1 Exercises DE-3

Topic of assignment:

Brune impedance, lattice transmission line analysis.

Brune Impedance

Problem *# 1(4 pts): Residue form*
A Brune impedance is defined as the ratio of the force $F(s)$ over the flow $V(s)$, and may be expressed in residue form as

$$Z(s) = c_0 + \sum_{k=1}^{K} \frac{c_k}{s - s_k} = \frac{N(s)}{D(s)} \tag{1.1}$$

with

$$D(s) = \prod_{k=1}^{K} (s - s_k) \qquad \text{and} \qquad c_k = \lim_{s \to s_k} (s - s_k) D(s) = \prod_{n'=1}^{K-1} (s - s_n).$$

The prime on index n' means that $n = k$ is not included in the product.

– 1.1(1 pts): Take the Laplace transform ($\mathcal{LT}$) of Eq. 1.2 and find the total impedance $Z(s)$ of the mechanical circuit.

$$M \frac{d^2}{dt^2} x(t) + R \frac{d}{dt} x(t) + K x(t) = f(t) \leftrightarrow (Ms^2 + Rs + K) X(s) = F(s). \tag{1.2}$$

– 1.2(1 pt): What are $N(s)$ and $D(s)$ (e.g. Eq. 1.1)?

Problem *# 2:(14 pts)* **Train-mission-line** *We wish to model the dynamics of a freight-train having N such cars, and study the* velocity transfer function *under various load conditions.*

As shown in Fig. 1, the train model consists of masses connected by springs.

Physical description:

Use the ABCD method to find the matrix representation of the system of Fig. 1. Define the force on the nth train car $f_n(t) \leftrightarrow F_n(\omega)$, and velocity $v_n(t) \leftrightarrow V_n(\omega)$.

Break the model into cells consisting of three elements: a series inductor representing half the mass ($M/2$), a shunt capacitor representing the spring ($C = 1/K$), and another series inductor representing half the mass ($L = M/2$), transforming the model into a cascade of symmetric ($\mathcal{A} = \mathcal{D}$) identical cell matrix $\mathcal{T}(s)$.

– 2.1(4 pt): Find the elements of the ABCD matrix $\mathcal{T}$ for the single cell that relate the input node 1 to output node 2

$$\begin{bmatrix} F \\ V \end{bmatrix}_1 = \mathcal{T} \begin{bmatrix} F(\omega) \\ -V(\omega) \end{bmatrix}_2 . \tag{1.3}$$

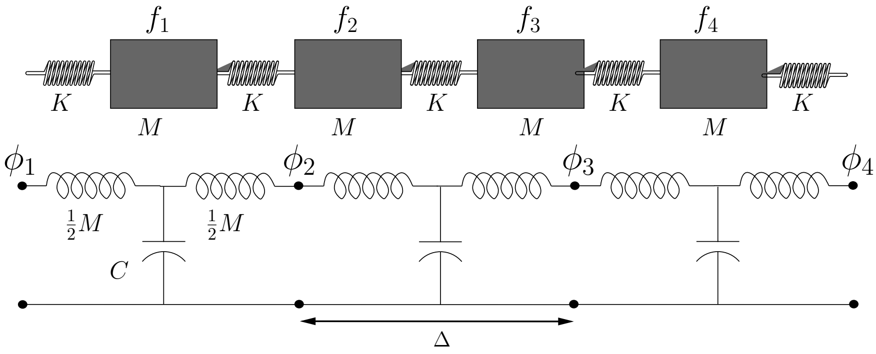

Figure 1: *Depiction of a train consisting of cars, treated as a mass M and linkages, treated as springs of stiffness K or compliance $C = 1/K$. Below it is the electrical equivalent circuit, for comparison. The mass is modeled as an inductor and the springs as capacitors to ground. The velocity is analogous to a current and the force $f_n(t)$ to the voltage $\phi_n(t)$. The length of each cell is Δ [m]. The train may be accurately modeled as a transmission line (TL), since the equivalent electrical circuit is a lumped model of a TL.*

– 2.2(6 pts): Express each element of $\mathcal{T}(s)$ in terms of the complex Nyquist ratio $s/s_c < 1$ ($s = 2\pi jf$, $s_c = 2\pi jf_c$). The Nyquist sampling cutoff frequency f_c is defined in terms the minimum number of cells (i.e., 2) of length Δ per wavelength:

The Nyquist sampling theorem says that there are at least two cars per wavelength (more than two time samples at the highest frequency). From the figure, the distance between cars $\Delta = c_o T_o$ [m] where

$$c_o = \frac{1}{\sqrt{MC}}. \qquad \text{[m/s]}$$

The cutoff frequency obeys $f_c \lambda_c = c_o$ where the Nyquist wavelength is $\lambda_c = 2\Delta$. Therefore the Nyquist sampling condition is

$$\omega < \omega_c = 2\pi f_c \equiv \frac{2\pi c_o}{\lambda_c} = \frac{2\pi c_o}{2\Delta} = \frac{\pi}{\Delta\sqrt{MC}}. \qquad \text{[Hz]} \tag{1.4}$$

– 2.3(4 pt): Use the property of the Nyquist sampling frequency *$f < f_c$ (Eq. 1.4) to remove higher order powers of frequency*

$$1 + \cancelto{0}{\left(\frac{s}{s_c}\right)^2} \approx 1 \tag{1.5}$$

to determine a band-limited approximation of $\mathcal{T}(s)$.