Chapter 4

Vector differential equations

4.1 Problems VC-1

4.1.1 Topics of this homework:

Vector algebra and fields in $\mathbb{R}^3$, gradient and scalar Laplacian operators, definitions of divergence and curl, Gauss’s (divergence) and Stokes’s (curl) laws, system classification (postulates).

4.1.2 Scalar fields and the $\nabla$ operator

Problem # 1: Let $T(x, y) = x^2 + y$ be an analytic scalar temperature field in two dimensions (single-valued $\in \mathbb{R}^2$).

$-1.1$: Find the gradient of $T(x)$ and make a sketch of $T$ and the gradient.

Ans:

$-1.2$: Compute $\nabla^2 T(x)$ to determine whether $T(x)$ satisfies Laplace’s equation.

Ans:

$-1.3$: Sketch the iso-temperature contours at $T = -10, 0, 10$ degrees.
– 1.4: The heat flux\(^1\) is defined as \( \mathbf{J}(x, y) = -\kappa(x, y) \nabla T \), where \( \kappa(x, y) \) is a constant that denotes thermal conductivity at the point \((x, y)\). Given that \( \kappa = 1 \) everywhere (the medium is homogeneous), plot the vector \( \mathbf{J}(x, y) = -\nabla T \) at \( x = 2, \ y = 1 \). Be clear about the origin, direction, and length of your result.

\( \text{Ans:} \)

– 1.5: Find the vector \( \perp \) to \( \nabla T(x, y) \)—that is, tangent to the iso-temperature contours. Hint: Sketch it for one \((x, y)\) point (e.g., \( 2, 1 \)) and then generalize.

\( \text{Ans:} \)

– 1.6: The thermal resistance \( R_T \) is defined as the potential drop \( \Delta T \) over the magnitude of the heat flux \(|\mathbf{J}|\). At a single point the thermal resistance is

\[ R_T(x, y) = -\nabla T / |\mathbf{J}|. \]

How is \( R_T(x, y) \) related to the thermal conductivity \( \kappa(x, y) \)?

\( \text{Ans:} \)

\[ \text{Problem # 2: Acoustic wave equation} \]

Note: In this problem, we will work in the frequency domain.

\(^{1}\)The heat flux is proportional to the change in temperature times the thermal conductivity \( \kappa \) of the medium.
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– 2.1: The basic equations of acoustics in one dimension are

\[-\frac{\partial}{\partial x} P = \rho_0 s \psi \quad \text{and} \quad -\frac{\partial}{\partial x} \psi = \frac{s}{\eta_0 P_0} P.\]

Here \(P(x,\omega)\) is the pressure (in the frequency domain), \(\psi'(x,\omega)\) is the volume velocity (the integral of the velocity over the wavefront with area \(A\)), \(s = \sigma + \omega \mathbf{j}\), \(\rho_0 = 1.2\) is the specific density of air, \(\eta_0 = 1.4\), and \(P_0\) is the atmospheric pressure (i.e., \(10^5\) Pa). Note that the pressure field \(P\) is a scalar (pressure does not have direction), while the volume velocity field \(\psi\) is a vector (velocity has direction).

We can generalize these equations to three dimensions using the \(\nabla\) operator

\[-\nabla P = \rho_0 s \psi \quad \text{and} \quad -\nabla \cdot \psi = \frac{s}{\eta_0 P_0} P.\]

– 2.2: Starting from these two basic equations, derive the scalar wave equation in terms of the pressure \(P\),

\[\nabla^2 P = \frac{s^2}{c_0^2} P,\]

where \(c_0\) is a constant representing the speed of sound.

Ans:

– 2.3: What is \(c_0\) in terms of \(\eta_0\), \(\rho_0\), and \(P_0\)?

Ans:

– 2.4: Rewrite the pressure wave equation in the time domain using the time derivative property of the Laplace transform \([\text{e.g., } \frac{dx}{dt} \leftrightarrow sX(s)]\). For your notation, define the time–domain signal using a lowercase letter, \(p(x,y,z,t) \leftrightarrow \mathcal{P}\).

Ans:
4.1.3 Vector fields and the \( \nabla \) operator

4.1.4 Vector algebra

Problem # 3: Let \( \mathbf{R}(x, y, z) \equiv x(t)\mathbf{\hat{x}} + y(t)\mathbf{\hat{y}} + z(t)\mathbf{\hat{z}}. \)

- 3.1: If \( a, b, \) and \( c \) are constants, what is \( \mathbf{R}(x, y, z) \cdot \mathbf{R}(a, b, c). \)
  \textbf{Ans:}

- 3.2: If \( a, b, \) and \( c \) are constants, what is \( \frac{d}{dt} (\mathbf{R}(x, y, z) \cdot \mathbf{R}(a, b, c)). \)
  \textbf{Ans:}

Problem # 4: Find the divergence and curl of the following vector fields:

- 4.1: \( \mathbf{v} = \mathbf{\hat{x}} + \mathbf{\hat{y}} + 2\mathbf{\hat{z}} \)
  \textbf{Ans:}

- 4.2: \( \mathbf{v}(x, y, z) = x\mathbf{\hat{x}} + xy\mathbf{\hat{y}} + z^2\mathbf{\hat{z}} \)
  \textbf{Ans:}

- 4.3: \( \mathbf{v}(x, y, z) = x\mathbf{\hat{x}} + xy\mathbf{\hat{y}} + \log(z)\mathbf{\hat{z}} \)
  \textbf{Ans:}
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- 4.4: $\mathbf{v}(x, y, z) = \nabla(1/x + 1/y + 1/z)$

Ans:

4.1.5 Vector and scalar field identities

Problem # 5: Find the divergence and curl of the following vector fields:

- 5.1: $\mathbf{v} = \nabla \phi$, where $\phi(x, y) = xe^y$

Ans:

- 5.2: $\mathbf{v} = \nabla \times \mathbf{A}$, where $\mathbf{A} = x\hat{x} + y\hat{y} + z\hat{z}$

Ans:

- 5.3: $\mathbf{v} = \nabla \times \mathbf{A}$, where $\mathbf{A} = y\hat{x} + x^2\hat{y} + z\hat{z}$

Ans:

- 5.4: For any differentiable vector field $\mathbf{V}$, write two vector calculus identities that are equal to zero.

Ans:
– 5.5: What is the most general form a vector field may be expressed in, in terms of scalar \( \Phi \) and vector \( \mathbf{A} \) potentials?

Ans:

Problem # 6: Perform the following calculations. If you can state the answer without doing the calculation, explain why.

– 6.1: Let \( \mathbf{v} = \sin(x)\mathbf{\hat{x}} + y\mathbf{\hat{y}} + z\mathbf{\hat{z}} \). Find \( \nabla \cdot (\nabla \times \mathbf{v}) \).

Ans:

– 6.2: Let \( \mathbf{v} = \sin(x)\mathbf{\hat{x}} + y\mathbf{\hat{y}} + z\mathbf{\hat{z}} \). Find \( \nabla \times (\nabla \sqrt{\mathbf{v} \cdot \mathbf{v}}) \)

Ans:

– 6.3: Let \( \mathbf{v}(x, y, z) = \nabla(x + y^2 + \sin(\log(z))) \). Find \( \nabla \times \mathbf{v}(x, y, z) \).

Ans:

4.1.6 Integral theorems

Problem # 7: For each of the following problems, in a few words, identify either Gauss’s or Stokes’s law, define what it means, and explain the formula that follows the question.

– 7.1: What is the name of this formula?

\[
\int_S \mathbf{n} \cdot \mathbf{v} \, dA = \int_V \nabla \cdot \mathbf{v} \, dV.
\]
- 7.2: What is the name of this formula?
\[ \int_S (\nabla \times \mathbf{V}) \cdot d\mathbf{S} = \oint_C \mathbf{V} \cdot d\mathbf{R} \]
Give one important application. \textbf{Ans:}

- 7.3: Describe a key application of the vector identity
\[ \nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}. \]
\textbf{Ans:}