1.3 Problems NS-3

Topic of this homework:  Pythagorean triplets, Pell’s equation, Fibonacci sequence

Pythagorean triplets

**Problem # 1:** Euclid’s formula for the Pythagorean triplets $a, b, c$ is $a = p^2 - q^2$, $b = 2pq$, and $c = p^2 + q^2$.

– 1.1: What condition(s) must hold for $p$ and $q$ such that $a$, $b$, and $c$ are always positive and nonzero?

**Ans:**

– 1.2: Solve for $p$ and $q$ in terms of $a$, $b$, and $c$.

**Ans:**

**Problem # 2:** The ancient Babylonians (ca. 2000 BCE) cryptically recorded $(a, c)$ pairs of numbers on a clay tablet, archeologically denoted Plimpton-322 (see 2.8).

– 2.1: Find $p$ and $q$ for the first five pairs of $a$ and $c$ shown here from Plimpton-322.

<table>
<thead>
<tr>
<th>$a$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>119</td>
<td>169</td>
</tr>
<tr>
<td>3367</td>
<td>4825</td>
</tr>
<tr>
<td>4601</td>
<td>6649</td>
</tr>
<tr>
<td>12709</td>
<td>18541</td>
</tr>
<tr>
<td>65</td>
<td>97</td>
</tr>
</tbody>
</table>

Find a formula for $a$ in terms of $p$ and $q$.

**Ans:**
2.2: Based on Euclid’s formula, show that \( c > (a, b) \).

**Ans:**

2.3: What happens when \( c = a \)?

**Ans:**

2.4: Is \( b + c \) a perfect square? Discuss.

**Ans:**
1.3. PROBLEMS NS-3

Pell’s equation:

**Problem # 3:** Pell’s equation is one of the most historic (i.e., important) equations of Greek number theory because it was used to show that $\sqrt{2} \in \mathbb{Q}$. We seek integer solutions of

$$x^2 - Ny^2 = 1.$$  

As shown in §2.5.2, the solutions $x_n, y_n$ for the case of $N = 2$ are given by the linear $2 \times 2$ matrix recursion

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = 1 \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

with $[x_0, y_0]^T = [1, 0]^T$ and $1 = e^{\pi i/2}$. It follows that the general solution to Pell’s equation for $N = 2$ is

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = (e^{\pi i/2})^n \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

To calculate solutions to Pell’s equation using the matrix equation above, we must calculate

$$A^n = e^{\pi n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = e^{\pi n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix},$$

which becomes tedious for $n > 2$.

– 3.1: Find the companion matrix and thus the matrix $A$ that has the same eigenvalues as Pell’s equation. Hint: Use Matlab’s function $[E, Lambda] = eig(A)$ to check your results! 

**Ans:**

– 3.2: Solutions to Pell’s equation were used by the Pythagoreans to explore the value of $\sqrt{2}$. Explain why Pell’s equation is relevant to $\sqrt{2}$. 

**Ans:**

– 3.3: Find the first three values of $(x_n, y_n)^T$ by hand and show that they satisfy Pell’s equation for $N = 2$. **Ans:** By hand, find the eigenvalues $\lambda_{\pm}$ of the $2 \times 2$ Pell’s equation matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$  

**Ans:**
- 3.4: By hand, show that the matrix of eigenvectors, $E$, is

$$E = \begin{bmatrix} e_+ & e_- \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}.$$

**Ans:**

- 3.5: Using the eigenvalues and eigenvectors you found for $A$, verify that

$$E^{-1}AE = \Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$$

**Ans:**

- 3.6: Once you have diagonalized $A$, use your results for $E$ and $\Lambda$ to solve for the $n = 10$ solution $(x_{10}, y_{10})^T$ to Pell’s equation with $N = 2$.

**Ans:**
Problem #4: Here we seek the general formula for $x_n$. Like Pell’s equation, the Fibonacci equation has a recursive eigenanalysis solution. To find it we must recast $x_n$ as a $2 \times 2$ matrix relationship and then proceed, as we did for the Pell case.

- 4.1: Show that the Fibonacci sequence $x_n = x_{n-1} + x_{n-2}$ may be generated by

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad \text{(NS-3.1)}$$

- 4.2: What is the relationship between $y_n$ and $x_n$?

\textbf{Ans:}

- 4.3: Write a Matlab/Octave program to compute $x_n$ using the matrix equation above. Test your code using the first few values of the sequence. Using your program, what is $x_{40}$? Note: Consider using the eigenanalysis of $A$, described by Eq. 2.5.18 of the text.

\textbf{Ans:}

- 4.4: Using the eigenanalysis of the matrix $A$ (and a lot of algebra), show that it is possible to obtain the general formula for the Fibonacci sequence

$$x_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right]. \quad \text{(NS-3.2)}$$

- 4.5: What are the eigenvalues $\lambda_{\pm}$ of the matrix $A$?

\textbf{Ans:}

- 4.6: How is the formula for $x_n$ related to these eigenvalues? Hint: Find the eigenvectors.

\textbf{Ans:}
4.7: What happens to each of the two terms

\[
\left( \frac{1 \pm \sqrt{5}}{2} \right)^{n+1}
\]

**Ans:**

4.8: What happens to the ratio \( x_{n+1}/x_n \)?

**Ans:**

**Problem #5:** Replace the Fibonacci sequence with

\[
x_n = \frac{x_{n-1} + x_{n-2}}{2},
\]

such that the value \( x_n \) is the average of the previous two values in the sequence.

5.1: What matrix \( A \) is used to calculate this sequence?

**Ans:**

5.2: Modify your computer program to calculate the new sequence \( x_n \). What happens as \( n \to \infty \)?

**Ans:**

5.3: What are the eigenvalues of your new \( A \)? How do they relate to the behavior of \( x_n \) as \( n \to \infty \)? Hint: You can expect the closed-form expression for \( x_n \) to be similar to Eq. NS-3.2.

**Ans:**
Problem # 6: Consider the expression
\[ \sum_{n=1}^{N} f_n^2 = f_N f_{N+1}. \]

– 6.1: Find a formula for \( f_n \) that satisfies this relationship. Hint: It holds for only the Fibonacci recursion formula.

\textbf{Ans:}