4: If time were complex, how might the world be different?
Sol: As best we know, time is real, thus it has the order property: the is a past, present and future. If time were complex this would not be the case. Thus if time were complex, the past could be the future.

Problem # 6: It is sometimes necessary to consider a function \( w(z) = u + vj \) in terms of the real functions \( u(x, y) \) and \( v(x, y) \) (e.g. separate the real and imaginary parts). Similarly, we can consider the inverse \( z(w) = x + yj \), where \( u(x, v) \) and \( y(u, v) \) are real functions.

– 6.1: Find \( u(x, y) \) and \( v(x, y) \) for \( w(z) = 1/z \).
Sol: Multiply by the complex conjugate \( x - yj \)
\[
\begin{align*}
\frac{1}{x + yj} &= \frac{x - yj}{x^2 + y^2} \\
&= u + vj
\end{align*}
\]
Therefore \( u(x, y) = \frac{x}{x^2 + y^2} \) and \( v(x, y) = \frac{-y}{x^2 + y^2} \).
2.3. PROBLEMS AE-3

**Problem # 7:** Find \( u(x, y) \) and \( v(x, y) \) for \( w(z) = e^z \) with complex constant \( c \in \mathbb{C} \) for questions 7.1, 7.2, and 7.3:

- 7.1: \( c = e \)

**Sol:** Since \( u + iv = e^z = e^{x+iy} = e^x (\cos y + j \sin y) \),

\[ u = e^x \cos y \]

and

\[ v = e^x \sin y. \]

- 7.2: \( c = 1 \) (recall that \( 1 = e^{j2\pi k} \) for \( k = 0, 1, 2, \ldots \))

**Sol:** From the general formula with \( c = 1 \)

\[ 1^z = e^{z \log 1} e^{j2\pi z} = e^{0} e^{j2\pi z} = e^{-yj2\pi} e^{-jxj2\pi} \]

where \( k \) is any integer. Thus \( u = e^{-j2\pi y} \cos k2\pi x \) and \( v = e^{-j2\pi y} \sin k2\pi x \).

- 7.3: \( c = j \). Hint: \( j = e^{\pi/2 + j2\pi m} \), \( m \in \mathbb{Z} \).

**Sol:** \( j^z = (e^{\pi/2 + j2\pi m})^z = e^{jz\pi/2 + jz2\pi m} = e^{-\pi/2} e^{-2\pi m} = 0.2079 e^{-2\pi m} \).

Thus for \( m = 0 \), \( j^z = (e^{\pi/2})^z = e^{jz\pi/2} = e^{j(x+y)\pi/2} = e^{-\pi y/2} (\cos(x\pi/2) + j\sin(x\pi/2)) \).

- 7.4: Find \( u(x, y) \) for \( w(z) = \sqrt{z} \).

**Sol:** The simple solution is to work in polar coordinates, which gives \( w(z) = \sqrt{|z|} e^{j\phi/2} \). The straightforward method is to use the identity

\[ w(s) = e^{\log w} = e^{\log \sqrt{z}} = e^{j\phi/2} \log s \]

But in polar coordinates \( z = s e^{\phi} \), thus \( \log s = \log |s| + j\phi \). Simple algebra gives the final result.

Using the complex analytic log function is the proper way to compute \( \sqrt{z} \) for all sheets of \( w(z) \), not \( \tan^{-1}(x, y) \) function as taught in highschool, which is limited to the principal value, or even worse \( \tan^{-1}(y/x) \), which is limited to \( -\pi/2 \leq \phi \leq \pi/2 \).

**Second method:** It is interesting that we can start with the inverse \( w^2 = z \). But then the solution requires much more algebra.

Define \( w = u + jv \) and \( z = x + jy \). Breaking \( w^2 = z \) into real and imaginary parts we find \( x = u^2 - v^2 \) and \( y = 2uv \). Removing \( v = y/2u \) from the first equation gives

\[ u^2 = x + (y/2u)^2 \]

\[ = x + (y/2)^2/u^2. \]

\[ u^2 - xu^2 = (y/2)^2 \]

Completing the square:

\[ (u^2 - x/2)^2 = (x/2)^2 + (y/2)^2 \]

\[ u^2 = \frac{x}{2} \pm \frac{x^2 + y^2}{2} \]

\[ (u, v) = \left( \pm \sqrt{\frac{x + |z|^2}{2}}, \frac{y}{2u(x, y)} \right) \]

Note that \( v \) depends on \( u(x, y) \), which is explicitly provided in the last equation, above.

Thus we have an explicit functional form for \( \sqrt{z} \) in terms of \( (x, y) \). We know that CR-1, CR-2 hold for this function except at the origin. This is then the equation that shows as the contour lines for the Matlab command \( \text{W=sqrt(Z)} \). This is more informative if you scale \( z \) by 5 or even 10 when using \( z = \sqrt{z} \).

- **Problem # 8:** Convolution of an impedance \( z(t) \) and its inverse \( y(t) \):

In the frequency domain a Brune impedance is defined as the ratio of a numerator polynomial \( N(s) \) to a denominator polynomial \( D(s) \).
Since

In the above figure we see vectors

From Eq.

Show your derivation, not the answer

Figure 2.5: This figure shows how to derive the Schwarz inequality, by finding the value of \( \alpha = \alpha^* \) corresponding to \( \min_{\alpha} |E(\alpha)| \). It is identical to Fig. ?? on page ??.

- 8.1: Consider a Brune impedance defined by the ratio of numerator and denominator polynomials, \( Z(s) = N(s)/D(s) \). Since the admittance \( Y(s) \) is defined as the reciprocal of the impedance, the product must be 1. If \( z(t) \leftrightarrow Z(s) \) and \( y(t) \leftrightarrow Y(s) \), it follows that \( z(t) * y(t) = \delta(t) \). What property must \( n(t) \leftrightarrow N(s) \) and \( d(t) \leftrightarrow D(s) \) obey for this to be true?

\[ 1 = Z(s)Y(s) = \frac{N(s)D(s)}{D(s)N(s)} \]

it follows that \( N(s)D(s) = D(s)N(s) \) thus \( n(t) * d(s) = d(t) * n(t) \). Namely the convolution of \( n(t) \) and \( d(t) \) commute (are independent of order).

- 8.2: The definition of a minimum phase function is that it must have a causal inverse. Show that every impedance is minimum phase.

\[ z(t) \text{ is causal and has a causal inverse } y(t), \text{ by definition every impedance must be minimum phase.} \]

Schwarz inequality

Problem # 9: The above figure shows three vectors for an arbitrary value of \( \alpha \in \mathbb{R} \) and a specific value of \( \alpha = \alpha^* \).

- 9.1: Find the value of \( \alpha \in \mathbb{R} \) such that the length (norm) of \( \vec{E} \) (i.e., \( ||\vec{E}|| \geq 0 \)) is minimum. Show your derivation, not the answer \( (\alpha = \alpha^*) \).

\[ ||E(\alpha)||^2 = \vec{E} \cdot \vec{E} = (\vec{V} + \alpha \vec{U}) \cdot (\vec{V} + \alpha \vec{U}) \geq 0. \] (AE-3.4)

Minimize with respect to \( \alpha \).

When \( U \) is scaled by \( \alpha^* \), length \( ||E(\alpha^*)|| \) is minimum, and \( (V - \alpha^*U) \perp U \), namely vector \( E(\alpha^*) \) is \( \perp \) to vector \( U \). This follows from \( \frac{1}{2M} ||\vec{E}||^2 = \frac{1}{2M} ((\vec{V} + \alpha \vec{U}) \cdot (\vec{V} + \alpha \vec{U})) = 2(\vec{V} + \alpha \vec{U}) \cdot \vec{U} = 0. \) Thus

\[ \alpha^* = -\frac{\vec{V} \cdot \vec{U}}{||\vec{U}||^2} \]

- 9.2: Find the formula for \( ||E(\alpha^*)||^2 \geq 0 \). Hint: Substitute \( \alpha^* \) into Eq. ?? (p. ??) and show that this results in the Schwarz inequality

\[ ||\vec{U} \cdot \vec{V}| \leq ||\vec{U}|| ||\vec{V}||. \]

\[ ||\vec{V}||^2 + 2\alpha^* \vec{V} \cdot \vec{U} + (\alpha^*)^2 ||\vec{U}||^2 \geq 0 \]
Substituting $\alpha^*$ gives

$$||V||^2||U||^2 - 2(V \cdot U)^2 + |U \cdot V|^2 \geq 0.$$  

Simplifying

$$||V||^2||U||^2 \geq |U \cdot V|^2$$

and taking the square root (and swap order), gives the Schwarz inequality

$$|\vec{U} \cdot \vec{V}| \leq ||\vec{U}||||\vec{V}||.$$  

**Problem # 10: Geometry and scaler products**

- **10.1:** *What is the geometrical meaning of the dot product of two vectors?*
  
  **Sol:** The dot product of two vectors is the length of the $\perp$ projection of one vector on the other. According to the Schwarz inequality, this project length must be less than the product of the lengths of the two vectors. 

- **10.2:** *Give the formula for the dot product of two vectors. Explain the meaning based on Fig. ?? (page ??).*
  
  **Sol:** $\vec{V} \cdot \vec{U} = ||\vec{V}||||\vec{U}|| \cos \theta_{\vec{V}, \vec{U}}$. It represents the amount of one vector going in the direction of the other. In a drawing, it is a projection of the one on the other, found by dropping the $\perp$ from the tip of one, on the other. 

- **10.3:** *Write the formula for the dot product of two vectors $\vec{U} \cdot \vec{V}$ in $\mathbb{R}^n$ in polar form (e.g., assume the angle between the vectors is $\theta$).*
  
  **Sol:** $\vec{U} \cdot \vec{V} = \sum_{i=1}^{n} a_i b_i (= ||\vec{U}|| ||\vec{V}|| \cos(\theta))$. This last relationship defines the angle between two vectors. 

- **10.4:** *How is the Schwarz inequality related to the Pythagorean theorem?*
  
  **Sol:** It says that for a right triangle, the case when $a = a^*$, the lengths of the two vectors must be greater than the projection of one on the other, unless they are co-linear (i.e., the angle between them is zero). 

- **10.5:** *Starting from $||\vec{U} + \vec{V}||$, derive the triangle inequality*

  $$||\vec{U} + \vec{V}|| \leq ||\vec{U}|| + ||\vec{V}||.$$  

  **Sol:** $||\vec{U} + \vec{V}||^2 = (\vec{U} + \vec{V}) \cdot (\vec{U} + \vec{V}) = ||\vec{U}||^2 + ||\vec{V}||^2 + 2\vec{U} \cdot \vec{V} \leq ||\vec{U}||^2 + ||\vec{V}||^2 + 2||\vec{U} \cdot \vec{V}||$ Using the Schwarz inequality we find $||\vec{U} + \vec{V}||^2 \leq ||\vec{U}||^2 + ||\vec{V}||^2 + 2||\vec{U} \cdot \vec{V}||$. Completing the square on the right gives $||\vec{U} + \vec{V}||^2 \leq (||\vec{U}|| + ||\vec{V}||)^2$. Final taking the square root gives the triangle inequality. 

- **10.6:** *The triangle inequality $||\vec{U} + \vec{V}|| \leq ||\vec{U}|| + ||\vec{V}||$ is true for two and three dimensions: Does it hold for five-dimensional vectors?*
  
  **Sol:** It is true in any number of dimensions. 

- **10.7:** *Show that the wedge product $\vec{U} \wedge \vec{V} \perp \vec{U} \cdot \vec{V}$.*
  
  **Sol:** $\vec{V} \wedge \vec{U} = ||\vec{V}|| ||\vec{U}|| \sin \theta_{\vec{V}, \vec{U}}$. See the discussion in the text on the wedge product. This is true in any number of dimensions. 

**Probability**

**Problem # 11: Basic terminology of experiments**

- **11.1:** *What is the mean of a trial, and what is the average over all trials?*
  
  **Sol:** The mean and average are the same. What is different in these two questions is the size of the set being averaged.
– 11.2: What is the expected value of a random variable $X$?

**Sol:** This is a mathematical definition of how to compute the mean, as an inner product.

– 11.3: What is the standard deviation about the mean?

**Sol:** This is the expected value of the second moment of the random variable.

– 11.4: What is the definition of information of a random variable?

**Sol:** The information is $I = 1/P(X_k)$.

– 11.5: How do you combine events? Hint: If the event is the flip of a biased coin, the events are $H = p$, $T = 1 - p$, so the event is $\{p, 1 - p\}$. To solve the problem, you must find the probabilities of two independent events.

**Sol:** If one event has probability $p$ it may be captured by a vector $[p, 1 - p]^T$. Two uncorrelated (independent) events then have probability $[p, 1 - p] \ast [p, 1 - p] = [p^2, 2p(1 - p), (1 - p^2)]$. Here $\ast$ represents convolution (Section ??, p. ??). Three events have four outcomes $[p, 1 - p] \ast [p, 1 - p] \ast [p, 1 - p]$. Pascal’s triangle is a related structure defined by recursive convolutions of $[1, 1]$, assuming $p = 1/2$.

– 11.6: What does the term independent mean in the context of question 11.5? Give an example.

**Sol:** This term means that one event (flip of a coin) has no influence on the next (or any other flip) of that same coin. An example of non-independent events might be that upon flipping the coin, it bent, thus changing the probability for any following flips.

– 11.7: Define odds.

**Sol:** The odds are the ratio of the two outcomes. Namely the odds are $p/(1 - p)$, or equivalently $(1 - p)/p = 1/p - 1$. 