2.3   Problems AE-3

Topics of this homework:

Visualizing complex functions, bilinear/Möbius transformation, Riemann sphere.
   Deliverables: Answers to problems

Two-port network analysis

Problem *# 1: Perform an analysis of electrical two-port networks, shown in Fig. 3.8 (page 105). This can be a mechanical system if the capacitors are taken to be springs and inductors taken as mass, as in the suspension of the wheels of a car. In an acoustical circuit, the low-pass filter could be a car muffler. While the physical representations will be different, the equations and the analysis are exactly the same.*

The definition of the ABCD *transmission matrix* ($\mathcal{T}$) is

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \tag{AE-3.1}$$

The *impedance matrix*, where the determinant $\Delta_T = AD - BC$, is given by

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{\mathcal{C}} \begin{bmatrix} \mathcal{A} & \Delta_T \\ 1 & \mathcal{D} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{AE-3.2}$$

   – 1.1: Derive the formula for the impedance matrix (Eq. AE-3.2) given the transmission matrix definition (Eq. AE-3.1). Show your work.
   Ans:

Problem *# 2: Consider a single circuit element with impedance $Z(s)$.*

   – 2.1: What is the ABCD matrix for this element if it is in series?
   Ans:

   – 2.2: What is the ABCD matrix for this element if it is in shunt?
   Ans:

Problem # 3: *Find the ABCD matrix for each of the circuits of Fig. 3.8.*

For each circuit, (i) show the cascade of transmission matrices in terms of the complex frequency $s \in \mathbb{C}$, then (ii) substitute $s = 1\jmath$ and calculate the total transmission matrix at this single frequency.

    – 3.1: Left circuit (let $R_1 = R_2 = 10$ kilo-ohms and $C = 10$ nano-farads)

   Ans:

    – 3.2: Right circuit (use L and C values given in the figure), where the pressure P is analogous to the voltage V, and the velocity U is analogous to the current I.

  Ans:

    – 3.3: Convert both transmission (ABCD) matrices to impedance matrices using Eq. AE-3.2. Do this for the specific frequency $s = 1\jmath$ as in the previous part (feel free to use Matlab/Octave for your computation).

  Ans:

    – 3.4: Right circuit: Repeat the analysis as in question 3.3.

  Ans:

Algebra

Problem # 4: *Fundamental theorem of algebra (FTA).*

    – 4.1: State the fundamental theorem of algebra (FTA).

  Ans:

(13 pts) Algebra with complex variables

Problem # 5: (7 pts) Order and complex numbers:

One can always say that $3 < 4$—namely, that real numbers have order. One way to view this is to take the difference and compare it to zero, as in $4 - 3 > 0$. Here we will explore how complex variables may be ordered. In the following define $\{x, y\} \in \mathbb{R}$ and complex variable $z = x + y\jmath \in \mathbb{C}$.

   – 5.1: Explain the meaning of $|z_1| > |z_2|$.

Ans:

   – 5.2: If $x_1, x_2 \in \mathbb{R}$ (are real numbers), define the meaning of $x_1 > x_2$.

Ans:

   – 5.3: Explain the meaning of $z_1 > z_2$.

Ans:

   – 5.4: (2 pts) What is the meaning of $|z_1 + z_2| > 3$?

Ans:

   – 5.5: (2 pts) If time were complex, how might the world be different?

Ans:

Problem # 6: *(1 pt) It is frequently necessary to consider a function $w(z) = u + v\jmath$ in terms of the real functions $u(x, y)$ and $v(x, y)$ (e.g. separate the real and imaginary parts). Similarly, we can consider the inverse $z(w) = x + y\jmath$, where $x(u, v)$ and $y(u, v)$ are real functions.*

    – 6.1: (1 pts) Find $u(x, y)$ and $v(x, y)$ for $w(z) = 1/z$.

Ans:

Problem # 7: *(5 pts) Find $u(x, y)$ and $v(x, y)$ for $w(z) = c^z$ with complex constant $c \in \mathbb{C}$ for questions 7.1, 7.2, and 7.3:*

    – 7.1: $c = e$

Ans:

    – 7.2: $c = 1$ (recall that $1 = e^{\pm\jmath 2\pi k}$ for $k \in \mathbb{Z}$

Ans:

    – 7.3: $c = \jmath$. Hint: $\jmath = e^{\jmath\pi/2 + \jmath 2\pi k}, \quad k \in \mathbb{Z}$.

Ans:

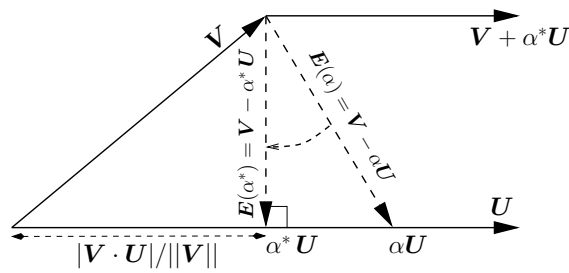   *– 7.4: (2 pts) What is $\jmath^{\jmath}$?*

Ans:

Figure 2.2: *This figure shows how to derive the Schwarz inequality, by finding the value of $\alpha = \alpha^*$ corresponding to $\min_{\alpha}[\boldsymbol{E}(\alpha)]$. It is identical to Fig. 3.5 on page 89.*

Schwarz inequality

Problem # 8: *The above figure shows three vectors for an arbitrary value of $\alpha \in \mathbb{R}$ and a specific value of $\alpha = \alpha^*$.*

– 8.1: Find the value of $\alpha \in \mathbb{R}$ such that the length (norm) of $\boldsymbol{E}$ (i.e., $||\boldsymbol{E}|| \geq 0$) is minimum. Show your derivation, not the answer ($\alpha = \alpha^*$).

Ans:

– 8.2: Find the formula for $||\boldsymbol{E}(\alpha^*)||^2 \geq 0$. Hint: Substitute α^* into Eq. 3.5.9 (p. 90) and show that this results in the Schwarz inequality

$$|\boldsymbol{U} \cdot \boldsymbol{V}| \leq ||\boldsymbol{U}||||\boldsymbol{V}||.$$

Ans:

Problem # 9: *Geometry and scaler products*

– 9.1: What is the geometrical meaning of the dot product of two vectors?
Ans:

– 9.2: Give the formula for the dot product of two vectors. Explain the meaning based on Fig. 3.4 (page 85).

Ans:

– 9.3: Write the formula for the dot product of two vectors $U \cdot V$ in $\mathbb{R}^n$ in polar form (e.g., assume the angle between the vectors is θ).

Ans:

– 9.4: How is the Schwarz inequality related to the Pythagorean theorem?

Ans:

– 9.5: Starting from $||U + V||$, derive the triangle inequality

$$||U + V|| \leq ||U|| + ||V||.$$

Ans:

– 9.6: The triangle inequality $||U + V|| \leq ||U|| + ||V||$ is true for two and three dimensions: Does it hold for five-dimensional vectors?

Ans:

– *9.7: Show that the wedge product* $U \wedge V \perp U \cdot V$.

Ans:

Probability

Problem *# 10: Basic terminology of experiments*

– *10.1: What is the* mean *of a trial, and what is the average over all trials?*
Ans:

– *10.2: What is the* expected value *of a random variable* X *?*
Ans:

– *10.3: What is the* standard deviation *about the mean?*
Ans:

– *10.4: What is the definition of* information *of a random variable?*
Ans:

– 10.5: How do you combine events? Hint: If the event is the flip of a biased coin, the events are $H = p$, $T = 1 - p$, so the event is $\{p, 1 - p\}$. To solve the problem, you must find the probabilities of two independent events.

Ans:

– 10.6: What does the term independent *mean in the context of question 10.5? Give an example.*

Ans:

– 10.7: Define odds.

Ans: