Chapter 3

Differential equations

3.0.1 Exercises DE-1

Topic of this homework:
Complex numbers and functions (ordering and algebra); Complex power series; Fundamental theorem of calculus (real and complex); Cauchy-Riemann conditions; Multivalued functions (branch cuts and Riemann sheets)

Complex Power Series

Problem # 1: In each case derive (e.g. using Taylor’s formula) the power series of \( w(s) \) about \( s = 0 \) and state the ROC of your series. If the power series doesn’t exist, state why! Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at \( s = 0 \).

–Q 1.1: \( 1/(1 - s) \)

Sol: \( 1/(1 - s) = \sum_{n=0}^{\infty} s^n \) which converges for \( |s| < 1 \) (e.g., the ROC is \( |s| < 1 \)).

–Q 1.2: \( 1/(1 - s^2) \)

Sol: \( 1/(1 - s^2) = \sum_{n=0}^{\infty} s^{2n} \) which converges for \( |s^2| < 1 \). (e.g., the ROC is \( |s| < 1 \)). One can also factor the polynomial, thus write it as: \( \frac{1}{(1-s)(1+s)} \). There are two poles, at \( s = \pm 1 \), and each has an ROC of 1.

–Q 1.3: \( 1/(1 - s^2) \)

Sol: To show this note that \( -\frac{d}{ds} (1 - s)^{-1} = \frac{1}{(1-s)^2} \). Expanding this gives \( \frac{1}{(1-s)^2} = -\frac{d}{ds} \sum_{n=0}^{\infty} s^n = \sum_{n=1}^{\infty} n s^{n-1} = \sum_{n=0}^{\infty} (n+1) s^n \), which converges for \( |s| < 1 \). A second way is to factor \( 1 - s^2 \) and then convolve the coefficients of the \( \infty \) series of \( 1/(1 \pm is) \).

–Q 1.4: \( 1/(1 + s^2) \). Hint: This series will be very ugly to derive if you try to take the derivatives \( \frac{d^n}{ds^n} [1/(1 + s^2)] \). Using the results of our previous homework, you should represent this function as \( w(s) = -0.5i/(s - i) + 0.5i/(s + i) \).

Sol: The resulting series is \( 1/(1 + s^2) = 0.5 \sum_{n=0}^{\infty} s^n ((-i)^n + (i)^n) \). The ROC is \( |s| < 1 \). We can see this by considering the poles of the function at \( s = \pm i \); both poles are 1 from \( s = 0 \), the point of expansion. An alternative is to write the function as \( 1/(1 - (is)^2) = \sum (is)^n \).

–Q 1.5: \( 1/s \)

Sol: If you try to do a Taylor expansion at \( s = 0 \), the first term, \( w(0) \rightarrow \infty \). Thus, the Taylor series expansion in \( s \) does not exist.
-Q 1.6: $1/(1 - |s|^2)$
**Sol:** The imaginary part is zero. Thus the derivative of the imaginary part is zero. Thus the CR conditions cannot be obeyed.

-Q 1.7: Consider the function $w(s) = 1/s$

1. Expand this function as a power series about $s = 1$. **Hint:** Let $1/s = 1/(1 - (1 - s))$. **Sol:** The power series is

$$w(s) = \sum_{n=0}^{\infty} (-1)^n (s - 1)^n$$

which converges for $|s - 1| < 1$. To convince you this is correct, use the Matlab/Octave command `syms s; taylor(1/s,s,'ExpansionPoint',1)`, which is equivalent to the shorthand `syms s; taylor(1/s,s,1)`. What is missing is the logic behind this expansion, given as follows: First move the pole to $z = -1$ via the Möbius “translation” $s = z + 1$, and expand using the Taylor series

$$\frac{1}{s} = \frac{1}{1 + z} = \sum_{n=0}^{\infty} (-z)^n.$$

Next back-substitute $z = s - 1$ giving

$$\frac{1}{s} = \sum (-1)^n (s - 1)^n.$$

It follows that the RoC is $|z| = |s - 1| < 1$, as provided by Matlab/Octave.

2. What is the ROC?

3. Expand $w(s) = 1/s$ as a power series in $s^{-1} = 1/s$ about $s^{-1} = 1$.
   **Sol:** Let $z = s^{-1}$ and expand about 1:

$$\frac{1}{1 - s^{-1}} = \frac{s}{s - 1} = -\frac{s}{1 - s} = s(1 + s^2 + s^3 \cdots) = s + s^2 + s^3 \cdots.$$

which has a zero at 0 and a pole at 1.

4. What is the ROC? **Sol:** $|s| < 1$.

5. What is the residue of the pole? **Sol:** -0.

-Q 1.8: Consider the function $w(s) = 1/(2 - s)$

1. Expand $w(s)$ as a power series in $s^{-1} = 1/s$. State the ROC as a condition on $|s^{-1}|$. **Hint:** Multiply top and bottom by $s^{-1}$.
   **Sol:** $1/(2 - s) = -s^{-1}/(1 - 2s^{-1}) = -s^{-1} \sum 2^n s^{-n}$. The ROC is $|2/s| < 1$, or $|s| > 2$.

2. Find the inverse function $s(w)$. Where are the poles and zeros of $s(w)$, and where is it analytic? **Sol:** Solving for $s(w)$ we find $2 - s = 1/w$ and $s = 2 - 1/w = (2w - 1)/w$. This has a pole at 0 and a zero at $w = 1/2$. The ROC is therefore from the expansion point out to, but not including $w = 0$.

If $a = 0.1$ what is the value of

$$x = 1 + a + a^2 + a^3 \cdots?$$

Show your work. **Sol:** To sum this series, use the fact that

$$x - ax = (1 + a + a^2 + a^3 \cdots) - a(1 + a + a^2) = 1 + a(1 - 1) + a^2(1 - 1) + \cdots$$

This gives $x(1 - a) = 1$, or $x = 1/(1 - a)$. Now since $a = .1$, the sum is $1/(1 - 0.1) = 1.11$.  

**End of natural text.**
If $a = 10$ what is the value of
\[ x = 1 + a + a^2 + a^3 \cdots ? \]

**Sol:** In this case the series clearly does not converge. To make it converge we need to write a formula for $y = 1/x$ rather than for $x$.

\[ y - y/a = (1 + 1/a + 1/a^2 + 1/a^3 \cdots) - 1/a(1 + 1/a + a1/2) = 1 + (1 - 1)/a + (1 - 1)/a^2 + \cdots \]

This gives $y(1 - 1/a) = 1$, or $y = 1/(1 - 1/a)$. Now since $a = 10$, the sum is $y = 1/(1 - 0.1) = 9$. We might conclude that since $x = 1/y$, $x = 1/9$. Does this make sense?

**Two fundamental theorems of calculus**

**Fundamental Theorem of Calculus (Leibniz):** According to the Fundamental Theorem of (Real) Calculus (FTC)

\[ F(x) = F(a) + \int_a^x f(\xi)d\xi, \quad (3.1) \]

where $x, a, \xi, F \in \mathbb{R}$. This is an **indefinite integral** (since the upper limit is unspecified). It follows that

\[ \frac{dF(x)}{dx} = \frac{d}{dx} \int_a^x f(x)dx = f(x). \]

This justifies also calling the indefinite integral the **anti-derivative**.

For a closed interval $[a, b]$, the FTC is

\[ \int_a^b f(x)dx = F(b) - F(a), \quad (3.2) \]

thus the integral is independent of the path from $x = a$ to $x = b$.

**Fundamental Theorem of Complex Calculus:** According to the Fundamental Theorem of Complex Calculus (FTCC)

\[ f(z) = f(z_0) + \int_{z_0}^z F(\xi)d\xi, \quad (3.3) \]

where $z_0, z, \xi, F \in \mathbb{C}$. It follows that

\[ \frac{df(z)}{dz} = \frac{d}{dz} \int_{z_0}^z F(\xi)d\xi = F(z). \]

**Problem #2: To do**

–Q 2.1: *Consider Equation 3.1. What is the condition on $f(x)$ for which this formula is true?*

**Sol:** The sufficient condition is that the integrand $f(x)$ is be **analytic**, namely $f(x) = \sum_{x=0}^{\infty} a_n x^n$. This assures that $f(x)$ is single valued and may be integrated, since it may integrated term by term. It follows that as long as $x < \text{ROC}$, this integral exists. Thus the integral equals $F(x) - F(a)$. Note that if the integrand has a Taylor series, all of its derivatives exist within the ROC, because the coefficients depend on derivatives of $f(x)$.

–Q 2.2: *Consider Equation 3.8. What is the condition on $f(z)$ for which this formula is true?*

**Sol:** The sufficient condition is that the integrand $f(z)$ must be **complex analytic**, namely $f(z) = \sum_{z=0}^{\infty} c_n z^n$, with $c \in \mathbb{C}$.

–Q 2.3: *Perform the following integrals ($z = x + iy \in \mathbb{C}$):*

1. $I = \int_0^{1+j} zdz$ **Sol:** $I = \int_0^{1+j} z^1 dz = \frac{1}{2}(1 + j)^2 = \frac{1}{2}(1 - 1 + 2j) = j$
2. $I = \int_0^{1+j} zdz$, but this time make the path explicit: from 0 to 1, with $y=0$, and then to $y=1$, with $x=1$. **Sol:**

$$I = \int_{x=0}^1 (x + 0j) \, dx + \int_{y=0}^1 (1 + yj) \, dy j$$

$$I = \frac{1}{2} x^2 \bigg|_{x=0}^1 + \int_{y=0}^1 (j - y) \, dy$$

$$= \frac{1}{2} + \left( yj - \frac{1}{2}y^2 \right) \bigg|_{y=0}^1$$

$$= \frac{1}{2} + j - \frac{1}{2}$$

$$= j$$

We conclude that the integration of $z$ is independent of the path. This is true for any integrand $z^n$ with $n \in \mathbb{Z}$.

3. Do your results agree with Equation 3.10? **Sol:** Yes the two integrals must agree, because the function is analytic, and the integral must be the same, independent of the path.

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**Q 2.4:** Perform the following integrals on the closed path $C$, which we define to be the unit circle. You should substitute $z = e^{i\theta}$ and $dz = ie^{i\theta}d\theta$, and integrate from $\{-\pi, \pi\}$ to go once around the unit circle.

1. $\int_C zdz$  
   **Sol:** $\int_C zdz = \int_{-\pi}^{\pi} e^{i\theta}de^{i\theta} = \int_{-\pi}^{\pi} e^{i2\theta}d\theta = e^{i2\theta}|_{-\pi}^{\pi} = 0$.

2. $\int_C \frac{1}{z}dz$ **Sol:** ??

   **Sol:** $\int_{-\pi}^{\pi} id\theta = 2\pi i$.

3. Do your results agree with Equation 3.10? If not, do you know why not? **Sol:** (a) obeys the FTCC because $f(z) = z$ is analytic everywhere, (b) does not obey the FTCC because $f(z)=1/z$ is not analytic at $z=0$ (inside C).

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**Cauchy-Riemann Equations**

For the following problem: $i = \sqrt{-1}$, $s = \sigma + i\omega$, and $F(s) = u(\sigma, \omega) + iv(\sigma, \omega)$.

In class I showed that the integration of a complex analytic function is independent of the path, formally known as the *Fundamental theorem of complex calculus*. The derivative of $F(s)$ is defined as

$$\frac{df}{ds} = \frac{d}{ds} \left[ u(\sigma, \omega) + jv(\sigma, \omega) \right].$$  \hspace{1cm} (3.4)

If the integral is independent of the path, then the derivative must also be independent of direction

$$\frac{df}{ds} = \frac{\partial f}{\partial \sigma} = \frac{\partial f}{\partial j\omega}. \hspace{1cm} (3.5)$$

1. The Cauchy-Riemann (CR) conditions

$$\frac{\partial u(\sigma, \omega)}{\partial \sigma} = \frac{\partial v(\sigma, \omega)}{\partial \omega} \quad \text{and} \quad \frac{\partial u(\sigma, \omega)}{\partial \omega} = -\frac{\partial v(\sigma, \omega)}{\partial \sigma}$$

may be used to show where Equation 3.5 holds.
To do:

(a) Assuming Equation 3.5 is true, use it to derive the CR equations. **Sol:** This was derived in class. First form the partial derivatives as indicated and then set the real and imaginary parts equal. This results in the two CR equations.

(b) Merge the CR equations to show that \( u \) and \( v \) obey Laplace’s equation \( \nabla^2 u(\sigma, \omega) = 0 \) and \( \nabla^2 v(\sigma, \omega) = 0 \). One may conclude that the real and imaginary parts of complex analytic functions must obey these conditions. **Sol:** Take partial derivatives with respect to \( \sigma \) and \( \omega \) and solve for one equation in each of \( u \) and \( v \).

2. Apply the CR equations to the following functions. State for which values of \( s = \sigma + i\omega \) the CR conditions do or do not hold (e.g. where the function \( F(s) \) is or is not analytic). **Hint: Recall your answers to problem 1.2 of this assignment.**

(a) \( F(s) = e^s \) **Sol:** CR conditions hold everywhere.

(b) \( F(s) = 1/s \) **Sol:** CR conditions are violated at \( s = 0 \). The function is analytic everywhere except \( s = 0 \).

Branch cuts and Riemann sheets

1. Consider the function \([w(z)]^2 = z\). This function can also be written as \( w(z) = \sqrt{z} \). Define \( z = r e^{i\phi} \) and \( w(z) = \rho e^{i\theta} = \sqrt{r} e^{i\phi/2} \).

(a) How many Riemann sheets do you need in the domain \((z)\) and the range \((w)\) to fully represent this function as single valued? **Sol:** There are two sheets for \( z \) and one sheet for \( w = \sqrt{z} \). When the point in domain \( z \) (being mapped to \( w(z) \)) crosses the \( z \) branch cut, the range \((w)\) switches from the \( z^+ \) sheet to the \( z^- \) sheet. \( w(z) \) remains analytic on the cut, since it is analytic everywhere. The function \( w(z) = \sqrt{z} \) is analytic everywhere, even at \( z = 0 \). Indicate (e.g. using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range. **Sol:** Above we show the mapping for the square root function \( w(z) = \sqrt{z} = \sqrt{re^{i\phi/2}} \).

(b) Use \texttt{zviz.m} to plot the positive and negative square roots \( +\sqrt{z} \) and \(-\sqrt{z} \). Describe what you see. **Sol:** The sheet for the positive root is shown in Fig. 3.2 (page 106 of the Oct 24 version of the class notes.) Two view the two sheets use Matlab command \texttt{zviz sqrt(W)} \texttt{-sqrt(t(W))}.

(c) Where does \texttt{zviz.m} place the branch cut for this function? **Sol:** Typically the cut is placed along the negative real \( z \) axis \( \phi = \pm \pi \). This is Matlab’s/Octave’s default location. In the figure above, it has been placed along the positive real axis, \( \phi = 0 = 2\pi \).

(d) Must it necessarily be in this location? **Sol:** No, it can be moved, at will. It must start from \( z = 0 \) and end at \(|z| \to \infty \). The cut may be move when using \texttt{zviz.m} by multiplying \( z \) by \( e^{i\phi} \). For example, \texttt{zviz W = sqrt(j*Z)} rotates the cut by \( \pi/2 \). The colors of \( w(z) \) (angle maps to color) always ‘jump’ at the branch cut, as you make the transition across the cut.

2. Consider the function \( w(z) = \log(z) \). As before define \( z = re^{i\phi} \) and \( w(z) = \rho e^{i\theta} \).

(a) Describe with a sketch, and then discuss the branch cut for \( f(z) \). **Sol:** From the plot of \texttt{zviz w(z) = log(z)} of Lecture 18, we see a branch cut going from \( w = 0 \) to \( w = -\infty \). If we express \( z \) in polar coordinates \((z = re^{i\phi})\), then \( w(z) = \log(r) + \phi j = u(x,y) + v(x,y)j \), where \( r(x,y) = |z| = \sqrt{x^2 + y^2} \) and \( \phi = \angle z = \phi(x,y) \). Thus a zero in \( w(z) \) appears at \( z = 1 + 0j \), and only appears on the principle sheet of \( z \) (between \([-\pi < \angle z = \phi < \pi]\)), because this is the only place where \( \phi = 0 \). As the angle \( \phi \) increases, the imaginary part of \( w = \angle z \), which increases without bound. Thus \( w \) is like a spiral stair case, or cork-screw. If \( \rho = 1 \) and \( \phi \neq 0 \), \( w(r) = \log(1) + \phi j \) is not zero, since the angle is not zero.
(b) What is the inverse of this function, \( z(f) \)? Does this function have a branch cut (if so, where is it)? \textbf{Sol:} \( z(w) = e^w \) is a single valued function, so a branch cut is not appropriate. Only multi-valued functions require a branch cut.

3. Using \texttt{zviz.m}, show that
\[
\tan^{-1}(z) = -\frac{j}{2} \log \frac{j - z}{j + z}. \tag{3.6}
\]

In Fig. ?? (p. ??) these two functions are shown to be identical. \textbf{Sol:} Use the Matlab commands \( \text{atan}(Z) \) and \( -(j/2)*\log((j+Z)/(j-Z)) \).

4. Algebraically justify Eq. ??. Hint: Let \( w(z) = \tan^{-1}(z) \), \( z(w) = \tan w = \sin w / \cos w \), then solve for \( e^{w_j} \). \textbf{Sol:} Following the hint gives
\[
z(w) = -j \frac{e^{w_j} - e^{-w_j}}{e^{w_j} + e^{-w_j}} = -j \frac{e^{2w_j} - 1}{e^{2w_j} + 1}.
\]

Solving this bilinear equation for \( e^{2w_j} \) gives
\[
e^{2w_j} = \frac{1 + zj}{1 - zj} = \frac{j - z}{j + z}.
\]

Taking the log and using our definition of \( w(z) \) we find
\[
w(z) = \tan^{-1}(z) = -\frac{j}{2} \log \frac{j - z}{j + z}.
\]

Cauer synthesis given an impedance

This section needs work. Transcribed from page 162 of 0.97.06 of Feb 26, 2019

\textbf{Problem #3:} The continued fraction method can be generalized from a residue expansion of \( Z(s) = \frac{N(s)}{D(s)} \), by a transmission line network synthesis. In this problem we shall explore this method. \textit{This is new material: Appendix ??, page ??}

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\textbf{Q 3.1}

Starting from the impedance
\[
Z(s) = \frac{1}{s + 1}.
\]

Find the impedance network as a ladder network. \textbf{Sol:} Taking the reciprocal we find the sum of two admittances
\[
Y(s) = s + 1.
\]

The the impedance is \( Z(s) = 1/(s + 1) \).

\textbf{Q 3.2:} Use a residue expansion to mimic the CVA floor function for polynomial expansions. Find the expansion of \( H(s) = s^2/(1 + s) \). \textbf{Sol:}
\[
Z(s) = A + Bs + C/(1 + s) = -1 + s + \frac{1}{1 + s}. \tag{3.7}
\]

Thus the Cauer synthesis is a series \(-1 + s\) and a shunt \(1||s\) (i.e., \( Y(s) = 1 + s \) I verified this is correct: series \(-1 + s\) and shunt \(1 + s\)

It seems this proves that Cauer synthesis may be implemented with the residue expansion replacing the floor function in the CFA. Very cool. This solves Burne’s network synthesis problem.