1 Problems DE-1

Topics of this homework:

Complex Taylor Series; quadratic forms, complex numbers and functions (ordering and algebra), Cauchy-Riemann conditions, multivalued functions (branch cuts and Riemann sheets), Cauer synthesis

Complex Power Series

Problem *# 1: (6 pts) In each case derive the power series of $w(s)$ about $s = 0$ and state the ROC of the series, and the residues. If the power series doesn't exist, state why!* Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at $s = 0$.

– *1.1(1pt):* $1/(1 - s^2)$

– *1.2(1pt):* $1/(1 - |s|^2)$

– *1.3(2pt): Find the RoC for the Riemann zeta function $\zeta(s)$*

Given the definition of the $\zeta(s)$ function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + \cdots \quad \text{for } \Re s = \sigma > 1.,$$

where $n \in \mathbb{N}$ and $s = \sigma + \jmath\omega$ (s is the Laplace frequency). As stated, the RoC is the RHP $\Re s = \sigma > 1$.

– *1.4(2pt): Derive the first term of Euler's product formula by removing all the even terms (multiples of 2)*

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = \frac{1}{1^s} + \cancelto{0}{\frac{1}{2^s}} + \frac{1}{3^s} + \cancelto{0}{\frac{1}{4^s}} + \frac{1}{5^s} + \cancelto{0}{\frac{1}{6^s}} + \frac{1}{7^s} + \cancelto{0}{\frac{1}{8^s}} + \frac{1}{9^s} + \cancelto{0}{\frac{1}{10^s}} + \frac{1}{11^s} + \cancelto{0}{\frac{1}{12^s}} + \cdots \quad \text{for } \Re s = \sigma > 1,$$

results in the first term in *Euler's product formula*, for the zeta function.

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = \frac{1}{1^s} + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{10^s} + \frac{1}{11^s} + \cdots \quad \text{for } \Re s = \sigma > 1.$$

Show how to modify the formula to remove all the multiples of 3.

Problem *# 2(2 pt): Consider the function $w(s) = 1/s$*

– *2.1(1pt): Expand this function as a power series about $s = i$.*

– *2.2(1pt): What is the residue of the pole?*

Problem *# 3(1 pt): Consider the function* $w(s) = 1/(2-s)$

– 3.1(1pt): Expand $w(s)$ *as a power series in* $s^{-1} = 1/s$*. State the ROC as a condition on* $|s^{-1}|$. Hint: Multiply top and bottom by s^{-1}.

Problem *# 4(2 pt):Summing the series*
Taylor series of functions have more than one region of convergence.

– 4.1(1pt): If $a = 0.1$ *what is the value of*

$$x = 1 + a + a^2 + a^3 \cdots ?$$

Show your work.

– 4.2(1pt): If $a = 10$ *what is the value of*

$$x = 1 + a + a^2 + a^3 \cdots ?$$

Quadratic forms

A matrix that has positive eigenvalues is said to be positive-definite. The eigenvalues are real if the matrix is symmetric, so this is a necessary condition for the matrix to be positive-definite. This condition is related to conservation of energy, since the power is the voltage times the current. Given an impedance matrix

$$\mathbf{V} = \mathbf{Z}\mathbf{I},$$

the power $\mathcal{P}$ is

$$\mathcal{P} = \mathbf{I} \cdot \mathbf{V} = \mathbf{I} \cdot \mathbf{Z}\mathbf{I},$$

which must be positive-definite for the system to obey conservation of energy.

Problem *# 5: In this problem, consider the* 2×2 *impedance matrix*

$$\mathbf{Z} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}.$$

– 5.1: Solve for the power $\mathcal{P}(i_1, i_2)$ *by multiplying out this matrix equation (which is in* quadratic form*) (*$\mathbf{I} \equiv \begin{bmatrix} i_1 & i_2 \end{bmatrix}^T$*):*

$$\mathcal{P}(i_1, i_2) = \mathbf{I}^T \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \mathbf{I}.$$

– 5.2: Is the impedance matrix positive-definite? Show your work by finding the eigenvalues of the matrix $\mathbf{Z}$*.*

– 5.3: Should an impedance matrix always be positive-definite? Explain.

Cauchy-Riemann Equations

For the following problem: $i = \sqrt{-1}$, $s = \sigma + i\omega$, and $F(s) = u(\sigma, \omega) + iv(\sigma, \omega)$.

Problem # 6(2pt): *According to the* Fundamental theorem of complex calculus *the integration of a complex analytic function is independent of the path.*

If the integral is independent of the path, then the derivative must also be independent of direction

$$\frac{dF}{ds} = \frac{\partial F}{\partial \sigma} = \frac{\partial F}{\partial \jmath\omega}. \tag{DE-1.1}$$

– 6.1: (2 pts) Apply the CR equations to the following functions. State for which values of $s = \sigma + i\omega$ the CR conditions do or do not hold (e.g. where the function $F(s)$ is or is not analytic).

1. (2pts) $F(s) = e^s$

Branch cuts and Riemann sheets

Problem # 7: *(4pts) Consider the function $w^2(z) = z$. This function can also be written as $w(z) = \sqrt{z_\pm}$. Define $z_+ = re^{\phi\jmath}$, $z_- = re^{\jmath(\phi+2\pi)}$ and $w(z) = \rho e^{\theta\jmath} = \sqrt{r}e^{\jmath\phi}$.*

– 7.1: (1pt) How many Riemann sheets do you need in the domain *(z) and the* range *(w) to fully represent this function as single valued?*

– 7.2: (2pt) Indicate (e.g. using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.

– 7.3: (1pt) Can a branch cut be moved?

Problem # 8: *(1pt) Consider the function $w(z) = \log(z)$. As before define $z = re^{\phi\jmath}$ and $w(z) = \rho e^{\theta\jmath}$.*

– 8.1: (1pt) What is the inverse function $z(w)$? Does this function have a branch cut (if so, where is it)?

A Cauer synthesis of any Brune impedance

Problem # 9: *One may synthesize a transmissison line (ladder network) from a Brune (positive real) impedance $Z(s)$ by using the continued fraction method. To obtain the series and shunt impedance values, one may use residue expansion.*

– 9.1: Starting from the Brune impedance $Z(s) = \frac{1}{s+1}$, find the impedance network as a ladder network.

– 9.2: Use a residue expansion to mimic the CFA floor function for polynomial expansions. Find the residue expansion of $H(s) = s^2/(s+1)$ and express it as a ladder netwrok.

– 9.3

Discuss how the series impedance $Z(s, x)$ and shunt admittance $Y(s, x)$ determine the wave velocity $\kappa(s, x)$ and the characteristic impedance $z_o(s, x)$ when

1. $Z(s)$ and $Y(s)$ are both independent of x
2. $Z(s, x)$ and $Y(s)$ ($Y(s)$ is independent of x, $Z(s, x)$ depends on x)
3. $Z(s)$ and $Y(s, x)$ ($Z(s)$ is independent of x, $Y(s, x)$ depends on x)
4. $Z(s, x)$ and $Y(s, x)$ (both $Y(s, x), Z(s, x)$ depend on x)

This shows that a Cauer synthesis may be implemented with the residue expansion replacing the floor function in the CFA. This seems to solve Burne's network synthesis problem.