1 Exercises AE-1

Topic of this homework: Fundamental theorem of algebra, polynomials, analytic functions and their inverse, convolution, roots.

Deliverable: Answers to problems

Note: The term 'analytic' is used in two different ways. (1) An analytic function is a function that may be expressed as a locally convergent power series; (2) analytic geometry refers to geometry using a coordinate system.

Polynomials and the fundamental theorem of algebra (FTA) (6pt)

Problem *# 1:(2pt) A polynomial of degree N is defined as*

$$P_N(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N$$

– *1.1:(1pt) How many coefficients a_n does a polynomial of degree N have?*

– *1.2:(1pt) How many roots does $P_N(x)$ have?*

Problem *# 2:(2pt) The* fundamental theorem of algebra *(FTA)*

– *2.1:(1pt) State and then explain the* Fundamental Theorem of Algebra.

– *2.2:(1pt) Using the FTA,* prove *your answer to Q 1.2.*

Hint: Apply the FTA to *prove* how many roots a polynomial $P_N(x)$ of order N has.

Problem *# 3:(1pt) Consider the polynomial function $P_2(x) = 1 + x^2$ of degree $N = 2$, and its reciprocal $F(x) = 1/P_2(x)$.*

– *3.1:(1pt) What are the roots (e.g. 'zeros') $x_\pm$ of $P_2(x)$?*

Problem *# 4:(1pt) $F(x) = 1/P_2(x)$ may be expressed as ($A, B, x_\pm \in \mathbb{C}$)*

$$F(x) = \frac{A}{x - x_+} + \frac{B}{x - x_-}, \tag{1.1}$$

where $x_\pm$ are the roots (zeros) of $P_2(x)$, which become the *poles* of $F(x)$, and A, B are the *residues*. The expression for $F(x)$ is sometimes called a 'partial fraction expansion' or 'residue expansion,' and it appears frequently in engineering applications.

– *4.1:(1pt) Find $A, B \in \mathbb{C}$ in terms of the roots $x_\pm$ of $P_2(x)$.*

Analytic functions (13 pt)

A classic series is the *geometric series*

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n, \tag{1.2}$$

with Taylor coefficients $a_n = 1$.

Problem *# 5:(5 pt) The geometric series*

– 5.1:(1pt) What is the region of convergence *(RoC) for the power series of $1/(1-x)$ given above Namely, where does the power series $P(x)$ converge to $1/(1-x)$? State your answer as a condition on x.*

– 5.2:(1pt) How does the RoC relate to the location of the pole of $1/(1-x)$?

– 5.3:(1pt) Where are the zeros, if any, in Eq. 1.2?

– 5.4:(1pt) Assuming x is in the RoC, prove that the geometric series correctly represents $1/(1-x)$, by multiplying both sides of Eq. 1.2 by $(1-x)$.

– 5.5:(1pt) Describe the Taylor series having expansion point $x_0 = \infty$.

Problem *# 6:(5pt) We may use the geometric series to study the polynomial*

$$P_N(x) = 1 + x + x^2 + \ldots + x^N = \sum_{n=0}^{N} x^n. \tag{1.3}$$

– 6.1:(1pt) What is the RoC for Eq. 1.3?

– 6.2:(1pt) Does Eq. 1.3 have both poles and zeros? Explain.

– 6.3:(1pt) Prove that

$$P_N(x) = \frac{1 - x^{N+1}}{1-x} \tag{1.4}$$

– 6.4:(1pt) What is the RoC for Eq. 1.4?

– 6.5:(1pt) Is the function $1/(1-x)$ analytic outside of the RoC?

Problem *# 7:(3 pt) The exponential series is*

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{1.5}$$

with Taylor coefficients $a_n = 1/n!$, which may be derived from the Taylor formula.

– 7.1:(1pt) What is the region of convergence (RoC) for the exponential series given above (e.g. where does the power series $P(x)$ converge to the function value $f(x)$)?

– 7.2:(1pt) What is the RoC for Eq. 1.5?

– 7.3:(1pt) Let $x = \jmath$ in Eq. 1.5, and write out the series expansion of e^x in terms of its real and imaginary parts.

Inverse analytic functions and composition (8 pt)

Overview: It may be surprising, but every analytic function has an inverse function. Starting from the function ($x, y \in \mathbb{C}$)

$$y(x) = \frac{1}{1-x}$$

the inverse is

$$x = \frac{y-1}{y} = 1 - \frac{1}{y}.$$

Problem *# 8:(2 pt) Consider the inverse function described above.*

– 8.1:(1pt) Where are the poles and zeros of $x(y)$?

– 8.2:(1pt) Where (for what condition on y) is $x(y)$ analytic?

Problem *# 9:(4 pt) Consider the exponential function $z(s) = e^s$ ($s, z \in \mathbb{C}$).*

– 9.1:(1pt) Find the inverse $s(z)$.

– 9.2:(1pt) Next define $y(s) = 1/(1-s)$ and $z(s) = e^s$ ($s = \sigma + \omega\jmath \in \mathbb{C}$). Compose these two functions (i.e., evaluate $(y \circ z)(s)$)

– 9.3:(3pt) Where are the poles and zeros of $(y \circ z)(x)$?

– 9.4:(1pt) Where (for what condition on x) is $(y \circ z)(x)$ analytic?

Convolution (2pt)

Multiplying two polynomials, when they are short or simple, is not demanding. However if they have many terms, it can become tedious. For example, multiplying two 10^{th} degree polynomials is not trivial. An alternative is a method called convolution.

Problem *# 10:(1pt) Convolution of sequences.*

– 10.1:(1pt) Calculate $\{1, 1\} \star \{1, 1\} \star \{1, 1\}$

Problem *# 11:(1pt) Multiplying two polynomials is the same as convolving their coefficients. Let*

$$f(x) = x^3 + 3x^2 + 3x + 1$$
$$g(x) = x^3 + 2x^2 + x + 2$$

– 11.1:(1pt) Use convolution to find $h(x) = f(x) \cdot g(x)$?

Newton's root-finding method (6 pt)

Newton's method provides and iterative algorithm to find the roots of any polynomial $P_N(s)$ where $s \in \mathbb{C}$, of the form

$$s_{n+1} = s_n - \frac{P_N(s_n)}{P'_N(s_n)},$$

where $P'_N(s) = \frac{d}{ds}P_N(s)$, $s_n \in \mathbb{C}$ and $n, N \in \mathbb{N}$.

Problem *# 12: (6 pt) Use Newton's iteration to find roots of the polynomial*

$$P_3(x) = 1 - x^3.$$

– 12.1:(1pt) Starting with $x_0 = \jmath 3/2$, describe the first two steps of the iteration.

Hint: Start with the complex plane (as the coordinate system) and label (plot) the poles and zeros of the "update term" (on far right).

– 12.2:(3pt) Calculate x_1 and x_2. What root is the iteration approaching?

– 12.3:(2pt) Does Newton's method work for $P_2(x) = 1 + x^2$?

If so, why? Hint: What are the roots in this case?

Riemann zeta function $\zeta(s)$

Definitions and preliminary analysis:

The zeta function $\zeta(s)$ is defined by the complex analytic power series

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots. \tag{1.6}$$

This series converges, and thus is valid, only in the region of convergence (ROC) given by $\Re s = \sigma > 1$ since there $|n^{-\sigma}| < 1$. To determine its formula in other regions of the s plane one must extend the series via *analytic continuation*.

Euler product formula: As was first published by Euler in 1737, one may recursively factor out the leading prime term, resulting in Euler's product formula.[1] Multiplying $\zeta(s)$ by the factor $1/2^s$, and subtracting from $\zeta(s)$, removes all the terms $1/(2n)^s$ (e.g., $1/2^s + 1/4^s + 1/6^s + 1/8^s + \cdots$)

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} \cdots - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \cdots\right), \tag{1.7}$$

which results in

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \cdots. \tag{1.8}$$

[1] This is known as *Euler's sieve*, as distinguish from the *Eratosthenes sieve*.

Problem *# 13:*

– 13.1: What is the RoC for Eq. 1.8

– 13.2: Repeat this with a lead factor $1/3^s$ applied to Eq. 1.8.

– 13.3: What is the RoC for Eq. **??**

– 13.4: Repeat this process, with all prime scale factors (i.e., $1/5^s, 1/7^s, \cdots, 1/\pi_k^s, \cdots$), and show that

$$\zeta(s) = \prod_{\pi_k \in \mathbb{P}} \frac{1}{1-\pi_k^{-s}} = \prod_{\pi_k \in \mathbb{P}} \zeta_k(s) \tag{1.9}$$

where π_p represents the p^{th} prime.

– 13.5: Given the product formula we may identify the poles of $\zeta_p(s)$ ($p \in \mathbb{Z}$), which is important for defining the ROC of each factor.
For example, the p^{th} factor of Eq. 1.9, expressed as an exponential, is

$$\zeta_p(s) \equiv \frac{1}{1-\pi_p^{-s}} = \frac{1}{1-e^{-sT_p}}, \tag{1.10}$$

where $T_p \equiv \ln \pi_p$.

– 13.6: Plot $\zeta_p(s)$ using `zviz` for $p = 1$. Describe what you see.