1 Exercises DE-2

Topic of this assignment:
Integration of complex functions; Cauchy’s theorem, integral formula, residue theorem; power series; Riemann sheets and branch cuts; inverse Laplace transforms

Fundamental theorem of complex calculus (?? pts)

Problem # 1: (7 pts)
FTCC and integration in the complex plane
According to the Fundamental Theorem of Complex Calculus (FTCC)

\[ f(z) = f(z_0) + \int_{z_0}^{z} F(\zeta) d\zeta, \]  \hspace{1cm} (1.1)

where \( z_0, z, \zeta, F \in \mathbb{C} \). It follows that

\[ F(z) = \frac{d}{dz} f(z). \]  \hspace{1cm} (1.2)

Eq. 1.5 is thus known as the \textit{anti-derivative} of \( f(z) \).

- 1.1: (1 pts)
For a closed interval \( \{a, b\} \), the FTCC can be stated as

\[ \int_{a}^{b} F(z) dz = f(b) - f(a), \]  \hspace{1cm} (1.3)

meaning that the result of the integral is independent of the path from \( x = a \) to \( x = b \). What condition(s) on the integrand \( f(z) \) is (are) sufficient to assure that Eq. 1.3 holds?

- 1.2: (1 pts)
For the function \( f(z) = c^z \), where \( c \in \mathbb{C} \) is an arbitrary complex constant, use the Cauchy-Riemann (CR) equations to show that \( f(z) \) is analytic for all \( z \in \mathbb{C} \).

Problem # 2: (5 pts)
In the following problems, find the integral

\[ I = \int_{C} F(z) dz \]

for the given path \( C \). In some cases this might be the \textit{definite} integral (e.g., Eq. 1.3).

Let the function \( F(z) = c^z \), where \( c \in \mathbb{C} \) is given for each problem below. \textit{Hint: Can you apply the FTCC?}
- 2.1: (1 pts)
Find the anti-derivative of $F(z)$.

- 2.2: (1 pts)
$c = 1/e = 1/2.7183\ldots$ where $C$ is $\zeta = 0 \to i \to z$

- 2.3: (1 pts)
$c = 2$ where $C$ is $\zeta = 0 \to (1 + i) \to z$

- 2.4: (1 pts)
$c = i$ where the path $C$ is an inward directed spiral described by $z(t) = 0.99^t e^{i2\pi t}$, for $t$ from 0 to $\infty$.

- 2.5: (1 pts)
$c = e^{t-\tau_0}$ where $\tau_0 > 0$ is a real number, and $C$ is $z = (1 - i\infty) \to (1 + i\infty)$. Hint: Do you recognize this integral? If not, I suggest you ignore the problem.
Two fundamental theorems of calculus

**Fundamental Theorem of Calculus (Leibniz):**

According to the Fundamental Theorem of (Real) Calculus (FTC)

\[
f(x) = f(a) + \int_a^x F(\xi) d\xi,
\]

where \( x, a, \xi, F, f \in \mathbb{R} \).

**Fundamental Theorem of Complex Calculus:**

According to the Fundamental Theorem of Complex Calculus (FTCC)

\[
f(z) = f(z_0) + \int_{z_0}^z F(\zeta) d\zeta,
\]

where \( z_0, z, \zeta, f, F \in \mathbb{C} \).

**Problem # 3: (2pts) FTC and FTCC**

- **3.1: (1 pts)**
  Consider Equation 1.4. What is the condition on \( F(x) \) for which this formula is true?

- **3.2: (1 pts)**
  Consider Equation 1.5. What is the condition on \( F(\zeta) \) for which this formula is true?
Problem # 4(12 pts): In the following problems, solve the integral
\[ I = \int_C F(z)dz \]
for a given path \( C \) in the complex \( z \in \mathbb{C} \) plane.

- 4.1(3pt): Let \( F(z) = \sum_{k=0}^{\infty} c_k z^k \).

- 4.2(3pt): Let
\[ F(z) = \frac{\sum_{k=0}^{\infty} c_k z^k}{z-j}. \]

- 4.3(3 pts): Perform the following complex integrals (\( z = x + iy \in \mathbb{C} \), from \( \{a,b\} \in \mathbb{C} \). Let
\[ I = \int_C F(z)dz \]
for the given path \( C \).

1. \( I = \int_0^{1+j} zdz \)
2. \( I = \int_0^{1+j} zdz \), but this time make the path explicit: from 0 to 1, with \( y = 0 \), and then to \( y = 1 \), with \( x = 1 \). Draw a picture of the path.
3. Discuss whether your results agree with the FTCC (Eq. 1.5).

- 4.4(3 pts): Perform the following integrals on the closed path \( C \), which we define to be the unit circle. You should substitute \( z = e^{i\theta} \) and \( dz = ie^{i\theta} d\theta \), and integrate from \( \{-\pi, \pi\} \) to go once around the unit circle.

1. \( \int_C zdz \)
2. \( \frac{1}{z}dz \)
3. Discuss whether your results agree with the FTCC (Eq. 1.5).

Problem # 5(2pt): FTCC and integration in the complex plane
Let the function \( F(z) = c^z \), where \( c \in \mathbb{C} \) is given for each problem below. Find the anti-derivative \( f(z) \) for each \( F(z) \)
\[ f(z) = f(a) + \int_a^z F(\zeta)d\zeta. \]

Hint: Apply the FTCC?

- 5.1(1pt): Find the anti-derivative of \( F(z) \).

- 5.2(1pt): \( c = 1/e = 1/2.7183 \ldots \), where \( C \) is \( \zeta = 0 \rightarrow i \rightarrow z \)

Problem # 6(2 pts): Cauchy’s theorems for integration in the complex plane
There are three basic definitions related to Cauchy’s integral formula CT-1, CT-2 and CT-3. They are all related, and can greatly simplify integration in the complex plane. When a function depends on a complex variable we shall use uppercase notation, consistent with the engineering literature for the Laplace transform.
1. **Cauchy’s (Integral) Theorem** CT-1:

\[
\oint_C F(z) \, dz = 0
\]

2. **Cauchy’s Integral Formula** CT-2:

\[
\frac{1}{2\pi j} \oint_C \frac{F(z)}{z - z_0} \, dz = \begin{cases} 
F(z_0), & z_0 \in \mathbb{C} \text{ (inside)} \\
0, & z_0 \notin \mathbb{C} \text{ (outside)} 
\end{cases}
\]

3. **(Cauchy’s) Residue Theorem** CT-3:

\[
\oint_C F(z) \, dz = 2\pi j \sum_{k=1}^{K} \text{Res}_k
\]

– 6.1(2pt): **Describe the relationships between the three theorems CT-1, CT-2, CT-3:**

1. (CT-1) and (CT-2)
2. (CT-1) and (CT-3)

**Problem # 7(3 pts):** Apply Cauchy’s theorems to solve the following integrals. In the following

\[
F(z) = \frac{1}{1 + z^2}
\]

State which theorem(s) you used, and show your work.

– 7.1(1pt): \(\oint_C F(z) \, dz\) where \(C\) is a circle centered at \(z = 0\) with a radius of \(\frac{1}{2}\).

– 7.2(1pt): \(\oint_C F(z) \, dz\) where \(C\) is a circle centered at \(z = j\) with a radius of 1.

sol

– 7.3(1pt): \(\oint_C F(z) \, dz\) where \(C\) is a circle centered at \(z = 0\) with a radius of 2.

**Problem # 8(7 pts):** Integration in the complex plane

In the following questions, you must integrate \(F(s) = u(\sigma, \omega) + iv(\sigma, \omega)\) around the contour \(C\) for complex \(s = \sigma + i\omega\),

\[
I = \oint_C F(s) \, ds.
\]

Follow the directions carefully for each question. When asked to state where the function is and is not analytic, you are not required to use the Cauchy-Riemann equations (but you should if you can’t answer the question ‘by inspection’).

– 8.1(2 pt): \(F(s) = \frac{1}{s^2}\)

1. State where the function is and is not analytic.

2. Explicitly evaluate the integral when \(C\) is the unit circle, defined as \(s = e^{i\theta}, 0 \leq \theta \leq 2\pi\).

– 8.2(2pt): \(F(s) = e^{it}\)
1. State where the function is and is not analytic.

2. Evaluate the unit-circle integral using Cauchy’s theorem and/or the residue theorem.

\[-8.3(3\text{pt}): \ F(s) = \frac{1}{s+2}\]

1. State where the function is and is not analytic.

2. Let \( \mathcal{C} \) be the unit circle, defined as \( s = e^{i\theta}, \ 0 \leq \theta \leq 2\pi \). Evaluate the integral using Cauchy’s theorem and/or the residue theorem.

3. Let \( \mathcal{C} \) be a circle of radius 3, defined as \( s = 3e^{i\theta}, \ 0 \leq \theta \leq 2\pi \). Evaluate the integral using Cauchy’s theorem and/or the residue theorem.

**Problem #9: With the help of a computer**

In the following problems, we will look at some of the concepts from this homework using Matlab/Octave. We are using the `syms` function which requires Matlab’s/Octave’s symbolic math toolbox. Or you may use the EWS lab’s Matlab. Alternative symbolic-math tool, such as Wolfram Alpha.

**Example:** To find the Taylor series expansion about \( s = 0 \) of

\[ F(s) = -\log(1 - s), \]

first consider the derivative and its Taylor series (about \( s = 0 \))

\[ F'(s) = \frac{1}{1 - s} = \sum_{n=0}^{\infty} s^n. \]

Then, integrate this series term by term

\[ F(s) = -\log(1 - s) = \int s F'(s) \, ds = \sum_{n=0}^{\infty} \frac{s^n}{n}. \]

Alternatively you may use Matlab/Octave commands:

```
syms s
taylor(-log(1-s),’order’,7)
```

\[-9.1: \text{Use } \text{Octave’s } \text{taylor}(-\log(1-s)) \text{ to 7th order, as in the example above.}\]

1. Try the above Matlab/Octave commands. Give the first 7 terms of the Taylor series (confirm that Matlab/Octave agrees with the formula derived above).

2. What is the inverse Laplace transform of this series? Consider the series term by term.

\[-9.2: \text{The function } 1/\sqrt{z} \text{ has a branch point at } z = 0, \text{ thus it is singular there.}\]

1. Can you apply Cauchy’s integral theorem when integrating around the unit circle?

2. Below is a Matlab/Octave code that computes \( \int_0^{4\pi} \frac{dz}{\sqrt{z}} \) using Matlab’s/Octave’s symbolic analysis package:

\[\text{https://www.wolframalpha.com/}\]
3. Modify this code to integrate $f(z) = \frac{1}{z^2}$ once around the unit circle. What answers do you get for $I$ and $J$?

- **9.3:** Bessel functions can describe waves in a cylindrical geometry

The Bessel function has a Laplace transform with a branch cut

$$J_0(t)u(t) \leftrightarrow \frac{1}{\sqrt{1+s^2}}.$$ 

Draw a hand sketch showing the nature of the branch cut. Hint: Use zviz.

**Problem # 10: Matlab/Octave exercises:**

- **10.1:** Comment on the following Matlab/Octave exercises

1. Try the following Matlab/Octave commands, and then comment on your findings.

```matlab
%Take the inverse LT of 1/sqrt(1+sˆ2)
syms s
I=ilaplace(1/(sqrt((1+sˆ2))));
disp(I)

%Find the Taylor series of the LT
T = taylor(1/sqrt(1+sˆ2),10); disp(T);

%Verify this
syms t
J=laplace(besselj(0,t));
disp(J);

%plot the Bessel function
b=besselj(0,t);
plot(t/pi,b);
grid on;
```

- **10.2:** When did Friedrich Bessel live?

- **10.3:** What did he use Bessel functions for?

- **10.4:** Using zviz, for each of the following functions

1. Describe the plot generated by zviz $S=Z$.

2. Are the functions defined below legal Brune impedances? (i.e., Do they function obey $\Re Z(\sigma > 0) \geq 0$)? Hint: Consider the phase (color). Plot zviz $Z$ for a reminder of the colormap.

1. zviz 1./sqrt(1+S.^2)
2. zviz 1./sqrt(1-S.^2)
3. zviz 1./(1+sqrt(S)))
Problem #11: Find the $\mathcal{LT}^{-1}$ of one factor of the Riemann zeta function $\zeta_p(s)$
where $\zeta_p(s) \leftrightarrow z_p(t)$ and describe your results in words.

Hint: Consider the geometric series representation

$$
\zeta_p(s) = \frac{1}{1 - e^{-sT_p}} = \sum_{k=0}^{\infty} e^{-skT_p},
$$

for which you can look up the $\mathcal{LT}^{-1}$ transform of each term.

Problem #12: Inverse transform of products:

The time domain version of Eq. 1.6 (p. 8) may be written as the convolution of all the $z_k(t)$ factors

$$
z(t) \equiv z_2(t) \ast z_3(t) \ast z_5(t) \ast z_7(t) \cdots \ast z_p(t) \ast \cdots,
$$

where $\ast$ represents time convolution.

$$
v(t) \quad \rightarrow \quad q(t) \quad \rightarrow \quad i(t)
$$

$$
\begin{align*}
q(t) &= \alpha q(n - T_p) + v(t) \\
i(t) &= q(t) - (1/\alpha)q(t - T_p)
\end{align*}
$$

Figure 1: This feedback network is described by a time-domain difference equation with delay $T_p$, has an all-pole transfer function $\zeta_p(s) \equiv Q(s)/I(s)$ given by Eq. 1.8, which physically corresponds to a stub of a transmission line, with the input at one end and the output at the other. To describe the $\zeta(s)$ function we must take $\alpha = -1$. A transfer function $Y(s) = V(s)/I(s)$ that has the same poles as $\zeta_p(s)$, but with zeros as given by Eq. 1.9, is the input admittance $Y(s) = I(s)/V(s)$ of the transmission line, defined by the ratio of the Laplace transform of the current $i(t) \leftrightarrow I(s)$ over the voltage $v(t) \leftrightarrow V(s)$.

Explain what this means in physical terms. Start with two terms (e.g., $z_1(t) \ast z_2$).

Physical interpretation: Such functions may be generated in the time domain as shown in Fig. 1 (p. 8), using a feedback delay of $T_p$ seconds described by the two equations in the figure with a unity feedback gain $\alpha = -1$. Taking the Laplace transform of the system equation we see that the transfer function between the state variable $q(t)$ and the input $x(t)$ is given by $\zeta_p(s)$, which is and all-pole function, since

$$
Q(s) = e^{-sT_p}Q(s) + V(s), \quad \text{or} \quad \zeta_p(s) \equiv \frac{Q(s)}{V(s)} = \frac{1}{1 - e^{-sT_p}}.
$$

Closing the feed-forward path gives a second transfer function $Y(s) = I(s)/V(s)$, namely

$$
Y(s) \equiv \frac{I(s)}{V(s)} = \frac{1 - e^{-sT_p}}{1 + e^{-sT_p}}.
$$

If we take $i(t)$ as the current and $v(t)$ as the voltage at the input to the transmission line, then $y_p(t) \leftrightarrow \zeta_p(s)$ represents the input impedance at the input to the line. The poles and zeros of the impedance interleave along the $j\omega$ axis. By a slight modification $\zeta_p(s)$ may alternatively be written as

$$
y_p(s) = \frac{e^{sT_p/2} + e^{-sT_p/2}}{e^{sT_p/2} - e^{-sT_p/2}} = j \tan(s T_p/2).
$$

(1.10)
Every impedance $Z(s)$ has a corresponding reflectance function given by a Möbius transformation, which may be read off of Eq. 1.9 as

$$\Gamma(s) \equiv \frac{1 + Z(s)}{1 - Z(s)} = e^{-sT_p}$$

(1.11)

since impedance is also related to the round-trip delay $T_p$ on the line. The inverse Laplace transform of $\Gamma(s)$ is the round trip delay $T_p$ on the line

$$\gamma(t) = \delta(t - T_p) \leftrightarrow e^{-sT_p}.$$  

(1.12)

Working in the time domain provides a key insight, as it allows us to parse out the best analytic continuation of the infinity of possible continuations, that are not obvious in the frequency domain. Transforming to the time domain is a form of analytic continuation of $\zeta(s)$, that depends on the assumption that $z(t)$ is one-sided in time (causal).