3.0.2 Exercises DE-2

Topic of this homework: Cauchy-Riemann conditions; Integration of complex functions; Cauchy’s theorem, integral formula, residue theorem; power series; Riemann sheets and branch cuts; inverse Laplace transforms

Problem #1: FTCC and integration in the complex plane

Recall that, according to the Fundamental Theorem of Complex Calculus (FTCC),

\[ f(z) = f(z_0) + \int_{z_0}^{z} F(\zeta) d\zeta, \]  

(3.8)

where \( z_0, z, \zeta, F \in \mathbb{C} \). It follows that

\[ f(z) = \frac{d}{dz} F(z). \]  

(3.9)

Thus Eq. 3.8 is also known as the anti-derivative of \( f(z) \).

–Q 1.1: For a closed interval \( \{a, b\} \), the FTCC can be stated as

\[ \int_{a}^{b} F(z) dz = f(b) - f(a), \]  

(3.10)

meaning that the result of the integral is independent of the path from \( x = a \) to \( x = b \). What condition(s) on the integrand \( f(z) \) is (are) sufficient to assure that Eq. 3.10 holds? Sol: If the integrand is complex analytic for all \( z \in \mathbb{C} \), meaning it can be represented as a convergent power series,

\[ f(z) = \sum_{k=0}^{\infty} c_k z^k, \]

then the theorem holds. For the integrand to have a convergent power series, it must satisfy the Cauchy-Riemann (CR) equations, which assure that the derivatives are independent of the path (i.e., direction) of integration.

–Q 1.2: For the function \( f(z) = c^z \), where \( c \in \mathbb{C} \) is an arbitrary complex constant, use the Cauchy-Riemann (CR) equations to show that \( f(z) \) is analytic for all \( z \in \mathbb{C} \).

Sol: We may rewrite this function as \( f(z) = e^{\ln(c)z} \), where \( z = x + iy \) and \( f = u + iv \). Thus

\[ u(x, y) = e^{\ln(c)x} \cos(\ln(c)y), \]
\[ v(x, y) = e^{\ln(c)x} \sin(\ln(c)y) \]

\[ \frac{\partial u}{\partial x} = \ln(c) e^{\ln(c)x} \cos(\ln(c)y) = \frac{\partial v}{\partial y} = \ln(c) e^{\ln(c)x} \cos(\ln(c)y) \]

\[ \frac{\partial u}{\partial y} = -\ln(c) e^{\ln(c)x} \sin(\ln(c)y) = -\frac{\partial v}{\partial x} = -\ln(c) e^{\ln(c)x} \sin(\ln(c)y) \]

Thus the CR conditions are satisfied everywhere and the function is analytic for all \( z \in \mathbb{C} \).

–Q 1.3: In the following problems, solve the integral

\[ I = \int_{C} F(z) dz \]

for a given path \( C \). In some cases this might be the definite integral (Eq. 3.10).

Let the function \( F(z) = c^z \), where \( c \in \mathbb{C} \) is given for each problem below. Hint: Can you apply the FTCC?
1. Find the anti-derivative of \( F(z) \). **Sol:** Since \( e^z \), the indefinite integral (anti-derivative) is 
\[
I(z) = \frac{1}{\ln c} e^{\ln(c)z}
\]
since \( \frac{d}{dz} I(z) = \frac{d}{dz} \frac{1}{\ln c} e^{\ln(c)z} = e^{\ln(c)z} = F(z) \).

2. \( c = 1/e = 1/2.7183\ldots \) where \( C \) is \( \zeta = 0 \rightarrow i \rightarrow z \) **Sol:** The integrand is \( F(z) = e^{-z} \), which is entire. Thus the integral is independent of the path (i.e., \( C \) is not relevant to the final answer).
\[
I(z) = \int_0^z e^{-\zeta} d\zeta + \int_i^z e^{-\zeta} d\zeta = F(z) - F(i) + F(i) - F(0)
\]
\[
= \int_0^z e^{-\zeta} d\zeta = -e^{-z} \bigg|_0^z = -e^{-z} - 1
\]

3. \( c = 2 \) where \( C \) is \( \zeta = 0 \rightarrow (1 + i) \rightarrow z \) **Sol:** The integrand is \( F(z) = 2z \), where \( 2 = e^{\ln 2} \). The path \( C \) is not relevant to the final answer.
\[
I(z) = \int_0^z 2\zeta d\zeta = \int_0^z e^{\zeta \ln 2} d\zeta = \left. \frac{e^{\zeta \ln 2}}{\ln 2} \right|_0^z = (e^{z \ln 2} - 1)/\ln 2 \approx 1.443(e^{0.693z} - 1)
\]

4. \( c = i \) where the path \( C \) is an inward spiral described by \( z(t) = 0.99t e^{2\pi i t} \) for \( t = 0 \rightarrow t_0 \rightarrow \infty \) **Sol:** \( i = e^{i\pi/2} e^{i\pi n} \). We have already proved that the path doesn’t matter for any \( F(z) = c^z \), so we just need to evaluated \( z(t) \) for \( t = 0 \) and \( t \rightarrow \infty \). This gives \( z(0) = 1 \) and \( z(t \rightarrow \infty) = 0 \).
\[
I = \int_{z(0)}^{z(t \rightarrow \infty)} i^z dz = \int_{z(0)}^{z(t \rightarrow \infty)} e^{i\pi z/2} dz = \left. \frac{2e^{i\pi z/2}}{i\pi} \right|_0^1 = \frac{2}{\pi} (-e^{i\pi/2} - 1) = -2(i + 1) / \pi
\]

5. \( c = e^{t - \tau_0} \) where \( \tau_0 > 0 \) is a real number, and \( C \) is \( z = (1 - i\infty) \rightarrow (1 + i\infty) \). *Hint: Do you recognize this integral? If you do not recognize the integral, please do not spend a lot of time trying to solve it via the 'brute force' method.*
**Sol:** This is the basically the inverse Laplace transform of \( e^{-\tau_0 z} \), we are just missing the scale factor \( \frac{1}{2\pi i} \).
\[
I(t) = \int_{1-i\infty}^{1+i\infty} e^{(t-\tau_0)z} dz - \int_{1+i\infty}^{1-i\infty} e^{-\tau_0 z} e^{zt} dz = 2\pi i \delta(t - \tau_0)
\]

**Problem #2:** Cauchy’s theorems for integration in the complex plane

There are three basic definitions related to Cauchy’s integral formula. They are all related, and can greatly simplify integration in the complex plane. When a function depends on a complex variable we shall use uppercase notation, consistent with the engineering literature for the Laplace transform.

1. **Cauchy’s (Integral) Theorem** (Stillwell, p. 319; Boas, p. 45)
\[
\oint_C F(z) dz = 0,
\]
if and only if \( F(z) \) is complex-analytic inside of \( C \).

This is related to the Fundamental Theorem of Complex Calculus (FTCC)
\[
f(z) = f(a) + \int_a^z F(z) dz,
\]
where \( f(z) \) is the anti-derivative of \( F(z) \), namely \( F(z) = df/dz \). The FTCC requires \( F(z) \) to be complex-analytic for all \( z \in \mathbb{C} \). By closing the path (contour \( C \), Cauchy’s theorem (and the following theorems) allows us to integrate functions that may not be complex-analytic for all \( z \in \mathbb{C} \).
2. **Cauchy’s Integral Formula** (Boas, p. 51; Stillwell, p. 220)

\[
\frac{1}{2\pi j} \oint_C \frac{F(z)}{z - z_0} \, dz = \begin{cases} 
F(z_0), & z_0 \in C \text{ (inside)} \\
0, & z_0 \notin C \text{ (outside)} 
\end{cases}
\]

Here \( F(z) \) is required to be analytic everywhere within (and on) the contour \( C \). \( F(z_0) \) is called the *residue* of the pole.

3. **(Cauchy’s) Residue Theorem** (Boas, p. 72)

\[
\oint_C F(z) \, dz = 2\pi j \sum_{k=1}^{K} \text{Res}_k,
\]

where \( \text{Res}_k \) are the *residues* of all poles of \( F(z) \) enclosed by the contour \( C \).

**How to calculate the residues:** The residues can be rigorously defined as

\[
\text{Res}_k = \lim_{z \to z_k} [(z - z_k)f(z)].
\]

This can be related to *Cauchy’s integral formula*: Consider the function \( F(z) = w(z)/(z - z_k) \), where we have factored \( F(z) \) to isolate the first-order pole at \( z = z_k \). If the remaining factor \( w(z) \) is analytic at \( z_k \), then the residue of the pole at \( z = z_k \) is \( w(z_k) \).

---

**–Q 2.1:** Describe the relationships between the three theorems:

1. (1) and (2) **Sol:** When \( z_0 \) falls outside of \( C \), (2) reduces to (1)

2. (1) and (3) **Sol:** When there are no poles inside \( C \), all the residues are zero, and (3) reduces to (1).

3. (2) and (3) **Sol:** Case (2) has only one induced pole at \( z = z_0 \), having residue \( F(z_0) \). Thus (3) is the same as (2) when \( K = 1 \), the pole at \( z_0 \) is within contour \( C \), and the single residue is \( F(z_0) \).

---

**–Q 2.2:** Consider the function with poles at \( z = \pm j \)

\[
F(z) = \frac{1}{1 + z^2} = \frac{1}{(z - j)(z + j)}
\]

Find the residue expansion.

**Sol:**

\[
F(z) = \frac{j}{2} \left( \frac{1}{z + j} - \frac{1}{z - j} \right).
\]

---

**–Q 2.3:** Apply Cauchy’s theorems to solve the following integrals.

**State which theorem(s) you used, and show your work.**

1. \( \oint_C F(z) \, dz \) where \( C \) is a circle centered at \( z = 0 \) with a radius of \( \frac{1}{2} \).

**Sol:** Because the contour \( C \) does not include the poles, \( F(z) \) is analytic everywhere inside \( C \). Using *Cauchy’s integral theorem*, the integral is 0.

2. \( \oint_C F(z) \, dz \) where \( C \) is a circle centered at \( z = j \) with a radius of 1.

**Sol:** Since we only enclose the pole at \( z = j \), use the *integral formula* with \( F(z) = 1/(z + j) \):

\[
\oint_C F(z) \, dz = 2\pi j \text{Res}_j = 2\pi j \left[ \frac{1}{z + j} \right]_{z=j} = 2\pi j \cdot \frac{1}{2j} = \pi
\]

---
3. $\oint_C F(z)dz$ where $C$ is a circle centered at $z = 0$ with a radius of 2.

   **Sol:** Since we enclose both poles, using the residue theorem:
   \[
   \oint_C F(z)dz = 2\pi j(\text{Res}_j + \text{Res}_{-j}) = 2\pi j\left(\frac{1}{2j} - \frac{1}{2j}\right) = 0
   \]

   As a side note, the inverse Laplace transform for $F(z)$ is $\sin(t)$, which is zero for $t = 0$, consistent with this result.

**Problem # 3: Integration in the complex plane**

In the following questions, you’ll be asked to integrate $F(s) = u(\sigma, \omega) + iv(\sigma, \omega)$ around the contour $C$ for complex $s = \sigma + i\omega$,

\[
\oint_C F(s)ds.
\]

Follow the directions carefully for each question. When asked to state where the function is and is not analytic, you are not required to use the Cauchy-Riemann equations (but you should if you can’t answer the question ‘by inspection’).

- **Q 3.1:** $F(s) = \sin(s)$

   **Sol:** Analytic everywhere. $- \cos(s) = \int_0^{2\pi} \sin(s)ds = 0$. This function is *entire* (i.e., has no poles) so the integral must be zero.

- **Q 3.2:** Given function $F(s) = \frac{1}{s}$

   1. State where the function is and is not analytic. **Sol:** Analytic everywhere except at $s = 0$, where it has a pole.

   2. Explicitly evaluate the integral when $C$ is the unit circle, defined as $s = e^{i\theta}$, $0 \leq \theta \leq 2\pi$. **Sol:**
      \[
      \oint_C F(s)ds = \int_0^{2\pi} \frac{1}{e^{i\theta}}i e^{i\theta}d\theta = \int_0^{2\pi} id\theta = 2\pi i
      \]

   3. Evaluate the same integral using Cauchy’s theorem and/or the residue theorem. **Sol:** The residue is 1 so the integral is $2\pi i$.

- **Q 3.3:** $F(s) = \frac{1}{s^2}$

   1. State where the function is and is not analytic. **Sol:** Analytic everywhere except at $s = 0$, where it has a $2^{nd}$ order pole.

   2. Explicitly evaluate the integral when $C$ is the unit circle, defined as $s = e^{i\theta}$, $0 \leq \theta \leq 2\pi$. **Sol:**
      \[
      \oint_C F(s)ds = \int_0^{2\pi} \frac{1}{e^{2i\theta}}i e^{i\theta}d\theta = \int_0^{2\pi} i e^{-i\theta}d\theta = i \left[ \frac{1}{-i} e^{-i\theta} \right]_0^{2\pi} = 1(e^{-i2\pi} - e^0) = 0
      \]

   3. What does your result imply about the residue of the $2^{nd}$ order pole at $s = 0$? **Sol:** The residue is 0.

- **Q 3.4:** $F(s) = e^{st}$

   1. State where the function is and is not analytic. **Sol:** Analytic everywhere.

   2. Explicitly evaluate the integral when $C$ is the square $(\sigma, \omega) = (1, 1) \rightarrow (-1, 1) \rightarrow (-1, -1) \rightarrow (1, -1) \rightarrow (1, 1)$. **Sol:** When you perform this integral piece-wise, you will find that all terms cancel out and the result is 0.
3. Evaluate the same integral using Cauchy’s theorem and/or the residue theorem. **Sol:** The function is analytic everywhere, so the integral is 0 by Cauchy’s theorem.

\[-Q \, 3.5: \, F(s) = \frac{1}{s+2}\]

1. State where the function is and is not analytic. **Sol:** Analytic everywhere except at \( s = -2 \), where it has a pole.

2. Let \( C \) be the unit circle, defined as \( s = e^{i\theta}, \, 0 \leq \theta \leq 2\pi \). Evaluate the integral using Cauchy’s theorem and/or the residue theorem.

**Sol:** The function is analytic everywhere inside \( C \), so the integral is 0 by Cauchy’s theorem.

3. Let \( C \) be a circle of radius 3, defined as \( s = 3e^{i\theta}, \, 0 \leq \theta \leq 2\pi \). Evaluate the integral using Cauchy’s theorem and/or the residue theorem.

**Sol:** This contour contains the pole. The residue is 1, therefore the integral is equal to \( 2\pi i \).

\[-Q \, 3.6: \, F(s) = \frac{1}{2\pi i} \frac{e^{it}}{(s+4)}\]

1. State where the function is and is not analytic. **Sol:** Analytic everywhere except at \( s = -2 \), where it has a pole.

2. Let \( C \) be a circle of radius 3, defined as \( s = 3e^{i\theta}, \, 0 \leq \theta \leq 2\pi \). Evaluate the integral using Cauchy’s theorem and/or the residue theorem.

**Sol:** This contour contains the pole. The residue is \( \frac{1}{2\pi i} e^{-2t} \), therefore the integral is equal to \( e^{-2t} \).

3. Let \( C \) contain the entire left-half \( s \)-plane. Evaluate the integral using Cauchy’s theorem and/or the residue theorem. Do you recognize this integral? **Sol:** This contour contains the pole. The residue is \( \frac{1}{2\pi i} e^{-2t} \), therefore the integral is equal to \( e^{-2t} \). This contour is the inverse Laplace transform.

\[-Q \, 3.7: \, F(s) = \pm \frac{1}{\sqrt{\pi}} (e.g. \, F^2 = \frac{1}{\pi})\]

1. State where the function is and is not analytic. **Sol:** Analytic everywhere except \( s = 0 \), where there is a pole.

2. This function is multivalued. How many Riemann sheets do you need in the domain \((s)\) and the range \((f)\) to fully represent this function? Indicate (e.g. using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range. **Sol:** There are 2 sheets in the domain (for the \( \pm \) square root) which map to 1 sheet in the range.

3. Explicitly evaluate the integral

\[\int_C \frac{1}{\sqrt{z}} \, dz\]

when \( C \) is the unit circle, defined as \( s = e^{i\theta}, \, 0 \leq \theta \leq 2\pi \). Is this contour ‘closed’? State why or why not. **Sol:** The solution is

\[2\sqrt{2} \int_{\theta=0}^{2\pi} \left. 2e^{i\theta/2} \right|_0^{2\pi} = 2(e^{j\pi} - e^{0}) = -4.\]
In polar coordinates
\[
\int_0^{2\pi} \frac{ds}{\sqrt{s}} = \int_0^{2\pi} \frac{de^{i\theta}}{e^{i\theta/2}}
\]
\[
= i \int_0^{2\pi} \frac{e^{i\theta}}{e^{i\theta/2}} d\theta
\]
\[
= i \int_0^{2\pi} e^{i\theta/2} d\theta
\]
\[
= 2 e^{i\theta/2} \bigg|_0^{2\pi}
\]
\[
= 2[e^{i\pi} - 1] = 2(-2) = -4.
\]

This contour is not closed. One way to determine this is to see if going once around the unit circle returns \( F(s) \) to its original value.

\[
F(e^{i0}) = 1 \neq F(e^{i2\pi}) = e^{-i\pi} = -1.
\]

4. Explicitly evaluate the integral
\[
\int_C \frac{1}{\sqrt{z}} dz
\]
when \( C \) is twice around the unit circle, defined as \( s = e^{i\theta}, \ 0 \leq \theta \leq 4\pi \). Is this contour ‘closed’? State why or why not. Hint: Note that
\[
\sqrt{e^{i(\theta+2\pi)}} = \sqrt{e^{i2\pi}e^{i\theta}} = e^{i\pi}\sqrt{e^{i\theta}} = -1\sqrt{e^{i\theta}}
\]

**Sol:**
\[
\int_0^{4\pi} \frac{ds}{\sqrt{s}} = \int_0^{4\pi} \frac{de^{i\theta}}{e^{i\theta/2}}
\]
\[
= i \int_0^{4\pi} \frac{e^{i\theta}}{e^{i\theta/2}} d\theta
\]
\[
= i \int_0^{4\pi} e^{i\theta/2} d\theta
\]
\[
= 2 e^{i\theta/2} \bigg|_0^{4\pi}
\]
\[
= 2[e^{i2\pi} - 1] = 2(0) = 0.
\]

This contour is closed. One way to determine this is to see if going twice around the unit circle returns \( F(s) \) to its original value. \( F(e^{i0}) = 1 = F(e^{i4\pi}) = e^{-i2\pi} = 1 \).

5. What does your result imply about the residue of the (twice-around \( \frac{1}{2} \) order) pole at \( s = 0 \)? **Sol:** The residue is 0.

6. Show that the residue is zero. Hint: apply the definition of the residue. **Sol:** \( c_{-1} = \lim_{z \to z_k} z/\sqrt{z} = \lim_{z \to z_k} \sqrt{z} = 0 \).

**Problem #4: A two-port network application for the Laplace transform**

Recall that the Laplace transform (LT) \( f(t) \leftrightarrow F(s) \) of a causal function \( f(t) \) is
\[
F(s) = \int_0^\infty f(t)e^{-st} dt,
\]
where \( s = \sigma + j\omega \) is complex frequency\(^2\) in [radians] and \( t \) is time in [seconds]. Causal functions and the Laplace transform are particularly useful for describing systems, which have no response until a signal enters the system (e.g. at \( t = 0 \)).

\(^1\)Many loosely adhere to the convention that the frequency domain uses upper-case [e.g. \( F(s) \)] while the time domain uses lower case [\( f(t) \)]

\(^2\)While radians are useful units for calculations, when providing physical insights in discussions of problem solutions, it is easier to work with Hertz, since frequency in [Hz] and time in [s] are mentally more more natural units than radians. The same is true of degrees vs. radians. Boas (p. 10) recommends the use degrees over radians. He gives the example of \( 3\pi/5 \) [radians], which is more easily visualize as 108°.
The definition of the inverse Laplace transform ($\mathcal{L}^{-1}$) requires integration in the complex plane:

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}ds = \frac{1}{2\pi j} \oint_C F(s)e^{st}ds.$$ 

The Laplace contour $C$ actually includes two pieces

$$\oint_C = \int_{\sigma - j\infty}^{\sigma + j\infty} + \int_{C_\infty},$$

where the path represented by ‘$C_\infty$’ is a semicircle of infinite radius with $\sigma \to -\infty$. It is somewhat tricky to do, but it may be proved that the integral over the contour $C_\infty$ goes to zero. For a causal, ‘stable’ (e.g. doesn’t blow up over time) signal, all of the poles of $F(s)$ must be inside of the Laplace contour, in the left-half $s$-plane.

**Transfer functions** Linear, time-invariant systems are described by an ordinary differential equations. For example, consider the first-order linear differential equation

$$a_1 \frac{dy(t)}{dt} = b_1 \frac{dx(t)}{dt} + b_0x(t). \quad (3.11)$$

This equation describes the relationship between the input ($x(t)$) and output ($y(t)$) of the system. If we define Laplace transforms $y(t) \leftrightarrow Y(s)$ and $x(t) \leftrightarrow X(s)$, then this equation may be written in the frequency domain as

$$a_1sY(s) = b_1sX(s) + b_0X(s).$$

The transfer function for this system is defined as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_1s + b_0}{a_1s} = \frac{b_1}{a_1} + \frac{b_0}{a_1s}.$$

In this problem, we will look at the transfer function of a simple two-port network, shown in Figure 3.1. This network is an example of a RC low-pass filter, which acts as a leaky integrator.

![Figure 3.1: This three-element electrical circuit is a system that acts to low-pass filter the signal voltage $V_1(\omega)$, to produce signal $V_2(\omega)$.](image)

**Problem #5: ABCD method**

**Q 5.1: Low-pass RC filter**

1. Use the ABCD method to find the matrix representation of Fig. 3.1.

**Sol:**

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} (1 + R_1Cs) & R_1 \\ sC & 1 \end{bmatrix} \begin{bmatrix} 1 & R_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} (1 + R_1Cs) & (R_1 + R_2 + R_1R_2Cs) \\ sC & (1 + R_2Cs) \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}.$$
2. Assuming that \( I_2 = 0 \), find the transfer function \( H(s) \equiv V_2/V_1 \). From the results of the ABCD matrix you determined above, show that

\[
H(s) = \frac{1}{1 + R_1Cs}.
\]

**Sol:** Since \( I_2 = 0 \) the upper row of the ABCD matrix gives the relationship between \( V_1 \) and \( V_2 \) as

\[
V_1 = (1 + R_1C)V_2
\]

Thus the ratio is as desired.

3. The transfer function \( H(s) \) has one pole. Where is the pole? Find the residue of this pole. **Sol:** If we rewrite \( H(s) \) in the standard form, the pole \( s_p \) and residue \( A \) may be easily identified:

\[
H(s) = \frac{A}{s - s_p} = \frac{1}{1 + R_1Cs} = \frac{1/(R_1C)}{s + 1/(R_1C)}
\]

Thus the pole is \( s_p = -1/R_1C \) and the residue is \( A = 1/R_1C \).

4. Find \( h(t) \), the inverse Laplace transform of \( H(s) \). **Sol:**

\[
h(t) = \oint_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} \frac{e^{st}}{1 + R_1Cs} \frac{ds}{2\pi j} = \frac{1}{R_1C}e^{-t/R_1C}u(t)
\]

The integral follows from the Residue Theorem. The pole is at \( s_p = -1/RC \) and the residue is \( 1/RC \).

5. Assuming that \( V_2 = 0 \) find \( Y_{12}(s) \equiv I_2/V_1 \). **Sol:** Setting \( V_2 = 0 \) we may easily read off the requested function as

\[
V_1 = -(R_1 + R_2 + R_1R_2Cs)I_2
\]

thus

\[
Y_{12}(s) = -\frac{1}{R_1 + R_2 + R_1R_2Cs} = \frac{A}{s - s_p}
\]

with residue \( A = 1/R_1R_2C \) and pole \( s_p = -(R_1 + R_2)/(R_1R_2C) \).

6. Find the input impedance to the right-hand side of the system, \( Z_{22}(s) \equiv V_2/I_2 \) for two cases:

(a) \( I_1 = 0 \)

(b) \( V_1 = 0 \)

**Sol:** There are two cases. When \( I_1 = 0 \), \( Z_{22} = R_2 + 1/sC \). When \( V_1 = 0 \)

\[
Z_{22}(s) = R_2 + R_1 || \frac{1}{sC}
\]

\[
= R_2 + \frac{R_1/sC}{1 + R_1/sC} = R_2 + \frac{R_1}{1 + R_1Cs}
\]

\[
= \frac{R_1 + R_2 + R_1R_2Cs}{1 + R_1Cs}
\]

Reading this second case off of our matrix solution gives

\[
0 = (1 + R_1C)V_2 - (R_1 + R_2 + R_1R_2Cs)I_2
\]

or solving for \( Z_{22} \) gives the brute-force result.
7. Compute the determinant of the ABCD matrix. \textit{Hint: It is always 1.} 

\textbf{Sol:} 

\[ 
\begin{vmatrix}
1 + R_1 C s & R_1 + R_2 + R_1 R_2 C s \\
R_1 C & 1 + R_2 C s \\
\end{vmatrix} 
= 1 + (R_1 + R_2) C s + R_1 R_2 (C s)^2 - (r R_1 + R_2) C s - R_1 R_2 (C s)^2 
= 1 
\]

8. Compute the derivative of \( H(s) = \frac{V_2}{V_1} \mid_{t_2=0} \). \textbf{Sol:} From the result of the previous problem 2

\[ H(s) = \frac{1}{1 + R_1 C s} \]

Thus we wish to find

\[ \frac{d}{ds} H(s) = \frac{d}{ds} \left( \frac{1}{1 + R_1 C s} \right)^{-1} = \frac{-R_1 C}{(1 + R_1 C s)^2} \]

Here is a slightly easier way, using the log function:

\[ \frac{1}{H(s)} \frac{dH(s)}{ds} = \frac{d}{ds} \ln H(s) = - \frac{d}{ds} \ln(1 + R_1 C s) = \frac{-1}{1 + R_1 C s} \frac{d}{ds} (1 + R_1 C s) = \frac{-R_1 C}{(1 + R_1 C s)^2} \]

Thus

\[ \frac{dH(s)}{ds} = H(s) \frac{-R_1 C}{(1 + R_1 C s)} = \frac{-R_1 C}{(1 + R_1 C s)^2} \]

\textbf{Problem # 6: With the help of a computer}

In the following problems, we will look at some of the concepts from this homework using Matlab/Octave. We are using the \texttt{syms} function which requires Matlab’s/Octave’s symbolic math toolbox. Or you may use the EWS lab’s Matlab. Alternative symbolic-math tool, such as Wolfram Alpha.\(^3\)

\textbf{Example:} To find the Taylor series expansion about \( s = 0 \) of

\[ F(s) = - \log(1 - s), \]

first consider the derivative and its Taylor series (about \( s = 0 \))

\[ F'(s) = \frac{1}{1 - s} = \sum_{n=0}^{\infty} s^n. \]

Then, integrate this series term by term

\[ F(s) = - \log(1 - s) = \int_0^s F'(s) ds = \sum_{n=0}^{\infty} \frac{s^n}{n}. \]

Alternatively you may use Matlab/Octave commands:

\begin{verbatim}
syms s
taylor(-log(1-s),’order’,7)
\end{verbatim}

\textbf{Q 6.1: Use Octave’s taylor(-log(1-s)) to 7th order, as in the example above.}

\(^3\)https://www.wolframalpha.com/
1. Try the above Matlab/Octave commands. Give the first 7 terms of the Taylor series (confirm that Matlab/Octave agrees with the formula derived above). **Sol:**

\[ F(s) = \cdots + \frac{s^7}{7} + \frac{s^6}{6} + \frac{s^5}{5} + \frac{s^4}{4} + \frac{s^3}{3} + \frac{s^2}{2} + s \]

2. What is the inverse Laplace transform of this series? Consider the series term by term. **Sol:** \( f(t) = \sum \delta^{(n)}/n \)

**Q 6.2:** The function \( 1/\sqrt{z} \) has a branch point at \( z = 0 \), thus it is singular there.

1. Can you apply Cauchy’s integral theorem when integrating around the unit circle? **Sol:** No, one cannot apply the Cauchy Theorem since it is not analytic at \( z = 0 \). But the integral may be evaluated.

2. Below is a Matlab/Octave code that computes \( \int_0^{4\pi} \frac{dz}{\sqrt{z}} \) using Matlab’s/Octave’s symbolic analysis package:

   ```matlab
   syms z
   I=int(1/sqrt(z))
   J = int(1/sqrt(z),exp(-j*pi),exp(j*pi))
   eval(J)
   ``

   Run this script. What answers do you get for \( I \) and \( J \)? **Sol:** This script returns the answers \( I = 2 \sqrt{z} \) and \( J = 2.4493e-16 \), which is numerically the same as zero.

3. Modify this code to integrate \( f(z) = \frac{1}{z^2} \) once around the unit circle. What answers do you get for \( I \) and \( J \)? **Sol:** This function has a 2d order pole at \( s = 0 \). Thus from the CIT, the integral evaluates to zero.

Proof:

\[ I = \oint ds \frac{dz}{s^2} = -\frac{1}{s} \bigg|^{2\pi}_0 = -e^{-i\theta} \bigg|^{2\pi}_0 = -(1 - 1) = 0 \]

More generally \( I = \oint ds \frac{dz}{s^n} = 0 \) for \( n \neq 1 \). As best I know, this holds for any \( n \in \mathbb{Z}, \mathbb{Q}, \mathbb{F}, \mathbb{R}, \mathbb{C} \). For \( n = 1 \) it has a value of \( 2\pi j \).

**Q 6.3:** Bessel functions can describe waves in a cylindrical geometry

The Bessel function has a Laplace transform with a branch cut

\[ J_0(t)u(t) \leftrightarrow \frac{1}{\sqrt{1 + s^2}}. \]

Draw a hand sketch showing the nature of the branch cut. Hint: Use \texttt{zviz}. **Sol:** The roots are given by \( s_{\pm} = \pm j \). The branch cut connects the two roots, or can go from each root to \( \infty \). Either choice is valid.

**Problem #7: Matlab/Octave exercises:**

**Q 7.1:** Comment on the following Matlab/Octave exercises

1. Try the following Matlab/Octave commands, and then comment on your findings.

   ```matlab
   %Take the inverse LT of 1/sqrt(1+s^2)
   syms s
   I=ilaplace(1/(sqrt((1+s^2))));
   disp(I)
   ```
\textbf{Sol: } \( I = J_0(t)u(t) \).

\%Find the Taylor series of the LT
\[ T = \text{taylor}(1/\sqrt{1+s^2},10); \]
disp(T);

\textbf{Sol: } 
\[ T = \cdots + \frac{3s^4}{8} - \frac{s^2}{2} + 1 \]

\%Verify this
\text{syms} t
J=\text{laplace(besselj(0,t))};
disp(J);

\textbf{Sol: } \( I = \frac{1}{\sqrt{1+s^2}} \).

\%plot the Bessel function
\text{t=0:0.1:10*pi;}
b=besselj(0,t);
\text{plot(t/pi,b);}
grid on;

\textbf{Sol: } Plot of \( J_0(t)u(t) \).

\textit{–Q 7.2: When did Friedrich Bessel live?}
\textbf{Sol: } 1784-1846, in Königsberg Germany.

\textit{–Q 7.3: What did he use Bessel functions for?}
\textbf{Sol: } Solving the Bessel equation, which is the wave equation in 2D. Bessel functions were first introduced by the Daniel Bernoulli.

\textit{–Q 7.4: Using zviz, for each of the following functions}

1. Describe the plot generated by \text{zviz S=Z}. \textbf{Sol: } It is a polar plot of the function, with intensity coding the magnitude and color coding the phase. Red is a positive real number while and blue is a negative real number.

2. Are the functions defined below legal Brune impedances? (i.e., Do they function obey \( \Re Z(\sigma > 0) \geq 0 \))? \textit{Hint: Consider the phase (color). Plot zviz Z for a reminder of the colormap.}

\textit{–Q 7.5: Comment on the colorized plots of these functions}

1. \text{zviz 1./sqrt(1+S.^2)}
   \textbf{Sol: } No. The RHP has blue near the branch cut, in the RHP.

2. \text{zviz 1./sqrt(1-S.^2)}
   \textbf{Sol: } NO, there is a branch cut in the RHP.

3. \text{zviz 1./(1+sqrt(S)))}
   \textbf{Sol: } Yes, its red almost everywhere even though it has a branch cut from \( [-\infty < \sigma \leq -10] \). Since \( 1/\sqrt{s} \) has an \( \mathcal{L}T^{-1} \), this function must as well. Matlab found
   \[ \frac{1}{\sqrt{1+s}} \leftrightarrow \frac{e^{-t}}{\sqrt{\pi} \sqrt{t}} u(t), \]
   however Octave failed to find the inverse transform, (but was able to find the forward transform).
**Problem #8:** Find the inverse Laplace transform of the zeta function $\zeta_p(s)$

$\zeta_p(s) \leftrightarrow z_p(t)$ (Eq. 2.10, p. 33) and describe the result in words.

*Hint:* Consider the geometric series representation

$$\zeta_p(s) = \frac{1}{1 - e^{-skT_p}} = \sum_{k=0}^{\infty} e^{-skT_p}, \quad (3.12)$$

for which you can easily look up (or may have memorized) the inverse Laplace transform of each term.

*Sol:* Since each term in the series is a pure delay, giving

$$z_p(t) = \delta(t)T_p \equiv \sum_{k=0}^{\infty} \delta(t - kT_p) \leftrightarrow \frac{1}{1 - e^{-skT_p}}. \quad (3.13)$$

**Problem #9:** Inverse transform of Product of factors:

The time domain version of Eq. ?? (p. ??) may be written as the convolution of all the $z_k(t)$ factors

$$z(t) \equiv z_2(t) * z_3(t) * z_5(t) * z_7(t) \cdots * z_p(t) * \cdots, \quad (3.14)$$

where $*$ represents time convolution.

![Feedback Network Diagram](fig:imped:DE2)

Figure 3.2: This feedback network is described by a time-domain difference equation with delay $T_p$, has an all-pole transfer function $\zeta_p(s) \equiv Q(s)/I(s)$ given by Eq. 3.15, which physically corresponds to a stub of a transmission line, with the input at one end and the output at the other. To describe the $\zeta(s)$ function we must take $\alpha = -1$. A transfer function $Y(s) = V(s)/I(s)$ that has the same poles as $\zeta_p(s)$, but with zeros as given by Eq. 3.16, is the input admittance $Y(s) = I(s)/V(s)$ of the transmission line, defined at the ratio of the Laplace transform of the current $i(t) \leftrightarrow I(s)$ over the voltage $v(t) \leftrightarrow V(s)$. Fig:imped:DE2

Explain what this means in physical terms. Start with two terms (e.g., $z_1(t) * z_2$). *Sol:* In terms of the physics, these transmission line equations are telling us that $\zeta(s)$ may be decomposed into an infinite cascade of transmission lines (Eq. 3.14), each having a delay given by $T_p = \ln \tau_p$. The input admittance of this cascade may be interpreted as an analytic continuation of $\zeta(s)$ which defines the eigen-modes of that cascaded impedance function.

**Problem #10:** Physical interpretation:

Such functions may be generated in the time domain as shown in Fig. 3.2 (p. 70), using a feedback delay of $T_p$ seconds described by the two equations in the figure with a unity feedback gain $\alpha = -1$. Taking the Laplace transform of the system equation we see that the transfer function between the state variable $q(t)$ and the input $x(t)$ is given by $\zeta_p(s)$, which is and all-pole function, since

$$Q(s) = e^{-sT_p}Q(s) + V(s), \quad \text{or} \quad \zeta_p(s) \equiv \frac{Q(s)}{V(s)} = \frac{1}{1 - e^{-sT_p}}. \quad (3.15)$$

Closing the feed-forward path gives a second transfer function $Y(s) = I(s)/V(s)$, namely

$$Y(s) \equiv \frac{I(s)}{V(s)} = \frac{1 - e^{-sT_p}}{1 + e^{-sT_p}}. \quad (3.16)$$

---

4Here we use a shorthand double-parentheses notation to define the infinite (one-sided) sum $f(t)T_p = \sum_{k=0}^{\infty} f(t - kT)$. 
If we take $i(t)$ as the current and $v(t)$ as the voltage at the input to the transmission line, then $y_p(t) \leftrightarrow \zeta_p(s)$ represents the input impedance at the input to the line. The poles and zeros of the impedance interleave along the $j\omega$ axis. By a slight modification $\zeta_p(s)$ may alternatively be written as

$$Y_p(s) = \frac{e^{sT_p/2} + e^{-sT_p/2}}{e^{sT_p/2} - e^{-sT_p/2}} = j \tan(sT_p/2).$$  \hspace{1cm} (3.17)$$

Every impedance $Z(s)$ has a corresponding reflectance function given by a Möbius transformation, which may be read off of Eq. 3.16 as

$$\Gamma(s) \equiv \frac{1 + Z(s)}{1 - Z(s)} = e^{-sT_p}$$  \hspace{1cm} (3.18)$$

since impedance is also related to the round-trip delay $T_p$ on the line. The inverse Laplace transform of $\Gamma(s)$ is the round trip delay $T_p$ on the line

$$\gamma(t) = \delta(t - T_p) \leftrightarrow e^{-sT_p}. \hspace{1cm} (3.19)$$

In terms of the physics, these transmission line equations are telling us that $\zeta(s)$ may be decomposed into an infinite cascade of transmission lines (Eq. 3.14), each having a delay given by $T_p = \ln \pi_p$. The input admittance of this cascade may be interpreted as an analytic continuation of $\zeta(s)$ which defines the eigen-modes of that cascaded impedance function.

Working in the time domain provides a key insight, as it allows us to parse out the best analytic continuation of the infinity of possible continuations, that are not obvious in the frequency domain. Transforming to the time domain is a form of analytic continuation of $\zeta(s)$, that depends on the assumption that $z(t)$ is one-sided in time (causal).