1.3 Problems NS-3

Topic of this homework: Pythagorean triplets, Pell's equation, Fibonacci sequence

Pythagorean triplets

Problem # *1: Euclid's formula for the Pythagorean triplets* a, b, c *is* $a = p^2 - q^2$, $b = 2pq$, *and* $c = p^2 + q^2$.

– 1.1: What condition(s) must hold for p *and* q *such that* a, b, *and* c *are always positive and nonzero?*

Ans:

– 1.2: Solve for p *and* q *in terms of* a, b, *and* c.

Ans:

Problem # *2: The ancient Babylonians (ca. 2000 BCE) cryptically recorded* (a, c) *pairs of numbers on a clay tablet, archeologically denoted* Plimpton-322 *(see* **??***).*

– 2.1: Find p *and* q *for the first five pairs of* a *and* c *shown here from Plimpton-322.*

a	c
119	169
3367	4825
4601	6649
12709	18541
65	97

Find a formula for a in terms of p and q.

Ans:

– *2.2: Based on Euclid's formula, show that $c > (a, b)$.*

Ans:

– *2.3: What happens when $c = a$?*

Ans:

– *2.4: Is $b + c$ a perfect square? Discuss.*

Ans:

Pell's equation:

Problem # *3: Pell's equation is one of the most historic (i.e., important) equations of Greek number theory because it was used to show that $\sqrt{2} \in \mathbb{I}$. We seek integer solutions of*

$$x^2 - Ny^2 = 1.$$

As shown on page **??**, the solutions x_n, y_n for the case of $N = 2$ are given by the linear 2×2 matrix recursion

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = 1\jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

with $[x_0, y_0]^T = [1, 0]^T$ and $1\jmath = \sqrt{-1} = e^{j\pi/2}$. It follows that the general solution to Pell's equation for $N = 2$ is

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = (e^{\jmath\pi/2})^n \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

To calculate solutions to Pell's equation using the matrix equation above, we must calculate

$$A^n = e^{\jmath\pi n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n = e^{\jmath\pi n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix},$$

which becomes tedious for $n > 2$.

– *3.1: Find the companion matrix and thus the matrix A that has the same eigenvalues as Pell's equation.* Hint: Use Matlab's function `[E,Lambda] = eig(A)` to check your results!
Ans:

– *3.2: Solutions to Pell's equation were used by the Pythagoreans to explore the value of $\sqrt{2}$. Explain why Pell's equation is relevant to $\sqrt{2}$.*
Ans:

– *3.3: Find the first three values of $(x_n, y_n)^T$ by hand and show that they satisfy Pell's equation for $N = 2$.* **Ans:** *By hand, find the eigenvalues $\lambda_\pm$ of the 2×2 Pell's equation matrix*

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

Ans:

– *3.4: By hand, show that the matrix of eigenvectors, E, is*

$$E = [\vec{e}_+ \quad \vec{e}_-] = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}.$$

Ans:

– 3.5: Using the eigenvalues and eigenvectors you found for A, verify that

$$E^{-1}AE = \Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$$

Ans:

*– 3.6: Now that you have diagonalized A (Equation **??**), use your results for E and Λ to solve for the $n = 10$ solution $(x_{10}, y_{10})^T$ to Pell's equation with $N = 2$.*

Ans:

Problem # *4: Here we seek the general formula for x_n. Like Pell's equation, the Fibonacci equation has a recursive eigenanalysis solution. To find it we must recast x_n as a 2×2 matrix relationship and then proceed, as we did for the Pell case.*

– 4.1: Show that the Fibonacci sequence $x_n = x_{n-1} + x_{n-2}$ may be generated by

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \qquad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{NS-3.1}$$

– 4.2: What is the relationship between y_n and x_n?

Ans:

– 4.3: Write a Matlab/Octave program to compute x_n using the matrix equation above. Test your code using the first few values of the sequence. Using your program, what is x_{40}? Note: Consider using the eigenanalysis of A, described by Eq. **??** of the text.

Ans:

– *4.4: Using the eigenanalysis of the matrix A (and a lot of algebra), show that it is possible to obtain the general formula for the Fibonacci sequence*

$$x_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]. \tag{NS-3.2}$$

– *4.5: What are the eigenvalues $\lambda_\pm$ of the matrix A?*

Ans:

– *4.6: How is the formula for x_n related to these eigenvalues?* Hint: Find the eigenvectors.

Ans:

– *4.7: What happens to each of the two terms*

$$\left[\left(1 \pm \sqrt{5}\right)/2\right]^{n+1}?$$

Ans:

– *4.8: What happens to the ratio x_{n+1}/x_n?*

Ans:

Problem # *5: Replace the Fibonacci sequence with*

$$x_n = \frac{x_{n-1} + x_{n-2}}{2},$$

such that the value x_n is the average of the previous two values in the sequence.

– *5.1: What matrix A is used to calculate this sequence?*
Ans:

– *5.2: Modify your computer program to calculate the new sequence x_n. What happens as $n \to \infty$?*
Ans:

– *5.3: What are the eigenvalues of your new A? How do they relate to the behavior of x_n as $n \to \infty$?* Hint: You can expect the closed-form expression for x_n to be similar to Eq. NS-3.2.
Ans:

– *5.4: What matrix A is used to calculate this sequence?*
Ans:

– *5.5: Modify your computer program to calculate the new sequence x_n. What happens as $n \to \infty$?*
Ans:

– 5.6: What are the eigenvalues of your new A? How do they relate to the behavior of x_n as $n \to \infty$? Hint: You can expect the closed-form expression for x_n to be similar to Eq. NS-3.2.
Ans:

Problem # *6: Consider the expression*

$$\sum_{1}^{N} f_n^2 = f_N f_{N+1}.$$

– 6.1: Find a formula for f_n that satisfies this relationship. Hint: It holds for only the Fibonacci recursion formula.
Ans:

CFA as a matrix recursion

Problem # *7: The CFA may be writen as a matrix recursion. For this we adopt a special notation, unlike other matrix notations,[1] with $k \in \mathbb{N}$:*

$$\begin{bmatrix} n \\ x \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & \lfloor x_k \rfloor \\ 0 & \frac{1}{x_k - \lfloor x_k \rfloor} \end{bmatrix} \begin{bmatrix} n \\ x \end{bmatrix}_k .$$

This equation says that $n_{k+1} = \lfloor x_k \rfloor$ and $x_{k+1} = 1/(x_k - \lfloor x_k \rfloor)$. It does *not* mean that $n_{k+1} = \lfloor x_k \rfloor x_k$, as would be implied by standard matrix notation. The lower equation says that $r_k = x_k - \lfloor x_k \rfloor$ is the *remainder*—namely, $x_k = \lfloor x - k \rfloor + r_k$ (Octave/Matlab's `rem(x,floor(x))` function), also known as `mod(x,y)`.

– 7.1: Start with $n_0 = 0 \in \mathbb{N}$, $x_0 \in \mathbb{I}$, $n_1 = \lfloor x_0 \rfloor \in \mathbb{N}$, $r_1 = x - \lfloor x \rfloor \in \mathbb{I}$, and $x_1 = 1/r_1 \in \mathbb{I}$, $r_n \neq 0$. For $k = 1$ this generates on the left the next CFA parameter $n_2 = \lfloor x_1 \rfloor$ and $x_2 = 1/r_2 = 1/(x_0 - \lfloor x_0 \rfloor)$ from n_0 and x_0. Find $[n, x]^T_{k+1}$ for $k = 2, 3, 4, 5$.
Ans:

[1] This notation is highly nonstandard due to the nonlinear operations. The matrix elements are *derived* from the vector rather than multiplying them. These calculation may be done with the help of Matlab/Octave.