Chapter 4. Stream 3A: Scalar Calculus (11 Lectures)

4.5.4 Exercises DE-3

Brune Impedance

Problem #1: Residue form

A Brune impedance is defined as the ratio of the force \( F(s) \) over the flow \( V(s) \), and may be expressed in residue form as

\[
Z(s) = c_0 + \sum_{k=1}^{K} \frac{c_k}{s - s_k} = \frac{N(s)}{D(s)}
\]  

(4.50)

with

\[
D(s) = \prod_{k=1}^{K} (s - s_k) \quad \text{and} \quad c_k = \lim_{s \rightarrow s_k} (s - s_k)D(s) = \prod_{n'=1}^{K-1} (s - s_n).
\]

The prime on index \( n' \) means that \( n = k \) is not included in the product.

–Q 1.1: Find the Laplace transform (LT) of a 1) spring, 2) dashpot and 3) mass. Express these in terms of the force \( F(s) \) and the velocity \( V(s) \), along with the electrical equivalent impedance:

1. Hooke’s Law \( f(t) = Kx(t) \).
2. Dash-pot resistance \( f(t) = Rv(t) \).
3. Newton’s Law for Mass \( f(t) = Mdv(t)/dt \).

–Q 1.2: Take the Laplace transform (LT) of Eq. 3.49, and find the total impedance \( Z(s) \) of the mechanical circuit.

–Q 1.3: What are \( N(s) \) and \( D(s) \) (e.g. Eq. 4.50)?

–Q 1.4: Assume that \( M = R = K = 1 \), find the residue form of the admittance \( Y(s) = 1/Z(s) \) (e.g. Eq. 4.50) in terms of the roots \( s_{\pm} \).

You may check your answer with the Matlab’s residue command.

–Q 1.5: By applying the CRT, find the inverse Laplace transform (LT^{-1}). Use the residue form of the expression that you derived in the previous exercise.
4.5. LECTURE 29: INVERSE LAPLACE TRANSFORM ($T < 0$): CAUCHY RESIDUE THEOREM

Figure 4.10: Depiction of a train consisting of cars, treated as a mass $M$ and linkages, treated as springs of stiffness $K$ or compliance $C = 1/K$. Below it is the electrical equivalent circuit, for comparison. The mass is modeled as an inductor and the springs as capacitors to ground. The velocity is analogous to a current and the force $f_n(t)$ to the voltage $v_n(t)$. The train may be accurately modeled as a transmission line (TL), since the equivalent electrical circuit is a lumped model of a TL.

Train-mission-line

This problem is both important and difficult. Thus the solution has been turned on. There is a lot going on here, with several important key results. Do the problems in order, and please come to office hours.

We wish to model the dynamics of a freight-train having $N$ such cars, and study the velocity transfer function under various load conditions. As shown in Fig. 4.10, the train model consists of masses connected by springs.

The velocity transfer function for this system is defined as the ratio of the $N$th output velocity $V_N$ to the input velocity $V_1$. Consider the engine on the left pulling the train at velocity $v_1(t) \leftrightarrow V_1(s)$ and each car responding with a velocity of $V_n(s)$. Then

$$H_{N,1}(s) = \frac{V_N(s)}{V_1(s)}$$

is the frequency domain transfer function of the $N$th car, having velocity $V_N$, to the velocity of the engine, $V_1$.

Equations for eigenvalues, eigenvectors and eigenmatrix of $T$. Given a $T$ (ABCD) transmission matrix, the eigenvalues are and vectors are (see Appendix E for details).

Cell matrix:

$$T(s) = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix}.$$ 

Eigenvalues:

$$\begin{bmatrix} \lambda_+ \\ \lambda_- \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (A + D) - \sqrt{(A - D)^2 + 4BC} \\ (A + D) + \sqrt{(A - D)^2 + 4BC} \end{bmatrix}$$

Due to symmetry, $A = D$, this simplifies to $\lambda_\pm = A \mp \sqrt{BC}$ so that the eigenmatrix is

$$\Lambda_{A=D} = \begin{bmatrix} A - \sqrt{BC} & 0 \\ 0 & A + \sqrt{BC} \end{bmatrix}$$

Eigenvectors: The eigenvectors simplify even more

$$[E_\pm] = \begin{bmatrix} \frac{1}{\sqrt{BC}} \left( A - D \mp \sqrt{(A - D)^2 + 4BC} \right) \\ 1 \end{bmatrix} = \begin{bmatrix} \mp \sqrt{\frac{B}{C}} \\ 1 \end{bmatrix}$$

Eigenmatrix:

$$E = \begin{bmatrix} -\sqrt{\frac{B}{C}} & +\sqrt{\frac{B}{C}} \\ 1 & 1 \end{bmatrix}, \quad E^{-1} = \frac{1}{2} \begin{bmatrix} -\sqrt{\frac{C}{B}} & 1 \\ +\sqrt{\frac{C}{B}} & 1 \end{bmatrix}$$
Problem # 2: Transfer functions

Use the ABCD method to find the matrix representation of Fig. 4.10. At each node define the force $f_n(t) \leftrightarrow F_n(\omega)$ and the velocity $v_n(t) \leftrightarrow V_n(\omega)$ at junction $n$.

Consistent with the figure, break the model into cells consisting of three elements: a series inductor representing half the mass ($L = M/2$), a shunt capacitor representing the spring ($C = 1/K$), and another series inductor representing half the mass ($L = M/2$). Making the model a cascade of symmetric $A = D$ identical cell matrix $T(s)$ (P7, p. 135).

\[Q2.1:\] Write the ABCD matrix $T$ for a single cell, composed of series mass $M/2$, shunt compliance $C$ and series mass $M/2$, that relates the input node 1 to node 2.

Here

\[
\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = T \begin{bmatrix} F_2(\omega) \\ -V_2(\omega) \end{bmatrix}.
\]

Note that here the mechanical force $F$ is analogous to electrical voltage, and the mechanical velocity $V$ is analogous to electrical current. **Solution:**

\[
T = \begin{bmatrix} 1 & sM/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC & 1 \end{bmatrix} \begin{bmatrix} 1 & sM/2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 + 2s^2MC/2 & (sM)(1 + s^2MC/4) \\ sC & 1 + s^2MC/2 \end{bmatrix}
\] (4.51a)

\[
= \begin{bmatrix} 1 + 2(\frac{s}{s_c})^2 & sM(1 + (\frac{s}{s_c})^2) \\ sC & 1 + 2(\frac{s}{s_c})^2 \end{bmatrix}
\] (4.51b)

\[Q2.2:\] Define the wave velocity:

\[c_o = \frac{1}{\sqrt{MC}} < \omega_c/2 < \omega_c\sqrt{2}/2,\] (4.52)

and the wave delay $T_o$, the wavelength $\lambda = c_o/f_c$, and the the distance between cars as $\Delta_o = c_oT_o$, we may define the Nyquist sampling rate as approximation Eq. 4.51 follows when $\omega < \omega_c$.

**Solution:** Such a system is a transmission line having a wave speed of $c_o = 1/\sqrt{MC}$ and characteristic impedance $r_o = \sqrt{M/C}$. Each cell, composed of 2 masses connected by one spring, is taken to have a length $\Delta$. 

We wish to define the Nyquist frequency $f_c$ such that the wavelength $\lambda > n\Delta$, where $\Delta$ is the cell length. Using the formula for the wavelength in terms of the wave velocity and frequency we find

\[\lambda = c_o/f_c = n\Delta.\]

Nyquist sampling requires $n > 2$. For Nyquist sampling ($n > 2$), we conclude that

\[f_c < \frac{c_o}{2\Delta} = \frac{1}{2\Delta\sqrt{MC}}.\]

If we wish to have the system be accurate for a given frequency we may make the cell length $\Delta$ smaller, while keeping the velocity constant ($MC$ is held constant). Thus the characteristic resistance [ohms/unit length] $r_o$ must change as $f_c \to \infty$ and $\Delta \to 0$. We can either let $M \to \infty$ and $C \to 0$ (their product remains constant), or the other way around. In one case $r_o \to \infty$ and in the other case it goes to 0.

\[Q2.3:\] Rewrite Eq. 4.51b in terms of the Nyquist frequency $s_c \equiv 2\pi f_c$

\[
T = \begin{bmatrix} 1 + 2(s/s_c)^2 & sM(1 + (s/s_c)^2) \\ sC & 1 + 2(s/s_c)^2 \end{bmatrix}
\]

\[
\approx \begin{bmatrix} 1 & sM \\ sC & 1 \end{bmatrix}
\]

The approximation is valid below the Nyquist cutoff frequency $s < s_c$. 


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**Q 2.4:** The matrix has eigenvalues defined as:

$$\lambda_\pm = 1 \mp 2s/s_c \approx e^{\pm 2s/s_c} = e^{\mp sT_c}.$$  

From this we can interpret the eigenvalues as the cell delay $T_c = 2/s_c$. Hint: the formulae for the eigenvalues, eigenvectors and eigenmatrix are given in the problem setup. **Solution:** The eigenvectors are

$$E_\pm = \begin{bmatrix} \pm \sqrt{M/C} \\ 1 \end{bmatrix},$$

with the characteristic impedance defined as $r_o = \sqrt{M/C}$.

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**Q 2.5:** Assuming that $N = 2$ and that $F_2 = 0$ (two mass problem), find the transfer function $H(s) \equiv V_2/V_1$.

From the results of the $T$ matrix you determined above, find

$$H_{21}(s) = \frac{V_2}{V_1}\bigg|_{F_2=0}$$

**Solution:** From the lower equation we see that $V_1 = sCF_2 - (s^2MC/2 + 1)V_2$. Recall that $F_2 = 0$, thus

$$\frac{V_2}{V_1} = -\frac{1}{s^2MC/2 + 1} = \left(\frac{c_+}{s - s_+} + \frac{c_-}{s - s_-}\right),$$

with complex eigen-frequencies $s_\pm = \pm j\sqrt{2MC} = \pm \sqrt{2}s_c$ and complex residues $c_\pm = \pm j\sqrt{2MC} = \pm s_c/\sqrt{2}$.

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**Q 2.6:** Find $h_{21}(t)$, the inverse Laplace transform $H_{21}(s)$.

**Solution:**

$$h(t) = \int_{s_0-j\infty}^{s_0+j\infty} \frac{e^{st}}{s^2MC/2 + 1} \frac{ds}{2\pi j} = c_+e^{-s_+t}u(t) + c_-e^{-s_-t}u(t).$$

The integral follows from the CRT. The poles are at $s_\pm = \pm j\sqrt{\frac{2}{MC}}$ and the residues are $c_\pm = \pm j/\sqrt{2MC}$.

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**Q 2.7:** What is the input impedance $Z_2 = F_2/V_2$ if $F_3 = -r_0V_3$?

**Solution:** Starting from $T$ calculated above, find $Z_2$

$$Z_2(s) = \frac{F_2}{V_2} = T \begin{bmatrix} F_3 \\ -V_3 \end{bmatrix} = \frac{-(1 + s^2CM/2)r_0V_3 - sM(1 + s^2CM/4)V_3}{-sCr_0V_3 - (1 + s^2CM/2)V_3}$$

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**Q 2.8:** Simplify the expression for $Z_2$ with $N \to \infty$ by defining the characteristic impedance as $r_o = \sqrt{M/C}$ and again assuming that: 1) $F_3 = -r_0V_3$ (i.e., $-V_3$ cancels), 2) $|s/s_c| \to 0$ (Nyquist approximation).

**Solution:** Applying the Nyquist approximation (i.e., $s/s_c \to 0$)

$$Z_2(s) = \frac{r_o(1 + s^2CM/2) + sM(1 + s^2CM/4)}{r_osC + (1 + s^2CM/2)}$$

$$\approx \frac{r_o + sM}{1 + r_osC} = \frac{MC}{M} \frac{r_o + sM}{1 + r_osC} = \frac{M}{C} \frac{r_oC + sMC}{M + r_osMC} = \frac{r_oC + sMC}{M + r_osMC}$$

$$\approx \frac{r_o^2 r_oC + s/s_c}{M + r_os/s_c} = \frac{r_o^3 C}{M}$$

$$= r_o.$$  

We may conclude that below the Nyquist cutoff frequency, the system approximates a transmission line terminated in its characteristic impedance.
–Q 2.9: State the ABCD matrix relationship between the first and $N$th node in terms of the cell matrix.
Write out the transfer function for one cell: $H_{21}$? **Solution:**

\[
T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]

Now use the formulae for the eigen-values and vectors to obtain $T$ for $N = 1$:

\[
T = E \Lambda E^{-1} = E \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} E^{-1}.
\]

–Q 2.10: What is the velocity transfer function $H_{N1} = \frac{V_N}{V_1}$?

Hint: Use an eigenmatrix diagonalization, as we did for the Pell equation (Appendix C). **Solution:**

\[
\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = T^N \begin{bmatrix} F_N(\omega) \\ -V_N(\omega) \end{bmatrix}
\]

along with the eigenvalue expansion

\[
T^N = E \Lambda^N E^{-1} = E \begin{bmatrix} \lambda_+^N & 0 \\ 0 & \lambda_-^N \end{bmatrix} E^{-1}.
\]

We may conclude that as we add more cells, the delay linearly increases with $N$, since each eigenvalue represents the delay of one cell, and the delay adds.