

1.3 Problems NS-3

Topic of this homework: Pythagorean triplets, Pell’s equation, Fibonacci sequence

Pythagorean triplets

Problem # 1: Euclid’s formula for the Pythagorean triplets \(a, b, c\) is \(a = p^2 - q^2\), \(b = 2pq\), and \(c = p^2 + q^2\).

– 1.1: What condition(s) must hold for \(p\) and \(q\) such that \(a\), \(b\), and \(c\) are always positive and nonzero?

Ans:

– 1.2: Solve for \(p\) and \(q\) in terms of \(a\), \(b\), and \(c\).

Ans:

Problem # 2: The ancient Babylonians (ca. 2000 BCE) cryptically recorded \((a, c)\) pairs of numbers on a clay tablet, archeologically denoted Plimpton-322 (see 2.8).

– 2.1: Find \(p\) and \(q\) for the first five pairs of \(a\) and \(c\) shown here from Plimpton-322.

<table>
<thead>
<tr>
<th>(a)</th>
<th>(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>119</td>
<td>169</td>
</tr>
<tr>
<td>3367</td>
<td>4825</td>
</tr>
<tr>
<td>4601</td>
<td>6649</td>
</tr>
<tr>
<td>12709</td>
<td>18541</td>
</tr>
<tr>
<td>65</td>
<td>97</td>
</tr>
</tbody>
</table>

Find a formula for \(a\) in terms of \(p\) and \(q\).

Ans:
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- 2.2: Based on Euclid’s formula, show that $c > (a, b)$.  
\textbf{Ans:}

- 2.3: What happens when $c = a$?  
\textbf{Ans:}

- 2.4: Is $b + c$ a perfect square? Discuss.  
\textbf{Ans:}

\textbf{Pell’s equation:}

\textbf{Problem} # 3: Pell’s equation is one of the most historic (i.e., important) equations of Greek number theory because it was used to show that $\sqrt{2} \in \mathbb{I}$. We seek integer solutions of

$$x^2 - Ny^2 = 1.$$ 

As shown on page 42, the solutions $x_n, y_n$ for the case of $N = 2$ are given by the linear $2 \times 2$ matrix recursion

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = 1_j \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

with $[x_0, y_0]^T = [1, 0]^T$ and $1_j = \sqrt{-1} = e^{j\pi/2}$. It follows that the general solution to Pell’s equation for $N = 2$ is

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = (e^{j\pi/2})^n \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$ 

To calculate solutions to Pell’s equation using the matrix equation above, we must calculate

$$A^n = e^{j\pi n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n = e^{j\pi n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix},$$

which becomes tedious for $n > 2$. 
– 3.1: Find the companion matrix and thus the matrix $A$ that has the same eigenvalues as Pell’s equation. Hint: Use Matlab’s function $[E, \text{Lambda}] = \text{eig}(A)$ to check your results!

Ans:

– 3.2: Solutions to Pell’s equation were used by the Pythagoreans to explore the value of $\sqrt{2}$. Explain why Pell’s equation is relevant to $\sqrt{2}$.

Ans:

– 3.3: Find the first three values of $(x_n, y_n)^T$ by hand and show that they satisfy Pell’s equation for $N = 2$. Ans: By hand, find the eigenvalues $\lambda_{\pm}$ of the $2 \times 2$ Pell’s equation matrix

\[ A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}. \]

Ans:

– 3.4: By hand, show that the matrix of eigenvectors, $E$, is

\[ E = \begin{bmatrix} \bar{e}_+ & \bar{e}_- \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}. \]

Ans:
1.3. PROBLEMS NS-3

– 3.5: Using the eigenvalues and eigenvectors you found for \( A \), verify that

\[
E^{-1}AE = \Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}
\]

Ans:

– 3.6: Now that you have diagonalized \( A \) (Equation 2.5.18), use your results for \( E \) and \( \Lambda \) to solve for the \( n = 10 \) solution \((x_{10}, y_{10})^T\) to Pell’s equation with \( N = 2 \).

Ans:

Problem # 4: Here we seek the general formula for \( x_n \). Like Pell’s equation, the Fibonacci equation has a recursive eigenanalysis solution. To find it we must recast \( x_n \) as a \( 2 \times 2 \) matrix relationship and then proceed, as we did for the Pell case.

– 4.1: Show that the Fibonacci sequence \( x_n = x_{n-1} + x_{n-2} \) may be generated by

\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{NS-3.1}
\]

– 4.2: What is the relationship between \( y_n \) and \( x_n \)?

Ans:

– 4.3: Write a Matlab/Octave program to compute \( x_n \) using the matrix equation above. Test your code using the first few values of the sequence. Using your program, what is \( x_{40} \)? Note: Consider using the eigenanalysis of \( A \), described by Eq. 2.5.18 of the text.

Ans:
4.4: Using the eigenanalysis of the matrix $A$ (and a lot of algebra), show that it is possible to obtain the general formula for the Fibonacci sequence

$$x_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right].$$ \hspace{1cm} (NS-3.2)

4.5: What are the eigenvalues $\lambda_{\pm}$ of the matrix $A$?

**Ans:**

4.6: How is the formula for $x_n$ related to these eigenvalues? Hint: Find the eigenvectors.

**Ans:**

4.7: What happens to each of the two terms

$$\left[ \left( \frac{1 \pm \sqrt{5}}{2} \right)^{n+1} \right]$$

**Ans:**

4.8: What happens to the ratio $x_{n+1}/x_n$?

**Ans:**

**Problem # 5:** Replace the Fibonacci sequence with

$$x_n = \frac{x_{n-1} + x_{n-2}}{2},$$

such that the value $x_n$ is the average of the previous two values in the sequence.
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5.1: What matrix $A$ is used to calculate this sequence?

Ans:

5.2: Modify your computer program to calculate the new sequence $x_n$. What happens as $n \to \infty$?

Ans:

5.3: What are the eigenvalues of your new $A$? How do they relate to the behavior of $x_n$ as $n \to \infty$? Hint: You can expect the closed-form expression for $x_n$ to be similar to Eq. NS-3.2.

Ans:

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Ans:
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Ans:

Problem # 6: Consider the expression

$$\sum_{n=1}^{N} f_n^2 = f_N f_{N+1}.$$  

- 6.1: Find a formula for $f_n$ that satisfies this relationship. Hint: It holds for only the Fibonacci recursion formula.

Ans:

CFA as a matrix recursion

Problem # 7: The CFA may be written as a matrix recursion. For this we adopt a special notation, unlike other matrix notations, with $k \in \mathbb{N}$:

$$\begin{bmatrix} n \\ x \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & [x_k] \\ 0 & \frac{1}{x_k - \lfloor x_k \rfloor} \end{bmatrix} \begin{bmatrix} n \\ x \end{bmatrix}_k.$$  

This equation says that $n_{k+1} = [x_k]$ and $x_{k+1} = 1/(x_k - \lfloor x_k \rfloor)$. It does not mean that $n_{k+1} = [x_k] x_k$, as would be implied by standard matrix notation. The lower equation says that $r_k = x_k - \lfloor x_k \rfloor$ is the remainder—namely, $x_k = [x - k] + r_k$ (Octave/Matlab’s $\text{rem}(x, \text{floor}(x))$ function), also known as $\text{mod}(x, y)$.

- 7.1: Start with $n_0 = 0 \in \mathbb{N}$, $x_0 \in \mathbb{I}$, $n_1 = [x_0] \in \mathbb{N}$, $r_1 = x - [x] \in \mathbb{I}$, and $x_1 = 1/r_1 \in \mathbb{I}$, $r_n \neq 0$. For $k = 1$ this generates on the left the next CFA parameter $n_2 = [x_1]$, and $x_2 = 1/r_2 = 1/(x_0 - \lfloor x_0 \rfloor)$ from $n_0$ and $x_0$. Find $[n, x]_{k+1}^T$ for $k = 2, 3, 4, 5$.

Ans:

$^{1\text{This notation is highly nonstandard due to the nonlinear operations. The matrix elements are derived from the vector rather than multiplying them. These calculation may be done with the help of Matlab/Octave.}}$