4.2 Problems VC-2

4.2.1 Topics of this homework:

Partial differential equations; fundamental theorem of vector calculus (Helmholtz's theorem); wave equation; Maxwell's equations (ME) and variables ($\boldsymbol{E}, \boldsymbol{D}; \boldsymbol{B}, \boldsymbol{H}$); Second-order vector differentials; Webster horn equation.

Notation: The following notation is used in this homework:

1. $s = \sigma + j\omega$ is the Laplace frequency, as used in the Laplace transform.
2. A Laplace transform pair is indicated by the symbol $\leftrightarrow$: for example, $f(t) \leftrightarrow F(s)$.
3. π_k is the kth prime; for example, $\pi_k \in \mathbb{P}$, $\pi_k = [2, 3, 5, 7, 11, 13, \ldots]$ for $k = 1, \ldots, 6$).

4.2.2 Partial differential equations (PDEs): Wave equation

Problem # *1: Solve the wave equation in one dimension by defining $\xi = t \mp x/c$.*

– 1.1: Show that d'Alembert's solution, $\varrho(x,t) = f(t - x/c) + g(t + x/c)$, is a solution to the acoustic pressure wave equation in one dimension:

$$\frac{\partial^2 \varrho(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \varrho(x,t)}{\partial t^2},$$

where $f(\xi)$ and $g(\xi)$ are arbitrary functions. **Ans:**

Problem # *2: Solving the wave equation in spherical coordinates (i.e., three dimensions)*

– 2.1: Write the wave equation in spherical coordinates $\varrho(r, \theta, \phi, t)$. Consider only the radial term r (i.e., dependence on angles θ and ϕ is assumed to be zero). Hint: The form of the Laplacian as a function of the number of dimensions is given in Eq. 5.1.9 (page 173). Alternatively, look it up on the internet or in a calculus book.
Ans:

– 2.2: Show that this equation is true:

$$\nabla_r^2 \varrho(r) \equiv \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\varrho(r) = \frac{1}{r}\frac{\partial^2}{\partial r^2} r\varrho(r). \tag{VC-2.1}$$

Hint: Expand both sides of the equation. **Ans:**

– 2.3: Use the results from Eq. VC-2.1 to show that the solution to the spherical wave equation is

$$\nabla_r^2\, \varrho(r,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\, \varrho(r,t) \tag{VC-2.2}$$

$$\varrho(r,t) = \frac{f(t-r/c)}{r} + \frac{g(t+r/c)}{r}. \tag{VC-2.3}$$

Ans:

– 2.4: Using $f(\xi) = \sin(\xi)u(\xi)$ and $g(\xi) = e^{\xi}u(\xi)$, write the solutions to the spherical wave equation, where $u(\xi)$ is the Heaviside step function.
Ans:

– 2.5: Sketch this $f(\xi)$ and $g(\xi)$ for several times (e.g., 0, 1, and 2 seconds), and describe the behavior of the pressure $\varrho(r,t)$ as a function of time t and radius r.
Ans:

– 2.6: What happens when the inbound wave reaches the center at $r = 0$?
Ans:

4.2.3 Helmholtz's formula

Every differentiable vector field may be written as the sum of a scalar potential ϕ and a vector potential $\boldsymbol{w}$. This relationship is best known as the fundamental theorem of vector calculus (also called Helmholtz's formula):

$$\boldsymbol{v} = -\nabla\phi + \nabla\times\boldsymbol{w}. \tag{VC-2.4}$$

This formula seems to be a natural extension of the algebraic products $\boldsymbol{A}\cdot\boldsymbol{B} \perp \boldsymbol{A}\times\boldsymbol{B}$, since $\boldsymbol{A}\cdot\boldsymbol{B} \propto \|\boldsymbol{A}\|\|\boldsymbol{B}\|\cos(\theta)$ and $\boldsymbol{A}\times\boldsymbol{B} \propto \|\boldsymbol{A}\|\|\boldsymbol{B}\|\sin(\theta)$, as developed in Appendix A.3.1, page 219. Thus these orthogonal components have magnitude 1 when we take the norm, due to Euler's identity ($\cos^2(\theta) + \sin^2(\theta) = 1$).

As shown in Table 5.1 (p. 171), Helmholtz's formula separates a vector field (i.e., $\mathbf{v}(\mathbf{x})$) into compressible and rotational parts:

1. The rotational (e.g., angular) part is defined by the vector potential $\boldsymbol{w}$, which requires that $\nabla\times\nabla\times\boldsymbol{w} \neq 0$. A field is irrotational (conservative) when $\nabla\times\boldsymbol{v} = 0$, meaning that the field $\boldsymbol{v}$ can be generated using only a scalar potential, $\boldsymbol{v} = \nabla\phi$ (note that this is how a conservative field is usually defined, by saying there exists some ϕ such that $\boldsymbol{v} = \nabla\phi$).[2]

2. The compressible (e.g., radial) part of a field is defined by the scalar potential ϕ, which requires that $\nabla\cdot\nabla\phi = \nabla^2\phi \neq 0$. A field is incompressible (solenoidal) when $\nabla\cdot\boldsymbol{v} = 0$, meaning that the field $\boldsymbol{v}$ can be generated using only a vector potential, $\boldsymbol{v} = \nabla\times\boldsymbol{w}$.

The definitions and generating potential functions of irrotational (conservative) and incompressible (solenoidal) fields naturally follow from two key vector identities: (1) $\nabla\cdot(\nabla\times\boldsymbol{w}) = 0$ and (2) $\nabla\times(\nabla\phi) = 0$.

Problem # *3: Define the following:*

– 3.1: A conservative vector field

Ans:

– 3.2: An irrotational vector field

Ans:

– 3.3: An incompressible vector field

Ans:

[2] A note about the relationship between the generating function and the test: You might imagine special cases where $\nabla\times\boldsymbol{w} \neq 0$ but $\nabla\times\nabla\times\boldsymbol{w} = 0$ (or $\nabla\phi \neq 0$ but $\nabla^2\phi = 0$). In these cases, the vector (or scalar) potential can be recast as a scalar (or vector) potential. For example, consider a field $\boldsymbol{v} = \nabla\phi_0 + \boldsymbol{b}$, where $\boldsymbol{b} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. Note that $\boldsymbol{b}$ can actually be generated by either a scalar potential ($\phi_1 = \frac{1}{2}[x^2 + y^2 + z^2]$, such that $\nabla\phi_1 = \boldsymbol{b}$) or a vector potential ($\boldsymbol{w}_0 = \frac{1}{2}[z^2\hat{\mathbf{x}} + x^2\hat{\mathbf{y}} + y^2\hat{\mathbf{z}}]$, such that $\nabla\times\boldsymbol{w}_0 = \boldsymbol{b}$). We find that $\nabla\times\boldsymbol{v} = 0$; therefore $\boldsymbol{v}$ must be irrotational. We say this irrotational field is generated by $\nabla\phi = \nabla(\phi_0 + \phi_1)$.

– *3.4: A solenoidal vector field*

Ans:

– *3.5: When is a conservative field irrotational?*

Ans:

– *3.6: When is an incompressible field irrotational?*

Ans:

Problem # *4: For each of the following, (i) compute $\nabla \cdot \boldsymbol{v}$, (ii) compute $\nabla \times \boldsymbol{v}$, and (iii) classify the vector field (e.g., conservative, irrotational, incompressible, etc.).*

– *4.1:* $\boldsymbol{v}(x, y, z) = -\nabla(3yx^3 + y\log(xy))$

Ans:

– *4.2:* $\boldsymbol{v}(x, y, z) = xy\hat{\mathbf{x}} - z\hat{\mathbf{y}} + f(z)\hat{\mathbf{z}}$

Ans:

– *4.3:* $\boldsymbol{v}(x, y, z) = \nabla \times (x\hat{\mathbf{x}} - z\hat{\mathbf{y}})$

Ans:

4.2.4 Maxwell's Equations

The variables have the following names and defining equations (see Table 5.4, p. 201):

Symbol	Equation	Name	Units
$\boldsymbol{E}$	$\nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$	Electric field strength	[volts/m]
$\boldsymbol{D} = \epsilon_o \boldsymbol{E}$	$\nabla \cdot \boldsymbol{D} = \rho$	Electric displacement (flux density)	[coul/m^2]
$\boldsymbol{H}$	$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \dot{\boldsymbol{D}}$	Magnetic field strength	[amps/m]
$\boldsymbol{B} = \mu_o \boldsymbol{H}$	$\nabla \cdot \boldsymbol{B} = 0$	Magnetic induction (flux density)	[webers/m^2]

Note that $\boldsymbol{J} = \sigma \boldsymbol{E}$ is the *current density* (which has units of [amps/m^2]). Furthermore, the *speed of light in vacuo* is $c_o = 3 \times 10^8 = 1/\sqrt{\mu_o \epsilon_o}$ [m/s], and the *characteristic resistance* of light $r_o = 377 = \sqrt{\mu_o/\epsilon_o}$ [Ω (i.e., ohms)].

4.2.5 Speed of light

Problem # *5: The* speed of light *in vacuo is* $c_o = 1/\sqrt{\mu_o \epsilon_o} \approx 3 \times 10^8$ *[m/s]. The* characteristic resistance *in vacuo is* $r_o = \sqrt{\mu_o/\epsilon_o} \approx 377\,[\Omega]$.

– *5.1: Find a formula for the in-vacuo* permittivity ϵ_o *and* permeability *in terms of* c_o *and* r_o. **Ans:** *Based on your formula, what are the numeric values of* ϵ_o *and* μ_o*?*

Ans:

– *5.2: In a few words, identify the law given by this equation, define what it means, and explain the formula:*

$$\int_S \hat{\mathbf{n}} \cdot \boldsymbol{v}\, dA = \int_{\mathcal{V}} \nabla \cdot \boldsymbol{v}\, dV.$$

Ans:

4.2.6 Application of Maxwell's equations

Problem # *6: The electric Maxwell equation is* $\nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$*, where* $\boldsymbol{E}$ *is the* electric field strength *and* $\dot{\boldsymbol{B}}$ *is the time rate of change of the* magnetic induction field*, or simply the magnetic flux density. Consider this equation integrated over a two-dimensional surface* S*, where* $\hat{\mathbf{n}}$ *is a unit vector normal to the surface (you may also find it useful to define the closed path* C *around the surface):*

$$\iint_S [\nabla \times \boldsymbol{E}] \cdot \hat{\mathbf{n}} dS = -\frac{\partial}{\partial t} \iint_S \boldsymbol{B} \cdot \hat{\mathbf{n}} dS.$$

– 6.1: Apply Stokes's theorem to the left-hand side of the equation.
Ans:

– 6.2: Consider the right-hand side of the equation. How is it related to the magnetic flux Ψ *through the surface* S*?*
Ans:

– 6.3: Assume the right-hand side of the equation is zero. Can you relate your answer in question 6.1 to one of Kirchhoff's laws?
Ans:

Problem # *7: The magnetic Maxwell equation is* $\nabla \times \boldsymbol{H} = \boldsymbol{C} \equiv \boldsymbol{J} + \dot{\boldsymbol{D}}$*, where* $\boldsymbol{H}$ *is the* magnetic field strength*,* $\boldsymbol{J} = \sigma \boldsymbol{E}$ *is the conductive (resistive) current density, and the* displacement current $\dot{\boldsymbol{D}}$ *is the time rate of change of the* electric flux density $\boldsymbol{D}$*. Here we defined a new variable* $\boldsymbol{C}$ *as the total current density.*

– 7.1: First consider the equation over a two-dimensional surface S*:*

$$\iint_S [\nabla \times \boldsymbol{H}] \cdot \hat{\mathbf{n}} dS = \iint_S [\boldsymbol{J} + \dot{\boldsymbol{D}}] \cdot \hat{\mathbf{n}} dS = \iint_S \boldsymbol{C} \cdot \hat{\mathbf{n}} dS.$$

Then apply Stokes's theorem to the left-hand side of this equation. In a sentence or two, explain the meaning of the resulting equation. Hint: What is the right-hand side of the equation? **Ans:**

Problem # *8: Consider the next equation in three dimensions. Take the divergence of both sides and integrate over a volume V (closed surface S):*

$$\iiint_V \nabla \cdot [\nabla \times \mathbf{H}] dV = \iiint_V \nabla \cdot \mathbf{C} dV.$$

– *8.1: What happens to the left-hand side of this equation?* Hint: Can you apply a vector identity? **Ans:** *Apply the divergence theorem (sometimes known as Gauss's theorem) to the right-hand side of the equation, and interpret your result.* Hint: Can you relate your result to one of Kirchhoff's laws?
Ans:

4.2.7 Second-order differentials

Problem # *9: This problem is about second-order vector differentials.*

– *9.1: If $\boldsymbol{v}(x, y, z) = \nabla\phi(x, y, z)$, then what is $\nabla \cdot \boldsymbol{v}(x, y, z)$?*
Ans:

– *9.2: Evaluate $\nabla^2\phi$ and $\nabla \times \nabla\phi$ for $\phi(x, y) = xe^y$.*
Ans:

– *9.3: Evaluate $\nabla \cdot (\nabla \times \boldsymbol{v})$ and $\nabla \times (\nabla \times \boldsymbol{v})$ for $\boldsymbol{v} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$.*
Ans:

– *9.4: When* $\boldsymbol{V}(x, y, z) = \nabla(1/x + 1/y + 1/z)$*, what is* $\nabla \times \boldsymbol{V}(x, y, z)$*?*
Ans:

– *9.5: When was Maxwell born and when did he die? How long did he live (within* ± 10 *years)?*
Ans:

4.2.8 Capacitor analysis

Problem # *10: Find the solution to the Laplace equation between two infinite*[3] *parallel plates separated by a distance* d*. Assume that the left plate at* $x = 0$ *is at voltage* $V(0) = 0$ *and the right plate at* $x = d$ *is at voltage* $V_d \equiv V(d)$*.*

– *10.1: Write Laplace's equation in one dimension for* $V(x)$*.*
Ans:

– *10.2: Write the general solution to your differential equation for* $V(x)$*.*
Ans:

– *10.3: Apply the boundary conditions* $V(0) = 0$ *and* $V(d) = V_d$ *to solve for the constants in your equation from question 10.2.*
Ans:

[3] We study plates that are infinite because this means the electric field lines are perpendicular to the plates, running directly from one plate to the other. However, we solve for per-unit-area characteristics of the capacitor.

– 10.4: Find the charge density per unit area ($\sigma = Q/A$, where Q is charge and A is area) on the surface of each plate. Hint: $\boldsymbol{E} = -\nabla V$, and Gauss's law states that $\iint_S \boldsymbol{D} \cdot \hat{\mathbf{n}} dS = Q_{\text{enc}}$.
Ans:

– 10.5: Determine the per-unit-area capacitance C of the system.
Ans:

4.2.9 Webster horn equation

Problem # *11: Horns illustrate an important generalization of the solution of the one dimensional wave equation in regions where the properties (i.e., area of the tube) vary along the axis of wave propagation. Classic applications of horns are in vocal tract acoustics, loudspeaker design, cochlear mechanics, and any case that has wave propagation. Write the formula for the Webster horn equation, and explain the variables.*
Ans: