Chapter 4

Vector differential equations

4.0.1 Exercises VC-1

Topic of this homework: Vector algebra and fields in \( \mathbb{R}^3 \); Gradient and scalar Laplacian operator; Definitions of Divergence and Curl; Gauss’s (divergence) & Stokes’ (Curl) Law; Schwarz inequality; Quadratic forms; System postulates

Vector algebra in \( \mathbb{R}^3 \).

Definitions of the vector scalar (aka dot) product \( \mathbf{A} \cdot \mathbf{B} \), cross product \( \mathbf{A} \times \mathbf{B} \) and triple product \( \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \) may be found in Appendix ?? (p. ??), where \( \mathbf{A}, \mathbf{B}, \mathbf{C} \) in \( \mathbb{R}^3 \subset \mathbb{C}^3 \). A fourth “double-cross” product is:

\[
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \alpha_o \mathbf{B} - \beta_o \mathbf{C}.
\]

where \( \alpha_o = \mathbf{A} \cdot \mathbf{C} \) and \( \beta_o = \mathbf{A} \cdot \mathbf{B} \) (Note: \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \)).

\[
\mathbf{A} \cdot \mathbf{B} = ||\mathbf{A}|| ||\mathbf{B}|| \cos \theta
\]

\[
\mathbf{A} \times \mathbf{B} = ||\mathbf{A}|| ||\mathbf{B}|| \sin \theta \hat{\mathbf{z}}
\]

Figure 4.1: Definitions of vectors \( \mathbf{A}, \mathbf{B}, \mathbf{C} \) (vectors in \( \mathbb{R}^3 \)) used in the definition of \( \mathbf{A} \cdot \mathbf{B}, \mathbf{A} \times \mathbf{B} \) and \( \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \). There are two algebraic vector products, the scalar (dot) product \( \mathbf{A} \cdot \mathbf{B} \in \mathbb{R} \) and the vector (cross) product \( \mathbf{A} \times \mathbf{B} \in \mathbb{R}^3 \). Note that the result of the dot product is a scalar, while the vector product yields a vector, which is \( \perp \) to the plane containing \( \mathbf{A}, \mathbf{B} \). This is figure ?? (p. ??), Sect. ??.

To Do:

1. Scalar product \( \mathbf{A} \cdot \mathbf{B} \)

   (a) If \( \mathbf{A} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z} \) and \( \mathbf{B} = b_x \hat{x} + b_y \hat{y} + b_z \hat{z} \), write out the definition of \( \mathbf{A} \cdot \mathbf{B} \). \textbf{Sol:} See the definition in the above figure. \( \mathbf{A} \cdot \mathbf{B} = a_x b_x + a_y b_y + a_z b_z \). In general: \( \mathbf{A} \cdot \mathbf{B} = \sum_k A_k B_k \).

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\(^1\)Greenberg p. 694, Eq. 8.
(b) The dot product is often defined as $||A|| \cdot ||B|| \cos(\theta)$, where $||A|| = \sqrt{A \cdot A}$ and $\theta$ is the angle between $A, B$. If $||A|| = 1$, describe how the dot product relates to the vector $B$. 

**Sol:** See the definition in the above figure. The vector product is the portion of $B$ in the direction of $A$.

2. **Vector (cross) product** $A \times B$

(a) If $A = a_x \hat{x} + a_y \hat{y} + a_z \hat{z}$ and $B = b_x \hat{x} + b_y \hat{y} + b_z \hat{z}$, write out the definition of $A \times B$. 

**Sol:**

$$A \times B \equiv \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{x} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{y} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{z} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}.$$  

(b) Show that the cross product is equal to the area of the parallelogram formed by $A, B$, namely $||A|| \cdot ||B|| \sin(\theta)$, where $||A|| = \sqrt{A \cdot A}$ and $\theta$ is the angle between $A$ and $B$. 

**Sol:** A parallelogram’s area is equal to its base times its height. Therefore, let’s say the base is length $||A||$, and the height $||B|| \sin(\theta)$, which is the portion of $B$ that is perpendicular to $A$.

3. **Triple product** $A \cdot (B \times C)$

Let $A = [a_1, a_2, a_3]^T$, $B = [b_1, b_2, b_3]^T$, $C = [c_1, c_2, c_3]^T$ be three vectors in $\mathbb{R}^3$.

(a) Starting from the definition of the dot and cross product, explain using a diagram and/or words, how one shows that: $A \cdot (B \times C) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

**Sol:** Using the determinate-definition of the cross product,

$$B \times C \equiv \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \hat{x} \begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} - \hat{y} \begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix} + \hat{z} \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix}.$$  

Let $D = B \times C$ and compute $A \cdot D = A \cdot (B \times C)$. Finally compute the requested right-hand side, and compare the two. It should be clear that they are the same, because the dot product transfers the elements of vector $A$ to cross product and reduces the product to the scalar.

(b) Describe why $|A \cdot (B \times C)|$ is the volume of parallelepiped generated by $A, B$ and $C$. 

**Sol:** Note that the norm of $B \times C$ is the area of the parallelogram generated by $C$ and $B$. Taking the dot product with $A$ results in the volume of the corresponding parallelepiped (prism). So the absolute value of triple product is volume of parallelepiped.

(c) Explain why three vectors $A, B, C$ are in one plane if and only if the triple product $A \cdot (B \times C) = 0$. 

**Sol:** (triple product is zero) if and only if: (volume is zero), if and only if: (they are in the same plane)

4. Given two vectors $A, \hat{B}$ in the $\hat{x}, \hat{y}$ plane (see Fig. 1), with $B = \hat{y}$ (i.e., $||\hat{B}|| = 1$). Show that $A$ may be split into two orthogonal parts, one in the direction of $\hat{B}$ and the other perpendicular ($\perp$) to $B$. Hint: Express the vector products of $A$ and $\hat{B}$ (dot and cross) in polar coordinates (Greenberg, 1988, problem 8, p. 695).

$$A = (A \cdot \hat{B})\hat{B} + \hat{B} \times (A \times \hat{B}) = A_\parallel + A_\perp.$$  

**Sol:**

$$A \cdot \hat{B} = ||A|| \cos(\theta) \quad \text{and} \quad A \times \hat{B} = ||A|| \sin(\theta)$$
The first quantity is in the direction of $\hat{B}$, while the second is in the direction $A \times \hat{B}$, which is \perp to $\hat{B}$. Thus 

$$A = ||A|| \left( \hat{B} \cos(\theta) + A \times \hat{B} \sin(\theta) \right)$$

$$= A_{||} + A_{\perp}.$$  

Scalar fields and the $\nabla$ operator

To Do:

1. Let $T(x, y) = x^2 + y$ be an analytic scalar temperature field in 2 dimensions (single-valued $\in \mathbb{R}^2$).
   
   (a) Find the gradient of $T(x)$ and make a sketch of $T$ and the gradient. **Sol:** $\nabla(x^2 + y) = 2x\hat{x} + \hat{y}$. The temperature is quadratic in $x$ and linear in $y$, which has the shape of a trough in $x$, linearly increasing in $y$. In the $y$ ($\hat{y}$) direction the gradient is constant, and in the $\hat{x}$ direction, it is linear, and goes through zero at $x = 0$, with $T(0) = 0$. Skiing in the $y$ direction would be a constant ride of slope 1. If the snow had no friction, you would accelerate, but the terminal velocity would be due to the friction of the snow on the skis. Along the $x$ direction, you would accelerate, at first, coming down, and at $x = 0$ you would stop accelerating, and begin slow down. This would be a more interesting problem if you treated it in terms of the forces on the skis and included friction as well as gravity.

   (b) Compute $\nabla^2 T(x)$, to determine if $T(x)$ satisfies Laplace’s equation. **Sol:** Forming this operation we find that

   $$\frac{\partial^2}{\partial x^2} x^2 + \frac{\partial^2}{\partial y^2} y = 2.$$  

   So $T(x)$ does not satisfy laplace’s equation, rather it satisfies the Poisson equation $\nabla^2 T(x) = 2$.

   (c) Sketch the iso-temperature contours at $T = -10, 0, 10$ degrees. **Sol:** The iso-potential contours are the concave parabolas $y = T_0 - x^2$.

   (d) The heat flux $^2$ is defined as $J(x, y) = -\kappa(x, y)\nabla T$ where $\kappa(x, y)$ is a constant denoting thermal conductivity at the point $(x, y)$. Assuming $\kappa = 1$ everywhere (the medium is homogenous), plot the vector $J(x, y) = \nabla T$ at $x = 2, y = 1$. Be clear about the origin, direction and length of your result. **Sol:** $J = \nabla T = -2x\hat{x} + 2\hat{y}$ thus $-\kappa \nabla T(2, 1) = J = -(4\hat{x} + 2\hat{y})$, which has a length of $\sqrt{17}$ and is pointed $1/\sqrt{17}$ unit down and $4/\sqrt{17}$ units to the left.

   (e) Find the vector $\perp \nabla T(x, y)$, namely tangent to the iso-temperature contours. Hint: Sketch it for one $(x, y)$ point (e.g., 2, 1) and then generalize. **Sol:** We may invoke the third dimension $\hat{z}$ to generate this vector: $\pm \hat{z} \times \nabla T = 0 0 \pm 1 2x 1 0 = \mp (1\hat{x} - 2x\hat{y} + 0\hat{z})$. Alternatively, rotate $\nabla T$ by $\pm \pi/2$ in the $(x, y)$ plane.

   (f) The thermal resistance $R_T$ is defined as the potential drop $\Delta T$ over the magnitude of the heat flux $|J|$. At a single point the thermal resistance is

   $$R_T(x, y) = -\Delta T/|J|.$$  

   How is $R_T(x, y)$ related to the thermal conductivity $\kappa(x, y)$?  

   **Sol:** $R_T(x, y) = 1/\kappa(x, y)$. In general, resistance is the reciprocal of conductivity (conductance). This is true for electrical and acoustic systems as well.

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2The heat flux is proportional to the change in temperature times the thermal conductivity $\kappa$ of the medium https://en.wikipedia.org/wiki/Heat_equation#Derivation_in_one_dimension.
2. **Acoustic wave equation:** Note: In the following problem, we will work in the frequency domain.

The basic equations of acoustics in 1 dimension are

\[
-\frac{\partial}{\partial x} P = \rho_0 s \mathcal{V} \quad \text{and} \quad -\frac{\partial}{\partial x} \mathcal{V} = \frac{s}{\eta_0 P_0} P.
\]

Here \( P(x, \omega) \) is the pressure (in the frequency domain), \( \mathcal{V}(x, \omega) \) is the volume velocity (integral of the velocity over the wave-front having area \( A \)), \( s = \sigma + \omega j \), \( \rho_0 = 1.2 \) is the specific density of air, \( \eta_0 = 1.4 \) and \( P_0 \) is the atmospheric pressure (i.e., \( 10^5 \text{ [Pa]} \)) (see the handout Appendix F.2 for details). Note that the pressure field \( P \) is a scalar (pressure does not have direction), while the volume velocity field \( \mathcal{V} \) is a vector (velocity has direction).

We can generalize these equations to 3 dimensions using the \( \nabla \) operator

\[
-\nabla P = \rho_0 s \mathcal{V} \quad \text{and} \quad -\nabla \cdot \mathcal{V} = \frac{s}{\eta_0 P_0} P.
\]

(a) Starting from these two basic equations, derive the scalar wave equation in terms of the pressure \( P \),

\[
\nabla^2 P = \frac{s^2}{c_0^2} P
\]

where \( c_0 \) is a constant representing the speed of sound. **Sol:** We wish to remove \( \mathcal{V} \) from the two equations, to obtain a single equation in pressure. If we take the partial wrt \( x \) of the pressure equation, and then substitute the velocity equation, to remove the velocity:

\[
\nabla^2 P = -\rho_0 s \nabla \cdot \mathcal{V} = \frac{s^2 \rho_0}{\eta_0 P_0} P = \frac{s^2}{c_0^2} P
\]

(b) What is \( c_0 \) in terms of \( \eta_0, \rho_0, \) and \( P_0 \)? **Sol:** Comparing the last two terms from the previous solution we see that

\[
c_0 = \sqrt{\eta_0 P_0/\rho_0}.
\]

(c) Rewrite the pressure wave equation in the time domain, using the time derivative property of the Laplace transform (e.g., \( dx/dt \leftrightarrow sX(s) \)). For your notation, define the time-domain signal using a lowercase letter, \( p(x, y, z, t) \leftrightarrow P \). **Sol:**

\[
\nabla^2 p(x, y, z, t) = \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} p(x, y, z, t)
\]

**Vector fields and the \( \nabla \) operator**

**To Do:**

**Vector Algebra**

1. Let \( \mathbf{R}(x, y, z) \equiv x(t) \hat{x} + y(t) \hat{y} + z(t) \hat{z} \):

(a) If \( a, b, c \) are constants, what is \( \mathbf{R}(x, y, z) \cdot \mathbf{R}(a, b, c) \)? **Sol:** Using the formula for a scalar dot-product:

\[
\mathbf{R}(x, y, z) \cdot \mathbf{R}(a, b, c) \equiv [x(t) \hat{x} + y(t) \hat{y} + z(t) \hat{z}] \cdot [a \hat{x} + b \hat{y} + c \hat{z}] = x(t)a + y(t)b + z(t)c.
\]

(4.1)
(b) If \(a, b, c\) are constants, what is \(\frac{d}{dt} [\mathbf{R}(x, y, z) \cdot \mathbf{R}(a, b, c)]\)?

**Sol:** \(\left( a \frac{d}{dt} x(t) + b \frac{d}{dt} y(t) + c \frac{d}{dt} z(t) \right)\).

2. Find the divergence and curl of the following vector fields:

   (a) \(\mathbf{v} = \hat{x} + \hat{y} + 2\hat{z}\) **Sol:** \(\nabla \cdot \mathbf{v} = 0, \nabla \times \mathbf{v} = 0\)

   (b) \(\mathbf{v}(x, y, z) = x\hat{x} + xy\hat{y} + z^2\hat{z}\) **Sol:** \(\nabla \cdot \mathbf{v} \equiv \partial_x x + \partial_y xy + \partial_z z^2 = 1 + x + 2z \nabla \times \mathbf{v} \equiv \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ x & xy & z^2 \end{vmatrix} = (0 - 0)\hat{x} + (0 - 0)\hat{y} + (y - 0)\hat{z} = y\hat{z}\)

   (c) \(\mathbf{v}(x, y, z) = x\hat{x} + xy\hat{y} + \log(z)\hat{z}\) **Sol:** Divergence: \(\partial_x x + \partial_y xy + \partial_z \log(z) = 1 + x + 1/z\), Curl: \(\hat{x} (\partial_y \log(z) - \partial_x xy) + \hat{y} (\partial_z x - \partial_x \log(z)) + \hat{z} (\partial_x xy - \partial_y x) = 2\hat{y}\)

   (d) \(\mathbf{v}(x, y, z) = \nabla(1/x + 1/y + 1/z)\) **Sol:** First find \(\mathbf{v} = -(\hat{x}/x^2 + \hat{y}/y^2 + \hat{z}/z^2)\). Divergence of \(\mathbf{v} = -(\partial_x 1/x^2 + \partial_y 1/y^2 + \partial_z 1/z^2) = 2(1/x^3 + 1/y^3 + 1/z^3)\), Curl of \(\mathbf{v} = 0\), because the curl of the gradient is always zero.

**Vector & scalar field identities**

1. Find the divergence and curl of the following vector fields:

   (a) \(\mathbf{v} = \nabla \phi\), where \(\phi(x, y) = xe^y\) **Sol:** \(\nabla \times \nabla \phi = xe^y\), and \(\nabla^2 \phi = 0\)

   (b) \(\mathbf{v} = \nabla \times \mathbf{A}\), where \(\mathbf{A} = x\hat{x} + y\hat{y} + z\hat{z}\) **Sol:** \(\nabla \cdot (\nabla \times \mathbf{A}) = 0\), and \(\nabla \times (\nabla \times \mathbf{A}) = 0\)

   (c) \(\mathbf{v} = \nabla \times \mathbf{A}\), where \(\mathbf{A} = y\hat{x} + x^2\hat{y} + z\hat{z}\) **Sol:** \(\nabla \cdot (\nabla \times \mathbf{A}) = 0\), and \(\nabla \times (\nabla \times \mathbf{A}) = -2\hat{y}\)

2. For any differentiable vector field \(\mathbf{V}\), write down two vector-calculus identities that are equal to zero. **Sol:** Curl of the gradient \(\nabla \times \nabla \Phi(x, y, z) = 0\) and the divergence of the curl \(\nabla \cdot \nabla \times \mathbf{V}(x, y, z) = 0\) are both zero. (Page 780, Stillwell)

3. What is the most general form of a vector field may be expressed in, in terms of scalar \(\Phi\) and vector \(\mathbf{A}\) potentials? **Sol:** \(\mathbf{V} = \nabla \Phi(x, y, z) + \nabla \times \mathbf{A}(x, y, z)\), where \(\Phi\) is the scalar potential and \(\mathbf{A}\) is the vector potential.

4. Perform the following calculations. If you can state the answer without doing the calculation, explain why.

   (a) Let \(\mathbf{v} = \sin(x)\hat{x} + y\hat{y} + z\hat{z}\). Find \(\nabla \cdot (\nabla \times \mathbf{v})\) **Hint:** Look at Lec 41 on page 83 of the notes, Eq. 58, 59. **Sol:** 0

   (b) Let \(\mathbf{v} = \sin(x)\hat{x} + y\hat{y} + z\hat{z}\). Find \(\nabla \times (\nabla \sqrt{\nabla \cdot \mathbf{v}})\) **Sol:** 0

   (c) Let \(\mathbf{v}(x, y, z) = \nabla[x + y^2 + \sin(\log(z))].\) Find \(\nabla \times \mathbf{v}(x, y, z).\) **Sol:** It is zero because \(\nabla \times \nabla f(x, y, z)\) is always zero.

**Integral theorems**

1. In a few words, identify the law, define what it means, and explain the following formula:

   \[\int_S \hat{n} \cdot \mathbf{v}\ dA = \int_V \nabla \cdot \mathbf{v}\ dV.\]

   **Sol:** This is the integral form of Gauss’ law. The unit normal vector is \(\perp\) to the surface \(S\) having area \(A \equiv \int_S dA\). The integral represents the total flow normal to the surface. The surface integral is equal to the integral of the divergence of the vector field \(\nabla \cdot \mathbf{v}\) over the volume contained by the surface, and defined as \(V\).

2. What is the name of this formula?

   \[\int_S (\nabla \times \mathbf{V}) \cdot dS = \oint_C \mathbf{V} \cdot d\mathbf{R}\]

   Give one important application. **Sol:** Stokes Theorem, which relates the differential to the integral form of Maxwell’s equations.
3. Describe a key application of the vector identity
\[
\nabla \times (\nabla \times V) = \nabla (\nabla \cdot V) - \nabla^2 V.
\]

**Sol:** When we wish to reduce Maxwell’s two curl equations to the vector wave equation, we must use this identity.

**Schwarz inequality**

Below is a picture of these three vectors for an arbitrary value of \( a \) and a specific \( a = a^* \).

1. Find the value of \( a^* \in \mathbb{R} \) such that the length (norm) of \( E \) (i.e., \( ||E|| \geq 0 \)) is minimum? Hint minimize
\[
||E||^2 = E \cdot E = (V + aV) \cdot (V + aU) \geq 0 \tag{4.3}
\]
with respect to \( a \).

**Sol:** In Fig. 4.0.1 we see vectors \( V, U \), and for reference, \( V + 0.5U \). Also shown are scaled values of \( U, aU \) and \( a^*U \). These point in the same direction, but are shorter by amounts \( a \) and \( a^* \). When \( U \) is scaled by \( a^* \), length \( ||E(a^*)|| \) is minimum, and \( (V - a^*U) \perp U \), namely vector \( E(a^*) \) is \( \perp \) to vector \( U \). This follows from
\[
\frac{d}{da}||E||^2 = \frac{d}{da}((V + aU) \cdot (V + aU)) = 2(V + aU) \cdot U = 0.
\]
Thus
\[
a^* = -\frac{V \cdot U}{||U||^2}
\]

2. Find the formula for \( ||E(a^*)||^2 \geq 0 \). Hint: Substitute \( a^* \) into Eq. 4.3, and show that this results in the **Schwarz inequality**
\[
||U \cdot V| \leq ||U|| ||V||.
\]

**Sol:** From Eq. 4.3
\[
||V||^2 + 2a^*V \cdot U + (a^*)^2 ||U||^2 \geq 0
\]
Substituting \( a^* \) gives
\[
||V||^2||U||^2 - 2(V \cdot U)^2 + ||U||^2 \geq 0.
\]
Simplifying
\[
||V||^2||U||^2 \geq ||U \cdot V||^2
\]
and taking the square root (and swap order), gives the **Schwarz inequality**
\[
||U \cdot V| \leq ||U|| ||V||.
\]

3. What is the geometrical meaning of the dot product of two vectors? **Sol:** The dot product of two vectors is the length of the \( \perp \) projection of one vector on the other. According to the Schwarz inequality, this project length must be less than the product of the lengths of the two vectors.

4. Give the formula for the dot product between two vectors. Explain the meaning based on Fig. 4.0.1. **Sol:** \( V \cdot U = ||V|| ||U|| \cos \theta_{V,U} \). It represents the amount of one vector going in the direction of the other. In a drawing, it is a projection of the one on the other, found by dropping the \( \perp \) from the tip of one, on the other.
5. Write the formula for the “dot product” between two vectors: \( \mathbf{U} \cdot \mathbf{V} \) in \( \mathbb{R}^n \) in polar form (e.g., assume the angle between the vectors is equal to \( \theta \)). \( \text{Sol:} \quad \mathbf{U} \cdot \mathbf{V} = \sum_{i=1}^{n} a_i b_i (= ||\mathbf{U}|| \ ||\mathbf{V}|| \cos(\theta)). \)

   This last relationship defines the angle between two vectors.

6. How is this related to the Pythagorean theorem? \( \text{Sol:} \quad \) It says that for a right triangle, the case when \( a = a^* \), the lengths of the two vectors must be greater than the projection of one on the other, unless they are co-linear (i.e., the angle between them is zero).

7. Starting from \( ||\mathbf{U} + \mathbf{V}|| \) derive the triangle inequality

   \[ ||\mathbf{U} + \mathbf{V}|| \leq ||\mathbf{U}|| + ||\mathbf{V}||. \]

   \( \text{Sol:} \quad ||\mathbf{U} + \mathbf{V}||^2 = (\mathbf{U} + \mathbf{V}) \cdot (\mathbf{U} + \mathbf{V}) = ||\mathbf{U}||^2 + ||\mathbf{V}||^2 + 2 \mathbf{U} \cdot \mathbf{V} \leq ||\mathbf{U}||^2 + ||\mathbf{V}||^2 + 2 ||\mathbf{U}|| \ ||\mathbf{V}||. \)

   Using the Schwarz inequality we find \( ||\mathbf{U} + \mathbf{V}||^2 \leq (||\mathbf{U}|| + ||\mathbf{V}||)^2 \). Final taking the square root gives the triangle inequality.

8. The triangular inequality \( ||\mathbf{U} + \mathbf{V}|| \leq ||\mathbf{U}|| + ||\mathbf{V}|| \) is true for 2 and 3 dimensions: Does it hold for 5 dimensional vectors? \( \text{Sol:} \quad \) It is true in any number of dimensions.

### Quadratic forms

A matrix that has positive eigenvalues is said to be positive-definite. The eigenvalues are real if the matrix is symmetric, so this is a necessary condition for the matrix to be positive-definite. This condition is related to conservation of energy since the power is the voltage times the current. Given an impedance matrix

\[ \mathbf{V} = \mathbf{ZI}, \]

the power \( \mathcal{P} \) is

\[ \mathcal{P} = \mathbf{I} \cdot \mathbf{V} = \mathbf{I} \cdot \mathbf{ZI}, \]

which must be positive definite for the system to obey conservation of energy. For the following problems, consider the \( 2 \times 2 \) \( \mathbf{Z} \) matrix

\[ \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}. \]

1. Solve for the power \( \mathcal{P}(i_1, i_2) \) by multiplying out the matrix equation below (which is in quadratic form) \( (\mathbf{I} \equiv \begin{bmatrix} i_1 & i_2 \end{bmatrix}^T) \)

   \[ \mathcal{P}(i_1, i_2) = \mathbf{I}^T \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \mathbf{I}. \]

   \( \text{Sol:} \quad \mathcal{P}(i_1, i_2) = \begin{bmatrix} i_1 & i_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = 2i_1^2 + 2i_1i_2 + 4i_2^2. \)

2. Is the impedance matrix positive definite? Show your work by finding the eigenvalues of the matrix \( \mathbf{Z} \). \( \text{Sol:} \quad \) Yes, as it is positive definite if the eigenvalues are both positive. You need to show that the eigenvalues are positive (not zero or negative). They are, so it is. How to do all this is worked on in Example 3, page 593.

\[ \begin{vmatrix} 2 - \lambda & 1 \\ 1 & 4 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda = 3 \pm \sqrt{2} > 0 \]

3. Should an impedance matrix always be positive definite? Explain. \( \text{Sol:} \quad \) Yes.
System Classification

Provide a one-sentence definition of the following properties:

L/NL : linear(L)/nonlinear(NL): **Sol:** Superposition and scaling hold

TI/TV : time-invariant(TI)/time varying(TV): **Sol:** The measurement time is irrelevant

P/A : passive(P)/active(A): **Sol:** An active system has a power source, a passive system does not.

C/NC : causal(C)/non-causal(NC): **Sol:** Responds only upon or after being driven.

Re/Clx : real(Re)/complex(Clx): **Sol:** The time function is real (or complex).

1. Along the rows of the table, classify the following systems: In terms of a table having 5 columns, labeled with the abbreviations: L/NL, TI/TV, P/A, C/NC, Re/Clx:

<table>
<thead>
<tr>
<th>#</th>
<th>Case:</th>
<th>L/NL</th>
<th>TI/TV</th>
<th>P/A</th>
<th>C/NC</th>
<th>Re/Clx</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Resistor $v(t) = r_0 i(t)$</td>
<td>Sol: L</td>
<td>Sol: TI</td>
<td>Sol: P</td>
<td>Sol: C</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>2</td>
<td>Inductor $v(t) = L \frac{di}{dt}$</td>
<td>Sol: L</td>
<td>Sol: TI</td>
<td>Sol: P</td>
<td>Sol: C</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>3</td>
<td>Switch $v(t) = \begin{cases} 0 &amp; t \leq 0 \ V_0 &amp; t &gt; 0 \end{cases}$</td>
<td>Sol: L</td>
<td>Sol: TV</td>
<td>Sol: P</td>
<td>Sol: C</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>5</td>
<td>Transistor $I_{out} = g_m (V_{in})$</td>
<td>Sol: NL</td>
<td>Sol: TI</td>
<td>Sol: P</td>
<td>Sol: C</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>7</td>
<td>“Resistor” $v(t) = r_0 i(t + 3)$</td>
<td>Sol: L</td>
<td>Sol: TI</td>
<td>Sol: P</td>
<td>Sol: NC</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>8</td>
<td>modulator $f(t) = e^{i2\pi t} g(t)$</td>
<td>Sol: L</td>
<td>Sol: TV</td>
<td>Sol: P</td>
<td>Sol: C</td>
<td>Sol: Clx</td>
</tr>
</tbody>
</table>

**Sol:** Notes:

- is a nonlinear system and is active system only when it is connected to a battery, similar to a diode.
- The current is non-causal since it has a 3 [s] negative time delay, specified in the time domain.
- is 1 Hz complex-modulation, so it is both complex and time-varying (TV)

2. Using the same classification scheme, characterize the following equations:

<table>
<thead>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A(x) \frac{dy(t)}{dx} + D(t) y(x, t) = 0$</td>
<td>Sol: L</td>
<td>Sol: TV</td>
<td>Sol: P</td>
<td>Sol: C</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{dy(t)}{dt} + \sqrt{t} y(t) = \sin(t)$</td>
<td>Sol: L</td>
<td>Sol: TV</td>
<td>Sol: ?</td>
<td>Sol: C</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>3</td>
<td>$y^2(t) + y(t) = \sin(t)$</td>
<td>Sol: NL</td>
<td>Sol: TI</td>
<td>Sol: ?</td>
<td>Sol: C</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{dy(t)}{dt} + x y(t + 1) + x^2 y = 0$</td>
<td>Sol: L</td>
<td>Sol: TI</td>
<td>Sol: P</td>
<td>Sol: NC</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{dy(t)}{dt} + (t - 1) y^2(t) = ie^t$</td>
<td>Sol: NL</td>
<td>Sol: TV</td>
<td>Sol: A?</td>
<td>Sol: C</td>
<td>Sol: Clx</td>
</tr>
</tbody>
</table>