1 Exercises AE-1

**Topic of this homework:** Fundamental theorem of algebra, polynomials, analytic functions and their inverse, convolution, roots.

**Deliverable:** Answers to problems

*Note: The term ‘analytic’ is used in two different ways. (1) An analytic function is a function that may be expressed as a locally convergent power series; (2) analytic geometry refers to geometry using a coordinate system.*

**Polynomials and the fundamental theorem of algebra (FTA) (6pt)**

**Problem # 1:** (2pt) A polynomial of degree $N$ is defined as

$$P_N(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N$$

- 1.1: (1pt) How many coefficients $a_n$ does a polynomial of degree $N$ have?

- 1.2: (1pt) How many roots does $P_N(x)$ have?

**Problem # 2:** (2pt) The fundamental theorem of algebra (FTA)

- 2.1: (1pt) State and then explain the Fundamental Theorem of Algebra.

- 2.2: (1pt) Using the FTA, prove your answer to Q 1.2.

*Hint: Apply the FTA to prove how many roots a polynomial $P_N(x)$ of order $N$ has.*

**Problem # 3:** (1pt) Consider the polynomial function $P_2(x) = 1 + x^2$ of degree $N = 2$, and its reciprocal $F(x) = 1/P_2(x)$.

- 3.1: (1pt) What are the roots (e.g. ‘zeros’) $x_{\pm}$ of $P_2(x)$?

**Problem # 4:** (1pt) $F(x) = 1/P_2(x)$ may be expressed as $(A, B, x_{\pm} \in \mathbb{C})$

$$F(x) = \frac{A}{x - x_+} + \frac{B}{x - x_-}, \quad (1.1)$$

where $x_{\pm}$ are the roots (zeros) of $P_2(x)$, which become the poles of $F(x)$, and $A, B$ are the residues. The expression for $F(x)$ is sometimes called a ‘partial fraction expansion’ or ‘residue expansion,’ and it appears frequently in engineering applications.

- 4.1: (1pt) Find $A, B \in \mathbb{C}$ in terms of the roots $x_{\pm}$ of $P_2(x)$.
Analytic functions (13 pt)

A classic series is the geometric series

\[ \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n, \]  

(1.2)

with Taylor coefficients \( a_n = 1 \).

**Problem # 5:** (5 pt) The geometric series

- 5.1: (1pt) What is the region of convergence (RoC) for the power series of \( 1/(1-x) \) given above? Namely, where does the power series \( P(x) \) converge to \( 1/(1-x) \)? State your answer as a condition on \( x \).

- 5.2: (1pt) How does the RoC relate to the location of the pole of \( 1/(1-x) \)?

- 5.3: (1pt) Where are the zeros, if any, in Eq. 1.2?

- 5.4: (1pt) Assuming \( x \) is in the RoC, prove that the geometric series correctly represents \( 1/(1-x) \), by multiplying both sides of Eq. 1.2 by \( 1-x \).

- 5.5: (1pt) Describe the Taylor series having expansion point \( x_0 = \infty \).

**Problem # 6:** (5pt) We may use the geometric series to study the polynomial

\[ P_N(x) = 1 + x + x^2 + \ldots + x^N = \sum_{n=0}^{N} x^n. \]  

(1.3)

- 6.1: (1pt) What is the RoC for Eq. 1.3?

- 6.2: (1pt) Does Eq. 1.3 have both poles and zeros? Explain.

- 6.3: (1pt) Prove that

\[ P_N(x) = \frac{1-x^{N+1}}{1-x} \]  

(1.4)

- 6.4: (1pt) What is the RoC for Eq. 1.4?

- 6.5: (1pt) Is the function \( 1/(1-x) \) analytic outside of the RoC?

**Problem # 7:** (3 pt) The exponential series is

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]  

(1.5)

with Taylor coefficients \( a_n = 1/n! \), which may be derived from the Taylor formula.
– 7.1:(1pt) What is the region of convergence (RoC) for the exponential series given above (e.g. where does the power series \( P(x) \) converge to the function value \( f(x) \))?

– 7.2:(1pt) What is the RoC for Eq. 1.5?

– 7.3:(1pt) Let \( x = j \) in Eq. 1.5, and write out the series expansion of \( e^x \) in terms of its real and imaginary parts.

**Inverse analytic functions and composition (8 pt)**

**Overview:** It may be surprising, but every analytic function has an inverse function. Starting from the function \((x, y) \in \mathbb{C}\)

\[
y(x) = \frac{1}{1 - x}
\]

the inverse is

\[
x = \frac{y - 1}{y} = 1 - \frac{1}{y}.
\]

**Problem # 8:(2 pt) Consider the inverse function described above.**

– 8.1:(1pt) Where are the poles and zeros of \( x(y) \)?

– 8.2:(1pt) Where (for what condition on \( y \)) is \( x(y) \) analytic?

**Problem # 9:(4 pt) Consider the exponential function \( z(s) = e^s \) \( (s, z) \in \mathbb{C}) \).**

– 9.1:(1pt) Find the inverse \( s(z) \).

– 9.2:(1pt) Next define \( y(s) = 1/(1 - s) \) and \( z(s) = e^s \) \((s = \sigma + \omega j) \in \mathbb{C}) \). Compose these two functions (i.e., evaluate \((y \circ z)(s)\))

– 9.3:(3pt) Where are the poles and zeros of \((y \circ z)(x)\)?

– 9.4:(1pt) Where (for what condition on \( x \)) is \((y \circ z)(x)\) analytic?

**Convolution (2pt)**

Multiplying two polynomials, when they are short or simple, is not demanding. However if they have many terms, it can become tedious. For example, multiplying two 10\(^{th}\) degree polynomials is not trivial. An alternative is a method called convolution.

**Problem # 10:(1pt) Convolution of sequences.**

– 10.1:(1pt) Calculate \(\{1, 1\} \ast \{1, 1\} \ast \{1, 1\}\)
Problem #11: (1pt) Multiplying two polynomials is the same as convolving their coefficients. Let
\[ f(x) = x^3 + 3x^2 + 3x + 1 \]
\[ g(x) = x^3 + 2x^2 + x + 2 \]

– 11.1: (1pt) Use convolution to find \( h(x) = f(x) \cdot g(x) \) ?

Newton’s root-finding method (6 pt)

Newton’s method provides and iterative algorithm to find the roots of any polynomial \( P_N(s) \) where \( s \in \mathbb{C} \), of the form
\[ s_{n+1} = s_n - \frac{P_N(s_n)}{P_N'(s_n)}, \]
where \( P_N'(s) = \frac{d}{ds} P_N(s), s_n \in \mathbb{C} \) and \( n, N \in \mathbb{N} \).

Problem #12: (6 pt) Use Newton’s iteration to find roots of the polynomial
\[ P_3(x) = 1 - x^3. \]

– 12.1: (1pt) Starting with \( x_0 = j3/2 \), describe the first two steps of the iteration.

Hint: Start with the complex plane (as the coordinate system) and label (plot) the poles and zeros of the “update term” (on far right).

– 12.2: (3pt) Calculate \( x_1 \) and \( x_2 \). What root is the iteration approaching?

– 12.3: (2pt) Does Newton’s method work for \( P_2(x) = 1 + x^2 \)? If so, why? Hint: What are the roots in this case?

Riemann zeta function \( \zeta(s) \)

Definitions and preliminary analysis:

The zeta function \( \zeta(s) \) is defined by the complex analytic power series
\[ \zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots. \] (1.6)

This series converges, and thus is valid, only in the region of convergence (ROC) given by \( \Re s = \sigma > 1 \) since there \( |n^{-\sigma}| < 1 \). To determine its formula in other regions of the \( s \) plane one must extend the series via analytic continuation.

Euler product formula: As was first published by Euler in 1737, one may recursively factor out the leading prime term, resulting in Euler’s product formula.\(^1\) Multiplying \( \zeta(s) \) by the factor \( 1/2^s \), and subtracting from \( \zeta(s) \), removes all the terms \( 1/(2n)^s \) (e.g., \( 1/2^s + 1/4^s + 1/6^s + 1/8^s + \cdots \))
\[ \left(1 - \frac{1}{2^s}\right) \zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \cdots - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \cdots\right), \] (1.7)
which results in
\[ \left(1 - \frac{1}{2^s}\right) \zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \cdots. \] (1.8)
\(^1\)This is known as Euler’s sieve, as distinguish from the Eratosthenes sieve.
Problem # 13:

– 13.1: What is the RoC for Eq. 1.8

– 13.2: Repeat this with a lead factor $1/3^s$ applied to Eq. 1.8.

– 13.3: What is the RoC for Eq. ??

– 13.4: Repeat this process, with all prime scale factors (i.e., $1/5^s, 1/7^s, \ldots, 1/\pi_k^s, \ldots$), and show that

$$
\zeta(s) = \prod_{\pi_k \in P} \frac{1}{1 - \pi_k^{-s}} = \prod_{\pi_k \in P} \zeta_k(s)
$$

where $\pi_p$ represents the $p^{th}$ prime.

– 13.5: Given the product formula we may identify the poles of $\zeta_p(s)$ ($p \in \mathbb{Z}$), which is important for defining the ROC of each factor.

For example, the $p^{th}$ factor of Eq. 1.9, expressed as an exponential, is

$$
\zeta_p(s) \equiv \frac{1}{1 - \pi_p^{-s}} = \frac{1}{1 - e^{-s T_p}},
$$

where $T_p \equiv \ln \pi_p$.

– 13.6: Plot $\zeta_p(s)$ using *zviz* for $p = 1$. Describe what you see.