3.3 Problems DE-3

3.3.1 Topics of this homework: Brune impedance

lattice transmission line analysis

3.3.2 Brune Impedance

Problem *# 1: Residue form*

A Brune impedance is defined as the ratio of the force $F(s)$ to the flow $V(s)$ and may be expressed in residue form as

$$Z(s) = c_0 + \sum_{k=1}^{K} \frac{c_k}{s - s_k} = \frac{N(s)}{D(s)} \tag{DE-3.1}$$

with

$$D(s) = \prod_{k=1}^{K} (s - s_k) \qquad \text{and} \qquad c_k = \lim_{s \to s_k} (s - s_k) D(s) = \prod_{n'=1}^{K-1} (s - s_n).$$

The prime on the index n' means that $n = k$ is not included in the product.

– 1.1: Find the Laplace transform ($\mathcal{LT}$) of a (1) spring, (2) dashpot, and (3) mass.

Express these in terms of the force $F(s)$ and the velocity $V(s)$, along with the electrical equivalent impedance: (1) Hooke's law $f(t) = Kx(t)$, (2) dashpot resistance $f(t) = Rv(t)$, and (3) Newton's law for mass $f(t) = M dv(t)/dt$. **Ans:**

– 1.2: Take the Laplace transform ($\mathcal{LT}$) of Eq. DE-3.2 and find the total impedance $Z(s)$ of the mechanical circuit.

$$M \frac{d^2}{dt^2} x(t) + R \frac{d}{dt} x(t) + Kx(t) = f(t) \leftrightarrow (Ms^2 + Rs + K)X(s) = F(s). \tag{DE-3.2}$$

Ans:

– 1.3: What are $N(s)$ and $D(s)$ (see Eq. DE-3.1)?

Ans:

– 1.4: Assume that $M = R = K = 1$ and find the residue form of the admittance $Y(s) = 1/Z(s)$ (see Eq. DE-3.1) in terms of the roots $s_\pm$. Hint: Check your answer with Octave's/Matlab's `residue` *command.*

Ans:

– 1.5: By applying Eq. 4.5.3 (page 149), find the inverse Laplace transform ($\mathcal{LT}^{-1}$). Use the residue form of the expression that you derived in question 1.4.

Ans:

3.3.3 Transmission-line analysis

Problem *# 2:* **Train-mission-line** *We wish to model the dynamics of a freight train that has N such cars and study the* velocity transfer function *under various load conditions.*

As shown in Fig. 4.11, the train model consists of masses connected by springs.

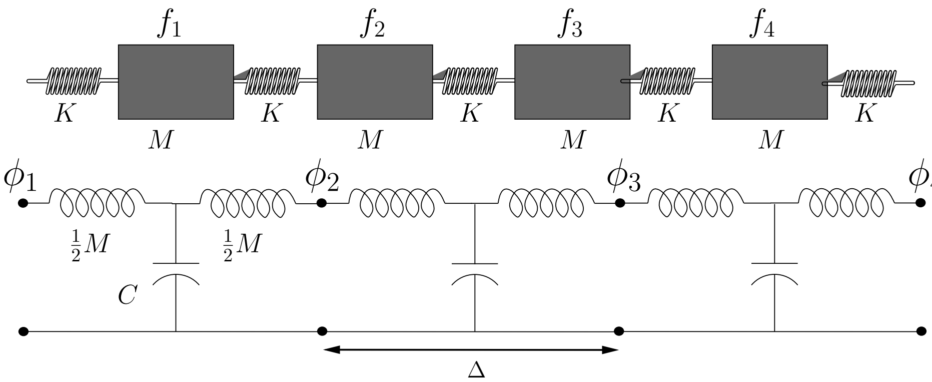

Figure 3.2: *Depiction of a train consisting of cars treated as masses M and linkages treated as springs of stiffness K or compliance $C = 1/K$. Below it is the electrical equivalent circuit for comparison. The masses are modeled as inductors and the springs as capacitors to ground. The velocity is analogous to a current and the force $f_n(t)$ to the voltage $\phi_n(t)$. The length of each cell is Δ [m]. The train may be accurately modeled as a transmission line (TL), since the equivalent electrical circuit is a lumped model of a TL. This method, called a* Cauer synthesis, *is based on the ABCD transmission line method of Sec. 3.8 (p. 105).*

Use the ABCD method (see the discussion in Appendix B.3, p. 228) to find the matrix representation of the system of Fig. 4.11. Define the force on the nth train car $f_n(t) \leftrightarrow F_n(\omega)$ and the velocity $v_n(t) \leftrightarrow V_n(\omega)$.

Break the model into cells consisting of three elements: a series inductor representing half the mass ($M/2$), a shunt capacitor representing the spring ($C = 1/K$), and another series inductor representing half the mass ($L = M/2$), transforming the model into a cascade of symmetric ($\mathcal{A} = \mathcal{D}$) identical cell matrices $\mathcal{T}(s)$.

– 2.1: Find the elements of the ABCD matrix $\mathcal{T}$ for the single cell that relate the input node 1 to output node 2

$$\begin{bmatrix} F \\ V \end{bmatrix}_1 = \mathcal{T} \begin{bmatrix} F(\omega) \\ -V(\omega) \end{bmatrix}_2 . \tag{DE-3.3}$$

Ans:

– 2.2: Express each element of $\mathcal{T}(s)$ in terms of the complex Nyquist ratio $s/s_c < 1$ ($s = 2\pi j f$, $s_c = 2\pi j f_c$). The Nyquist wavelength sampling condition is $\lambda_c > 2\Delta$. It says the critical wavelength $\lambda_c > 2\Delta$. Namely it is defined in terms the minimum number of cells 2Δ, per minimum wavelength λ_c.
The Nyquist wavelength sampling theorem says that there are at least two cars per wavelength.

Proof: From the figure, the distance between cars $\Delta = c_o T_o$ [m], where

$$c_o = \frac{1}{\sqrt{MC}} \qquad \text{[m/s]}.$$

The cutoff frequency obeys $f_c \lambda_c = c_o$. The Nyquist critical wavelength is $\lambda_c = c_o/f_c > 2\Delta$. Therefore the Nyquist sampling condition is

$$f < f_c \equiv \frac{c_o}{\lambda_c} = \frac{c_o}{2\Delta} = \frac{1}{2\Delta\sqrt{MC}} \qquad \text{[rad/sec]}. \tag{DE-3.4}$$

Finally, $s_c = \jmath 2\pi f_c$.

Ans:

– 2.3: Use the property of the Nyquist sampling frequency *$\omega < \omega_c$ (Eq. DE-3.4) to remove higher order powers of frequency*

$$1 + \cancelto{0}{\left(\frac{s}{s_c}\right)^2} \approx 1 \tag{DE-3.5}$$

to determine a band-limited approximation of $\mathcal{T}(s)$.

Ans:

Problem *# 3: Now consider the cascade of N such $\mathcal{T}(s)$ matrices and perform an eigenanalysis.*

– 3.1: Find the eigenvalues and eigenvectors of $\mathcal{T}(s)$ as functions of s/s_c.

Ans:

Problem *# 4: Find the velocity transferfunction $H_{12}(s) = V_2/V_1|_{F_2=0}$.*

– 4.1: Assuming that $N = 2$ and $F_2 = 0$ (two half-mass problem), find the transfer function $H(s) \equiv V_2/V_1$. From the results of the $\mathcal{T}$ matrix, find

$$H_{21}(s) = \left.\frac{V_2}{V_1}\right|_{F_2=0}$$

Express H_{12} in terms of a residue expansion. **Ans:**

– *4.2: Find $h_{21}(t) \leftrightarrow H_{21}(s)$.*
Ans:

– *4.3: What is the input impedance $Z_2 = F_2/V_2$, assuming $F_3 = -r_0 V_3$?*
Ans:

– *4.4: Simplify the expression for Z_2 as follows:*

1. Assuming the *characteristic impedance* $r_0 = \sqrt{M/C}$,
2. terminate the system in r_0: $F_2 = -r_0 V_2$ (i.e., $-V_2$ cancels).
3. Assume higher-order frequency terms are less than 1 ($|s/s_c| < 1$).
4. Let the number of cells $N \to \infty$. Thus $|s/s_c|^N = 0$.

When a transmission line is terminated in its characteristic impedance r_0, the input impedance $Z_1(s) = r_0$. Thus, when we simplify the expression for $\mathcal{T}(s)$, it should be equal to r_0. Show that this is true for this setup.
Ans:

– *4.5: State the ABCD matrix relationship between the first and Nth nodes in terms of the cell matrix. Write out the transfer function for one cell, H_{21}.*
Ans:

– *4.6: What is the velocity transfer function* $H_{N1} = \frac{V_N}{V_1}$*?*

Ans: