Chapter 2

Algebraic Equations

2.1 Problems AE-1

Topics of this homework: Fundamental theorem of algebra, polynomials, analytic functions and their inverse, convolution, Newton's root finding method, Riemann zeta function. Deliverables: Answers to problems

Note: The term analytic is used in two different ways. (1) An analytic function is a function that may be expressed as a locally convergent power series; (2) analytic geometry refers to geometry using a coordinate system.

Polynomials and the fundamental theorem of algebra (FTA)

Problem *# 1: A polynomial of degree N is defined as*

$$P_N(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N.$$

– 1.1: How many coefficients a_n does a polynomial of degree N have?

Ans:

– 1.2: How many roots does $P_N(x)$ have?

Ans:

Problem *# 2: The* fundamental theorem of algebra *(FTA)*

– 2.1: State and then explain the FTA.

Ans:

– *2.2: Using the FTA,* prove *your answer to question 1.2.* Hint: Apply the FTA to *prove* how many roots a polynomial $P_N(x)$ of order N has.

Ans:

Problem *# 3: Consider the polynomial function* $P_2(x) = 1 + x^2$ *of degree* $N = 2$ *and the related function* $F(x) = 1/P_2(x)$*. What are the roots (e.g., zeros)* $x_\pm$ *of* $P_2(x)$*?* Hint: Complete the square on the polynomial $P_2(x) = 1 + x^2$ of degree 2, and find the roots.

Ans:

Problem *# 4:* $F(x)$ *may be expressed as (*$A, B, x_\pm \in \mathbb{C}$*)*

$$F(x) = \frac{A}{x - x_+} + \frac{B}{x - x_-}, \tag{AE-1.1}$$

where $x_\pm$ are the roots (zeros) of $P_2(x)$, which become the *poles* of $F(x)$; A and B are the *residues*. The expression for $F(x)$ is sometimes called a *partial fraction expansion* or *residue expansion*, and it appears in many engineering applications.

– *4.1:Find* $A, B \in \mathbb{C}$ *in terms of the roots* $x_\pm$ *of* $P_2(x)$*.*

Ans:

– *4.2: Verify your answers for* A *and* B *by showing that this expression for* $F(x)$ *is indeed equal to* $1/P_2(x)$*.*

Ans:

– *4.3: Give the values of the poles and zeros of* $P_2(x)$*.*

Ans:

– *4.4: Give the values of the poles and zeros of $F(x) = 1/P_2(x)$.*

Ans:

2.1.1 Analytic functions

Overview: Analytic functions are defined by infinite (power) series. The function $f(x)$ is said to be *analytic* at any value of constant $x = x_o$, where there exists a convergent power series

$$P(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n$$

such that $P(x_o) = f(x_o)$. The point $x = x_o$ is called the *expansion point*. The region around x_o such that $|x - x_o| < 1$ is called the *radius of convergence*, or region of convergence (RoC). The local power series for $f(x)$ about $x = x_o$ is defined by the Taylor series:

$$\begin{aligned} f(x) &\approx f(x_o) + \left.\frac{df}{dx}\right|_{x=x_o}(x - x_o) + \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=x_o}(x - x_o)^2 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n}{dx^n} f(x)\right|_{x=x_o} (x - x_o)^n. \end{aligned}$$

Two classic examples are the geometric series[1] where $a_n = 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \tag{AE-1.2}$$

and the exponential function where $a_n = 1/n!$, Eq. 3.2.11 (p. 68). The coefficients for both series may be derived from the Taylor formula.

Problem *# 5: The geometric series*

– *5.1: What is the* region of convergence *(RoC) for the power series Eq. AE-1.2 of $1/(1 - x)$ given above—for example, where does the power series $P(x)$ converge to the function value $f(x)$? State your answer as a condition on x.* Hint: What happens to the power series when $x > 1$?

Ans:

– *5.2: In terms of the pole, what is the RoC for the geometric series in Eq. AE-1.2?*

Ans:

[1] The geometric series is *not* defined as the function $1/(1 - x)$, it is defined as the series $1 + x + x^2 + x^3 + \cdots$, such that the ratio of consecutive terms is x.

– 5.3: How does the RoC relate to the location of the pole of $1/(1-x)$?
Ans:

– 5.4: Where are the zeros, if any, in Eq. AE-1.2?

Ans:

– 5.5: Assuming x is in the RoC, prove that the geometric series correctly represents $1/(1-x)$ by multiplying both sides of Eq. AE-1.2 by $(1-x)$.
Ans:

Problem # 6: *Use the geometric series to study the degree N polynomial. It is very important to note that all the coefficients c_n of this polynomial are 1.*

$$P_N(x) = 1 + x + x^2 + \cdots + x^N = \sum_{n=0}^{N} x^n. \tag{AE-1.3}$$

– 6.1: Prove that

$$P_N(x) = \frac{1 - x^{N+1}}{1 - x}. \tag{AE-1.4}$$

Ans:

– 6.2: What is the RoC for Eq. AE-1.3?
Ans:

– *6.3: What is the RoC for Eq. AE-1.4?*

Ans:

– *6.4: How many poles does $P_N(x)$ (Eq. AE-1.3) have? Where are they?*

Ans:

– *6.5: How many zeros does $P_N(x)$ (Eq. AE-1.4) have? State where are they in the complex plane.*

Ans:

– *6.6: Explain why Eqs. AE-1.3 and AE-1.4 have different numbers of poles and zeros.*

Ans:

– *6.7: Is the function $1/(1-x)$ analytic outside of the RoC?*

Ans:

– *6.8: Extra credit. Evaluate $P_N(x)$ at $x = 0$ and $x = 0.9$ for the case of $N = 100$, and compare the result to that from Matlab.*

```
%sum the geometric series and P_100(0.9)
clear all;close all;format long
N=100; x=0.9; S=0;
for n=0:N
S=S+x^n
```

```
end
P100=(1-x^(N+1))/(1-x);
disp(sprintf('S= %g, P100= %g, error= %g',S,P100, S-P100))
```

Ans:

Problem # 7: *The exponential series*

– 7.1: What is the RoC for the exponential series Eq. 3.2.11?

Ans:

– 7.2: Let $x = \jmath$ in Eq. 3.2.11, and write out the series expansion of e^x in terms of its real and imaginary parts.

Ans:

– 7.3: Let $x = \jmath\theta$ in Eq. 3.2.11, and write out the series expansion of e^x in terms of its real and imaginary parts. How does your result relate to Euler's identity ($e^{\jmath\theta} = \cos(\theta) + \jmath\sin(\theta)$)?

Ans:

2.1.2 Inverse analytic functions and composition

Overview: It may be surprising, but every analytic function has an inverse function. Starting from the function ($x, y \in \mathbb{C}$)

$$y(x) = \frac{1}{1-x}$$

the inverse is

$$x = \frac{y-1}{y} = 1 - \frac{1}{y}.$$

Problem # 8: *Consider the inverse function described above*

– 8.1: Where are the poles and zeros of $x(y)$?
Ans:

– 8.2: Where (for what condition on y) is $x(y)$ analytic?
Ans:

Problem *# 9 Consider the exponential function $z(x) = e^x$ ($x, z \in \mathbb{C}$).*

– 9.1: Find the inverse $x(z)$.
Ans:

– 9.2: Where are the poles and zeros of $x(z)$?
Ans:

Problem *# 10: Composition.*

– 10.1: If $y(s) = 1/(1-s)$ and $z(s) = e^s$, compose these two functions to obtain $(y \circ z)(s)$.
Give the expression for $(y \circ z)(s) = y(z(s))$. **Ans:**

– 10.2: Where are the poles and zeros of $(y \circ z)(s)$?
Ans:

– 10.3: Where (for what condition on x) is $(y \circ z)(x)$ analytic?
Ans:

Eigen-analysis

Problem *# 11: (4 pts) The vectorized eigen-equation for a matrix $\mathcal{A}$ is*

$$\boldsymbol{A}\boldsymbol{E} = \boldsymbol{E}\boldsymbol{\Lambda}. \tag{AE-1.5}$$

– 11.1: (4pt) Provide a formula for $\boldsymbol{A}^3$ in terms of the eigenvector $\boldsymbol{E}$ and eigenvalue $\boldsymbol{\Lambda}$ matricies.
Ans:

– 11.2: (4 pts) Find the eigenvalues of the matrix, and find the roots, by completing the square, where $a, b, c, d \in \mathbb{C}$, and

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Ans:

(4 pts) Convolution

Multiplying two short or simple polynomials is not demanding. However, if the polynomials have many terms, it can become tedious. For example, multiplying two 10th-degree polynomials is not something one would like to do every day.

An alternative is a method called *convolution*. The inverse of convolution is called *deconvolution*, which is equivalent to long-division of polynomials, also known as factoring polynomials (§3.4.1, p. 109-111). Newton's method is a reliable and accurately algorithm to extract roots from polynomials using term by term deconvolution. When the roots are well approximated by fractional numbers, the method is accurate to within computational accuracy. For example, if the root is $\pi \approx \hat{\pi}_{19} \equiv 817696623/260280919 \in \mathbb{F}$, as given by `rats(pi,19)`. $\hat{\pi}_{19}$ is the 64 bit machine's internal representation of π since $\pi - \hat{\pi}_{19} = 0$ (See text Fig. 2.6, p. 48).

Problem *# 12: (4 pts) Convolution of sequences. Practice convolution (by hand!!) using a few simple examples. Manually evaluate the following convolutions. Show your work!*

– 12.1: (2 pts) Multiplying two polynomials is the same as convolving their coefficients.
Given

$$f(x) = x^3 + 3x^2 + 3x + 1 \leftrightarrow [1; 3, 3, 1]$$
$$g(x) = x^3 + 2x^2 + x + 2 \leftrightarrow [1; 2, 1, 2].$$

show that

$$f(x)g(x) = x^6 + 5x^5 + 10x^4 + 12x^3 + 11x^2 + 7x + 2 \leftrightarrow [1; 3, 3, 1] \star [1; 2, 1, 2].$$

Ans:

– *12.2: (1 pts)* $[1; -1] \star [1; 2, 4, 7, 0]$

Ans:

– *12.3: (1 pts)* $[1; 2, 1] \star [1; -1]$

Ans:

Newton's root-finding method

Problem *# 13: Use Newton's iteration to find the roots of the polynomial*

$$P_3(x) = 1 - x^3.$$

– *13.1: Draw a graph describing the first step of the iteration starting with* $x_0 = (1/2, 0)$.

Ans:

– *13.2: Calculate* x_1 *and* x_2*. What number is the algorithm approaching?*

Ans:

– *13.3: Does Newton's method work for $P_2(x) = 1 + x^2$? If so, why?* Hint: What are the roots in this case?
Ans:

Problem *# 14: In this problem we consider the case of fractional roots, and take advantage of this fact during the itteration. Given that the roots are integers, composed of primes, we may uniquely identify the primes by factoring the numerator and denominator of the rational approximation of the root.*
The method is:

1. Start the Newton itteration
$$s_{n+1} = s_n - \frac{M(s_n)}{M'(s_n)}$$
2. Apply the CFA to the next output `rats`(s_{n+1})
3. Factor the Num and Dem of the CFA
4. Terminate when the factors converge

Using this method, show that we can find either the best possible fractional approximation to the roots (or even the exact roots, when the answer is within machine accuracy).

– *14.1: Find the roots of a Monic having coefficents $m_k \in \mathbb{F}$.*
Let
$$M_3(x) = (x - 254/17)(x - 2047/13)(x - 17/13)$$
In this case the root vector R becomes
$$R = [14.9412, 157.4615, 1.3077].$$
Verify that `rats(M)` returns the rational set of roots. **Ans:**