2.1.8 Exercises AE-2

**Topic of this homework:** Linear systems of equations; Gaussian elimination; Matrix permutations; Overspecified systems of equations; Analytic geometry; Ohm’s law; Two-port networks

**Deliverable:** Answers to problems

**Problem # 1: Nonlinear (quadratic) to linear equations**

In the following problems we deal with algebraic equations in more than one variable, that are not linear equations. For example, the circle $x^2 + y^2 = 1$ is just such an equation. It may be solved for $y(x) = \pm \sqrt{1 - x^2}$.

**Example:** If we let $z_+ = x + yj = x + j\sqrt{1 - x^2} = e^{\theta j}$, we obtain the equation for half a circle ($y > 0$). The entire circle is described by the magnitude of $z$, as $|z|^2 = (x + yj)(x - yj) = 1$.

–Q 1.1: Given the curve defined by the equation:

$$x^2 + xy + y^2 = 1$$

–Q 1.2: Find the function $y(x)$.

**Sol:** Completing the square in $y$ and solve for $y(x)$:

$$(y + x/2)^2 - x^2/4 + x^2 = 1$$

$$(y + x/2)^2 = 1 - \frac{3}{4}x^2$$

$$y + x/2 = \pm \sqrt{\frac{4 - 3x^2}{4}}$$

$$y = \frac{1}{2} \left( \pm \sqrt{4 - 3x^2} - x \right)$$

–Q 1.3: Using Matlab/Octave, plot $y(x)$, and describe the graph.

**Sol:**

Thus we find the equation is a rotated ellipse.

–Q 1.4: What is the name of this curve?

**Sol:** It is an ellipse, rotated by 45 degrees.
–Q 1.5: Find the solution (in \(x, p\), and \(q\)) to the following equations:

\[
\begin{align*}
x + y &= p \\
xy &= q \\
\end{align*}
\]

\textbf{Sol:} Solve the first equation for \(y\) as \(y = p - x\), and then substitute it into the second equation

\[
x(p - x) = -x^2 + px = q.
\]

Thus we find the quadratic

\[
x^2 - px + q = 0
\]

having roots given by completing the square

\[
(x - p/2)^2 = (p/2)^2 - q.
\]

resulting in \(x = p/2 \pm \sqrt{(p/2)^2 - q}\), \(y = p - x\).

\textbf{Summary:} Here we started with one linear and one quadratic (hyperbola). By the use of composition we found the roots.

–Q 1.6: Find an equation that is linear in \(y\) starting from equations that are quadratic (2\(^{nd}\) degree) in the two unknowns \(x, y\):

\[
\begin{align*}
x^2 + xy + y^2 &= 1 \\
4x^2 + 3xy + 2y^2 &= 3
\end{align*}
\]

\textbf{Sol:} The goal is to obtain a linear equation in \(y\). Thus we need to remove the quadratic term \(y^2\). Scale the upper equation by 2 and subtract it from the lower equation:

\[
2x^2 + xy = 1
\]

Solving for \(y\) gives \(y = (1 - 2x^2)/x\).

Note that matrix notation is not as useful for quadratic equations as it is for linear equations (but it can be used in some cases).

–Q 1.7: Compose the two quadratic equations

\[
\begin{align*}
x^2 + xy + y^2 &= 1 \\
2x^2 + xy &= 1
\end{align*}
\]

and describe the results. \textbf{Sol:} By isolating \(y\) from one of the two equations, we may remove it from the other equation, giving us a single 4\(^{th}\) degree equation in \(x\):

\[
x^2 + (1 - 2x^2) + (1 - 2x^2)^2/x^2 = 1
\]

or

\[
x^4 + x^2 - 2x^4 + 1 - 4x^2 + 4x^4 - x^2 = 0
\]

Collecting terms

\[
3x^4 - 4x^2 + 1 = 0
\]

This is a quartic, but is a quadratic in \(x^2\). Of course \(x\) may be complex, rendering this very difficult to deal with in any detail. \textbf{Conclusion:} We started with two ellipses (they have an \(xy\) term which can be removed by a rotation, as we showed in Problem 1.1) This again demonstrates that composition of \(m \times n\) gives degree of \(mn\). When one multiplies polynomials the degree is \(m + n\). Thus composition gives the product and multiplication gives the sum of the degrees of the individual polynomials.

\(5\)This problem is taken from Stillwell, Exercise 6.2.1 (p. 91).
Problem #1: Gaussian elimination

1. Find the inverse of

\[
A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.
\]

**Sol:**

\[
A^{-1} = \frac{1}{3-8} \begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}.
\]

2. Verify that \(A^{-1}A = AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). **Sol:** Multiply them to show this.

3. Find the solution to the following 3x3 matrix equation \(Ax = b\) by Gaussian elimination. Show your intermediate steps. You can check your work at each step using Octave/Matlab.

\[
\begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 8 \end{bmatrix}.
\]

(a) Show (i.e., verify) that the first GE matrix \(G_1\), which zeros out all entries in the first column, is given by

\[
G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}
\]

Identify the elementary row operations that this matrix performs.

**Sol:** Operate with GE matrix on \(A\)

\[
G_1[A|b] = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 4 \\ 0 & -2 & 5 \end{bmatrix}
\]

It scales the first row by -3 and adds it to the second row, and scales the first row by -1 and adds it to the third row.

(b) Find a second GE matrix, \(G_2\), to put \(G_1A\) in upper triangular form. Identify the elementary row operations that this matrix performs.

**Sol:**

\[
G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}
\]

which scales the second row by -1 and adds it to the third row. Thus we have

\[
G_2G_1[A|b] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 4 \\ 0 & 0 & 1 \end{bmatrix}
\]
(c) Find a third GE matrix, \( G_3 \), which scales each row so that its leading term is 1. Identify the elementary row operations that this matrix performs. *Sol:*

\[
G_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1/2 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

which scales the second row by -1/2. Thus we have

\[
G_3 G_2 G_1 [A|b] = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1/2 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
-3 & 1 & 0 \\
-1 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 & -1 \\
3 & 1 & 1 \\
1 & -1 & 4 \\
\end{bmatrix}
\]

(d) Finally, find the last GE matrix, \( G_4 \), that subtracts a scaled version of row 3 from row 2, and scaled versions of rows 2 and 3 from row 1, such that you are left with the identity matrix \( (G_4 G_3 G_2 G_1 A = I) \). *Sol:*

\[
G_4 = \begin{bmatrix}
1 & -1 & -1 \\
0 & 1 & 2 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

Thus we find \( G_4 G_3 G_2 G_1 [A|b] \) is

\[
= \begin{bmatrix}
1 & -1 & -1 \\
0 & 1 & 2 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & -1/2 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
-3 & 1 & 0 \\
-1 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 & -1 \\
3 & 1 & 1 \\
1 & -1 & 4 \\
\end{bmatrix}
\]

(e) Solve for \( [x_1, x_2, x_3]^T \) using the augmented matrix format \( G_4 G_3 G_2 G_1 [A|b] \) (where \( [A|b] \) is the augmented matrix). Note that if you’ve performed the preceding steps correctly, \( x = G_4 G_3 G_2 G_1 b \). *Sol:*

From the preceding problems, we see that \( [x_1, x_2, x_3]^T = [3, -1, 1]^T \)

**Permutations and Pivots**

(a) Find the pivot matrix \( G \) that rescales the third row of the augmented matrix \( A|b \) by 1/3.

*Sol:*

\[
G = \begin{bmatrix}
1 \\
1/3 \\
1 \\
\end{bmatrix}
\]

*Sol:*

\[
G_1 A = \begin{bmatrix}
1 \\
1/3 \\
1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 & -1 \\
3 & 1 & 1 \\
1 & -1 & 4 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 & -1 \\
1/3 & 1/3 & 3 \\
1 & -1 & 4 \\
\end{bmatrix}
\]

[: Gaussian elimination]

**Two linear equations**

In this exercise we transition from a general pair of equations

\[
\begin{align*}
f(x, y) &= 0 \\
g(x, y) &= 0
\end{align*}
\]
to the important case of two linear equations

\[ y = ax + b \\
\]
\[ y = \alpha x + \beta. \]

Note that, to help keep track of the variables, Roman coefficients \((a, b)\) are used for the first equation and Greek \((\alpha, \beta)\) for the second.

- What does it mean, graphically, if these two linear equations have
  - a unique solution,
  - a non-unique solution, or
  - no solution?

**Sol:** There are three possibilities:

- When they have different slopes, they meet at one \((x, y)\) point, which is the solution.
- If the two lines are identical, any point on the line is a solution.
- If they have the same slope but different intercepts (are parallel to each other) there is no solution.

- Assuming the two equations have a unique solution, find the solution for \(x\) and \(y\).

**Sol:** Since there must be one point where the two are equal, we may solve for that by setting the \(y\) values equal to each other:

\[ ax + b = \alpha x + \beta \]

Thus

\[ x = \frac{\beta - b}{a - \alpha} \]
\[ y = \frac{a \beta - b}{a - \alpha} + b \]

- When will this solution fail to exist (for what conditions on \(a, b, \alpha, \) and \(\beta\))?

**Sol:** As stated above, if they have the same slope \(\alpha = a\) but different intercepts \(\beta \neq b\), there is no solution. When \(\beta = b\) and \(\alpha = a\) every point on the line is a solution.

- Write the equations as a 2x2 matrix equation of the form \(A x = b\), where \(x = [x, y]^T\).

**Sol:**

\[
\begin{bmatrix}
1 & -a \\
1 & -\alpha
\end{bmatrix}
\begin{bmatrix}
y \\
x
\end{bmatrix}
= 
\begin{bmatrix}
b \\
\beta
\end{bmatrix}
\]

- Finding the inverse of the 2x2 matrix, and solve the matrix equation for \(x\) and \(y\).

**Sol:**

\[
\begin{bmatrix}
y \\
x
\end{bmatrix}
= 
\frac{1}{\Delta}
\begin{bmatrix}
-\alpha & a \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
b \\
\beta
\end{bmatrix}
= 
\frac{1}{\Delta}
\begin{bmatrix}
-\alpha b + a\beta \\
-b + \beta
\end{bmatrix}
\]

where the determinant is \(\Delta \equiv a - \alpha\).

- Discuss the properties of the determinant of the matrix \((\Delta)\) in terms of the slopes of the two equations \((a\) and \(\alpha)\).

**Sol:** When the slopes are the same there is no solution and \(\Delta = 0\). Thus the matrix solution is consistent with the geometry. This is our first result in analytic geometry.

- An application of linear functional relationships between two variables:

2x2 matrices are used to describe 2-port networks, as will be discussed in Lec 16. Transmission lines are a great example, where both voltage and current must be tracked as they travel along the line. Figure 2.3 shows an example segment of a transmission line.
Suppose you are given the following pair of linear relationships between the input (source) variables \( V_1 \) and \( I_1 \), and the output (load) variables \( V_2 \) and \( I_2 \) of the transmission line.

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix} \begin{bmatrix}
V_2 \\
I_2
\end{bmatrix}.
\]

- Let the output (the load) be \( V_2 = 1 \) and \( I_2 = 2 \) (i.e., \( V_2/I_2 = 1/2 \) [Ω]). Find the input voltage and current, \( V_1 \) and \( I_1 \). **Sol:** This case corresponds to

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix} \begin{bmatrix}
1 \\
2
\end{bmatrix} = \begin{bmatrix}
1 + 2 \\
1 - 2
\end{bmatrix}.
\]

Thus \( V_1 = 2 + 1j \) and \( I_1 = -1 \).

- Let the input (source) be \( V_1 = 1 \) and \( I_1 = 2 \). Find the output voltage and current \( V_2 \) and \( I_2 \). **Sol:** With the input specified the two equations are

\[
\begin{bmatrix}
1 \\
2
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix} \begin{bmatrix}
V_2 \\
I_2
\end{bmatrix}.
\]

To find the input we must invert the matrix \( \Delta = -j - 1 \)

\[
\begin{bmatrix}
V_2 \\
I_2
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}^{-1} \begin{bmatrix}
1 \\
2
\end{bmatrix}.
\]

Thus \( V_2 = 3/(1 + j) = 3(1 - j)/2, I_2 = (1 - 2j)/(1 + 1j) = -(1 + 3j)/2 \). The point of this exercise is that the two lines have a complex intersection point, not easily visualized.

**Linear equations with three unknowns**

This problem is similar to the previous problem, except we consider 3 dimensions. Consider two linear equations in unknowns \( x, y, z \), representing planes:

\[
\begin{align*}
a_1x + b_1y + z &= c_1 \\
a_2x + b_2y + z &= c_2
\end{align*}
\]

(2.13) (2.14)

- In terms of the geometry (i.e., think graphically), under what conditions do these two linear equations have (a) a unique solution, (b) a non-unique solution, or (c) no solution? **Sol:** This problem is virtually identical to the previous problem, except the solution is for the intersection of two planes in three dimensions \( z(x, y) \) rather than the intersection of two lines \( y(x) \), in 2 dimensions. One might picture a plane as a line in two dimensions. That is, if you “sweep” a line along a third dimension, it forms a plane.

In this case, 2 equations with 3 unknowns, there is no unique solution in \( x, y, z \). There are three possibilities:

- There is no unique solution - we require a third plane to have a single point of intersection.
There are two cases: (1) When the two planes have different slopes, they meet at along a line. The solution of the problem \((x, y)\) is then a line rather than a point. (non-unique). (2) If the two planes are identical (same slope and same intercept), all points on the plane(s) are solutions. (non-unique)

- If the planes have the same slope, but different intercepts (are parallel to each other) there is no solution. (no solution)

Do you know what is meant by the \textit{slope of a plane}? Can you define it? This is a concept that is natural in vector calculus. We have not gotten there yet, but this idea requires the concept of a gradient, which defines a vector perpendicular to the plane. We shall deal with this concept the third section of this course (i.e, stream 3).

- Given 2 equations in 3 unknowns, the closest we can come to a ‘unique’ solution is an equation in \((x, y)\), \((y, z)\), or \((x, z)\). Find a solution in terms of \(x\) and \(y\) by substituting one equation into the other. \textbf{Sol}: \((a_1 - a_2)x + (b_1 - b_2)y + (c_2 - c_1) = 0\)

- Now consider the intersection of the planes at some arbitrary constant height, \(z = z_0\). Write the modified plane equations as a 2x2 matrix equation in the form \(Ax = b\) where \(x = [x, y]^T\), and find the unique solution in \(x\) and \(y\) using matrix operations. \textbf{Sol}:

\[
\begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} = 
\begin{bmatrix}
c_1 - z_0 \\
c_2 - z_0
\end{bmatrix}
\]

\[
x = \frac{1}{a_1b_2 - a_2b_1} 
\begin{bmatrix}
b_2 & -a_2 \\
-b_1 & a_1
\end{bmatrix} 
\begin{bmatrix}
c_1 - z_0 \\
c_2 - z_0
\end{bmatrix}
\]

- When will this solution fail to exist (for what conditions on \(a_1, a_2, b_1, b_2, \) etc.)? \textbf{Sol}: As stated above, if the planes have the same “slope,” but different intercepts, there is no solution. The problem is that we don’t know what the slope or intercept of the plane means. But we do know how to apply gaussian elimination. We have shown before that if the determinant \(\Delta = a_1b_2 - a_2b_1 \neq 0\), then the ‘slopes’ of the planes are not equal (the planes are not parallel to each other at height \(z_0\)). Thus the geometry has the same meaning as for the case of lines, but in three rather than two dimensions. To prove this we need to apply GE (or the equivalent).

- Now, write the system of equations as a 3x3 matrix equation in \(x, y, z\) given the additional equation \(z = z_0\) \((\text{e.g. put it in the form } Ax = b\) where \(x = [x, y, z]^T\). \textbf{Sol}:

\[
\begin{bmatrix}
a_1 & b_1 & 1 \\
a_2 & b_2 & 1 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = 
\begin{bmatrix}
c_1 \\
c_2 \\
z_0
\end{bmatrix}
\]

- The determinant of a 3x3 matrix is given by

\[
\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix}
= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}
\]

For the 3x3 matrix equation you wrote in the previous part, find the determinant. How is the determinant related to the 2x2 case? Why?

\textbf{Sol}: The determinants are the same. In both cases, we require \(a_1b_2 - a_2b_1 \neq 0\) for the solution to exist. It makes sense that these answers are the same, as we haven’t changed the basic equations, just the format we presented them in.

- Put the following systems of equations in matrix form, and use Octave/Matlab to find (i) the determinant of the matrix, (ii) the matrix inverse, and (iii) the solution \((x, y, z)\). \textbf{If it is not possible to complete (i-iii), state why.}
1. 

\[ x + 3y + 2z = 1 \\
\[ x + 4y + z = 1 \\
\[ x + y = 1 \\

**Sol:**

\[
\begin{bmatrix}
1 & 3 & 2 \\
1 & 4 & 1 \\
1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

(i) The matrix determinant is -4. (ii) The inverse matrix is

\[
\begin{bmatrix}
.25 & -.5 & 1.25 \\
-.25 & .5 & -.25 \\
.75 & -.5 & -.25
\end{bmatrix}
\]

(iii) The solution is \([1, 0, 0]^T\)

2. 

\[ x + 3y + 2z = 1 \\
\[ 2x + 6y + 4z = 1 \\
\[ x + y = 1 \\

**Sol:**

\[
\begin{bmatrix}
1 & 3 & 2 \\
2 & 6 & 4 \\
1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

(i) The determinant is 0. (ii-iii) No solution, because the first two equations represent parallel planes. To show this, expand the determinant along the bottom row: 

\[
1(12 - 12 - 1(4 - 4) + 0(6 - 6)) = 0.
\]

2.1.9 **Integer equations: applications and solutions**

Any equation for which we seek only integer solutions is called a *Diophantine* equation.

**Problem #1: A practical example of using a Diophantine equation**

“A merchant had a 40-pound weight that broke into 4 pieces. When the pieces were weighed, it was found that each piece was a whole number of pounds and that the four pieces could be used to weigh every integral weight between 1 and 40 pounds. What were the weights of the pieces?” - Bachet de Bèziriac (1623 CE)

Here, weighing is performed using a balance scale having two pans, with weights being put on either pan. Thus, given weights of 1 and 3 pounds, one can weigh a 2-pound weight by putting the 1-pound weight in the same pan with the 2-pound weight, and the 3-pound weight in the other pan. Then, the scale will be balanced. A solution to the four weights for Bachet’s problem is \(1 + 3 + 9 + 27 = 40\) pounds.

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*Taken from: Joseph Rotman, “A first course in abstract algebra,” Chapter 1, Number Theory p. 50*
Q 1.1: Show how the combination of 1, 3, 9, & 27 pound weights may be used to weigh 1, 2, 3, ... , 8, 28, and 40 pounds of milk (or something else, such as flour). Assuming that the milk is in the left pan, provide the position of the weights using a negative sign ‘-’ to indicate the left pan and a positive sign ‘+’ to indicate the right pan. For example, if the left pan has 1 pound of milk, then 1 pound of milk in the right pan, ‘+1’ will balance the scales.

Hint: It is helpful to write the answer in matrix form. Set the vector of values to be weighed equal to a matrix indicating the pan assignments, multiplied by a vector of the weights \([1, 3, 9, 27]^T\). The pan assignments matrix should only contain the values -1 (left pan), +1 (right pan), and 0 (leave out). You can indicate these using ‘-‘, ‘+‘, and blank spaces.

Sol: Any integer between 1 and 40 may be expanded using the weights 1, 3, 9, 27. Here is the problem stated in matrix form:

\[
\begin{bmatrix}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
\ldots \\
28 \\
\ldots \\
40
\end{bmatrix}
= \begin{bmatrix}
+ \\
- \\
+ \\
- \\
- \\
- \\
+ \\
- \\
\ldots \\
+ \\
\ldots \\
+ \\
\end{bmatrix}
\begin{bmatrix}
1 \\
3 \\
9 \\
27
\end{bmatrix}
\]

The left column is the weight of the milk. The right-most column are the four weights. It should be clear that these four weights span the integers from 1-40 with binary weights. Each weight may be computed recursively from twice the sum of the previous weights +1, that is

\[W_{n+1} = 2W_n + 1 = 2^{n+1}\quad \text{since}\quad W_n = 2^n.\]

For example to get 26 we place weights 9+3+1 in the pan with 26, and get 27-1. For example 27 = 2*(9+3+1)+1 is the next weight. Recursively, the weights are 3=2*1+1, 9=2*(3+1)+1, 27=2*(9+3+1)+1. The next weight (not shown) would be: 81=2*(27+9+3+1)+1 = 2*40+1.

2.1.10 Ohm’s Law

Overview: In general, impedance is defined as the ratio of a force over a flow. For electrical circuits, the voltage is the ‘force’ and the current is the ‘flow.’ Ohm’s law states that the voltage across and the current through a circuit element are related by the impedance of that element (which may be a function of frequency). For resistors, the voltage over the current is called the resistance, and is a constant (e.g. the simplest case, \(V/I = R\)). For inductors and capacitors, the voltage over the current is a frequency-dependent impedance (e.g. \(V/I = Z(s)\), where \(s\) is the complex frequency \(s \in \mathbb{C}\)).

The impedance concept also holds in mechanics and acoustics. In mechanics, the ‘force’ is equal to the mechanical force on an element (e.g. a mass, dashpot, or spring), and the ‘flow’ is the velocity. In acoustics, the ‘force’ is pressure, and the ‘flow’ is the volume velocity or particle velocity of air molecules.

<table>
<thead>
<tr>
<th>Case</th>
<th>Force</th>
<th>Flow</th>
<th>Impedance</th>
<th>units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electrical</td>
<td>voltage (V)</td>
<td>current (I)</td>
<td>(Z)</td>
<td>Ohms [(\Omega)]</td>
</tr>
<tr>
<td>Mechanics</td>
<td>force (F)</td>
<td>velocity (V)</td>
<td>(Z)</td>
<td>Mechanical Ohms [(\Omega)]</td>
</tr>
<tr>
<td>Acoustics</td>
<td>pressure (P)</td>
<td>particle velocity (U)</td>
<td>(Z)</td>
<td>Acoustic Ohms [(\Omega)]</td>
</tr>
<tr>
<td>Thermal</td>
<td>temperature (T)</td>
<td>heat-flux (J)</td>
<td>(Z)</td>
<td>Thermal Ohms [(\Omega)]</td>
</tr>
</tbody>
</table>
**Problem #1**: The resistance of an incandescent (filament) lightbulb, measured cold, is about 100 ohms. As it lights up, the resistance of the metal filament increases. Ohm’s law says that the current

\[
\frac{V}{I} = R(T).
\]

where \(T\) is the temperature. In the United States, the voltage is 120 volts (RMS) at 60 [Hz].

- **Q 1.1**: Find the current when the light is first switched on.
  **Sol:** Thus the current is
  
  \[
  I = \frac{120}{R} = \frac{120}{100} = 1.2. \quad \text{[Amps]}
  \]

As the bulb heats up, the current rapidly drops, and the resistance increases. This typically takes less than a milliseconds [ms], which depends on the wattage of the light bulb. Such lightbulbs are nonlinear. These rules don’t apply to LED bulbs.

**Problem #2**: The power in Watts [W] is the product of the force and the flow.

- **Q 2.1**: What is the power of the light bulb of this example?
  **Sol:**
  
  \[
  P = V \cdot I = 120 \times 1.2 = 120 + 24 = 144 \text{ [W]}. 
  \]

**Problem #3**: State the impedance \(Z(s)\) of each of the following circuit elements:

- **Q 3.1**: A resistor with resistance \(R\)
  **Sol:** \(Z = R\)

- **Q 3.2**: An inductor with inductance \(L\)
  **Sol:** \(Z = sL\) with \(s = \sigma + \omega j\). Note the flux \(\psi(t) = Li(t)\). The voltage \(v(t)\) is the time derivative of the flux
  
  \[
  v(t) = \frac{d\psi(t)}{dt} = L\frac{di(t)}{dt}.
  \]

- **Q 3.3**: A capacitor with capacitance \(C\)
  **Sol:** \(Z = 1/sC\). Note the charge \(q(t) = Cv(t)\), thus the current \(i(t)\) is the time derivative of the charge
  
  \[
  i(t) = \frac{d}{dt}q(t) = C\frac{dv(t)}{dt}.
  \]

### 2.1.11 2-port network analysis

**Problem #1**: Perform an analysis of electrical 2-port networks, shown in Fig. 2.1. This can be a mechanical system

![Figure 2.1](image)

*Left: A low-pass RC electrical filter. The circuit elements \(R_1, R_2,\) and \(C\) are defined in the problems below. Right: A band-pass acoustic filter. Here, the pressure \(P\) is analogous to voltage, and the velocity \(U\) is analogous to current. The circuit elements are labeled with their \(L\) and \(C\) values as integers, to make the algebra simple.*

The definition of the ABCD transmission matrix \((T)\) is

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix}.
\] (2.15)
The impedance matrix, where the determinant \( \Delta_T = AD - BC \), is given by

\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} = \frac{1}{C} \begin{bmatrix}
A & \Delta_T \\
1 & D
\end{bmatrix} \begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}.
\]  

(2.16)

**Q 1.1:** Derive the formula for the impedance matrix (Eq. 2.16) given the transmission matrix definition (Eq. 2.15).

Show your work. **Sol:** The formula may be easily derived by re-arranging the equations from the matrix (Eq. 2.16). Begin with

\[ V_1 = AV_2 - BI_2 \]
\[ I_1 = CV_2 - DI_2 \]

From the second equation, we get

\[ V_2 = \frac{1}{C}I_1 + \frac{D}{C}I_2 \]

which gives (upon substitution)

\[ V_1 = \frac{A}{C}I_1 + \frac{AD}{C}I_2 - BI_2 = \frac{A}{C}I_1 + \left( \frac{AD}{C} - B \right)I_2 \]

which yields the matrix equation

\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} = \begin{bmatrix}
1/C & (AD/C - B) \\
1/C & D/C
\end{bmatrix} \begin{bmatrix}
I_1 \\
I_2
\end{bmatrix} = \frac{1}{C} \begin{bmatrix}
A & \Delta_T \\
1 & D
\end{bmatrix} \begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}.
\]  

(2.17)

**Problem # 2:** Consider a single circuit element with impedance \( Z(s) \)

**Q 2.1:** What is the ABCD matrix for this element if it is in ‘series’?

**Sol:**

\[
\begin{bmatrix}
1 & Z(s) \\
0 & 1
\end{bmatrix}
\]

**Q 2.2:** What is the ABCD matrix for this element if it is ‘shunt’?

**Sol:**

\[
\begin{bmatrix}
1 & 0 \\
1/Z(s) & 1
\end{bmatrix}
\]

**Problem # 3:** Find the ABCD matrix for each of the circuits of Figure 2.1.

For each circuit, (i) show the cascade of transmission matrices in terms of the complex frequency \( s \in \mathbb{C} \), then (ii) substitute \( s = 1j \) and calculate the total transmission matrix at this single frequency.

**Q 3.1:** Left circuit (let \( R_1 = R_2 = 10 \) k\( \Omega \) ‘kilo-ohms’, and \( C = 10 \) nF ‘nano-farads’)

**Sol:** Write the system in chain matrix form:

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} = \begin{bmatrix} 1 & Z_1 \\
0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\
Y_C & 1 \end{bmatrix} \begin{bmatrix} 1 & Z_3 \\
0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\
-I_2 \end{bmatrix} = \begin{bmatrix} 1 & Z_1 \\
0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\
sC & 1 \end{bmatrix} \begin{bmatrix} 1 & Z_3 \\
0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\
-I_2 \end{bmatrix}
\]

Now we substitute the given values:

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} = \begin{bmatrix} 1 & 10^4 \\
10^{-8} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\
1 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\
-I_2 \end{bmatrix} = \begin{bmatrix} 1 + j10^{-4} & 2 \times 10^4 + j \\
10^{-8} & 1 + j10^{-4} \end{bmatrix} \begin{bmatrix} V_2 \\
-I_2 \end{bmatrix}
\]

**Q 3.2:** Right circuit (use \( L \) and \( C \) values given in the figure), where the pressure \( P \) is analogous to the voltage \( V \), and the velocity \( U \) is analogous to the current \( I \).

**Sol:** Write the system in chain matrix form:

\[
\begin{bmatrix}
P_1 \\
U_1
\end{bmatrix} = \begin{bmatrix} 1 & sL_1 \\
0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\
sC_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{sC_3} \\
0 & 1 \end{bmatrix} \begin{bmatrix} P_2 \\
-U_2 \end{bmatrix}
\]
Now we substitute the given values:

\[
\begin{bmatrix} P_1 \\ U_1 \end{bmatrix} = \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2j & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/3j \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_2 \\ -U_2 \end{bmatrix} = \begin{bmatrix} -2/3 & 4j/3 \\ -19/12j & -5/3 \end{bmatrix} \begin{bmatrix} P_2 \\ -U_2 \end{bmatrix}
\]

I used Matlab/Octave to evaluate this script:

\[
a=[1 \; j; 0 \; 1]; b=[1 \; 0; 2j \; 1]; c=[1 \; 1/3j; 0 \; 1]; d=[1 \; 0; 1/4j \; 1]; T=a*b*c*d.
\]

Finally I found \( T(2,1) \) to be \( 19/12 \) using the Matlab/Octave command: \( \text{rats}(1.5833,6) \)

\(-Q\) 3.3: Convert both transmission (ABCD) matrices to impedance matrices using Equation 2.16. Do this for the specific frequency \( s = 1j \), as in the previous part (feel free to use Matlab/Octave for your computation).

**Sol:** Left circuit: Using the previous solution, and matlab:

\[
\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{j10^{-8}} \begin{bmatrix} 1+j10^{-4} & 1 \\ 1 & 1+j10^{-4} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}
\]

\(-Q\) 3.4: Right circuit: Using the previous solution, and matlab:

**Sol:**

\[
\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{1}{1.5833j} \begin{bmatrix} -2/3 & 1 \\ 1/5j & 1/3j \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}
\]