Chapter 3

Differential equations

3.1 Problems DE-1

3.1.1 Topics of this homework:
Complex numbers and functions (ordering and algebra), complex power series, fundamental theorem of calculus (real and complex); Cauchy-Riemann conditions, multivalued functions (branch cuts and Riemann sheets)

3.1.2 Complex Power Series

Problem # 1: In each case derive (e.g., using Taylor’s formula) the power series of \( w(s) \) about \( s = 0 \) and give the RoC of your series. If the power series doesn’t exist, state why! Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at \( s = 0 \).

\(-1.1: \) \( 1/ (1 - s) \)

**Ans:**

\(-1.2: \) \( 1/ (1 - s^2) \)

**Ans:**

\(-1.3: \) \( 1/ (1 + s^2) \).

**Ans:**
- 1.4: \(1/s\)

**Ans:**

- 1.5: \(1/(1 - |s|^2)\)

**Ans:**

**Problem #2:** Consider the function \(w(s) = 1/s\)

- 2.1: Expand this function as a power series about \(s = 1\). Hint: Let \(1/s = 1/(1 - 1 + s) = 1/(1 - (1 - s))\).

**Ans:**

- 2.2: What is the RoC?

**Ans:**

- 2.3: Expand \(w(s) = 1/s\) as a power series in \(s^{-1} = 1/s\) about \(s^{-1} = 1\).

**Ans:**
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- 2.4: What is the RoC?
Ans:

- 2.5: What is the residue of the pole?
Ans:

Problem # 3: Consider the function \( w(s) = \frac{1}{2 - s} \)

- 3.1: Expand \( w(s) \) as a power series in \( s^{-1} = 1/s \). State the RoC as a condition on \( |s^{-1}| \).
   Hint: Multiply top and bottom by \( s^{-1} \).
Ans:

- 3.2: Find the inverse function \( s(w) \). Where are the poles and zeros of \( s(w) \), and where is it analytic?
Ans:

Problem # 4: Summing the series
The Taylor series of functions have more than one region of convergence.

- 4.1: Given some function \( f(x) \), if \( a = 0.1 \), what is the value of

\[
f(a) = 1 + a + a^2 + a^3 + \cdots
\]

Show your work. Ans:
4.2: Let \( a = 10 \). What is the value of

\[
f(a) = 1 + a + a^2 + a^3 + \cdots.
\]

**Ans:**

### 3.1.3 Cauchy-Riemann Equations

**Problem # 5:** For this problem \( j = \sqrt{-1} \), \( s = \sigma + \omega j \), and \( F(s) = u(\sigma, \omega) + jv(\sigma, \omega) \). According to the fundamental theorem of complex calculus (FTCC), the integration of a complex analytic function is independent of the path. It follows that the derivative of \( F(s) \) is defined as

\[
\frac{dF}{ds} = \frac{d}{ds} [u(\sigma, \omega) + jv(\sigma, \omega)].
\]

If the integral is independent of the path, then the derivative must also be independent of the direction:

\[
\frac{dF}{ds} = \frac{\partial F}{\partial \sigma} = \frac{\partial F}{\partial \omega}.
\]

The Cauchy-Riemann (CR) conditions

\[
\frac{\partial u(\sigma, \omega)}{\partial \sigma} = \frac{\partial v(\sigma, \omega)}{\partial \omega} \quad \text{and} \quad \frac{\partial u(\sigma, \omega)}{\partial \omega} = -\frac{\partial v(\sigma, \omega)}{\partial \sigma}
\]

may be used to show where Equation DE-1.2 holds.

5.1: Assuming Equation DE-1.2 is true, use it to derive the CR equations.

**Ans:**

5.2: Merge the CR equations to show that \( u \) and \( v \) obey Laplace’s equations

\[
\nabla^2 u(\sigma, \omega) = 0 \quad \text{and} \quad \nabla^2 v(\sigma, \omega) = 0.
\]

**Ans:**

What can you conclude?

**Ans:**
3.1. PROBLEMS DE-1

Problem # 6: Apply the CR equations to the following functions. State for which values of $s = \sigma + i\omega$ the CR conditions do or do not hold (e.g., where the function $F(s)$ is or is not analytic). Hint: Review where CR-1 and CR-2 hold.

- 6.1: $F(s) = e^s$
  
  Ans:

- 6.2: $F(s) = 1/s$
  
  Ans:

3.1.4 Branch cuts and Riemann sheets

Problem # 7: Consider the function $w^2(z) = z$. This function can also be written as $w_{\pm}(z) = \sqrt{z_{\pm}}$. Assume $z = re^{i\phi}$ and $w(z) = \rho e^{i\theta} = \sqrt{r}e^{i\phi/2}$.

- 7.1: How many Riemann sheets do you need in the domain ($z$) and the range ($w$) to fully represent this function as single-valued?
  
  Ans:

- 7.2: Indicate (e.g., using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.
  
  Ans:
– 7.3: Use \texttt{zviz.m} to plot the positive and negative square roots $+\sqrt{z}$ and $-\sqrt{z}$. Describe what you see.

\textbf{Ans:}

– 7.4: Where does \texttt{zviz.m} place the branch cut for this function?

\textbf{Ans:}

– 7.5: Must the branch cut necessarily be in this location?

\textbf{Ans:}

\textbf{Problem # 8:} Consider the function $w(z) = \log(z)$. As in Problem 7, let $z = re^{\phi}$ and $w(z) = \rho e^{\theta}$.

– 8.1: Describe with a sketch and then discuss the branch cut for $f(z)$.

\textbf{Ans:}

– 8.2: What is the inverse of the function $z(f)$? Does this function have a branch cut? If so, where is it?

\textbf{Ans:}
– 8.3: Using *zviz.m*, show that
\[
\tan^{-1}(z) = -\frac{j}{2} \log\left(\frac{j-z}{j+z}\right).
\] (DE-1.3)

In Fig. 4.1 (p. 132) these two functions are shown to be identical.

**Ans:**

– 8.4: Algebraically justify Eq. DE-1.3. Hint: Let \( w(z) = \tan^{-1}(z) \) and \( z(w) = \tan w = \sin w / \cos w \); then solve for \( e^{wj} \).

**Ans:**

### 3.1.5 A Cauer synthesis of any Brune impedance

**Problem # 9:** One may synthesize a transmission line (ladder network) from a positive real impedance \( Z(s) \) by using the continued fraction method. To obtain the series and shunt impedance values, we can use a residue expansion. Here we shall explore this method.

– 9.1: Starting from the Brune impedance \( Z(s) = \frac{1}{s+1} \), find the impedance network as a ladder network.

**Ans:**

– 9.2: Use a residue expansion in place of the CFA floor function (Sec. ??, p. ??) for polynomial expansions. Find the residue expansion of \( H(s) = \frac{s^2}{s+1} \) and express it as a ladder network.

**Ans:**
9.3: Discuss how the series impedance \( Z(s, x) \) and shunt admittance \( Y(s, x) \) determine the wave velocity \( \kappa(s, x) \) and the characteristic impedance \( z_0(s, x) \) when (1) \( Z(s) \) and \( Y(s) \) are both independent of \( x \); (2) \( Y(s) \) is independent of \( x \), \( Z(s, x) \) depends on \( x \); (3) \( Z(s) \) is independent of \( x \), \( Y(s, x) \) depends on \( x \); and (4) both \( Y(s, x), Z(s, x) \) depend on \( x \).

\[ \text{Ans:} \]