4.4.2 Exercises DE-2

Topic of this homework: Cauchy-Riemann conditions; Integration of complex functions; Cauchy’s theorem, integral formula, residue theorem; power series; Riemann sheets and branch cuts; inverse Laplace transforms

Problem # 1: FTCC and integration in the complex plane
Recall that, according to the Fundamental Theorem of Complex Calculus (FTCC),

\[ f(z) = f(z_0) + \int_{z_0}^{z} F(\zeta) d\zeta, \quad (4.38) \]

where \( z_0, z, \zeta, F \in \mathbb{C} \). It follows that

\[ f(z) = \frac{d}{dz}F(z). \quad (4.39) \]

Thus Eq. 4.38 is also known as the anti-derivative of \( f(z) \).

–Q 1.1: For a closed interval \( \{a, b\} \), the FTCC can be stated as

\[ \int_{a}^{b} F(z) dz = f(b) - f(a), \quad (4.40) \]

meaning that the result of the integral is independent of the path from \( x = a \) to \( x = b \). What condition(s) on the integrand \( f(z) \) is (are) sufficient to assure that Eq. 4.40 holds?

–Q 1.2: For the function \( f(z) = c^z \), where \( c \in \mathbb{C} \) is an arbitrary complex constant, use the Cauchy-Riemann (CR) equations to show that \( f(z) \) is analytic for all \( z \in \mathbb{C} \).

–Q 1.3: In the following problems, solve the integral

\[ I = \int_{\mathcal{C}} F(z) dz \]

for a given path \( \mathcal{C} \). In some cases this might be the definite integral (Eq. 4.40).

Let the function \( F(z) = c^z \), where \( c \in \mathbb{C} \) is given for each problem below. Hint: Can you apply the FTCC?

1. Find the anti-derivative of \( F(z) \).
2. \( c = 1/e = 1/2.7183 \ldots \) where \( \mathcal{C} \) is \( \zeta = 0 \rightarrow i \rightarrow z \)
3. \( c = 2 \) where \( \mathcal{C} \) is \( \zeta = 0 \rightarrow (1 + i) \rightarrow z \)
4. \( c = i \) where the path \( \mathcal{C} \) is an inward spiral described by \( z(t) = 0.99t e^{i2\pi t} \) for \( t = 0 \rightarrow t_0 \rightarrow \infty \)
5. \( c = e^{t-\tau_0} \) where \( \tau_0 > 0 \) is a real number, and \( \mathcal{C} \) is \( z = (1 - i\infty) \rightarrow (1 + i\infty) \). Hint: Do you recognize this integral? If you do not recognize the integral, please do not spend a lot of time trying to solve it via the ‘brute force’ method.
Problem #2: Cauchy’s theorems for integration in the complex plane

There are three basic definitions related to Cauchy’s integral formula. They are all related, and can greatly simplify integration in the complex plane. When a function depends on a complex variable we shall use uppercase notation, consistent with the engineering literature for the Laplace transform.

1. **Cauchy’s (Integral) Theorem** (Stillwell, p. 319; Boas, p. 45)
   \[ \oint_{\mathcal{C}} F(z)\,dz = 0, \]
   if and only if \( F(z) \) is complex-analytic inside of \( \mathcal{C} \).
   
   This is related to the Fundamental Theorem of Complex Calculus (FTCC)
   \[ f(z) = f(a) + \int_{a}^{z} F(z)\,dz, \]
   where \( f(z) \) is the anti-derivative of \( F(z) \), namely \( F(z) = df/dz \). The FTCC requires \( F(z) \) to be complex-analytic for all \( z \in \mathbb{C} \). By closing the path (contour \( \mathcal{C} \)), Cauchy’s theorem (and the following theorems) allows us to integrate functions that may not be complex-analytic for all \( z \in \mathbb{C} \).

2. **Cauchy’s Integral Formula** (Boas, p. 51; Stillwell, p. 220)
   \[ \frac{1}{2\pi j} \oint_{\mathcal{C}} \frac{F(z)}{z - z_0}\,dz = \begin{cases} F(z_0), & z_0 \in \mathcal{C} \text{ (inside)} \\ 0, & z_0 \notin \mathcal{C} \text{ (outside)} \end{cases} \]
   Here \( F(z) \) is required to be analytic everywhere within (and on) the contour \( \mathcal{C} \). \( F(z_0) \) is called the residue of the pole.

3. **(Cauchy’s) Residue Theorem** (Boas, p. 72)
   \[ \oint_{\mathcal{C}} F(z)\,dz = 2\pi j \sum_{k=1}^{K} \text{Res}_k, \]
   where \( \text{Res}_k \) are the residues of all poles of \( F(z) \) enclosed by the contour \( \mathcal{C} \).
   **How to calculate the residues**: The residues can be rigorously defined as
   \[ \text{Res}_k = \lim_{z \to z_k} (z - z_k)f(z). \]
   This can be related to *Cauchy’s integral formula*: Consider the function \( F(z) = w(z)/(z - z_k) \), where we have factored \( F(z) \) to isolate the first-order pole at \( z = z_k \). If the remaining factor \( w(z) \) is analytic at \( z_k \), then the residue of the pole at \( z = z_k \) is \( w(z_k) \).

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**Q 2.1:** In one or two brief sentences, describe the relationships between the three theorems:
1. (1) and (2)
2. (1) and (3)
3. (2) and (3)

**Q 2.2:** Consider the function with poles at \( z = \pm j \)

\[ F(z) = \frac{1}{1 + z^2} = \frac{1}{(z - j)(z + j)} \]

Apply Cauchy’s theorems to solve the following integrals. **State which theorem(s) you used**, and **show your work**.
1. $\oint_{C} F(z)dz$ where $C$ is a circle centered at $z = 0$ with a radius of $\frac{1}{2}$.

2. $\oint_{C} F(z)dz$ where $C$ is a circle centered at $z = j$ with a radius of $1$.

3. $\oint_{C} F(z)dz$ where $C$ is a circle centered at $z = 0$ with a radius of $2$.

**Problem #3: Integration in the complex plane**

In the following questions, you’ll be asked to integrate $F(s) = u(\sigma, \omega) + iv(\sigma, \omega)$ around the contour $C$ for complex $s = \sigma + i\omega$,

$$\oint_{C} F(s)ds.$$ Follow the directions carefully for each question. When asked to state where the function is and is not analytic, you are not required to use the Cauchy-Riemann equations (but you should if you can’t answer the question ‘by inspection’).

- **Q 3.1:** $F(s) = \sin(s)$

- **Q 3.2:** Given function $F(s) = \frac{1}{s}$

1. State where the function is and is not analytic.

2. Explicitly evaluate the integral when $C$ is the unit circle, defined as $s = e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

3. Evaluate the same integral using Cauchy’s theorem and/or the residue theorem.

- **Q 3.3:** $F(s) = \frac{1}{s^2}$

1. State where the function is and is not analytic.

2. Explicitly evaluate the integral when $C$ is the unit circle, defined as $s = e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

3. What does your result imply about the residue of the 2nd order pole at $s = 0$?

- **Q 3.4:** $F(s) = e^{st}$

1. State where the function is and is not analytic.

2. Explicitly evaluate the integral when $C$ is the square $(\sigma, \omega) = (1, 1) \rightarrow (-1, 1) \rightarrow (-1, -1) \rightarrow (1, -1) \rightarrow (1, 1)$.

3. Evaluate the same integral using Cauchy’s theorem and/or the residue theorem.

- **Q 3.5:** $F(s) = \frac{1}{s+2}$

1. State where the function is and is not analytic.

2. Let $C$ be the unit circle, defined as $s = e^{i\theta}$, $0 \leq \theta \leq 2\pi$. Evaluate the integral using Cauchy’s theorem and/or the residue theorem.

3. Let $C$ be a circle of radius 3, defined as $s = 3e^{i\theta}$, $0 \leq \theta \leq 2\pi$. Evaluate the integral using Cauchy’s theorem and/or the residue theorem.

- **Q 3.6:** $F(s) = \frac{1}{2\pi i} \frac{e^{st}}{s+2}$
1. State where the function is and is not analytic.

2. Let \( C \) be a circle of radius 3, defined as \( s = 3e^{i\theta}, \ 0 \leq \theta \leq 2\pi \). Evaluate the integral using Cauchy’s theorem and/or the residue theorem.

3. Let \( C \) contain the entire left-half \( s \)-plane. Evaluate the integral using Cauchy’s theorem and/or the residue theorem. Do you recognize this integral?

\[ -Q \ \text{3.7:} \ F(s) = \pm \frac{1}{\sqrt{s}} \ (\text{e.g.} \ F^2 = \frac{1}{s}) \]

1. State where the function is and is not analytic.

2. This function is multivalued. How many Riemann sheets do you need in the domain (\( s \)) and the range (\( f \)) to fully represent this function? Indicate (e.g. using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.

3. Explicitly evaluate the integral

\[ \int_C \frac{1}{\sqrt{z}} \ dz \]

when \( C \) is the unit circle, defined as \( s = e^{i\theta}, \ 0 \leq \theta \leq 2\pi \). Is this contour ‘closed’? State why or why not.

4. Explicitly evaluate the integral

\[ \int_C \frac{1}{\sqrt{z}} \ dz \]

when \( C \) is twice around the unit circle, defined as \( s = e^{i\theta}, \ 0 \leq \theta \leq 4\pi \). Is this contour ‘closed’? State why or why not. \textit{Hint: Note that} \( \sqrt{e^{i(\theta+2\pi)}} = \sqrt{e^{i2\pi}e^{i\theta}} = e^{i\pi} \sqrt{e^{i\theta}} = -1\sqrt{e^{i\theta}} \)

5. What does your result imply about the residue of the (twice-around \( \frac{1}{2} \) order) pole at \( s = 0 \)?

6. Show that the residue is zero. \textit{Hint: apply the definition of the residue.}

\textbf{Problem # 4: A two-port network application for the Laplace transform}

Recall that the \textit{Laplace transform (LT)} \( f(t) \leftrightarrow F(s) \)\(^{18} \) of a causal function \( f(t) \) is

\[ F(s) = \int_0^\infty f(t)e^{-st} \ dt, \]

where \( s = \sigma + j\omega \) is complex frequency\(^{19} \) in [radians] and \( t \) is time in [seconds]. Causal functions and the Laplace transform are particularly useful for describing \textit{systems}, which have no response until a signal enters the system (e.g. at \( t = 0 \)).

The definition of the \textit{inverse Laplace transform (LT\(^{-1}) \)} requires integration in the complex plane:

\[ f(t) = \frac{1}{2\pi j} \int_{C_\sigma} F(s)e^{st} \ ds = \frac{1}{2\pi j} \oint_C F(s)e^{st} ds. \]

The Laplace contour \( C \) actually includes two pieces

\[ \oint_C = \int_{C_\sigma}^{\sigma_0+j\infty} + \int_{-C_\infty}^{\sigma_0-j\infty}, \]

\(^{18}\)Many loosely adhere to the convention that the frequency domain uses upper-case [e.g. \( F(s) \)] while the time domain uses lower case [\( f(t) \)]

\(^{19}\)While radians are useful units for calculations, when providing physical insight in discussions of problem solutions, it is easier to work with Hertz, since frequency in [Hz] and time in [s] are mentally more more natural units than radians. The same is true of degrees vs. radians. Boas (p. 10) recommends the use degrees over radians. He gives the example of \( 3\pi/5 \) [radians], which is more easily visualize as \( 108^\circ \).
where the path represented by \( C_{\infty} \) is a semicircle of infinite radius with \( \sigma \to -\infty \). It is somewhat tricky to do, but it may be proved that the integral over the contour \( C_{\infty} \) goes to zero. For a causal, ‘stable’ (e.g. doesn’t blow up over time) signal, all of the poles of \( F(s) \) must be inside of the Laplace contour, in the left-half \( s \)-plane.

**Transfer functions**  Linear, time-invariant systems are described by an ordinary differential equations. For example, consider the first-order linear differential equation

\[
a_1 \frac{d}{dt} y(t) = b_1 \frac{d}{dt} x(t) + b_0 x(t).
\]

This equation describes the relationship between the input \((x(t))\) and output \((y(t))\) of the system. If we define Laplace transforms \( y(t) \leftrightarrow Y(s) \) and \( x(t) \leftrightarrow X(s) \), then this equation may be written in the frequency domain as

\[
a_1 s Y(s) = b_1 s X(s) + b_0 X(s).
\]

The *transfer function* for this system is defined as

\[
H(s) = \frac{Y(s)}{X(s)} = \frac{b_1 s + b_0}{a_1 s} = \frac{b_1}{a_1} + \frac{b_0}{a_1 s}.
\]

5 In this problem, we will look at the transfer function of a simple two-port network, shown in Figure 4.7. This network is an example of a RC low-pass filter, which acts as a leaky integrator.

![Figure 4.7: This three-element electrical circuit is a system that acts to low-pass filter the signal voltage \( V_1(\omega) \), to produce signal \( V_2(\omega) \).](image)

**Problem # 5: ABCD method**

–Q 5.1: Low-pass RC filter

1. Use the ABCD method to find the matrix representation of Fig. 4.7.

2. Assuming that \( I_2 = 0 \), find the transfer function \( H(s) = V_2/V_1 \). From the results of the ABCD matrix you determined above, show that

\[
H(s) = \frac{1}{1 + R_1 C s}.
\]

3. The transfer function \( H(s) \) has one pole. Where is the pole? Find the *residue* of this pole.

4. Find \( h(t) \), the inverse Laplace transform of \( H(s) \).

5. Assuming that \( V_2 = 0 \) find \( Y_{12}(s) \equiv I_2/V_1 \).

6. Find the input impedance to the *right-hand side* of the system, \( Z_{22}(s) \equiv V_2/I_2 \) for two cases:
(a) \( I_1 = 0 \)
(b) \( V_1 = 0 \)

7. Compute the determinant of the ABCD matrix. *Hint: It is always 1.*

8. Compute the derivative of \( H(s) = \left. \frac{V_2}{V_1} \right|_{I_2=0} \).

**Problem #6: With the help of a computer**

In the following problems, we will look at some of the concepts from this homework using Matlab/Octave. We are using the `syms` function which requires Matlab’s/Octave’s symbolic math toolbox. Or you may use the EWS lab’s Matlab. Alternative symbolic-math tool, such as Wolfram Alpha.

**Example:** To find the Taylor series expansion about \( s = 0 \) of

\[
F(s) = -\log(1 - s),
\]

first consider the derivative and its Taylor series (about \( s = 0 \))

\[
F'(s) = \frac{1}{1 - s} = \sum_{n=0}^{\infty} s^n.
\]

Then, integrate this series term by term

\[
F(s) = -\log(1 - s) = \int s F'(s) ds = \sum_{n=0}^{\infty} \frac{s^n}{n}.
\]

Alternatively you may use Matlab/Octave commands:

```plaintext
syms s
TaylorSeries = taylor(-log(1-s), 'order', 7)
```

**Q 6.1:** Use the `taylor(-log(1-s))` to 7th order, as in the example above.

1. Try the above Matlab/Octave commands. Give the first 7 terms of the Taylor series (confirm that Matlab/Octave agrees with the formula derived above).

2. What is the inverse Laplace transform of this series? Consider the series term by term.

**Q 6.2:** The function \( 1/\sqrt{z} \) has a branch point at \( z = 0 \), thus it is singular there.

1. Can you apply Cauchy’s integral theorem when integrating around the unit circle?

2. Below is a Matlab/Octave code that computes \( \int_0^{4\pi} \frac{dz}{\sqrt{z}} \) using Matlab’s/Octave’s symbolic analysis package:

```plaintext
syms z
I = int(1/sqrt(z))
J = int(1/sqrt(z), exp(-j*pi), exp(j*pi))
eval(J)
```

To do: Run this script. What answers do you get for $I$ and $J$?

3. Modify this code to integrate $f(z) = 1/z^2$ once around the unit circle. What answers do you get for $I$ and $J$?

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**Q 6.3: Bessel functions can describe waves in a cylindrical geometry**

The Bessel function has a Laplace transform with a branch cut

$$J_0(t)u(t) \leftrightarrow \frac{1}{\sqrt{1+s^2}}.$$

Draw a hand sketch showing the nature of the branch cut. Hint: Use zviz.

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**Problem # 7: Matlab/Octave exercises:**

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**Q 7.1: Comment on the following Matlab/Octave exercises**

1. Try the following Matlab/Octave commands, and then comment on your findings.

```matlab
%Take the inverse LT of 1/sqrt(1+s^2)
syms s
I=ilaplace(1/(sqrt((1+s^2))));
disp(I)

%Find the Taylor series of the LT
T=taylor(1/sqrt(1+s^2),10);
disp(T);

%Verify this
syms t
J=laplace(besselj(0,t));
disp(J);

%plot the Bessel function
b=besselj(0,t);
plot(t/pi,b);
grid on;
```

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**Q 7.2: When did Friedrich Bessel live?**

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**Q 7.3: What did he use Bessel functions for?**

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**Q 7.4: Using zviz, for each of the following functions**

1. Describe the plot generated by zviz S=Z.

2. Are the functions defined below legal Brune impedances? (i.e., Do they function obey $\Re Z(\sigma > 0) \geq 0$)? Hint: Consider the phase (color). Plot zviz z for a reminder of the colormap.

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**Q 7.5: Comment on the colorized plots of these functions**

1. $\text{zviz } 1./\text{sqrt}(1+S.^2)$
2. \( \frac{1}{\sqrt{1-S^2}} \)
3. \( \frac{1}{1+\sqrt{S}} \)

**Problem #8**
Inverse Laplace transform of the zeta function \( \zeta(s) \) [\( \zeta_p(s) \leftrightarrow z_p(t) \) (Eq. R.14, p. 345) and describe the result in words.]

*Hint: Consider the geometric series representation*

\[
\zeta_p(s) = \frac{1}{1 - e^{-sT_p}} = \sum_{k=0}^{\infty} e^{-skT_p},
\]

(4.42)

for which you can easily look up (or may have memorized) the inverse Laplace transform of each term.

**Problem #9: Inverse transform of Product of factors:**
The time domain version of Eq. 3.8 (p. 102) may be written as the convolution of all the \( z_k(t) \) factors

\[
z(t) \equiv z_2 \star z_3(t) \star z_5(t) \star z_7(t) \cdots \star z_p(t) \cdots,
\]

(4.43)

where \( \star \) represents time convolution.

Figure 4.8: This feedback network is described by a time-domain difference equation with delay \( T_p \) seconds described by the two equations in the figure with a unity feedback gain \( \alpha = -1 \). A transfer function \( Y(s) = V(s)/I(s) \) that has the same poles as \( \zeta_p(s) \), but with zeros as given by Eq. 4.45, is the input admittance \( Y(s) = I(s)/V(s) \) of the transmission line, defined at the ratio of the Laplace transform of the current \( i(t) \leftrightarrow I(s) \) over the voltage \( v(t) \leftrightarrow V(s) \).

![Fig: impedance](DE2)

Explain what this means in physical terms. Start with two terms (e.g., \( z_1(t) \star z_2 \)).

**Problem #10: Physical interpretation:**
Such functions may be generated in the time domain as shown in Fig. 4.8 (p. 175), using a feedback delay of \( T_p \) seconds described by the two equations in the figure with a unity feedback gain \( \alpha = -1 \). Taking the Laplace transform of the system equation we see that the transfer function between the state variable \( q(t) \) and the input \( x(t) \) is given by \( \zeta_p(s) \), which is and all-pole function, since

\[
Q(s) = e^{-sT_p}Q(s) + V(s), \quad \text{or} \quad \zeta_p(s) \equiv \frac{Q(s)}{V(s)} = \frac{1}{1 - e^{-sT_p}}.
\]

(4.44)

Closing the feed-forward path gives a second transfer function \( Y(s) = I(s)/V(s) \), namely

\[
Y(s) = \frac{I(s)}{V(s)} = \frac{1 - e^{-sT_p}}{1 + e^{-sT_p}}.
\]

(4.45)
If we take $i(t)$ as the current and $v(t)$ as the voltage at the input to the transmission line, then $y_p(t) \leftrightarrow \zeta_p(s)$ represents the input impedance at the input to the line. The poles and zeros of the impedance interleave along the $j\omega$ axis. By a slight modification $\zeta_p(s)$ may alternatively be written as

$$Y_p(s) = \frac{e^{sT_p/2} + e^{-sT_p/2}}{e^{sT_p/2} - e^{-sT_p/2}} = j \tan(sT_p/2). \quad (4.46)$$

Every impedance $Z(s)$ has a corresponding reflectance function given by a Möbius transformation, which may be read off of Eq. 4.45 as

$$\Gamma(s) = \frac{1 + Z(s)}{1 - Z(s)} = e^{-sT_p} \quad (4.47)$$

since impedance is also related to the round-trip delay $T_p$ on the line. The inverse Laplace transform of $\Gamma(s)$ is the round trip delay $T_p$ on the line

$$\gamma(t) = \delta(t - T_p) \leftrightarrow e^{-sT_p}. \quad (4.48)$$

In terms of the physics, these transmission line equations are telling us that $\zeta(s)$ may be decomposed into an infinite cascade of transmission lines (Eq. R.16), each having a delay given by $T_p = \ln \pi_p$. The input admittance of this cascade may be interpreted as an analytic continuation of $\zeta(s)$ which defines the eigen-modes of that cascaded impedance function.

Working in the time domain provides a key insight, as it allows us to parse out the best analytic continuation of the infinity of possible continuations, that are not obvious in the frequency domain. Transforming to the time domain is a form of analytic continuation of $\zeta(s)$, that depends on the assumption that $z(t)$ is one-sided in time (causal).