## **Chapter 3**

# **Differential equations**

## 3.1 Problems DE-1

## 3.1.1 Topics of this homework:

Complex numbers and functions (ordering and algebra), complex power series, fundamental theorem of calculus (real and complex); Cauchy-Riemann conditions, multivalued functions (branch cuts and Riemann sheets)

## 3.1.2 Complex Power Series

**Problem #** 1: In each case derive (e.g., using Taylor's formula) the power series of w(s) about s=0 and give the RoC of your series. If the power series doesn't exist, state why! Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at s=0.

$$-1.1: 1/(1-s)$$
**Ans:**

$$-1.2$$
:  $1/(1-s^2)$  **Ans:**

$$-1.3$$
:  $1/(1+s^2)$ . **Ans:**

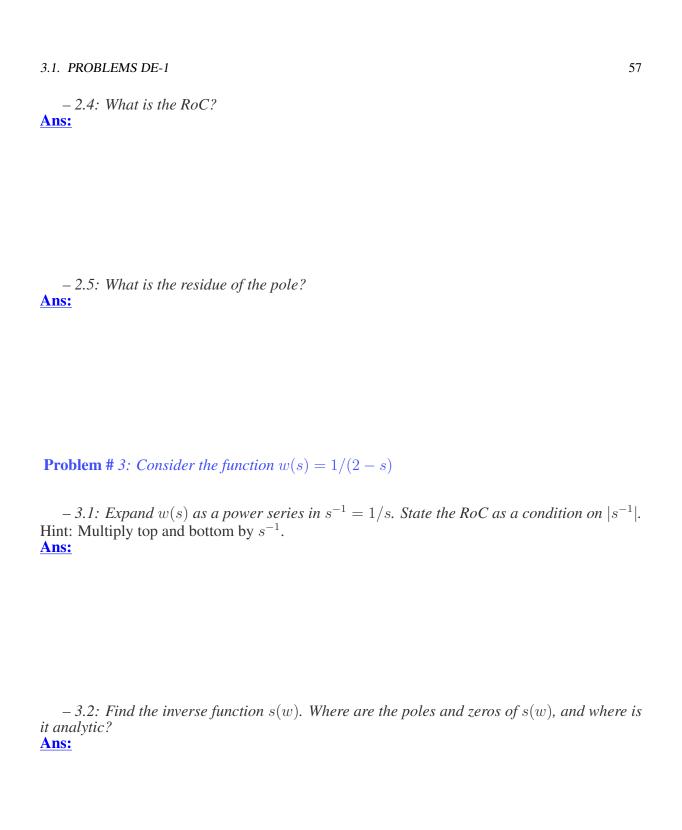
$$-1.5$$
:  $1/(1-|s|^2)$  **Ans:**

## **Problem #** 2: Consider the function w(s) = 1/s

-2.1: Expand this function as a power series about s=1. Hint: Let 1/s=1/(1-1+s)=1/(1-(1-s)). Ans:

-2.2: What is the RoC? **Ans:** 

$$-2.3$$
: Expand  $w(s)=1/s$  as a power series in  $s^{-1}=1/s$  about  $s^{-1}=1$ . Ans:



## **Problem** # 4:Summing the series

The Taylor series of functions have more than one region of convergence.

-4.1: Given some function f(x), if a = 0.1, what is the value of

$$f(a) = 1 + a + a^2 + a^3 + \cdots$$
?

Show your work. **Ans:** 

-4.2: Let a=10. What is the value of

$$f(a) = 1 + a + a^2 + a^3 + \cdots$$
?

Ans:

## 3.1.3 Cauchy-Riemann Equations

**Problem #** 5: For this problem  $j = \sqrt{-1}$ ,  $s = \sigma + \omega j$ , and  $F(s) = u(\sigma, \omega) + jv(\sigma, \omega)$ . According to the fundamental theorem of complex calculus (FTCC), the integration of a complex analytic function is independent of the path. It follows that the derivative of F(s) is defined as

$$\frac{dF}{ds} = \frac{d}{ds} \left[ u(\sigma, \omega) + \jmath v(\sigma, \omega) \right]. \tag{DE-1.1} \label{eq:def}$$

If the integral is independent of the path, then the derivative must also be independent of the direction:

$$\frac{dF}{ds} = \frac{\partial F}{\partial \sigma} = \frac{\partial F}{\partial \mu}.$$
 (DE-1.2)

The Cauchy-Riemann (CR) conditions

$$\frac{\partial u(\sigma,\omega)}{\partial \sigma} = \frac{\partial v(\sigma,\omega)}{\partial \omega} \quad \text{ and } \quad \frac{\partial u(\sigma,\omega)}{\partial \omega} = -\frac{\partial v(\sigma,\omega)}{\partial \sigma}$$

may be used to show where Equation DE-1.2 holds.

– 5.1: Assuming Equation DE-1.2 is true, use it to derive the CR equations.

Ans:

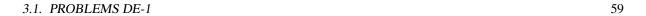
-5.2: Merge the CR equations to show that u and v obey Laplace's equations

$$\nabla^2 u(\sigma, \omega) = 0$$
 and  $\nabla^2 v(\sigma, \omega) = 0$ .

Ans:

What can you conclude?

Ans:



**Problem #** 6: Apply the CR equations to the following functions. State for which values of  $s = \sigma + i\omega$  the CR conditions do or do not hold (e.g., where the function F(s) is or is not analytic). Hint: Review where CR-1 and CR-2 hold.

$$-6.1$$
:  $F(s) = e^s$ 

Ans:

$$-6.2$$
:  $F(s) = 1/s$ 

Ans:

#### 3.1.4 Branch cuts and Riemann sheets

**Problem #** 7: Consider the function  $w^2(z) = z$ . This function can also be written as  $w_{\pm}(z) = \sqrt{z_{\pm}}$ . Assume  $z = re^{\phi_{\beta}}$  and  $w(z) = \rho e^{\theta_{\beta}} = \sqrt{r}e^{\phi_{\beta}/2}$ .

-7.1: How many Riemann sheets do you need in the domain (z) and the range (w) to fully represent this function as single-valued? **Ans:** 

- 7.2: Indicate (e.g., using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.

Ans:

$-7.3$ : Use zviz.m to plot the positive and negative square roots $+\sqrt{z}$ and $-\sqrt{z}$ . Describe what you see.  Ans:
-7.4: Where does zviz.m place the branch cut for this function?  Ans:
– 7.5: Must the branch cut necessarily be in this location?
Ans:
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<b>Problem #</b> 8: Consider the function $w(z) = \log(z)$ . As in Problem 7, let $z = re^{\phi j}$ and $w(z) = \rho e^{\theta j}$ .
- 8.1: Describe with a sketch and then discuss the branch cut for $f(z)$ .  Ans:

- 8.2: What is the inverse of the function z(f)? Does this function have a branch cut? If so, where is it?

Ans:



-8.3: Using zviz.m, show that

$$\tan^{-1}(z) = -\frac{1}{2}\log\frac{j-z}{j+z}.$$
 (DE-1.3)

In Fig. 4.1 (p. 132) these two functions are shown to be identical. **Ans:** 

-8.4: Algebraically justify Eq. DE-1.3. Hint: Let  $w(z) = \tan^{-1}(z)$  and  $z(w) = \tan w = \sin w/\cos w$ ; then solve for  $e^{wj}$ .

Ans:

## 3.1.5 A Cauer synthesis of any Brune impedance

**Problem #** 9: One may synthesize a transmission line (ladder network) from a positive real impedance Z(s) by using the continued fraction method. To obtain the series and shunt impedance values, we can use a residue expansion. Here we shall explore this method.

– 9.1: Starting from the Brune impedance  $Z(s) = \frac{1}{s+1}$ , find the impedance network as a ladder network.

**Ans:** 

– 9.2: Use a residue expansion in place of the CFA floor function (Sec. ??, p. ??) for polynomial expansions. Find the residue expansion of  $H(s) = s^2/(s+1)$  and express it as a ladder network.

Ans:

-9.3: Discuss how the series impedance Z(s,x) and shunt admittance Y(s,x) determine the wave velocity  $\kappa(s,x)$  and the characteristic impedance  $z_o(s,x)$  when (1) Z(s) and Y(s) are both independent of x; (2) Y(s) is independent of x, Z(s,x) depends on x; (3) Z(s) is independent of x, Y(s,x) depends on x; and (4) both Y(s,x), Z(s,x) depend on x. Ans: