An Invitation to Mathematical Physics
and Its History

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Chapter 1

Introduction

Much of early mathematics dating before 1600 BCE centered around the love of art and music, due to the sensations of light and sound. Our psychological senses of color and pitch are determined by the frequencies (i.e., wavelengths) of light and sound. The Chinese and later the Pythagoreans are well known for their early contributions to music theory. We are largely ignorant of exactly what the Chinese scholars knew. The best record comes from Euclid, who lived in the 3rd century, after Pythagoras. Thus we can only trace the early mathematics back to the Pythagoreans in the 6th century (580-500 BCE), which is centered around the Pythagorean theorem and early music theory.

Pythagoras strongly believed that "all is number," meaning that every number, and every mathematical and physical concept, could be explained by integral (integer) relationships, mostly based on either ratios, or the Pythagorean theorem. It is likely that his belief was based on Chinese mathematics from thousands of years earlier. It is also believed that his ideas about the importance of integers followed from the theory of music. The musical notes (pitches) obey natural integral ratio relationships, based on the octave (a factor of two in frequency). The western 12-tone scale breaks the octave into 12 ratios. Today this has been rationalized to be the 12 root of 2, which is approximately equal to 18/17. This innate sense of frequency ratios comes from the physiology of the auditory organ (the cochlea) which represents a fix distance along the Organ of Corti, the sensory organ of the inner ear.

As acknowledged by Stillwell (2010, p. 16), the Pythagorean view is relevant today:

> *With the digital computer, digital audio, and digital video coding everything, at least approximately, into sequences of whole numbers, we are closer than ever to a world in which "all is number."*

1.1 Early Science and Mathematics

While early Asian mathematics has been lost, it clearly defined the course for math for at least several thousand years. The first recorded mathematics were those of the Chinese (5000-1200 BCE) and the Egyptians (3,300 BCE). Some of the best early records were left by the people of Mesopotamia (Iraq, 1800 BCE).[1] While the first 5,000 years of math are not well documented, but the basic record is clear, as outlined in Fig. 1.1 (p. 21).

Thanks to Euclid, and later Diophantus (c250 CE), we have some basic (but vague) understanding of Chinese mathematics. For example, *Euclid's formula* (Eq. 2.11, p. 61) provides a method for computing Pythagorean triplets, a formula believed to be due to the Chinese (Stillwell, 2010, pp. 4-9).[2]

Chinese bells and stringed musical instruments were exquisitely developed with tonal quality, as documented by ancient physical artifacts (Fletcher and Rossing, 2008). In fact this development was so rich that one must ask why the Chinese failed to initiate the industrial revolution. Specifically, why

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[1] See Fig. 2.8, p. 62.

[2] One might reasonably view Euclid's role as that of a mathematical messenger.

did Europe eventually dominate with its innovation when it was the Chinese who did the extensive early invention?

It could have been for the wrong reasons, but perhaps our best insight into the scientific history from China may have come from an American chemist and scholar from Yale, Joseph Needham, who learned to speak Chinese after falling in love with a Chinese woman, and ended up researching early Chinese science and technology for the US government.

According to Lin (1995) this is known as the *Needham question*:

> "Why did modern science, the mathematization of hypotheses about Nature, with all its implications for advanced technology, take its meteoric rise only in the West at the time of Galileo[, but] had not developed in Chinese civilization or Indian civilization?"

Needham cites the many developments in China:[3]

> "Gunpowder, the magnetic compass, and paper and printing, which Francis Bacon considered as the three most important inventions facilitating the West's transformation from the Dark Ages to the modern world, were invented in China." (Lin, 1995)

> "Needham's works attribute significant weight to the impact of Confucianism and Taoism on the pace of Chinese scientific discovery, and emphasize what it describes as the 'diffusionist' approach of Chinese science as opposed to a perceived independent inventiveness in the western world. Needham held that the notion that the Chinese script had inhibited scientific thought was 'grossly overrated' " (Grosswiler, 2004).

Lin was focused on military applications, missing the importance of non-military contributions. A large fraction of mathematics was developed to better understand the solar system, acoustics, musical instruments and the theory of sound and light. Eventually the universe became a popular topic, as it still is today.

Regarding the "Needham question," I suspect the resolution is now clear. In the end, China withdrew itself from its several earlier expansions, based on internal politics (Menzies, 2004, 2008).

Chronological history pre 16th century

20th **BCE** *Chinese (Primes; quadratic equation; Euclidean algorithm (GCD))*

18th **BCE** *Babylonia (Mesopotamia/Iraq) (quadratic equation)*

6th **BCE** *Pythagoras (Thales) and the Pythagorean "tribe"*

4th **BCE** *Euclid (quadratic equation); Archimedes*

3^{d} **BCE** *Eratosthenes 276-194BCE*

3^{d} **CE** *Diophantus c250CE*

4th *Alexandria Library destroyed 391CE*

6th *Brahmagupta (negative numbers; quadratic equation) 598-670CE*

10th *al-Khwārizmī (algebra) 830CE Hasan Ibn al-Haytham (Alhazen) 965-1040CE*

12th *Bhaskara (calculus) 1114-1183 Marco Polo 1254-1324*

15th *Leonardo da Vinci 1452-1519; Michelangelo 1475-1564 Copernicus 1473-1543*

16th *Tartaglia (cubic eqs); Bombelli 1526-1572 Galileo Galilei 1564-1642*

1.1.1 Lec 1: The Pythagorean theorem

Thanks to Euclid's *Elements* (c323 BCE) we have an historical record, tracing the progress in geometry, as established by the Pythagorean theorem, which states that *for any right triangle* having sides of

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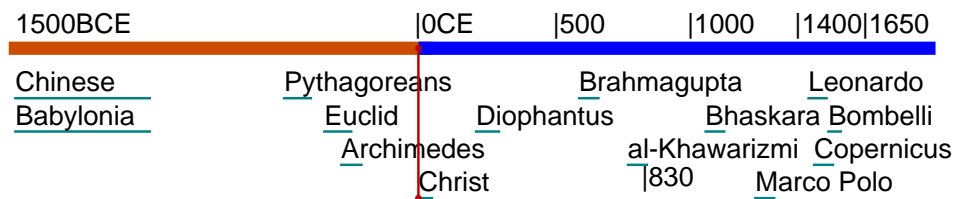[3]https://en.wikipedia.org/wiki/Joseph_Needham\#cite_note-11

1500BCE			0CE		500		1000		1400	1650

Chinese   Pythagoreans   Brahmagupta   Leonardo
Babylonia   Euclid   Diophantus   Bhaskara Bombelli
                Archimedes   al-Khawarizmi Copernicus
                Christ |830   Marco Polo

Figure 1.1: *Mathematical timeline between 1500* BCE *and 1650* CE*. The western renaissance is considered to have occurred between the 15ᵗʰ and 17ᵗʰ centuries. However, the Asian "renaissance" was likely well before the 1st century (i.e., 1500* BCE*). There is significant evidence that a Chinese 'treasure ship' visited Italy in 1434, initiating the Italian renaissance (Menzies, 2008). This was not the first encounter between the Italians and the Chinese, as documented in 'The travels of Marco Polo' (c1300* CE*).*

lengths $(a, b, c) \in \mathbb{R}$ that are either positive real numbers, or more interesting, integers $c > [a, b] \in N$ such that $a + b > c$,

$$c^2 = a^2 + b^2. \tag{1.1}$$

Early integer solutions were likely found by trial and error rather than by an algorithm.

If a, b, c are lengths, then a^2, b^2, c^2 are each the area of a square. Equation 1.1 says that the area a^2 plus the area b^2 equals the area c^2. Today a simple way to prove this is to compute the magnitude of the complex number $c = a + bȷ$, which forces the right angle

$$|c|^2 = (a + bȷ)(a - bȷ) = a^2 + b^2. \tag{1.2}$$

However, complex arithmetic was not an option for the Greek mathematicians, since complex numbers and algebra had yet to be discovered.

Almost 700 years after Euclid's *Elements*, the Library of Alexandria was destroyed by fire (391 CE), taking with it much of the accumulated Greek knowledge. As a result, one of the best technical records remaining is Euclid's *Elements*, along with some sparse mathematics due to Archimedes (c300 BCE) on geometrical series, computing the volume of a sphere, the area of the parabola, and elementary hydrostatics. In c1572 a copy Diophantus's *Arithmetic* was discovered by Bombelli in the Vatican library (Stillwell, 2010, p. 51). This book became an inspiration for Galileo, Descartes, Fermat and Newton.

Early number theory: Well before Pythagoras, the Babylonians (c1,800 BCE) had tables of triplets of integers $[a, b, c]$ that obey Eq. 1.1, such as $[3, 4, 5]$. However, the triplets from the Babylonians were larger numbers, the largest being $a = 12709, c = 18541$. A stone tablet (Plimpton-322) dating back to 1800 BCE (Fig. 2.8, p. 62) was found with integers for $[a, c]$. Given such sets of two numbers, which determined a third positive integer $b = 13500$ such that $b = \sqrt{c^2 - a^2}$, this table is more than convincing that the Babylonians were well aware of *Pythagorean triplets* (PTs), but less convincing that they had access to *Euclid's formula*, a formula for PTs. (Eq. 2.11 p. 61).

It seems likely that Euclid's *Elements* was largely the source of the fruitful era due to the Greek mathematician Diophantus (215-285) (Fig. 1.1), who developed the field of *discrete mathematics* now known as *Diophantine analysis*. The term means that the solution, not the equation, is integer. The work of Diophantus was followed by fundamental change in mathematics, possibly leading to the development of algebra, but at least including the discovery of

1. negative numbers,

2. quadratic equation (Brahmagupta, 7ᵗʰ CE),

3. algebra (al-Khwārizmī, 9ᵗʰ CE), and

4. complex arithmetic (Bombelli, 15ᵗʰ CE).

These discoveries overlapped with the European middle (aka, dark) ages. While Europe went "dark," presumably European intellectuals did not stop working during these many centuries.[4]

1.1.2   What is mathematics?

It seems strange when people complain they "can't learn math,"[5] but then claim to be good at languages. Pre-high-school students tend to confuse arithmetic with math. One does not need to be good at arithmetic to be good at math (but it doesn't hurt).

Math is a language, with the symbols taken from various languages, not so different from other languages. Today's mathematics is a written language with an emphasis on symbols and glyphs, biased toward Greek letters, obviously due to the popularity of Euclid's *Elements*. The specific evolution of these symbols is interesting (Mazur, 2014). Each symbol is dynamically assigned a meaning, appropriate for the problem being described. These symbols are then assembled to make sentences. It is similar to Chinese in that the spoken and written versions are different across dialects. Like Chinese, the sentences may be read out loud in any language (dialect), while the mathematical sentence (like Chinese characters) is universal.

Learning languages is an advanced social skill. However, the social outcomes of learning a language and math are very different. First learning a new language is fun because it opens doors to other cultures. Second math is different due to the rigor of the rules of the language, along with the way it is taught (e.g., not as a language). A third difference between math and language is that math evolved from physics, with important technical applications.

Like any language, the more mathematics you learn, the easier it is to understand, because mathematics is built from the bottom up. It's a continuous set of concepts, much like the construction of a house. If you try to learn calculus and differential equations, while skipping simple number theory, the lessons will be more difficult to understand. You will end up memorizing instead of understanding, and as a result you will likely forget it. When you truly understand something, it can never be forgotten. A nice example is the solution to a quadratic equation: If you learn how to *complete the square* (p. 77), you will never forget the quadratic formula.

The topics need to be learned in order, just as in the case of building the house. You can't build a house if you don't know about screws or cement (plaster). Likewise in mathematics, you will not learn to integrate if you have failed to understand the difference between integers, complex numbers, polynomials, and their roots.

A short list of topics for mathematics is numbers ($\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{I}, \mathbb{C}$), algebra, derivatives, anti-derivatives (i.e., integration), differential equations, vectors and the spaces they define, matrices, matrix algebra, eigenvalues and vectors, solutions of systems of equations, matrix differential equations and their eigen-solution. Learning is about understanding, not memorizing.

The rules of mathematics are formally defined by algebra. For example, the sentence $a = b$ means that the number a has the same value as the number b. The sentence is spoken as "a equals b." The numbers are nouns and the equal sign says they are equivalent, playing the role of a verb, or action symbol. Following the rules of algebra, this sentence may be rewritten as $a - b = 0$. Here the symbols for minus and equal indicate two types of actions (verbs).

Sentences can become arbitrarily complex, such as the definition of the integral of a function, or a differential equation. But in each case, the mathematical sentence is written down, may be read out loud, has a well-defined meaning, and may be manipulated into equivalent forms following the rules of algebra and calculus. This language of mathematics is powerful, with deep consequences, first known as algorithms, but eventually as theorems.

The writer of an equation should always translate (explicitly summarize the meaning of the expression), so the reader will not miss the main point, as a simple matter of clear writing.

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[4]It would be interesting to search the archives of the monasteries, where the records were kept, to determine exactly what happened during this religious blackout.

[5]"It looks like Greek to me."

Just as math is a language, so language may be thought of as mathematics. To properly write correct English it is necessary to understand the construction of the sentence. It is important to identify the subject, verb, object, and various types of modifying phrases. Look up the interesting distinction between *that* and *which*.[6] Thus, like math, language has rules. Most individuals use what "sounds right," but if you're learning English as a second language, it is necessary to understand the rules, which are arguably easier to master than the foreign speech sounds.

Context is everything, and the most important context is physics. Without a physical problem to solve, there can be no mathematics. People wanted to navigate the earth, and weigh things. This required an understand of gravity. Many questions about gravity were deep, such as "Where is the center of the universe?" Actually this answer is simple: Ask the Pope and he will tell you. But church dogma only goes so far. Mathematics, along with a heavy dose of physics, finally answered this huge question. Someone needed to perfect the telescope, and put satellites into space, and view the cosmos. Without mathematics none of this would have happened.

1.1.3 Early physics as mathematics: Back to Pythagoras

There is a second answer to the question What is mathematics? The answer comes from studying its history, which starts with the earliest record. This chronological view starts, of course, with the study of numbers. First there is the taxonomy of numbers. It took thousands of years to realize that numbers are more than the counting numbers $\mathbb{N}$, and to create a symbol for nothing (i.e., zero), and to invent negative numbers. With the invention of the abacus, a memory aid for the manipulation of complex sets of real integers, one could do very detailed calculations. But this required the discovery of *algorithms* (procedures) to add, subtract, multiply (many adds of the same number) and divide (many subtracts of the same number), such as the *Euclidean algorithm.* Eventually it became clear to the experts (early mathematicians) that there were natural rules to be discovered, thus books (e.g., Euclid's *Elements*) were written to summarize this knowledge.

The role of mathematics is to summarize algorithms (i.e., sets of rules), and formalize the idea as a theorem. Pythagoras and his followers, the Pythagoreans, believed that there was a fundamental relationship between mathematics and the physical world. The Pythagoreans may have been the first to capitalize on the relationship between science and mathematics, to use mathematics to design things for profit.[7] This may have been the beginning of capitalizing technology (i.e., engineering), based on the relationship between physics and math. This impacted commerce in many ways, such as map making, tools, implements of war (the wheel, gunpowder), art (music), water transport, sanitation, secure communication, food, etc. Of course the Chinese were the first to master many of these technologies.

Why is Eq. 1.1 called a *theorem*? Theorems require a proof. What exactly needs to be proved? We do not need to prove that (a, b, c) obey this relationship, since this is a condition that is observed. We do not need to prove that a^2 is the area of a square, as this is the definition of an area. What needs to be proved is that the relation $c^2 = a^2 + b^2$ holds *if, and only if,* the angle between the two shorter sides is $90°$. The Pythagorean theorem (Eq. 1.1) did not begin with Euclid or Pythagoras, rather they appreciated its importance, and documented its proof.

In the end the Pythagoreans, who instilled fear in the neighborhood, were burned out, and murdered, likely the fate of mixing technology with politics:

> "Whether the complete rule of number (integers) is wise remains to be seen. It is said that when the Pythagoreans tried to extend their influence into politics they met with popular resistance. Pythagoras fled, but he was murdered in nearby Mesopotamia in 497 BCE."

> –Stillwell (2010, p. 16)

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[6] https://en.oxforddictionaries.com/usage/that-or-which
[7] It seems likely that the Chinese and Egyptians also did this, but it is more difficult to document.

1.2   Modern mathematics is born

Modern mathematics (what we practice today) was born in the 15-16th centuries, in the minds of Leonardo da Vinci, Bombelli, Galileo, Descartes, Fermat, and many others (Stillwell, 2010). Many of these early masters were, like the Pythagoreans, secretive to the extreme about how they solved prob-
5   lems. This soon changed due to Galileo, Mersenne, Descartes and Newton, causing mathematics to blossom. The developments during this time may seemed hectic and disconnected. But this is a wrong impression, as the development was dependent on new technologies such as the telescope (optics) and more accurate time and frequency measurements, due to Galileo's studies of the pendulum, and a better understanding of the relation $f\lambda = c_o$ between frequency f, wavelength λ and the wave speed c_o.

Chronological history: 16th to 19th centuries
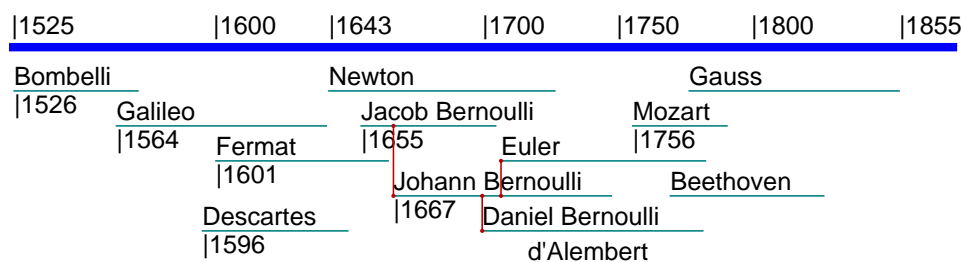

Figure 1.2:   *Timeline for the development of the theory of calculus, between 16th and 19th CE. Given the timeline, it seems likely that Bombelli's discovery of Diophantus's book "Arithmetic" in the Vatican library triggered many of the ideas presented by Galileo, Descartes and Fermat, followed by others (i.e., Newton). Bombelli's discovery might be considered as a magic moment in mathematics. The vertical red lines indicate mentor-student relationships. Mozart and Beethoven are shown as landmarks. Related timelines include (see index): Fig. 1.1 (p. 21) which gives the timeline from 1500 BCE to 1650 CE; Fig. 3.1 (p. 73) presents the 17-20 CE (Newton–Einstein) view from 1640–1950; Fig. 3.5 (p. 107) presents the period from Descartes to Gauss and Cauchy; Fig. 4.1 (p. 146) presents the period from Bombelli to Einstein (1525-1925).*

10   ### 1.2.1   Science meets mathematics

Galileo: In 1589 Galileo famously conceptualized an experiment where he suggested dropping two different weights from the Leaning Tower of Pisa, and he suggested that they must take the same time to hit the ground.

Conceptually this is a mathematically sophisticated experiment, driven by a mathematical argument
15   in which he considered the two weights to be connected by an elastic cord (a spring). His studies resulted in the concept of *conservation of energy*, one of the cornerstones of physical theory since that time.

Being joined with an elastic cord, the masses become one. If the velocity were proportional to the mass, as believed by Archimedes, the sum of the two weights would necessarily fall even faster. This results in a logical fallacy: How can two masses fall faster than either? This also violates the concept of
20   conservation of energy, as the total energy of two masses would be greater than that of the parts. In fact Galileo's argument may have been the first time that the principle of *conservation of energy* was clearly stated.

It seems likely that Galileo was attracted to this model of two masses connected by a spring because he was also interested in planetary motion, which consist of masses (sun, earth, moon), also mutually
25   attracted by gravity (i.e., the spring).

Galileo also performed related experiments on pendulums, where he varied the length l, mass m, and angle θ of the swing. By measuring the period (periods/unit time) he was able to formulate precise rules between the variables. This experiment also measured the force exerted by gravity, so the experiments were related, but in very different ways. The pendulum served as the ideal clock, as it needed very little
30   energy to keep it going, due to its very low friction (energy loss).[8]

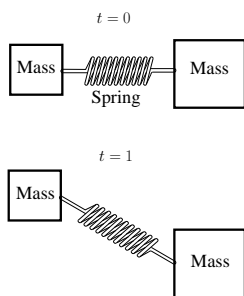[8]http://muse.tau.ac.il/museum/galileo/pendulum.html

Figure 1.3: *Depiction of the argument of Galileo (unpublished book of 1638) as to why weights of different masses (i.e., weight) must fall with the same velocity, contrary to what Archimedes had proposed c250 BCE.*

In a related experiment, Galileo measured the duration of a day by counting the number of swings of the pendulum in 24 hours, measured precisely by the daily period of a star as it crossed the tip of a church steeple. The number of seconds in a day is $24{\times}60{\times}60 = 86400 = 2^7 3^3 5^2$ [s/day]. Since 86,400 is the product of the first three primes, it is highly composite, and thus may be reduced in many equivalent ways. For example, the day can be divided evenly into 3, 5 or 6 parts, and remain exact in terms of the number of seconds that transpire. Factoring the number of days in a year (365=5*73) is not useful, since it may not be decomposed into many small primes.[9] Galileo also extended work on the relationship of wavelength and frequency of a sound wave in musical instruments. On top of these impressive accomplishments, Galileo greatly improved the telescope, which he needed for his observations of the planets.

Many of Galileo's contributions resulted in new mathematics, leading to Newton's discovery of the wave equation (c1687), followed 60 years later by its one-dimensional general solution by d'Alembert (c1747).

Mersenne: Marin Mersenne (1588–1648) also contributed to our understanding of the relationship between the wavelength and the dimensions of musical instruments, and is said to be the first to measure the speed of sound. At first Mersenne strongly rejected Galileo's views, partially due to errors in Galileo's reports of his results. But once Mersenne saw the significance of Galileo's conclusion, he became Galileo's strongest advocate, helping to spread the word (Palmerino, 1999).

This incident invokes an important theorem of nature: Is data more like bread or wine? The answer is, it depends on the data. Galileo's original experiments on pendulums and ball falling down slopes, were flawed by inaccurate data. This is likely because he didn't have good clocks. But he soon solved that problem and the data became more accurate. We don't know if Mersenne repeated Galileo's experiments, and then appreciated his theory, or if he communicated with Galileo. But resolution was that the early data were like bread (it rots), but when the experimental method was improved, with a better clock, the corrected data was like wine (which gets better with age). Galileo claimed that the time for the mass to drop a fixed distance was exactly proportional to the square of the time. This expression lead to $F = ma$. One follows from the other. If the mass varies then you get Newton's second law of motion (Eq. 3.2, p. 74).

He was also a decent mathematician, inventing (1644) the *Mersenne primes* (MP) π_m of the form

$$\pi_m = 2^{\pi_k} - 1,$$

where π_k ($k < m$) denotes the kth prime (p. 36). As of Dec 2018 51 MPs are known.[10] The first MP is $3 = \pi_2 = 2^{\pi_1} - 1$, and the largest known prime is a MP. Note that $127 = \pi_{31} = 2^{\pi_7} - 1$ is the MP of the MP π_7.

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[9]For example, if the year were $364 = 2^2 \cdot 7 \cdot 13$ days, it would would make for shorter years (by 1 day), 13 months per year (e.g., $28 = 2 \cdot 2 \cdot 7$ day vacation per year?), perfect quarters, and exactly 52 week. Every holiday would always fall on the same day, every year.

[10]http://mathworld.wolfram.com/MersennePrime.html

Newton: With the closure of Cambridge University due to the plague of 1665, Newton returned home to Woolsthorpe-by-Colsterworth (95 [mi] north of London), to work by himself for over a year. It was during this solitary time that he did his most creative work.

While Newton (1642–1726) may be best known for his studies on light, he was the first to predict the speed of sound. However, his theory was in error[11] by $\sqrt{c_p/c_v} = \sqrt{1.4} = 1.183$. This famous error would not be resolved for 140 years, awaiting the formulation of thermodynamics and the equi-partition theorem by Laplace in 1816.[12]

Just 11 years prior to Newton's 1687 *Principia*, there was a basic understanding that sound and light traveled at very different speeds, due to the experiments of Ole Rømer.[13] [14]

Ole Rømer first demonstrated in 1676 that light travels at a finite speed (as opposed to instantaneously) by studying the apparent motion of Jupiter's moon Io. In 1865, James Clerk Maxwell proposed that light was an electromagnetic wave, and therefore traveled at the speed c_o appearing in his theory of electromagnetism.[15]

The idea behind Rømer's discovery was that due to the large distance between Earth and Io, there was a difference between the period of the moon when Jupiter was closest to Earth vs. when it was farthest from Earth. This difference in distance caused a delay or advance in the observed eclipse of Io as it went behind Jupiter, delayed by the difference in time due to the difference in distance. It is like watching a video of a clock, delayed or sped up. When the video is slowed down, the time will be inaccurate (it will indicate an earlier time).

Studies of vision and hearing: Since light and sound (music) played such a key role in the development of the early science, it was important to fully understand the mechanism of our perception of light and sound. There are many outstanding examples where physiology impacted mathematics. Leonardo da Vinci (1452–1519) is well known for his early studies of the human body. Exploring our physiological senses requires a scientific understanding of the physical processes of vision and hearing, first considered by Newton (1687) (1643–1727), but first properly researched much later by Helmholtz (Stillwell, 2010, p. 261). Helmholtz's (1821–1894) studies and theories of music and the perception of sound are fundamental scientific contributions (Helmholtz, 1863a). His best known mathematical contribution is today known as the *fundamental theorem of vector calculus*, or simply *Helmholtz theorem.*

The amazing Bernoulli family: The first individual who seems to have openly recognized the importance of mathematics, enough to actually teach it, was Jacob Bernoulli (1654–1705) (Fig. 1.4). Jacob worked on what is now viewed as the standard package of analytic "circular" (i.e., periodic) functions: $\sin(x)$, $\cos(x)$, $\exp(x_j)$, $\log(x)$.[16] Eventually the full details were developed (for real variables) by Euler (Sections 3.1 p. 95 and R.1, p. 336).

From Fig. 1.2 we see that Jacob was contemporary with Descartes, Fermat, and Newton. Thus it seems likely that he was strongly influenced by Newton, who in turn was influenced by Descartes,[17] Viète and Wallis (Stillwell, 2010, p. 175).

Jacob Bernoulli, like all successful mathematicians of the day, was largely self-taught. Yet Jacob was in a new category of mathematicians because he was an effective teacher. Jacob taught his sibling Johann (1667–1748), who then taught his sibling Daniel (1700–1782). But most importantly, Johann taught Leonhard Euler (1707–1783) (Figs. 1.2 and 1.4), the most prolific (thus influential) of all mathematicians. This teaching resulted in an explosion of new ideas and understanding. It is most significant

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[11]The square root of the ratio of the *specific heat capacity* at constant pressure c_p to that at constant volume c_v.

[12]https://en.wikipedia.org/wiki/Pierre-Simon_Laplace#Speed_of_sound

[13]https://www.youtube.com/watch?v=b9F8Wn4vf5Y

[14]https://www.youtube.com/watch?v=rZ0wx3uD2wo

[15]https://en.wikipedia.org/wiki/Speed_of_light

[16]The log and tan functions are related by Eq. 4.2 p. 146.

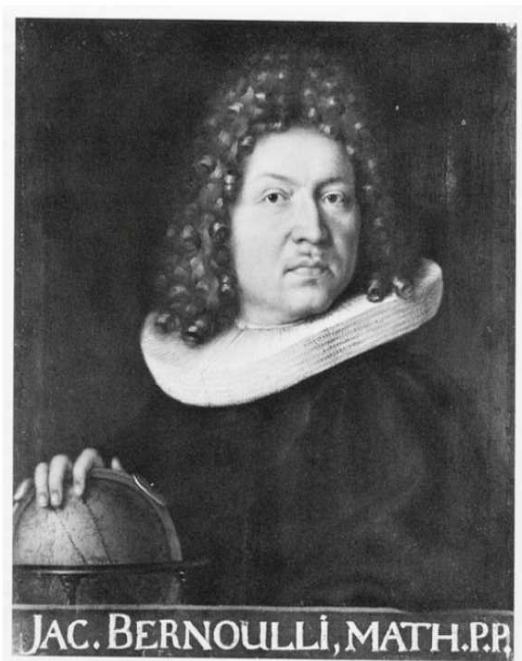[17]https://en.wikipedia.org/wiki/Early_life_of_Isaac_Newton

Figure 13.10: Portrait of Jakob Bernoulli by Nicholas Bernoulli

Figure 13.11: Johann Bernoulli

Figure 10.4: Leonhard Euler

Figure 1.4: *Above: Jakob (1655–1705) and Johann (1667–1748) Bernoulli. Below: Leonhard Euler (1707-1783) and Jean le Rond d'Alembert (1717-1783). Euler was blind in his right eye, hence the left portrait view. The figure numbers are from Stillwell (2010).*

that all four mathematicians published their methods and findings. Much later, Jacob studied with students of Descartes[18] (Stillwell, 2010, p. 268-9).

Euler: Euler went far beyond all the Bernoulli family (Stillwell, 2010, p. 315). A special strength of Euler was the degree to which he published. First he would master a topic, and then he would publish. His papers continued to appear long after his death (Calinger, 2015). It is also interesting that Leonhard Euler was a contemporary of Mozart (and James Clerk Maxwell of Abraham Lincoln).

Bach's mathematics: It turns out which I need to further document that JS Bach did a mathematical analysis of music. This seems to be about the same time that Euler was writing a book on music theory. Someone please help to fill in these details.[19]

more research.

d'Alembert: Another individual of that time of special note, who also published extensively, was d'Alembert (Fig. 1.4). Some of the most creative ideas were first proposed by d'Alembert. Unfortunately, and perhaps somewhat unfairly, his rigor was criticized by Euler, and later by Gauss (Stillwell, 2010). But once the tools of mathematics were finally openly published, largely by Euler, mathematics grew exponentially.[20]

Gauss: Figure 4.1 (p. 146) shows the timeline of the most famous mathematicians. It was one of the most creative times in mathematics. Gauss was born at the end of Euler's long and productive life. I suspect that Gauss owed a great debt to Euler: surely he must have been a scholar of Euler. One of Gauss's most important achievements may have been his contribution to solving the open question about the density of prime numbers.[21]

Table 1.1: *Three streams followed from Pythagorean theorem: number systems (Stream 1), geometry (stream 2) and infinity (stream 3).*

- *The Pythagorean theorem is the mathematical spring which bore the three streams.*

- *Several centuries per stream:*

    1) **Numbers:**

    6th BCE $\mathbb{N}$ *counting numbers*, $\mathbb{Q}$ *rationals*, $\mathbb{P}$ *primes*

    5th BCE $\mathbb{Z}$ *common integers*, $\mathbb{I}$ *irrationals*

    7th CE   *zero* $\in \mathbb{Z}$

    2) **Geometry:** *(e.g., lines, circles, spheres, toroids, . . .)*

    17th CE   *Composition of polynomials (Descartes, Fermat)*
      Euclid's geometry & algebra $\Rightarrow$ *analytic geometry*

    18th CE   *Fundamental theorem of algebra*

    3) **Infinity:** *($\infty \rightarrow$ Sets)*

    17-18th CE   *Taylor series, functions, calculus (Newton, Leibniz)*

    19th CE $\mathbb{R}$ *real*, $\mathbb{C}$ *complex 1851*

    20th CE   *Set theory*

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[18]It seems clear that Descartes was also a teacher.

[19]https://www.youtube.com/watch?v=6b5MM4x-3aQ,
https://www.uchicago.edu/features/combining_math_and_music/

[20]There are at least three useful exponential scales: Factors of 2, factors of $e \approx 2.7$, and factors of 10. The decibel uses factors of 2 (6 [dB]) and factors of 10 (20 [dB]). Information theory uses powers of 2 (1 [bit]), 4 (2 [bits]). Circuit theory uses factors of both e and 10.

[21]http://www-history.mcs.st-andrews.ac.uk/Biographies/Gauss.html.

1.2.2 Three Streams from the Pythagorean theorem

From the outset of his presentation, Stillwell (2010, p. 1) defines "three great *streams* of mathematical thought: *Numbers, Geometry and Infinity*" that flow from the Pythagorean theorem, as summarized in Table 1.1. This is a useful concept, based on reasoning not as obvious as one might think. Many factors are in play here. One of these is the strongly held opinion of Pythagoras that all mathematics should be based on integers.[22] The rest are tied up in the long, necessarily complex history of mathematics, as best summarized by the fundamental theorems (Table 2.1, p. 44), each of which is discussed in detail in a relevant chapter. Stillwell's concept of three streams, following from the Pythagorean theorem, is the organizing principle behind this book:

1. *Introduction* Section 1 is intended to be a self-contained survey of basic pre-college mathematics, as a detailed overview of the fundamentals, presented as three main streams:

   2 Number systems: Stream 1 (p. 35)

   3 Algebraic equations: Stream 2 (p. 73)

   4 Scalar calculus: Stream 3a (p. 145)

   5 Vector calculus: Stream 3b (p. 187)

   If you're a student, stick to these sections.

   Sections 2-5 are disorganized research ideas that have not yet found a home. Students, please stay out of these later chapters.

2. *Number Systems* (Stream 1, p. 319) Some uncertain ideas of number systems, starting with prime numbers, through complex numbers, vectors and matrices.

3. *Algebraic Equations* (Stream 2, p. 335) Algebra and its development, as we know it today. The theory of real and complex equations and functions of real and complex variables. Complex impedance $Z(s)$ of complex frequency $s = \sigma + \omega \jmath$ is covered with some care, developing the topic which is needed for engineering mathematics.

4. *Scalar Calculus* (Stream 2a) Ordinary differential equations. Integral theorems. Acoustics.

5. *Vector Calculus*: (Stream 2b) Vector partial differential equations. Gradient, divergence and curl differential operators. Stokes's and Green's theorems. Maxwell's equations.

---

[22]https://www.youtube.com/watch?v=gHSv94aTTnk

Thus one could factor a product of primes $N = \pi_k \pi_l$ by doing M divisions, where M is the number of primes less than N. This assumes the list of primes less than N are known. However, most integers are not a simple product of two primes

But the utility has to do with the sheer density of primes (the prime number theorem, i.e., Gauss's hypothesis). If we were simply looking up a few numbers from a short list of primes, it would be easy, but the density of primes among the integers is logarithmic ($\gg 1$ per octave).

This brings us to the *prime number theorem* (PNT). The security problem is the reason why these two theorems are so important: 1) Every integer has a unique representation as a product of primes, and 2) the density of primes is large (see the discussions on p. 45 and Section O.2, p. 319). Thus security reduces to the "needle in the haystack problem" due to the cost of a search.

Examples of the GCD: $k = \gcd(m, n)$

- *Examples (m, n, k $\in \mathbb{Z}$):*
    - $5 = \gcd(13 \cdot 5, 11 \cdot 5)$. *The GCD is the common factor 5.*
    - $\gcd(13 \cdot 10, 11 \cdot 10) = 10$ ($\gcd(130, 110) = 10 = 2 \cdot 5$, *is not prime*)
    - $\gcd(1234, 1024) = 2 (1234 = 2 \cdot 617, 1024 = 2^{10})$
    - $\gcd(\pi_k \pi_m, \pi_k \pi_n) = \pi_k$
    - $k = \gcd(m, n)$ *is the part that cancels in the fraction $m/n \in F$*
    - $m/\gcd(m, n) \in \mathbb{Z}$

- Coprimes *(m $\perp$ n) are numbers with no distinct common factors: i.e., gcd(m,n)=1*
    - *The GCD of two primes is always 1: gcd(13,11) = 1, $gcd(\pi_m, \pi_n)$=1 (m $\neq$ n)*
    - *$m = 7 \cdot 13, n = 5 \cdot 19 \Rightarrow (7 \cdot 13) \perp (5 \cdot 19)$*
    - *If $m \perp n$ then $gcd(m, n) = 1$*
    - *If $gcd(m, n) = 1$ then $m \perp n$*

- *The GCD may be extended to polynomials: e.g., $gcd(ax^2 + bx + c, \alpha x^2 + \beta x + \gamma)$*
    - *$gcd((x - 3)(x - 4), (x - 3)(x - 5))= (x - 3)$*
    - *$gcd(x^2 - 7x + 12, 3(x^2 - 8x + 15))= 3(x - 3)$*
    - *$gcd(x^2 - 7x + 12, (3x^2 - 24x + 45))=3(x - 3)$*
    - *$gcd((x - 2\pi)(x - 4), (x - 2\pi)(x - 5))=(x - 2\pi)$ (Needs long division)*

Figure 2.3: *The Euclidean algorithm for finding the GCD of two numbers is one of the oldest algorithms in mathematics, and is highly relevant today. It is both powerful and simple. It was used by the Chinese during the Han dynasty (Stillwell, 2010, p. 70) for reducing fractions. It may be used to find pairs of integers that are coprime (their GCD must be 1). The GCD may be used to identify factors of polynomials by long division. It has an important sister algorithm called the* continued fraction *algorithm (CFA), that generalizes the Euclidean algorithm.*

2.3.1   Lec 5: Greatest common divisor (Euclidean algorithm)

The *Euclidean algorithm* is a method to find the *greatest common divisor* (GCD) k between two integers n, m, denoted $k = gcd(n, m)$, where $n, m, k \in \mathbb{N}$ (Graham *et al.*, 1994). For example, $15 = \gcd(30, 105)$ since, when factored $(30, 105) = (2 \cdot 3 \cdot 5, 7 \cdot 3 \cdot 5) = 3 \cdot 5 \cdot (2, 7) = 15 \cdot (2, 7)$. The Euclidean algorithm is a method for finding the largest common divisor without factoring the two numbers. It was known to the Chinese (i.e., not discovered by Euclid) (Stillwell, 2010, p. 41). The GCD may be used to find *coprimes*, as described below in Fig. 2.3.

There are six closely related *numeric operators* which return integers $\in \mathbb{Z}$ (Sect. A.2, p. 248). In Matlab/Octave: `floor(x)`, `ceil(x)`, `round(x)` take $x \in \mathbb{R}$, and `mod(n,m)` (modulo), `rem(n,m)` (remainder), `gcd(n,m)` (greatest common divisor) take $n, m \in \mathbb{N}$. Rounding schemes are used in continued fractions and the others are widely used when doing arithmetic, such as division.

The remainder function (modulo) for real arguments is closely related to the phase of a pure delay, since the Laplace transform of a delay [Sect. I.2 (p. 282)] (delayed Dirac) is related to the phase is the

remainder (modular) function (mod 2π), since

$$\angle e^{-\jmath\omega T} = \texttt{rem}(\omega T, 2\pi).$$

Interestingly, in this instance the arguments of the remainder function are real ($\in \mathbb{R}$). Modular arithmetic may be alternatively written as $mod(t, T) = ((t))_T$. Thus modular functions are the group of periodic functions. In Matlab/Octave `unwrap()` is the inverse of `mod()`.

The Euclidean algorithm: The algorithm is best explained by a trivial example: Let the two numbers be $6, 9$. At each step the smaller number (6) is subtracted from the larger (9) and the smaller number and the difference (the remainder) are saved. This process continues until the two resulting numbers are equal, which is the GCD. For our example: $9 - 6 = 3$, leaving the smaller number 6 and the difference 3. Repeating this we get $6 - 3 = 3$, leaving the smaller number 3 and the difference 3. Since these two numbers are the same we are done and the GCD=3. We can easily verify this result by factoring [e.g., $(9, 6) = 3(3, 2)$]. The value may also be numerically verified using the Matlab/Octave GCD command $gcd(6, 9)$, which returns 3.

Direct matrix method: The GCD may be written as a matrix recursion given the starting vector $(m_0, n_0)^T$. The recursion is then

$$\begin{bmatrix} m_{k+1} \\ n_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_k \\ n_k \end{bmatrix}. \tag{2.7}$$

This recursion continues until $m_{k+1} < n_{k+1}$, at which point m and n must be swapped. The process is repeated until $m = n$, which equals the GCD. The direct method is inefficient because it recursively subtracts n many times until the resulting m is less than n, as shown in Fig. 2.4. It also must test for $m \leq n$ at each iteration, and then swap m and n if they are not equal.

Since this matrix may be inverted, the process may be reversed, which is closely related to the *continued fraction algorithm* (CFA) (p. 58).

Why is the GCD important?   The utility of the GCD algorithm arises directly from the fundamental difficulty in factoring large integers. Computing the GCD, using the Euclidean algorithm, is low cost, compared to factoring when finding the coprime factors, which is extremely expensive. This utility surfaces when the two numbers are composed of very large primes.

When two integers have no common factors they are said to be *coprimes*, and their GCD is 1. The ratio of two integers which are coprime is automatically in *reduced form* (they have no common factors). For example, $4/2 \in \mathbb{Q}$ is not reduced since $2 = gcd(4, 2)$ (with no remainder). Canceling out the common factor 2 gives the reduced form $2/1 \in \mathbb{F}$. Thus if we wish to form the ratio of two integers, first compute the GCD, then remove it from the two numbers to form the ratio. This assures the rational number is in its reduced form ($\in \mathbb{F}$ rather than $\in \mathbb{Q}$). If the GCD were 10^3 digits, it is obvious that any common factor would need to be be removed, thus greatly simplifying further computation. This will make a huge difference when using IEEE-754.

The floor function and the GCD are related in an important way, as discussed next.

Exercise: What does it mean to reach the turning point when using the Euclidean algorithm? **Solution:** When $m/n - \lfloor m/n \rfloor < n$ we have reached a turning point. When the remainder is zero (i.e., $m/n - \lfloor m/n \rfloor = 0$), we have reached the GCD.

The GCD and long division: When one learns how to divide a smaller number into a larger one, they must learn how to use the GCD. For example, suppose we which to compute $6 \div 110$. One starts by finding out how many times 6 goes into 11. Since $6 \times 2 = 12$, which is larger than 11, the answer is 1. We then subtract 6 from 11 to find the remainder 5. This is, of course, the floor function (e.g., $\lfloor 11/6 \rfloor = 1, \lfloor 11/5 \rfloor = 2$).

Example: Divide 10 into 99, the *floor* function (`floor(99/10)`) must be used, followed by the *remainder* function (`rem(99,10)`). **Solution:** When we divide a smaller number into a larger one, we must first find the floor followed by the remainder. For example $99/10 = 9 + 9/10$ has a floor of 9 and a remainder of 9/10.

Example: Start with the two integers $[873, 582]$. In factored form these are $[\pi_{25} \cdot 3^2, \pi_{25} \cdot 3 \cdot 2]$. Given the factors, we see that the largest common factor is $\pi_{25} \cdot 3 = 291$ ($\pi_{25} = 97$). When we take the ratio of the two numbers this common factor cancels

$$\frac{873}{582} = \frac{\pi_{25} \cdot \cancel{3} \cdot 3}{\pi_{25} \cdot \cancel{3} \cdot 2} = \frac{3}{2} = 1.5.$$

5   Of course if we divide 582 into 873 we will numerically obtain the answer $1.5 \in \mathbb{F}$.

Exercise: Show that in Matlab/Octave $\mathtt{rat(873/582)} = 1 + 1/(-2)$ gives the wrong answer. **Solution:** Since

$$\mathtt{factor(873)} = 3 \cdot 3 \cdot 97 \text{ and } \mathtt{factor(582)} = 2 \cdot 3 \cdot 97,$$

thus `3/2 = 1 + 1/2` is the correct answer. But due to rounding methods, it is not 3/2. As an example, in Matlab/Octave `rat(3/2)=2+1/(-2)`. Matlab's `rat()` command uses rounding rather than the floor function, which explains the difference. When the `rat()` function produces negative numbers, rounding is employed.
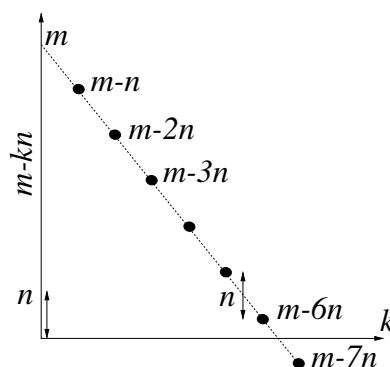
Figure 2.4: *The Euclidean algorithm recursively subtracts n from m until the remainder $m - kn$ is either less than n or zero. Note that this recursion is the same as $mod(m,n)$. Thus the GCD recursively computes $\mathtt{mod(m,n)}$ until the remainder $\mathtt{rem(m,n)}$ is less than n. It then swaps m, n, so that $n < m$. This repeats, until it terminates on the GCD. For the case depicted here the value of k, that renders the remainder $m - 6n < n$. If one more step were taken ($k = 7$) the remainder would become negative. By linear interpolation we can find that $m - \alpha n = 0, \alpha \in \mathbb{R}$, where the solid line cross the abscissa. For this example $k = 6.5$, thus $6 = \lfloor m/n \rfloor < n$. This is nonlinear (truncation) arithmetic, which is a property related to the GCD. The $\mathtt{floor()}$ functions finds the* turning point, *where we swap the two numbers, since by definition, $m > n$. In this example $k = 6 = \lfloor m/n \rfloor$.*

10   **Graphical description of the GCD:**   The Euclidean algorithm is very simple when viewed graphically. In Fig. 2.4 we show what is happening as one approaches the threshold. After reaching the threshold, the two numbers must be swapped, which is addressed by the upper row of Eq. 2.8.

Below is a simple Matlab/Octave code to find $k=\mathtt{gcd(m,n)}$ based on the Stillwell (2010) definition of the EA:

```
function k = gcd(m,n)
  while m ~=0
    A=m; B=n;
    m=max(A,B); n=min(A,B); %m>n
    m=m-n;
```

```
end
```

This program loops until $m = 0$.

Multiplication is simply recursive addition, and finding the GCD takes advantage of this fact. Let's take a trivial example, $(9, 6)$. Taking the difference $9 - 6$ (the larger from the smaller), and writing multiplication as sums, one can see what is going on. The GCD may be written two equivalent ways:

$$9 - 6 = (3 + 3 + 3) - (3 + 3) = 0 + 0 + 3 = 3,$$

and

$$9 - 6 = (3 + 3 + 3) - (2 + 2 + 2) = 1 + 1 + 1 = 3.$$

Written the first way, it is 3, because subtracting $(3+3)$ from $(3+3+3)$ leaves 3. Written the second way, each 3 is matched with a -2, leaving 3 ones, which add to 3. Of course the two decomposition must yield the same result because $2 \cdot 3 = 3 \cdot 2$. Thus finding the remainder of the larger number minus the smaller yields the GCD of the two numbers.

Coprimes: When the GCD of two integers is 1, the only common factor is 1. This is of key importance when trying to find common factors between the two integers. When `1=gcd(m,n)` they are said to be *coprime* or "relatively prime," which is frequently written as $m \perp n$. By definition, the largest common factor of coprimes is 1. But since 1 is not a prime ($\pi_1 = 2$), they have no common primes. It can be shown (Stillwell, 2010, p. 41-42) that when $a \perp b$, there exist $m, n \in \mathbb{Z}$ such that

$$am + bn = gcd(a, b) = 1.$$

A much more efficient method uses the `floor()` function, which is called *division with rounding*. Specifically the GCD may be written in one step as

$$\begin{bmatrix} m \\ n \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor \frac{m}{n} \rfloor \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix}_k. \tag{2.8}$$

This matrix is simply Eq. 2.7 to the power $\lfloor m/n \rfloor$, followed by swapping the inputs (the smaller must always be on the bottom).

2.3.2 Lec 6: Continued fraction algorithm

As shown in Fig. 2.5, the *continued fraction algorithm* (CFA) starts from a single real decimal number $x_o \in \mathbb{R}$, and recursively expands it as a fraction $x \in \mathbb{F}$ (Graham *et al.*, 1994). Thus the CFA may be used for forming rational approximations to any real number. For example, $\pi \approx 22/7$, and excellent approximation, well known to Chinese mathematicians.

The the Euclidean algorithm (i.e., GCD), on the other hand, operates on a pair of integers $m, n \in \mathbb{N}$ and returns their greatest common divisor $k \in \mathbb{N}$, such that $m/k, n/k \in \mathbb{F}$ are coprime, thus reducing the ratio to its irreducible form (i.e., $m/k \perp n/k$). Note this is done without factoring m, n.

Despite this seemingly irreconcilable difference between the GCD and CFA, the two are closely related, so close that Gauss called the Euclidean algorithm, for finding the GCD, as the continued fraction algorithm (CFA) (Stillwell, 2010, P. 48). At first glance it is not clear why Gauss would be so "confused." One is forced to assume that Gauss had some deeper insight into this relationship. If so, that insight would be valuable to understand.[22]

---

[22]In some sense, to be explored, the GCD and CFA are inverses.

Definition of the CFA

1. Start with $n = 0$ and the positive input target $x_o \in \mathbb{R}^+$. $n = 0, m_0 = 0, x_o = \pi$

2. <u>Rounding</u>: Let $m_n = \lfloor x_n \rfloor \in \mathbb{N}$. $m_0 = \lfloor \pi \rfloor = 3$

3. The input vector is then: $[x_n, m_n]^T$. $[\pi, 3]_0^T$.

4. <u>Remainder</u>: $r_n = x_n - m_n$ $(-0.5 \leq r_n \leq 0.5)$. $r_0 = \pi - 3 \approx 0.14159$

5. <u>Reciprocate</u>:

$$x_{n+1} = \begin{cases} 1/r_n, & \text{return to step 2} \quad r_n \neq 0 \\ 0, & \text{terminate} \quad\quad r_n = 0. \end{cases}$$

   $x_1 = 1/0.14159 = 7.062646$

6. The output vector is $[x_{n+1}, m_n]^T$. $[7.062646, 3]^T$

Figure 2.5: *The CFA of any positive number, $x_o \in \mathbb{R}^+$, is defined in this figure. Numerical values for $n = 0, x_o = \pi, m_0 = 0$ are given in blue on the far right. For $n = 1$ the input vector is $[x_1, m_1]^T = [7.062646, 3]^T$. If at any step the remainder is zero, the algorithm terminates (Step 5). Convergence is guaranteed. The recursion may continue to any desired accuracy. Alternative rounding schemes are given on p. 248.*

Notation: Writing out all the fractions can become tedious. For example, expanding $e = 2.7183 \cdots$ using the Matlab/Octave command `rat(exp(1))` gives the approximation

$$\exp(1) = 3 + 1/(-4 + 1/(2 + 1/(5 + 1/(-2 + 1/(-7))))) - o\left(1.75 \times 10^{-6}\right),$$
$$= [3; -4, 2, 5, -2, -7] - o(1.75 \times 10^{-6}).$$

Here we use a compact bracket notation, $\hat{e}_6 \approx [3; -4, 2, 5, -2, -7]$ where $o()$ indicates the error of the CFA expansion.

Since many entries are negative, we may deduce that rounding arithmetic is being used by Matlab/Octave (but this is not documented). Note that the leading integer part may be noted by an optional semicolon.[23] If the process is carried further, the values of $a_n \in \mathbb{N}$ give increasingly more accurate rational approximations.

Example: Let $x_o \equiv \pi \approx 3.14159\ldots$. Thus $a_o = 3, r_o = 0.14159, x_1 = 7.065 \approx 1/r_o$, and $a_1 = 7$. If we were to stop here we would have

$$\hat{\pi}_2 \approx 3 + \frac{1}{7 + 0.0625\ldots} \approx 3 + \frac{1}{7} = \frac{22}{7}. \tag{2.9}$$

This approximation of $\pi \approx 22/7$ has a relative error of 0.04%

$$\frac{22/7 - \pi}{\pi} = 4 \times 10^{-4}.$$

Example: For a second level of approximation we continue by reciprocating the remainder $1/0.0625 \approx 15.9966$ which rounds to 16 giving a negative remainder of $\approx -1/300$:

$$\hat{\pi}_3 \approx 3 + 1/(7 + 1/16) = 3 + 16/(7 \cdot 16 + 1) = 3 + 16/113 = 355/113.$$

---

[23]Unfortunately Matlab/Octave does not support the bracket notation.

With rounding the remainder is -0.0034, resulting in a much more rapid convergence. If floor rounding is used ($15.9966 = 15 - 0.9966$) the remainder is positive and close to 1, resulting in a much less accurate rational approximation for the same number of terms. It follows that there can be a dramatic difference depending on the rounding scheme, which, for clarity, should be specified, not inferred.

Rational approximation examples

$$\hat{\pi}_2 = \frac{22}{7} = [3; 7] \qquad\qquad\qquad \approx \hat{\pi}_2 + o(1.3 \times 10^{-3})$$

$$\hat{\pi}_3 = \frac{355}{113} = [3; 7, 16] \qquad\qquad\qquad \approx \hat{\pi}_3 - o(2.7 \times 10^{-7})$$

$$\hat{\pi}_4 = \frac{104348}{33215} = [3; 7, 16, -249] \qquad\qquad \approx \hat{\pi}_4 + o(3.3 \times 10^{-10})$$

Figure 2.6: The expansion of π to various orders, using the CFA, along with the order of the error of each rational approximation, with rounding. For example, $\hat{\pi}_2 = 22/7$ has an absolute error ($|22/7 - \pi|$) of about 0.13%.

CFA as a matrix recursion: Start with $n_0 = 0 \in \mathbb{N}$, $x_0 \in \mathbb{I}$, $n_1 = \lfloor x_0 \rfloor \in \mathbb{N}$, $r_1 = x - \lfloor x \rfloor \in \mathbb{I}$ and $x_1 = 1/r_1 \in \mathbb{I}$, $r_n \neq 0$.

5
May need work.

In matrix notation, with $k \in \mathbb{N}$

$$\begin{bmatrix} n \\ x \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & \lfloor x_0 \rfloor \\ 0 & \frac{1}{x_0 - \lfloor x_0 \rfloor} \end{bmatrix} \begin{bmatrix} n \\ x \end{bmatrix}_k$$

This generates the next CFA parameter n_1, and $x_1 = 1/r_1$ from n_0 and x_0.

Example: If $x_0 = \pi$, then $n_1 = \lfloor \pi \rfloor = 3$, $r_1 = \pi - n_1 = 0.141592656 \cdots$, and $x_1 = 1/r_1$:

$$\begin{bmatrix} 3 \\ 7.06251331 \end{bmatrix}_1 = \begin{bmatrix} 0 & \lfloor \pi \rfloor \\ 0 & \frac{1}{\pi - \lfloor \pi \rfloor} \end{bmatrix} \begin{bmatrix} 0 \\ \pi \end{bmatrix}_0$$

and for $n = 2$

$$\begin{bmatrix} 7 \\ 15.99659 \end{bmatrix}_2 = \begin{bmatrix} 7 \\ \frac{1}{0.06251331} \end{bmatrix}_2 = \begin{bmatrix} 0 & 7 \\ 0 & \frac{1}{7.0625 - 7} \end{bmatrix} \begin{bmatrix} 3 \\ 7.06251331 \end{bmatrix}_1$$

For $n = 3$, $\pi_3 = [n_1; n_2, n_3] = [3; 7, 15]$. Continuing $n_4 = \lfloor 1.003418 \rfloor = 1$ and $n_5 = 292$. Clean up indexing.

Exercise: Find the CFA using the floor function, to 12th order.
Solution: $\hat{\pi}_{12} = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1]$.

10

Example: Matlab/Octave's `rat(pi,1e-20)` gives:
```
3 + 1/(7 + 1/(16 + 1/(-294 + 1/(3 + 1/(-4 + 1/(5 + 1/(-15 + 1/(-3))))))))
```
In bracket notation

$$\hat{\pi}_9 = [3; 7, 16, -294, 3, -4, 5, -15, 03].$$

Because the sign changes, it is clear that Matlab/Octave use rounding rather than the floor function.

Exercise: Based on several examples, which rounding scheme is the most accurate? Explain why.
Solution: Rounding will result in a smaller remainder at each iteration, thus a smaller net error and thus improved convergence.

When the CFA is applied and the expansion terminates ($r_n = 0$), the target is rational. When the expansion does not terminate (which is not always easy to determine, as the remainder may be ill-conditioned due to small numerical rounding errors), the number is irrational. Thus the CFA has important theoretical applications regarding irrational numbers. You may explore this using Matlab's `rats(pi)` command.

Besides these five basic rounding schemes, there are two other important $\mathbb{R} \to \mathbb{N}$ functions (i.e., mappings), which will be needed later: `mod(x,y)`, `rem(x,y)` with $x, y \in \mathbb{R}$. The base 10 numbers may be generated from the counting numbers using `y=mod(x,10)`.

Exercise:

1. Show how to generate a base-10 real number $y \in \mathbb{R}$ from the counting numbers $\mathbb{N}$ using the `m=mod(n,10)`$+k$`10` with $n, k \in \mathbb{N}$. **Solution:** Every time n reaches a multiple of 10, m is reset to 0 and the next digit to the left is increased by 1, by adding 1 to k, generating the digit pair km. Thus the mod() function forms the underlying theory behind decimal notation.

2. How would you generate binary numbers (base 2) using the `mod(x,b)` function? **Solution:** Use the same method as in the first example above, but with $b = 2$.

3. How would you generate hexadecimal numbers (base 16) using the `mod(x,b)` function? **Solution:** Use the same method as in the first example above, but with $b = 16$.

4. Write out the first 19 numbers in hex notation, starting from zero. **Solution:** 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F, 10, 11, 12. Recall that $10_{16} = 16_{10}$ thus $12_{16} = 18_{10}$, resulting in a total of 19 numbers if we include 0.

5. What is FF_{16} in decimal notation? **Solution:** `hex2dec('ff')` $= 255_{10}$.

Symmetry: A *continued fraction expansion* can have a high degree of "recursive symmetry." For example, the CFA of

$$R_1 \equiv \frac{1 + \sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}} = 1.618033988749895 \cdots . \tag{2.10}$$

Here a_n in the CFA is *always* 1 ($R_1 \equiv [1., 1, 1, \cdots]$), thus the sequence cannot terminate, *proving* that $\sqrt{5} \in \mathbb{I}$. A related example is $R_2 \equiv$ `rat(1+sqrt(2))`, which gives $R_2 = [2., 2, 2, 2, \cdots]$.

When expanding a target irrational number ($x_o \in \mathbb{I}$), and the CFA is truncated, the resulting rational fraction approximates the irrational target. For the example above, if we truncate at three coefficients ($[1; 1, 1]$) we obtain

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + 0}} = 1 + 1/2 = 3/2 = 1.5 = \frac{1 + \sqrt{5}}{2} + 0.118 \cdots .$$

Truncation after six steps gives

$$[1., 1, 1, 1, 1, 1, 1] = 13/8 \approx 1.6250 = \frac{1 + \sqrt{5}}{2} + .0070 \cdots .$$

Because all the coefficients are 1, this example converges very slowly. When the coefficients are large (i.e., remainder small), the convergence will be faster. The expansion of π is an example of faster convergence.

Exercise: Discuss the relationship between the CFA and the transmission line modeling method of Sect. 3.5 (p. 123). **Solution:** The solution is detailed in Appendix P (p. 323).

$$\text{Disc: } \frac{1}{1 - z^{-\pi_k}} = 1 + z^{-\pi_k} + z^{-2\pi_k} \cdots$$

In summary: Every rational number $m/n \in \mathbb{F}$, with $m > n > 1$, may be uniquely expanded as a continued fraction, with coefficients a_k determined using the CFA. When the target number is irrational ($x_o \in \mathbb{Q}$), the CFA does not terminate; thus, each step produces a more accurate rational approximation, converging in the limit as $n \to \infty$.

5

Thus the CFA expansion is an algorithm that can, in theory, determine when the target is rational, but with an important caveat: one must determine if the expansion terminates. This may not be obvious. The fraction $1/3 = 0.33333 \cdots$ is an example of such a target, where the CFA terminates yet the fraction repeats. It must be that

$$1/3 = 3 \times 10^{-1} + 3 \times 10^{-2} + 3 \times 10^{-3} + \cdots .$$

Here 3*3=9. As a second example[24]

$$1/7 = 0.142857142857142857142857 \cdots = 142857 \times 10^{-6} + 142857 \times 10^{-12} + \cdots$$

There are several notations for repeating decimals such as $1/7 = 0.1\overline{142857}$ and $1/7 = 0.1((142857))$. Note that $142857 = 999999/7$. Related identities include $1/11 = 0.090909 \cdots$ and $11 \times 0.090909 = 999999$. When the sequence of digits repeats, the sequence is predictable, and it must be rational. But it is impossible to be sure that it repeats because the length of the repeat can be arbitrarily long.

10

One of the many useful things about the procedure is its generalizations to the expansion of transmission lines, as complex functions of the Laplace complex frequency s. In Assignment 4.5.4, [see Fig. 4.10 (p. 183)], a transmission line is analogized in terms of the CFA, but residue-expanded (p. 84) in terms of the Laplace frequency s, rather than as integers. This method, called a *Cauer synthesis*, is based on the ABCD transmission line method of Secs. 3.5 (p. 123).

15

2.3.3 Lec 7: Pythagorean triplets (Euclid's formula)

Euclid's formula is a method for finding three integer lengths $[a, b, c] \in \mathbb{N}$ that satisfy Eq. 1.1. It is important to ask: Which set are the lengths [a,b,c] drawn from? There is a huge difference, both practical and theoretical, if they are from the real numbers $\mathbb{R}$, or the counting numbers $\mathbb{N}$. Given $p, q \in \mathbb{N}$ with $p > q$, the three lengths $[a, b, c] \in \mathbb{N}$ of Eq. 1.1 are given by

$$a = p^2 - q^2, \qquad b = 2pq, \qquad c = p^2 + q^2. \tag{2.11}$$
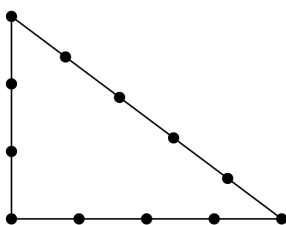
Figure 2.7: *Beads on a string form perfect right triangles when the number of unit lengths between beads for each side satisfies Eq. 1.1. For example, when $p = 2, q = 1$, the sides are $[3, 4, 5]$.*

This result may be directly verified, since

$$[p^2 + q^2]^2 = [p^2 - q^2]^2 + [2pq]^2$$

---

[24]Taking the Fourier transform of the target number, represented as a sequence, could help to identify an underlying periodic component. The number $1/7 \leftrightarrow [[1, 4, 2, 8, 5, 7]]_6$ has a 50 [dB] notch at 0.8π [rad] due to its 6 digit periodicity, carried to 15 digits (Matlab/Octave maximum precision), Hamming windowed, and zero padded to 1024 samples.

or

$$p^4 + q^4 + \cancel{2p^2q^2} = p^4 + q^4 - \cancel{2p^2q^2} + \cancel{4p^2q^2}.$$

Thus, Eq. 2.11 is easily proven, once given. Deriving Euclid's formula is obviously much more difficult, and is shown in Fig. 3.4 (p. 101).

A well-known example is the right triangle depicted in Fig. 2.7 (p. 61), defined by the integers $[3, 4, 5] \in N$, having angles $[0.54, 0.65, \pi/2]$ [rad], which satisfies Eq. 1.1 (p. 21). As quantified by Euclid's formula (Eq. 2.11), there are an infinite number of *Pythagorean triplets* (PTs). Furthermore the seemingly simple triangle, having angles of $[30, 60, 90] \in N$ [deg] (i.e., $[\pi/6, \pi/3, \pi/2] \in \mathbb{I}$ [rad]), has one irrational ($\mathbb{I}$) length ($[1, \sqrt{3}, 2]$).

a	c
119	169
3367	4825
4601	6649
12709	18541
65	97
319	481
2291	3541
799	1249
481	769
4961	8161
45	75
1679	2929
161	289
1771	3229
56	106

Figure 2.8: *"Plimpton-322" is a stone tablet from 1800 [BCE], displaying a and c values of the Pythagorean triplets $[a, b, c]$, with the property $b = \sqrt{c^2 - a^2} \in \mathbb{N}$. Several of the c values are primes, but not the a values. The stone is item 322 (item 3 from 1922) from the collection of George A. Plimpton.*

Table 2.2: *Table of Pythagorean triples computed from Euclid's formula Eq. 2.11 for various $[p, q]$. The last three columns are the first, fourth and next to last, from "Plimpton-322," along with their corresponding $[p, q]$. In all cases $c^2 = a^2 + b^2$ and $p = q + l$, where $l = \sqrt{c - b} \in \mathbb{N}$.*

q	1			2			3			5	125	27
p	2	3	4	3	4	5	4	5	6	12	54	50
a	3	8	15	5	12	21	7	16	27	119	12709	1771
b	4	6	8	12	16	20	24	30	36	120	13500	2700
c	5	10	17	13	20	29	25	34	45	169	18541	3229

The technique for proving Euclid's formula for PTs $[a, b, c] \in \mathbb{Q}$, derived in Fig. 3.4 (p. 101) is much more interesting than the PTs themselves.

The set from which the lengths $[a, b, c]$ are drawn was not missed by the Indians, Chinese, Egyptians, Mesopotamians, Greeks, etc. Any equation whose solution is based on integers is called a *Diophantine equation*, named after the Greek mathematician Diophantus of Alexandria (c250 CE) (Fig. 1.1, p. 21).

A stone tablet having the numbers engraved on it, as shown in Fig. 2.8, was discovered in Mesopotamia, from the 19th century [BCE], and cataloged in 1922 by George Plimpton.[25] [26] These numbers are a and c pairs from PTs [a,b,c]. Given this discovery, it is clear that the Pythagoreans were following those who

[25]http://www.nytimes.com/2010/11/27/arts/design/27tablets.html
[26]https://en.wikipedia.org/wiki/Plimpton_322

came long before them. Recently a second similar stone, dating between 350 and 50 [BCE] has been reported, that indicates early calculus on the orbit of Jupiter's moons, the very same moons that Rømer observed to show that the speed of light was finite (p. 26).[27]

2.3.4 Lec 8: Pell's equation

Pell's equation

$$x_n^2 - Ny_n^2 = (x_n - \sqrt{N}y_n)(x_n + \sqrt{N}y_n) = 1, \tag{2.12}$$

with non-square $N \in \mathbb{N}$ specified and $x, y \in \mathbb{N}$ unknown, has a venerable history in both physics (p. 35) and mathematics. Given its factored form it is obvious that every solution x_n, y_n has the asymptotic property

$$\left.\frac{x_n}{y_n}\right|_{n \to \infty} \to \pm\sqrt{N}.$$

It is believed that Pell's equation is directly related to the Pythagorean theorem, and therefore to the Euclidean algorithm, since they are both simple binomials having integer coefficients (Stillwell, 2010, 48), with Pell's equation being the hyperbolic version of Eq. 1.1. For example, with $N = 2$, a solution is $x = 17, y = 12$ (i.e., $17^2 - 2 \cdot 12^2 = 1$).

A 2×2 matrix recursion algorithm likely due to the Chinese, and used by the Pythagoreans to investigate $\sqrt{N}$, is

$$\begin{bmatrix} x \\ y \end{bmatrix}_{n+1} = \begin{bmatrix} 1 & N \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}_n, \tag{2.13}$$

where we indicate the index outside the vectors.

Starting with the trivial solution $[x_o, y_o]^T = [1, 0]^T$ (i.e., $x_o^2 - Ny_o^2 = 1$), additional solutions of Pell's equations are determined, having the propriety $\boldsymbol{x}_n/y_n \to \sqrt{N} \in \mathbb{F}$ (Stillwell, 2010, p. 44). This approach was motivated by the Euclidean algorithm (GCD, p. 54, 323), since $y_n/x_n \to \sqrt{2}$ (Stillwell, 2010, p. 37, 55).

Note that Eq. 2.13 is a 2x2 matrix composition method (see p. 105), since the output of one matrix multiply is the input to the next. Why Eqs. 2.13 and 2.12 are equivalent is explained in Section 3.2.3 (p. 94).

Asian solutions: The first solution of Pell's equation was published by Brahmagupta (c628), who independently discovered the equation (Stillwell, 2010, p. 46). Brahmagupta's novel solution also used the *composition method*, but different from Eq. 2.13 (Stillwell, 2010, p. 69). Then in 1150CE, Bhâskara II independently obtained solutions using Eq. 2.13 (Stillwell, 2010, p.69). This is the composition method we shall explore here, as summarized in Table E.1 (p. 268).

The best way to see how this recursion results in solutions to Pell's equation is by example. Initial-

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[27]http://www.nytimes.com/2016/01/29/science/babylonians-clay-tablets-geometry-astronomy-jupiter.html

izing the recursion with the trivial solution $[x_o, y_o]^T = [1, 0]^T$ gives

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad\qquad 1^2 - 2 \cdot 1^2 = -1$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad\qquad 3^2 - 2 \cdot 2^2 = 1$$

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad\qquad (7)^2 - 2 \cdot (5)^2 = -1$$

$$\begin{bmatrix} x_4 \\ y_4 \end{bmatrix} = \begin{bmatrix} 17 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} \qquad\qquad 17^2 - 2 \cdot 12^2 = 1$$

$$\begin{bmatrix} x_5 \\ y_5 \end{bmatrix} = \begin{bmatrix} 41 \\ 29 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 17 \\ 12 \end{bmatrix} \qquad\qquad (41)^2 - 2 \cdot (29)^2 = -1$$

Thus the recursion results in a modified version of Pell's equation

$$x_n^2 - 2y_n^2 = (-1)^n, \tag{2.14}$$

where only even values of n are solutions. This sign change had no effect on the Pythagoreans' goal, since they only cared about the ratio $y_n/x_n \to \pm\sqrt{2}$.

Modified recursion: The following summarizes the solution ($\in \mathbb{C}$) of Pell's equation for $N = 2$ using a slightly modified matrix recursion. Note that $x_n/y_n \to \sqrt{2}$ as $n \to \infty$, which was what the Pythagoreans were pursuing, and that the odd solutions are pure-imaginary.

As shown below the recursion may be slightly modified to fix the $(-1)^n$ problem, by multiplying the 2x2 matrix by $1\jmath = \sqrt{-1}$. This results in Pell's equation for every step of the recursion:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \jmath \begin{bmatrix} 1 \\ 1 \end{bmatrix}_1 = \jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_0 \qquad\qquad \jmath^2 - 2 \cdot \jmath^2 = 1$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_2 = \jmath^2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}_2 = \jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \jmath \begin{bmatrix} 1 \\ 1 \end{bmatrix}_1 \qquad\qquad 3^2 - 2 \cdot 2^2 = 1$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_3 = \jmath^3 \begin{bmatrix} 7 \\ 5 \end{bmatrix}_3 = \jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \jmath^2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}_2 \qquad\qquad (7\jmath)^2 - 2 \cdot (5\jmath)^2 = 1$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_4 = \begin{bmatrix} 17 \\ 12 \end{bmatrix}_4 = \jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \jmath^3 \begin{bmatrix} 7 \\ 5 \end{bmatrix}_3 \qquad\qquad 17^2 - 2 \cdot 12^2 = 1$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_5 = \jmath \begin{bmatrix} 41 \\ 29 \end{bmatrix}_5 = \jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 17 \\ 12 \end{bmatrix}_4 \qquad\qquad (41\jmath)^2 - 2 \cdot (29\jmath)^2 = 1$$

Solution to Pell's equation: By multiplying the matrix by $1\jmath$, all the solutions ($x_k \in \mathbb{C}$) to Pell's equation are determined. The $1\jmath$ factor corrects the alternation in sign, so every iteration yields a solution. Fig. E.1 (p. 268) shows that every output of this slightly modified matrix recursion gives solutions to Pell's equation. For $n = 0$ (the initial solution) $[x_0, y_0]$ is $[1, 0]_0$, $[x_1, y_1] = \jmath[1, 1]_1$, and $[x_2, y_2] = -[3, 2]_2$. These are easily checked using this recursion.

At each iteration, the ratio x_n/y_n approaches $\sqrt{2}$ with increasing accuracy, coupling it to the CFA which may also be used to find approximations to $\sqrt{N}$. The value of $41/29 \approx \sqrt{2}$, with a relative error of $<0.03\%$. The solution for $N = 3$ is given in Appendix E.1.1 (p. 268).

2.3.5   Lec 9: Fibonacci sequence

Another classic problem, also formulated by the Chinese, was the Fibonacci sequence, generated by the relation

$$f_{n+1} = f_n + f_{n-1}. \tag{2.15}$$

Here the next number $f_{n+1} \in \mathbb{N}$ is the sum of the previous two. If we start from $[0, 1]$, this difference equation leads to the Fibonacci sequence $f_n = [0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots]$. Alternatively, if we define $y_{n+1} = x_n$, then Eq. 2.15 may be compactly represented by a 2x2 *companion matrix* recursion (see Exercise on p. 70)

$$\begin{bmatrix} x \\ y \end{bmatrix}_{n+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}_n, \tag{2.16}$$

which has eigenvalues $(1 \pm \sqrt{5})/2$.

The correspondence of Eqs. 2.15 and 2.16 is easily verified. Starting with $[x, y]_0^T = [0, 1]^T$ we obtain for the first few steps

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}_0, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_1, \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_2, \quad \begin{bmatrix} 3 \\ 2 \end{bmatrix}_4 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}_3, \quad \cdots.$$

From the above $x_n = [0, 1, 1, 2, 3, 5, \ldots]$ is the Fibonacci sequence, since the next x_n is the sum of the previous two, and the next y_n is x_n.
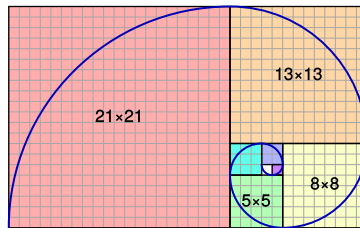
Figure 2.9: *This construction is called the Fibonacci spiral. Note how it is constructed out of squares having areas given by the square of the Fibonacci numbers. In this way, the spiral is smooth and the radius increases as the Fibonacci numbers (e.g., 8=3+5, 13 = 5+8, etc.). (Adapted from* `https://en.wikipedia.org/wiki/Golden_spiral`.)

Exercise: Use the Octave/Matlab command `compan(c)` to find the companion matrix of the polynomial coefficients defined by Eq. 2.15. **Solution:** Using the Matlab/Octave code: `c=[1, -1, -1]; C=compan(c);` returning

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.17}$$

5

Exercise: Find the eigenvalues of matrix C. **Solution:** The characteristic equation is

$$\det \begin{bmatrix} 1 - \lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0$$

or $\lambda^2 - \lambda - 1 = (\lambda - 1/2)^2 - 1/4 - 1 = 0$, which has roots $\lambda_\pm = (1 \pm \sqrt{5})/2 \approx \{1.618, -0.618\}$.

The mean-Fibonacci sequence: Suppose that the Fibonacci sequence recursion is replaced by the mean of the last two values, namely let

$$f_{n+1} = \frac{f_n + f_{n-1}}{2}. \tag{2.18}$$

This seems like a small change. But how does the solution differ? To answer this question it is helpful to look at the corresponding 2x2 matrix.

Exercise: Find the 2x2 matrix corresponding to Eq. 2.18. The 2x2 matrix may be found using the *companion matrix* method (p. 83). **Solution:** Using Matlab/Octave code:

```
A=[1, -1/2, -1/2];
C=compan(A);
```

which returns

$$A = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$$

Exercise: Find the sequence from the mean-Fibonacci starting from $[1, 0]_0$. **Solution:** By inspection one steady-state solution is $[1, 1]_\infty$. What is the second steady-state in the solution?

Relations to digital signal processing: Today we recognize Eq. 2.15 as a *difference equation,* which
is a pre-limit (pre Stream 3) form of a differential equation. The Greek 2x2 matrix form of Eq. 2.15 is an early precursor to 17th and 18th century developments in linear algebra. Thus the Greeks' recursive solution for the $\sqrt{2}$ and Bhâskara's (1030 CE) solution of Pell's equation are early precursors to discrete-time processing, as well as to calculus.

There are strong similarities between Pell's equation and the Pythagorean theorem. As we shall
see in Appendix O, Pell's equation is related to the geometry of a hyperbola, just as the Pythagorean equation is related to the geometry of a circle. We shall show, as one might assume, that there is a Euclidean formula for the case of Pell's equations, since these are all conic sections with closely related conic geometry. As we have seen, the solutions involve $\sqrt{-1}$. The derivation is a trivial extension of that for the Euclidean algorithm. The early solution of Brahmagupta was not related to such a simple
formula. Rather it was based on the recursion Eq. 2.15 (Stillwell, 2010).

It makes one wonder if Eq. 1.1, written as $x_n^2 + y_n^2 = z_n^2$, with $x_n, y_n, z_n \in \mathbb{N}$, has a 2x2 matrix composition solution.

Exercise: Work out the details of this conjecture. Hint: $z^2 = x^2 + y^2 = (x + y\jmath)(x - y\jmath)$. **Solution:** The solution may be found in Allen (2019).

Chapter 3

Algebraic Equations: Stream 2

The era of 1640 to 1900 (Fig. 3.1) produced a continuous stream of fundamental theorems. A few of the individuals who played notable roles in this development, in chronological (birth) order, include: Galileo, Mersenne, Newton, d'Alembert, Fermat, Huygens, Descartes and Helmholtz. These individuals were some of the first to develop the basic ideas, in various forms, that were then later reworked into the proofs, that today we recognize as *the fundamental theorems of mathematics*.

Chronological history from the 17th century

17th *Newton 1642-1727, Bernoulli, Johann 1667–1748*

18th *Bernoulli, Daniel, Cauchy 1789–1857, Euler 1707–1783, d'Alembert 1717–1783, Gauss 1777–1855*

19th *Helmholtz 1821–1894, Kirchhoff 1824–87, Kelvin 1824–1907, Riemann 1826–1866, Maxwell 1831–1879, Rayleigh 1842–1919, Heaviside 1850–1925, Poincare 1854–1912*

20th *Harvey Fletcher 1884–1981, Arnold Sommerfeld 1886–1951, Albert Einstein 1879–1955, Léon Brillouin 1889–1969, Harry Nyquist 1889–1976 . . .*

21^{21} *Hendrik Wade Bode 1905–1982,*

Time Line
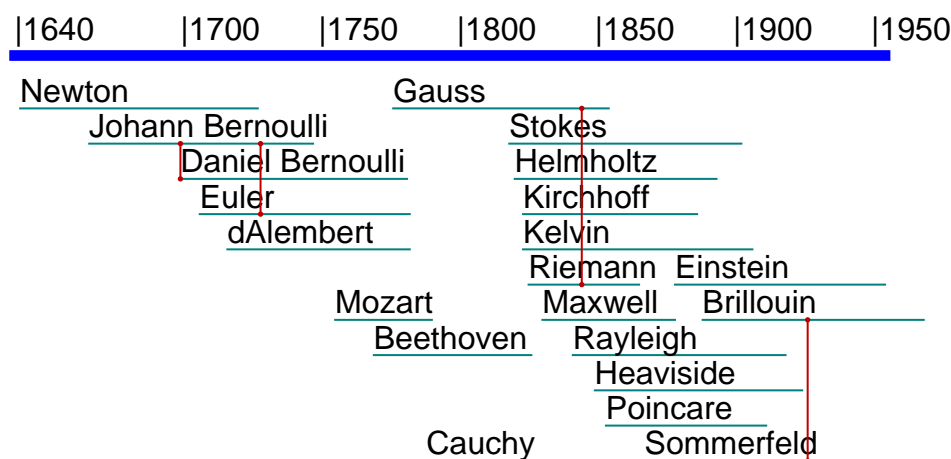
Figure 3.1: *timeline of the three centuries from the mid 17th to 20th CE, one of the most productive times of all, because mathematicians were sharing information. Figure 1.2 (p. 24) (Bombelli-Gauss) provides a closer look at the 16–19 CE, and Fig. 4.1 (p.146) (Bombelli-Einstein) provides the full view 16–20 CE.*

3.1   Lec 10: The physics behind nonlinear Algebra (Euclidean geometry)

Following Stillwell's history of mathematics, Stream 2 is geometry, which led to the merging of Euclid's geometrical methods and the 9th century development of algebra by al-Khwarizmi (830 CE). This integration of ideas led Descartes and Fermat to develop *analytic geometry*.

The mathematics up to the time of the Greeks, documented and formalized by Euclid, served students of mathematics for more than two thousand years. Algebra and geometry were, at first, independent lines of thought. When merged, the focus returned to the Pythagorean theorem. Algebra generalized the analytic conic section, greatly extending the geometrical approach as taught in Euclid's *Elements*. With the introduction of algebra, numbers, rather than lines, could be used to represent a geometrical length. Thus the appreciation for geometry grew, given the addition of the rigorous analysis using numbers.

Physics inspires algebraic mathematics: The Chinese used music, art, and navigation to drive mathematics. Unfortunately much of their knowledge has been handed down either as artifacts, such as musical bells and tools, or mathematical relationships documented, but not created, by scholars such as Euclid, Archimedes, Diophantus, and perhaps Brahmagupta. With the invention of algebra by al-Khwārizmī (830CE), mathematics became more powerful, and blossomed. During the 16th and 17th century, it had become clear that differential equations (DEs), such as the wave equation, can characterize a law of nature, at a single point in space and time. This principle was not obvious. A desire to understand motions of objects and planets precipitated many new discoveries. This period is illustrated in Fig. 1.2 (p. 24).

As previously described, the law of gravity was first formulated by Galileo using the concept of conservation of energy, which determines how masses are accelerated when friction is not considered and the mass is constant. Kepler investigated the motion of the planets. While Kepler was the first to observe that the orbits of planets are described by ellipses, it seems he under-appreciated the significance of his finding, and continued working on his incorrect epicycle planetary model. Following up on Galileo's work (Galileo, 1638), Newton (c1687) went on to show that there must be a gravitational potential between two masses (m_1, m_2) of the form

$$\phi_{\text{New}}(r(t)) \propto \frac{m_1 m_2}{r(t)}, \tag{3.1}$$

where $r = |x_1 - x_2|$ is the Euclidean distance between the two point masses at locations x_1 and x_2. Using algebra and his calculus, Newton formalized the equation of gravity, forces and motion (Newton's three laws), the most important being

$$f(t) = \frac{d}{dt} M v(t), \tag{3.2}$$

and showed that Kepler's discovery of planetary elliptical motion naturally follows from these laws (see p. 11). With the discovery of Uranus (1781) "Kepler's theory was ruined" (i.e., proven wrong) (Stillwell, 2010, p. 23).

Once Newton proposed the basic laws of gravity (Eq. 3.2), and resolved the elliptical motion of the planets, he proceeded to calculate, for the first time, the speed of sound. This required some form of the pressure *wave equation*

$$\frac{\partial^2}{\partial x^2} \varrho(x, t) = \frac{1}{c_o^2} \frac{\partial^2}{\partial t^2} \varrho(x, t), \tag{3.3}$$

a key equation in mathematical physics. The speed of sound is

$$c_o = \sqrt{\frac{\eta_o P_o}{\rho_o}} = 343, \quad [\text{m/s}]$$

which is a function of the density $\rho_o = 1.12$ [kg/m^3] and the dynamic stiffness ηP_o of air.[1] In other units the speed of sound is 767 [mph], 1125.3 [ft/s], 1.125 [ft/ms] and 1234.8 [km/h]. One of the earliest measurements was by Marin Mersenne in 1630 as 1,380 [Parisian feet per second]. 1 English foot is equal to 1.06575 Paris feet.

Exercise: How much smaller is one from Paris France than one from London England? **Solution:** The French seem to have been about 94[%] smaller than the English, according to this measure.

If we substitute for the pressure

$$\varrho(x, t) = e^{\jmath(\omega t \pm 2\pi k x)}, \tag{3.4}$$

where t is time and x is position, we find that $2\pi k = 2\pi/\lambda = 2\pi f/c_o$, because $f\lambda = c_o$.

Case of dispersive wave propagation: This classic relation $\lambda f = c$ is deceptively simple, thus confusing, because $k = 2\pi/\lambda$ becomes complex (has both real and imaginary parts) in dispersive media (e.g., acoustic waves in tubes) when losses are considered (Kirchhoff, 1868; Mason, 1928). A more important example is the case of electron waves in silicon crystals (Brillouin, 1953). In these more general cases, the *wave number* $k(f) = 2\pi f/c$ is replaced with the complex analytic function we call the *propagation function* $\kappa(s)$ of s, i.e., the wave becomes the eigen-function of the wave equation

$$p^{\pm}(x, t) = P_o(s)e^{st}e^{\pm\kappa(s)x}. \tag{3.5}$$

In these more general cases the wave number $\kappa(s)$ must be a *complex analytic function* of the *Laplace frequency* $s = \sigma + \omega\jmath$, as used with the Laplace transform. This is because electron "waves" in a dispersive semi-conductor, such as silicon, are "filtered" in magnitude and phase. Silicon is a highly dispersive "wave-filter," forcing the wavelength to be a complex function of frequency. This view is elegantly explained by Brillouin (1953, Chap. 1) in his historic text. Modern acoustics contains a rich source of related examples (Morse, 1948; Beranek, 1954; Fletcher and Rossing, 2008; Ramo *et al.*, 1965).

Newton's *Principia* was finally published in 1687, and the general solution to Newton's wave equation [i.e., $p(x, t) = G(t \pm x/c)$], where G is *any* function, was first published 60 years later by d'Alembert (c1747), which showed that for single frequency sounds, the wavelength λ and frequency f were related by

$$f\lambda = c. \tag{3.6}$$

Today d'Alembert's single-frequency analytic wave solution must be written as Eq. 3.5 having a complex wave number $\kappa(s) = 2\pi/\lambda(s)$ [m^{-1}]. This formulation led to the frequency domain concept of the Laplace analysis, based on the *linearity* (i.e., superposition) property of the wave equation (Postulate P2: Lec. 3.7.1, p. 133).

Newton's value for the speed of sound in air c_o was incorrect by the thermodynamic constant $\sqrt{\eta_o}$, a problem that would take 200+ years to formulate. What was needed was the *adiabatic process* (the concept of constant-heat). For audio frequencies (0.02-20 [kHz]), the small temperature gradients cannot diffuse the distance of a wavelength in one cycle (Pierce, 1981; Boyer and Merzbach, 2011), "trapping" the heat energy in the wave.[2] The eventual fix to Newton's formula for the sound speed was to define the *dynamic stiffness* of air $\eta_o P_o$, where P_o (1 [atm] or 10^5 [Pa]) is the static stiffness of air. But this fix still ignores losses (Kirchhoff, 1868).

Newton's success was important because it quantified the physics behind the speed of sound, and demonstrated that momentum (mv), not mass m, was transported by the wave. His concept was correct, and his formulation using algebra and calculus represented a milestone in science. In air, assuming no visco-elastic losses, it is constant (i.e., $c_o = \sqrt{\eta_o P_o/\rho_o}$). When including losses the wave number

---

[1] $\eta_o = C_p/C_v = 1.4$ is the ratio of two thermodynamic constants and $P_o = 10^5$ [Pa] is the barometric pressure of air.

[2] There were other physical enigmas, such as the observation that sound disappears in a vacuum or that a vacuum cannot draw water up a column by more than 34 feet.

becomes a complex function of frequency, leading to Eq. 3.5. In periodic structures, again the wave number becomes complex due to diffraction, as commonly observed in optics (e.g., diffraction gratings) and acoustics. Thus Eq. 3.6 only holds for the simplest cases, but in general it must be considered as a complex analytic function of s, as $\kappa(s)$ in Eq. 3.5.

5   The corresponding discovery for the formula for the speed of light was made 174 years after *Principia*, by Maxwell (c1861). Maxwell's formulation also required great ingenuity, as it was necessary to hypothesize an experimentally unmeasured term in his equations, to get the mathematics to correctly predict the speed of light.

    It is somewhat amazing that to this day, we have failed to understand gravity significantly better than 10 Newton's theory. Perhaps this is too harsh, given the 1905 work of Einstein.[3]

3.1.1 The first Algebra

Prior to the invention of algebra, people worked out problems as sentences using an obtuse description of the problem (Stillwell, 2010, p. 93). Algebra changed this approach, resulting in a compact language of mathematics, where numbers are represented as abstract symbols (e.g., x and α). The problem to be solved could be formulated in terms of sums of powers of smaller terms, the most common being powers of some independent variable (i.e., time or frequency). If we set $a_n = 1$

$$P_n(z) \equiv z^n + a_{n-1}z^{n-1} + \cdots + a_o z^0 = z^m + \sum_{k=0}^{n-1} a_k z^k = \prod_{k=0}^{n} (z - z_k). \tag{3.7}$$

is called a *monic polynomial*. The coefficient a_n cannot be zero, or the polynomial would not be of degree n. The resolution is to force $a_n = 1$, since this simplifies the expression and does not change the roots.

15   The key question is: What values of $z = z_k$ result in $P_n(z_k) = 0$. In other words, what are the roots z_k of the polynomial? Answering this question consumed thousands of years, with intense efforts by many aspiring mathematicians. In the earliest attempts, it was a competition to evaluate mathematical acumen. Results were held as a secret to the death bed. It would be fair to view this effort as an obsession. Today the roots of any polynomial may be found, to high accuracy, by numerical methods. Finding roots 20 is limited by the numerical limits of the representation, namely by IEEE-754 (p. 41). There are also a number of important theorems.

    Of particular interest is composing a circle with a line, where the line does not touch the circle, and finding the intersection (root). There was no solution to this problem using geometry. The resolution of this problem is addressed in the assignments below.

25 3.1.2 Finding roots of polynomials

The problem of factoring polynomials has a history more than a millennium in the making. While the quadratic (degree $N = 2$) was solved by the time of the Babylonians (i.e., the earliest recorded history of mathematics), the cubic solution was finally published by Cardano in 1545. The same year, Cardano's student solved the quartic ($N = 4$). In 1826 (281 years later) it was proved that the quintic ($N = 5$) 30 could not be factored by analytic methods.

    As a concrete example we begin with the important but trivial case of the quadratic

$$P_2(x) = ax^2 + bx + c. \tag{3.8}$$

First note that if $a = 0$, the quadratic reduces to the monomial $P_1(x) = bx + c$. Thus we have the necessary condition that $a \neq 0$. The best way to proceed is to divide a out and work directly with the normalized quadratic $\widehat{P}_2(x) = \frac{1}{a}P_2(x)$. In this way we do not need to worry about the exception, $a = 0$.

---

[3]Gravity waves were experimentally measured for the first time while I was formulating Section R.

The *roots* are those values of x such that $\widehat{P}_2(x_k) = 0$. One of the first results (recorded by the Babylonians, c2000 BCE) was the factoring of this equation by *completing the square* (Stillwell, 2010, p. 93). One may isolate x by rewriting Eq. 3.8 as

$$\frac{1}{a}P_2(x) = (x + b/2a)^2 - (b/2a)^2 + c/a. \tag{3.9}$$

This is easily verified by expanding the squared term and canceling $(b/2a)^2$

$$\frac{1}{a}P_2(x) = [x^2 + (b/a)x + \cancel{(b/2a)^2}] - \cancel{(b/2a)^2} + c/a.$$

Setting Eq. 3.9 to zero and solving for the two roots $x_\pm$ gives the *quadratic formula*

$$x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{3.10}$$

The roots of $\widehat{P}_2(x)$, with $a = 1$, simplify to

$$x_\pm = -\frac{1}{2}b \pm \sqrt{(b/2)^2 - c}. \tag{3.11}$$

This can be further simplified. The term $b^2 - c > 0$ under the square root is called the *discriminant*, and in physics and engineering problems, $>99.9\%$ of the time it is negative. Finally $b/2 \ll \sqrt{c}$; thus, the most natural way (i.e., corresponding to the most common physical cases) of writing the solution is[4]

$$x_\pm = -\frac{1}{2}b \pm \jmath\sqrt{c - (b/2)^2} \approx -b/2 \pm \jmath\sqrt{c}. \tag{3.12}$$

This form separates the real and imaginary parts of the solution in a natural way. The term $b/2$ is called the damping, which accounts for losses in a resonant circuit, while the term $\sqrt{c}$, for mechanical, acoustical and electrical networks, is called the resonant frequency, typically written as ω_o. The last approximation ignores the (typically) minor correction to the resonant frequency, which in engineering practice is typically always ignored. Knowing that there is a correction is highlighted by this formula, making one aware it exists.

It is not required that $a, b, c \in \mathbb{R}$, but for physical problems of interest, this is always true ($>99.9\%$ of the time).

Summary: The quadratic equation and its solution are ubiquitous in physics and engineering. It seems obvious that instead of memorizing the meaningless Eq. 3.10, one should learn the physically meaningful solution, Eq. 3.12, obtained via Eq. 3.9, with $a = 1$. Arguably, the factored and normalized form (Eq. 3.9) is easier to remember, as a method (completing the square), rather than as a formula to be memorized.

Additionally, the real ($b/2$) and imaginary $\pm\sqrt{c}$ parts of the two roots have physical significance as the damping and resonant frequency. Equation 3.10 has none.

No insight is gained by memorizing the quadratic formula. To the contrary, an important concept is gained by learning how to complete the square, which is typically easier than identifying a, b, c and blindly substituting them into Eq. 3.10. Thus it's worth learning the alternate solution (Eq. 3.12) since it is more common in practice and requires less algebra to interpret the final answer.

---

[4]This is the case for mechanical and electrical circuits having small damping. Physically $b > 0$ is the damping coefficient and $\sqrt{c} > 0$ is the resonant frequency. One may then simplify the form as $x^2 + 2bx + c^2 = (x + b + \jmath c)(x + b - \jmath c)$.

Exercise: By direct substitution demonstrate that Eq. 3.10 is the solution of Eq. 3.8. Hint: Work with $\widehat{P}_2(x)$. **Solution:** Setting $a = 1$ the quadratic formula may be written

$$x_\pm = \frac{-b \pm 1\jmath\sqrt{4c - b^2}}{2}.$$

Substituting this into $\widehat{P}_2(x)$ gives

$$\begin{aligned}
\widehat{P}_\pm(x_\pm) &= x_\pm{}^2 + bx_\pm + c \\
&= \left(\frac{-b \pm \sqrt{b^2 - 4c}}{2} \right)^2 + b \left(\frac{-b \pm \sqrt{b^2 - 4c}}{2} \right) + c \\
&= \frac{1}{4} \left(b^2 \mp 2b\sqrt{b^2 - 4c} + (b^2 - 4c) \right) + \frac{1}{4} \left(-2b^2 \pm 2b\sqrt{b^2 - 4c} \right) + c \\
&= 0.
\end{aligned}$$

Exercise: By direct substitution, show that Eq. 3.12 satisfies Eq. 3.10. **Solution:** Putting Eq. 3.10 into Eq. 3.10 in factored form gives

$$\begin{aligned}
\widehat{P}_2(x) &= (x - x_+)(x - x_-) \\
&= \left(x + \frac{1}{2}b + \jmath\sqrt{c - (b/2)^2} \right) \left(x + \frac{1}{2}b - \jmath\sqrt{c - (b/2)^2} \right) \\
&= \left(x + \frac{b}{2} \right)^2 + \left(c - b^2/4 \right) \\
&= \left(x^2 + bx + b^2/4 \right) + \left(c - b^2/4 \right) \\
&= x^2 + bx + c.
\end{aligned}$$

In third grade I learned the times-table trick for 9:

$$9 \cdot n = (n - 1) \cdot 10 + (10 - n). \tag{3.13}$$

With this simple rule I did not need to depend on my memory for the 9 times tables.[5] Learning an algorithm is much more powerful than memorization of the 9 times tables. Note that the two terms $(n - 1)$ and $(10 - n)$, add to 9. How one thinks about a problem can have great impact.

Newton's method for finding roots of $P_n(s)$ Newton is well known for an approximate but efficient method to find the roots of a polynomial.[6] Consider polynomial $P_n(s) \in \mathbb{C}$ for $s \in \mathbb{C}$

$$P_n(s) = c_n(s - s_0)^n + c_{n-1}(s - s_0)^{n-1} + \cdots + c_1(s - s_0) + c_0,$$

where we may use Taylor's formula (p. 89) to determine the coefficients

$$c_n = \frac{1}{n!} \frac{d^n}{ds^n} P_n(s) \bigg|_{s=s_0}.$$

---

[5]E.g.: $9 \cdot 7 = (7 - 1) \cdot 10 + (10 - 7) = 60 + 3$ and $9 \cdot 3 = (3 - 1) \cdot 10 + (9 - 3) = 20 + 7$. By expanding the above, one can see why it works: $9n = n10 - 10 + 10 - n = n(10 - 1)$.

[6]https://en.wikipedia.org/wiki/Newton's_method

If an initial guess for the root s_0 is close to a root, then $|(s - s_0)|^n \ll |(s - s_0)|$ for $n \geq 2 \in \mathbb{N}$. Thus we may truncate $P_n(s - s_o)$ to its linear term

$$P_n(s) \approx (s - s_0) \left. \frac{d}{ds} P_n(s) \right|_{s=s_0} + P_n(s_0)$$
$$= (s - s_0) P'_n(s_0) + P_n(s_0),$$

where $P'_n(s)$ is shorthand for $dP_n(s)/ds$.

Root s_0 may be found by iteration, defining a sequence $s_n \to s_0, n \in \mathbb{N}$, such that $P_n(s_n) = 0$ as $n \to \infty$. Replacing s with s_n, and s_0 with s_{n-1}, the formula becomes a recursion for the root s_n

$$(s_n - s_{n+1}) P'_n(s_{n+1}) + P_n(s_{n+1}) = P_n(s_n) \to 0,$$

as $n \to \infty$. Solving for s_n gives Newton's famous root-finding formula, based on the reciprocal of the *logarithmic derivative* P'_n / P_n

$$s_n = s_{n-1} - \frac{P_n(s_{n-1})}{P'_n(s_{n-1})}. \tag{3.14}$$

Here s_{n-1} is the expansion point and s_n is the nth approximation to the root. With every step, the expansion point moves closer to the root, converging to the root in the limit. As it comes closer, the linearity assumption becomes more accurate, ultimately resulting in the convergence to the root.

This expression is related to the log-derivative $d \log P(x)/dx = P'(x)/P(x)$. It follows that even for cases where fractional derivatives of roots are involved (p. 164),[7] Newton's method should converge, since the log-derivative linearizes the equation.[8]
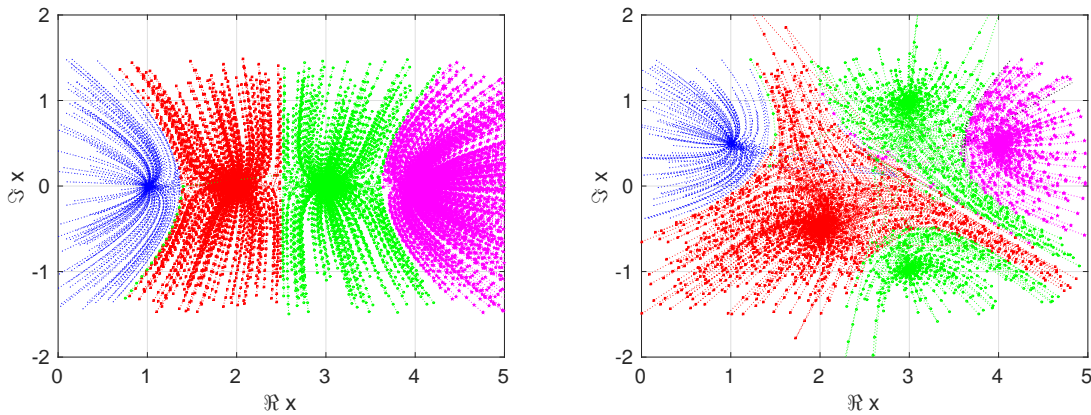
Figure 3.2: Newton's method applied to the polynomial having real roots [1, 2, 3, 4] (left) and 5 complex roots (right). A random starting point was chosen, and each curve shows the values of s_n as Newton's method converges to the root. Different random starting points converge to different roots. The method always results in convergence to a root. Claims to the contrary (Stewart, 2012, p. 347) are a result of forcing the roots to be real. For convergence, one must work with $s_n \in \mathbb{C}$.

Newton's view: Newton believed that imaginary roots and numbers have no meaning (p. 147), thus he sought only real roots (Stillwell, 2010, p. 119). In this case Newton's relation may be explored as a graph, which puts Newton's method in the realm of analytic geometry. The function $P'_n(x)$ is the slope of the polynomial $P_n(x)$ at x_n. The ratio $P_n(x)/P'_n(x)$ has poles at the roots of P' and zeros at the roots we seek.

---

[7] This seems like an obvious way toward understanding fractional, even irrational, roots.

[8] https://en.wikipedia.org/wiki/Argument_principle

Example: When the polynomial is $P_2 = 1 - x^2$, and $P_2'(x) = -2x$, Newton's iteration becomes

$$x_{n+1} = x_n + \frac{1 - x_n^2}{2x_n}.$$

Note the roots are $x_\pm = \pm 1$). For the case of $N = 2$ the root of $P_2'(x)$ is always the average of the roots of $P_2(x)$.

   To start the iteration ($n = 0$) we need an initial guess for x_0, which is an "initial random guess" of where a root might be. For $P_2(x) = 1 - x^2$,

$$x_1 = x_0 + \frac{1 - x_0^2}{2x_0} = x_0 + \frac{1}{2}\left(-x_0 + 1/x_0\right).$$

1. Let $P_2(x) = 1 - x^2$, and $x_0 = 1/2$. Draw a graph describing the first step of the iteration.
   Solution: Start with an (x, y) coordinate system and put points at $x_0 = (1/2, 0)$ and the vertex of $P_2(x)$, i.e.: $(0, 1)$ ($P_2(0) = 1$). Then draw $1 - x^2$, along with a line from x_0 to x_1.

2. Calculate x_1 and x_2. What number is the algorithm approaching? Is it a root of P_2? **Solution:** First we must find $P_2'(x) = -2x$. Thus the equation we will iterate is

$$x_{n+1} = x_n + \frac{1 - x_n^2}{2x_n} = \frac{x_n^2 + 1}{2x_n}.$$

   By hand

$$x_0 = 1/2$$
$$x_1 = \frac{(1/2)^2 + 1}{2(1/2)} = \frac{1}{4} + 1 = 5/4 = 1.25$$
$$x_2 = \frac{(5/4)^2 + 1}{2(5/4)} = \frac{(25/16) + 1}{10/4} = \frac{41}{40} = 1.025.$$

   These estimates rapidly approach the positive real root $x = 1$. Note that if one starts at the root of $P'(x) = 0$ (i.e., $x_0 = 0$), the first step is indeterminate.

3. Write an Octave/Matlab script to check your answer for part (a). **Solution:**
   ```
   x=1/2;
   for n = 1:3
   x = x+(1-x*x)/(2*x);
   end
   ```

   (a) For $n = 4$, what is the absolute difference between the root and the estimate, $|x_r - x_4|$?
       Solution: 4.6E-8 (very small!)

   (b) What happens if $x_0 = -1/2$? **Solution:** You converge on the negative root, $x = -1$.

4. Does Newton's method work for $P_2(x) = 1 + x^2$? Why?[9] Hint: What are the roots in this case?
   Solution: In this case $P_2'(x) = +2x$ thus the iteration gives

$$x_{n+1} = x_n - \frac{1 + x_n^2}{2x_n}.$$

   In this case the roots are $x_\pm = \pm 1\jmath$, namely purely imaginary. Obviously *Newton's method fails* because there is no way for the answer to become complex. If like Newton, you didn't believe in complex numbers, your method would fail to converge to the complex roots (i.e., real in, real out). This is because Octave/Matlab assumes $x \in \mathbb{C}$, since it is initialized as $\mathbb{C}$.

---

[9] `https://en.wikipedia.org/wiki/Newton's_method#Complex_functions`

5. What if you let $x_0 = (1 + \jmath)/2$ for the case of $P_2(x) = 1 + x^2$? **Solution:** By starting with a complex initial value, we fix the Real in = Real out problem.

Fractal diagrams: When the initial guess is real, but the nearest root is complex, Newton's iteration fails, as discussed in the exercises above. This failure to converge gives rise to fractal patterns, called the *Mandelbrot set*.[10]

5

Example: Assume that polynomial $P_3(s) = (s - a)^2/(s - b)^\pi$. Then

$$\ln P_3(s) = 2\ln(s - a) - \pi\ln(s - b)$$

and

$$\frac{d}{ds}\ln P_3(s) = \frac{2}{s - a} + \frac{\pi}{s - b}.$$

Reduction by the logarithmic derivative to simple poles: As shown by the above trivial example, any polynomial, having zeros of arbitrary degree (i.e., π in the example), may be reduced to the ratio of two polynomials, by taking the *logarithmic derivative*, since

$$L_N(s) = \frac{N(s)}{D(s)} = \frac{d}{ds}\ln P_n(s) = \frac{P_n'(s)}{P_n(s)}. \tag{3.15}$$

Here the starting polynomial is the denominator $D(s) = P_n(s)$ while the numerator $N(s) = P_n'(s)$ is the derivative of $D(s)$. Thus the logarithmic derivative can play a key role in analysis of complex analytic functions, as *it reduces higher order poles, even those of irrational degree, to simple poles*.

The logarithmic derivative $L_N(s)$ has a number of special properties:

1. $L_N(s)$ has simple poles s_p and zeros s_z.

10

2. The poles of $L_N(s)$ are the zeros of $P_n(s)$.

3. The zeros of $L_N(s)$ (i.e., $P_n'(s_z) = 0$) are the zeros of $P_n'(s)$.

4. $L_N(s)$ is analytic everywhere other than its poles.

5. Since the zeros of $P_n(s)$ are simple (no second-order poles), it is obvious that the zeros of $L_N(s)$ always lie close to the line connecting the two poles. One may easily demonstrate the truth of the statement numerically, and has been quantified by the *Gauss-Lucas theorem* which specifies the relationship between the roots of a polynomial and those of its derivative. Specifically, the roots of P_{N-1}' lie inside the *convex hull* of the roots of P_n.

15

6. Newton's method may be expressed in terms of the reciprocal of the logarithmic derivative, since

$$s_{k+1} = s_k + \epsilon_o/L_N(s),$$

where ϵ_o is called the *step size*, which is used to control the rate of convergence of the algorithm. If the step size is too large, the root-finding path may jump to a different domain of convergence, thus a different root of $P_n(s)$.

20

7. Not surprisingly, given all the special proprieties, $L_N(x)$ plays an key role in mathematical physics.

---

[10]`https://en.wikipedia.org/wiki/Mandelbrot_set`

Euler's product formula: Counting may be written as a recursion, simply by adding 1 to the previous
value starting from 0. The even numbers may be generated by adding 2, starting from 0.   Multiples
of 3 may be similarly generated by adding 3 to the previous value, starting from 0. Such recursions
are fundamentally related to prime numbers $n \in \mathbb{P}$, as first shown by Euler, starting from the counting
numbers $n \in \mathbb{N}$. The logic is based on the properties of the sieve (Lec. 2.3, p. 52). The basic idea is both
simple and basic, taking almost everyone by surprise, likely even Euler. It is related on the old idea that
the integers may be generated by the geometric series, when viewed as a recursion.

Example: Let's look at counting modulo prime numbers. For example

$$n = k \mod (2) \cdot k \mod (3) \cdot k \mod (5).$$

$$N_n = \alpha N_{n-M} + u_n$$
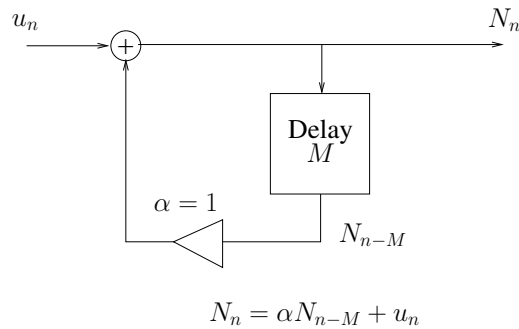
Figure 3.3:   *This feedback network is described by a time-domain difference equation with delay M [s], has an all-pole
transfer function, given by Eq. R.17.* THICKEN LINES

To see this define the *step function* $u_n = 0$ for $n < 0$ and $u_n = 1$ for $n \geq 0$ and the *counting number
function* $N_n = 0$ for $n < 0$. The counting numbers may be recursively generated from the recursion

$$N_{n+1} = N_{n-M} + u_n. \tag{3.16}$$

which for $M = 1$ gives $N_n = n$. For $M = 2$ $N_n = 0, 2, 4, \cdots$ are the even numbers.
    As was first published by Euler in 1737, one may recursively factor out the leading prime term,
resulting in Euler's product formula. Based on the argument given the discussion of the Sieve (Lec. 2.3,
52), one may automate the process and create a recursive procedure to identify multiples of the first item
on the list, and then remove the multiples of that prime. The lowest number on this list is the next prime.
One may then recursively generate all the multiples of this new prime, and remove them from the list.
Any numbers that remain are candidates for primes.
    The observation that this procedure may be automated with a recursive filter, such as that shown in
Fig. 3.3, implies that it may be transformed into the frequency domain, and described in terms of its
poles and zeros. He was successful in finding the function that describes these poles, which are related
to the primes. For example, the poles of the filter shown in Fig. 3.3 may be determined by taking the z-
transform of the recursion equation, and solving for the roots of the resulting polynomial. The recursion
equation is the time-domain equivalent to Riemann's zeta function $\zeta(s)$, which the frequency domain
equivalent representation.

Exercise: Show that $N_n = n$ follows from the above recursion. **Solution:** If $n = -1$ we have $N_n{=}0$
and $u_n = 0$. For $n = 0$ the recursion gives $N_1 = N_0 + u_0$, thus $N_1 = 0 + 1$. When $n = 1$ we have
$N_2 = N_1 + 1 = 1 + 1 = 2$. For $n = 2$ the recursion gives $N_3 = N_2 + 1 = 3$. Continuing the recursion
we find that $N_n = n$. Today we denote such recursions of this form as *digital filters*. The state diagram
for N_n is given in Fig. 3.3.
    To start the recursion define $u_n = 0$ for $n < 0$. Thus $u_0 = u_{-1} + 1$. But since $u_{-1} = 0$, $u_0 = 1$.
The counting numbers follow from the recursion. Thus we have

Exercise: Write an Octave/Matlab program that generates removes all the even number, to give the odds $N_n = \{1, 0, 3, 0, 5, 0, 7, 0, 9, \cdots\}$.

   Solution:

```
M=50; N=(0:M-1);
u=ones(1,M); u(1)=0;
Dem=[1 0 -1]; Num=[0 0 1];
n=filter(Num,Dem,u);
y2=n.*N; F1=N-y2
```

which gives: $F1 = [0, 1, 0, 3, 0, 5, 0, 7, 0, \cdots]$.

Exercise: Write a program to recursively down-sample N_n by 2:1. **Solution:**

```
N=[1 0 3 0 5 0 7 0 9 0 11 0 13 0 15]
k=1;
while 2*k-1 < length(N)
2*k-1;
M(k)=N(2*k+1);
k=k+1;
end
M
```

which gives: $M = [3, 5, 7, 9, 11, 13, 15]$

Here is a 1 line code that does the same thing:

```
N=1:15; M=N(1:2:15)
```

which returns the odd numbers between 1 and 15 (i.e., 1:2:15)

For the next step in the sieve, generate all the multiples of 3 (the second prime) and subtract these from the list. This will either zero out these number from the list, or create negative items which are then removed. Numbers are negative when the number has already been removed because it has a second factor. For example, 6 is already removed because it is a multiple of 2, thus was removed when removing the multiples of prime number 2.

3.1.3 Matrix formulation of the polynomial

There is a one-to-one relationship between every constant coefficient differential equation, its characteristic polynomial and the equivalent matrix form of that differential equation, defined by the *companion matrix* (Horn and Johnson, 1988, p. 146).

The companion matrix: The $N \times N$ *companion matrix*

Can Newton's method be formulated as a CM with eigen-value roots?

$$C_N = \begin{bmatrix} 0 & & & & & & -c_0 \\ 1 & 0 & & & \mathbf{0} & & -c_1 \\ 0 & 1 & 0 & & & & -c_2 \\ \vdots & 0 & 1 & 0 & \cdots & & \vdots \\ & & \cdots & \ddots & 0 & & \vdots \\ & \mathbf{0} & & & 1 & 0 & -c_{N-2} \\ & & & & 0 & 1 & -c_{N-1} \end{bmatrix}_{N \times N} \tag{3.17}$$

is derived from the monic polynomial of degree N

$$P_N(s) = s^N + c_{N-1}s^{N-1} \cdots + c_2 s^2 + c_1 s + c_o$$
$$= s^N + \sum_{n=0}^{N-1} c_n s^n,$$

having coefficient vector

$$\mathbf{c}_{N-1}^T = [1, c_{N-1}, c_{N-2}, \cdots c_o]^T.$$

The roots of the monic polynomial are the eigenvalues of the companion matrix C_N. The *degrees of freedom* (DoF) of a polynomial, and thus of its companion matrix, are the number of non-zero eigenvalues.

Exercise: Show that the eigenvalues of the 3x3 companion matrix are the same as the roots of $P_3(s)$. **Solution:** Expanding the determinant of $C_3 - sI_3$ along the right-most column:

$$- \begin{vmatrix} -s & 0 & -c_0 \\ 1 & -s & -c_1 \\ 0 & 1 & -(c_2 + s) \end{vmatrix} = c_0 + c_1 s + (c_2 + s)s^2 = s^3 + c_2 s^2 + c_1 s + c_0.$$

This is the negative of the characteristic polynomial $(-P_3(s))$, which has the same roots.

Exercise: Find the companion matrix for the Fibonacci sequence, defined by the recursion (i.e., difference equation)

$$f_{n+1} = f_n + f_{n-1},$$

initialized with $f_n = 0$ for $n < 0$ and $f_0 = 1$. **Solution:** Taking the "Z transform" gives the polynomial $z^1 = z^0 - z^{-1} = 0$ having coefficient vector $Z = [1, -1, -1]$, resulting in the Fibonacci companion matrix

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

as discussed Section 2.3.5 on page 65.

Example: Matlab/Octave: A polynomial is represented in Matlab/Octave in terms of its coefficient vector. When the polynomial vector for the poles of a differential equation is

$$\mathbf{c}_N = [1, c_{N-1}, c_{N-2}, \cdots c_0]^T,$$

the coefficient $c_N = 1$. This normalization guarantees that the leading term is not zero, and the number of roots (N) is equal to the degree of the *monic polynomial*.

3.1.4 Working with polynomials in Matlab/Octave:

In Matlab/Octave there are seven related functions you must become familiar with:

1. `R=roots(A)` Vector $A = [a_N, a_{N-1}, \ldots, a_o] \in \mathbb{C}$ are the complex coefficients of polynomial $P_N(z) = \sum_{n=0}^{N} a_n z^n \in \mathbb{C}$, where $N \in \mathbb{N}$ is the degree of the polynomial. It is convenient to force $a_N = 1$, corresponding to dividing the polynomial by this value, when it is not 1, guaranteeing it cannot be zero. Further R is the vector of roots, $[z_1, z_2, \cdots, z_n] \in \mathbb{C}$ such that `polyval(A,`z_k`)=0`.

   Example: `roots([1, -1])=1`.

2. `y=polyval(A,x)`: This evaluates the polynomial defined by vector `A` $\in \mathbb{C}^N$ evaluated at `x` $\in \mathbb{C}$, returning vector `y(x)` $\in \mathbb{C}$.

   Example: `polyval([1 -1],1)=0`, `polyval([1, 1],3)=4`.

3. `P=poly(R)`: This is the inverse of `root()`, returning a vector of polynomial coefficients `P` $\in \mathbb{C}^N$ of the corresponding *characteristic polynomial*, starting from either a vector of roots R, or a matrix A, for example, defined with the roots on the diagonal. The characteristic polynomial is defined as the determinant of $|A - \lambda I| = 0$ having roots R.   5

   Example: `poly([1])=[1, -1]i`, `poly([1,2])=[1,-3,2]`.

   Due to IEEE-754 scaling issues, this can give strange results that are numerically correct, but only within the limits of IEEE-754 accuracy.   10

4. `R=polyder(C)` This routine takes the N coefficients of polynomial C and returns the $N-1$ coefficients of the derivative of the polynomial. This is useful when working with Newton's method, where each step is proportional to $P_N(x)/P'_{N-1}(x)$.

   Example: `polyder([1,1])= [1]`

5. `[K,R]=residue(N,D)`: Given the ratio of two polynomials `N,D`, `residue(N,D)` returns vectors `K,R` such that
$$\frac{N(s)}{D(s)} = \sum_k \frac{K_k}{s - s_k}, \tag{3.18}$$
   where $s_k \in \mathbb{C}$ are the roots of the denominator `D` polynomial and `K` $\in \mathbb{C}$ is a vector of *residues*,   15
   which characterize the roots of the numerator polynomial $N(s)$. The use of `residue()` will be discussed in Sect. 4.4 (p. 166), and I, p. 284. This is one of the most valuable time-saving routine I know of.

   Example: `Residue(2, [1 0 -1])= [1 -1]`

6. `C=conv(A,B)`: Vector `C` $\in \mathbb{C}^{N+M-1}$ contains the polynomial coefficients of the convolution of   20
   the two vector of coefficients of polynomials `A,B` $\in \mathbb{C}^N$ and `B` $\in \mathbb{C}^M$.

   Example: `[1, 2, 1]=conv([1, 1], [1, 1])`.

7. `[C,R]=deconv(N,D)`: Vectors `C, N, D` $\in \mathbb{C}$. This operation uses long division of polynomials to find $C(s) = N(s)/D(s)$ with remainder $R(s)$, where `N = conv(D,C)+R`, namely

$$C = \frac{N}{D} \quad \text{remainder} \quad R. \tag{3.19}$$

   Example: Defining the coefficients of two polynomials as $A = [1, a_1, a_2, a_3]$ and $B = [1, b_1, b + 2]$, one may find the coefficients of the product from `C=conv(A,B)`, and recover B from C with `B=deconv(C,A)`.   25

8. `A=compan(D)`: Vector $D = [1, d_{N-1}, d_{N-2}, \cdots, d_0]^T \in \mathbb{C}$ contains the coefficients of the monic polynomial
$$D(s) = s^N + \sum_{k=1}^N d_{N-k} s^k,$$
   and A is the companion matrix of vector D (Eq. I.3, p. 284). The eigenvalues of A are the roots of monic polynomial $D(s)$.

   Example: `compan([1 -1 -1])= [1 1; 1 0]`

Exercise: Practice the use of Matlab's/Octave's related functions, which manipulate roots, polynomials and residues: `root()`, `conv()`, `deconv()`, `poly()`, `polyval()`, `polyder()`, `residue()`, `compan()`.

   Solution: Try Newton's method for various polynomials. Use `N=poly(R)` to provide the coefficients of a polynomial given the roots R. Then use `root()` to factor the resulting polynomial. Then use Newton's method and show that the iteration converges to the nearest root.[11]

   Solution: Write more examples similar to the ones provided above, and manually verify them.

Exercise: Write an Octave/Matlab script that finds the companion matrix of the Fibonacci recursion formula, and verify it manually. **Solution:**   `Z=[1, -1, -1]; C=compan(Z)`

3.2   Lec 11: Eigenanalysis I: eigenvalues of a matrix

At this point we turn a corner in the discussion, to discuss the important topic of *eigenanalysis*, which starts with the computation of the eigenvalues and their eigenvectors of a matrix. As briefly discussed in Lecture 2.2 (p. 43), eigenvectors are mathematical generalizations of resonances, or *modes*, naturally found in physical systems.

   When you pluck the string of a violin or guitar, or hammer a bell or tuning fork, there are natural resonances that occur. These are the eigenmodes of the instrument. The frequency of each mode is related to the eigenvalue, which in physical terms is the frequency of the mode. But this idea goes way beyond simple acoustical instruments. Wave-guides and atoms can be resonant systems. The resonances of the hydrogen atom are called the Lyman series,[12] a special case of the Rydberg series and Rydberg atom (p. 352).

   Thus this topic runs deep in both physics and, eventually, mathematics. In some real sense, eigenanalysis was what the Pythagoreans were seeking to understand. This relationship is rarely spoken about in the open literature, but once you see it, it can never be forgotten, as it colors your entire view of all aspects of modern physics.

3.2.1   Eigenvalues of a matrix

The method for finding eigenvalues is best described by an example.[13] Starting from the matrix Eq. 2.3.5 (p. 66), the eigenvalues are defined by the eigenmatrix equation

$$\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \lambda \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

The unknowns here are the eigenvalue λ and the eigenvector $\boldsymbol{e} = [e_1, e_2]^T$. First find λ by subtracting the right from the left

$$\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} - \lambda \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1-2\lambda & 1 \\ 2 & -2\lambda \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0 \tag{3.20}$$

   The only way that this equation for $\boldsymbol{e}$ can have a solution is if the matrix is singular.

Example: The determinant of the above 2x2 example is the product of the diagonal elements, minus the product of the off-diagonal elements, which results in the quadratic equation

$$-2\lambda(1 - 2\lambda) - 2 = 4\lambda^2 - 2\lambda - 2 = 0.$$

---

[11]A Matlab/Octave program that does this may be downloaded from `http://jontalle.web.engr.illinois.edu/uploads/493/M/NewtonJPD.m`.

[12]`https://en.wikipedia.org/wiki/Lyman_series`

[13]Appendix C (p. 259) is an introduction to the topics of eigenanalysis for 2x2 matrices.

Completing the square gives
$$(\lambda - 1/4)^2 - (1/4)^2 - 1/2 = 0, \tag{3.21}$$

thus the roots (i.e., eigenvalues) are $\lambda_\pm = \frac{1 \pm 3}{4} = \{1, -1/2\}$.

Exercise: Expand Eq. 3.21 and recover the quadratic equation. **Solution:**

$$(\lambda - 1/4)^2 - (1/4)^2 - 1/2 = \lambda^2 - \lambda/2 + \cancel{(1/4)^2} - \cancel{(1/4)^2} - 1/2 = 0.$$

Thus completing the square is equivalent to the original equation.

Exercise: Find the eigenvalues of matrix Eq. 2.12. (Hint: see p. 260) **Solution:** This is a minor variation on the previous example. Briefly

$$\det \begin{bmatrix} 1 - \lambda & N \\ 1 & 1 - \lambda \end{bmatrix} = (1 - \lambda)^2 - N = 0.$$

Thus $\lambda_\pm = 1 \pm \sqrt{N}$.

Exercise: Starting from $[x_n, y_n]^T = [1, 0]^T$ compute the first 5 values of $[x_n, y_n]^T$. **Solution:** Here is a Matlab/Octave code for computing x_n:

```
x(1:2,1)=[1;0];
A=[1 1;2 0]/2;
for k=1:10; x(k+1)=A*x(:,k); end
```

which gives the rational ($x_n \in \mathbb{Q}$) sequence: $1, 1/2, 3/4, 5/8, 11/2^4, 21/2^5, 43/2^6, 85/2^7, 171/2^8, 341/2^9, 683/2^{10}, \cdots$.

Exercise: Show that the solution to Eq. 2.18 is bounded, unlike that of the divergent Fibonacci sequence. Explain what is going on. **Solution:** Because the next value is the mean of the last two, the sequence is bounded. To see this one needs to compute the eigenvalues of the matrix Eq. 2.3.5 (p. 66).

Eigenanalysis: The key to the analysis of such equations is called the *eigenanalysis*, or *modal-analysis* method. These are also known as *resonant modes* in the physics literature. Eigenmodes describe the naturally occurring "ringing" found in physical wave-dominated boundary value problems. Each mode's "eigenvalue" quantifies the mode's natural frequency. Complex eigenvalues result in damped modes, which decay in time due to energy losses. Common examples include tuning forks, pendulums, bell, and strings of musical instruments, all of which have a characteristic frequency.

Two modes with the same frequency are said to be *degenerate.* This is a very special condition, with a high degree of symmetry.

Cauchy's residue theorem is used to find the time-domain response of each frequency-domain complex eigenmode. Thus eigenanalysis and eigenmodes of physics are the same thing (see Sect. 4.3, p. 156), but are described using different (i.e., mutually unrecognizable) notional methods.[14] The "eigen method" is summarized in Appendix D, p. 263.

Taking a simple example of a 2x2 matrix $\boldsymbol{T} \in \mathbb{C}$, we start from the definition of the two eigen-equations

$$\boldsymbol{T}\boldsymbol{e}_\pm = \lambda_\pm \boldsymbol{e}_\pm, \tag{3.22}$$

corresponding to two eigenvalues $\lambda_\pm \in \mathbb{C}$ and two 2x1 eigenvectors $\boldsymbol{e}_\pm \in \mathbb{C}$.

---

[14]During the discovery or creation of quantum mechanics, two alternatives were developed: Schrödinger's differential equation method and Heisenberg's matrix method. Eventually it was realized the two were equivalent.

Example: Assume that T is the Fibonacci matrix Eq. 2.16.

The eigenvalues $\lambda_{\pm}$ may be merged into a 2x2 diagonal eigenvalue matrix

$$\Lambda = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix},$$

while the two eigenvectors e_+ and e_- are merged into a 2x2 eigenvector matrix

$$E = [e_+, e_-] = \begin{bmatrix} e_1^+ & e_1^- \\ e_2^+ & e_2^- \end{bmatrix}, \tag{3.23}$$

corresponding to the two eigenvalues. Using matrix notation, this may be compactly written as

$$TE = E\Lambda. \tag{3.24}$$

Note that while $\lambda_{\pm}$ and $E_{\pm}$ commute, $E\Lambda \neq \Lambda E$.

From Eq. 3.24 we may obtain two very important forms:

1. the diagonalization of T

$$\Lambda = E^{-1}TE, \tag{3.25}$$

   and

2. the eigen-expansion of T

$$T = E\Lambda E^{-1}, \tag{3.26}$$

   which is useful for computing power of T (i.e., $T^{100} = E^{-1}\Lambda^{100}E$).

Example: If we take

$$T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

then the eigenvalues are given by $(1 - \lambda_{\pm})(1 + \lambda_{\pm}) = -1$, thus $\lambda_{\pm} = \pm\sqrt{2}$. This method of eigenanalysis is discussed in Section Q.2 (p. 328) and Appendix E (p. 267).

Exercise: Use the formula for the generalized diagonalization of a matrix to find the general solution of the mean-Fibonacci sequence. **Solution:** The eigenvalues are given by the roots of

$$0 = -\lambda(1/2 - \lambda) - 1/2 = (\lambda - 1/4)^2 - 9/16$$

which are $[1, -1/2]$.

By studying the eigenvalues of Eq. 2.3.5 one finds that the steady state solution approaches 1. Namely $f_n \to 1 = (f_{n-1} + f_{n-2})/2$ is the solution, as $n \to \infty$. Namely the average of the last two values must approach 1 for large n.

Exercise: Show that the geometric series formula holds for 2x2 matrices. Starting with the 2x2 identity matrix I_2 and $a \in \mathbb{C}$, with $|a| < 1$, show that

$$I_2(I_2 - aI_2)^{-1} = I_2 + aI_2 + a^2 I_2^2 + a^3 I_2^3 + \cdots.$$

Solution: Since $a^k I_2^k = a^k I_2$, we may multiply both sides by $I_2 - aI_2^k$ to obtain

$$\begin{aligned} I_2 &= I_2 + aI_2 + a^2 I_2^2 + a^3 I_2^3 + \cdots - aI_2(aI_2 + a^2 I_2^2 + a^3 I_2^3 + \cdots) \\ &= [1 + (a + a^2 + a^3 + \cdots) - (a + a^2 + a^3 + a^4 + \cdots)]I_2 \\ &= I_2 \end{aligned}$$

Exercise: Show that when $\boldsymbol{T}$ is not a square matrix, Eq. 3.22 can be generalized to

$$\boldsymbol{T}_{m,n} = \mathbf{U}_{m,m}\Lambda_{m,n}\mathbf{V}_{n,n}^{\dagger}.$$

This important generalization of eigenanalysis is called a *singular value decomposition* (SVD).

Summary: The GCD (Euclidean algorithm), Pell's equation and the Fibonacci sequence may all be written as compositions of 2x2 matrices. Thus Pell's equation and the Fibonacci sequence are special cases of 2x2 matrix composition

$$\begin{bmatrix} x \\ y \end{bmatrix}_{n+1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}_{n}.$$

This is an important and common thread of these early mathematical findings. It will turn out that this 2x2 matrix recursion plays a special role in physics, mathematics and engineering, because such equations are solved using the eigenanalysis method. More than several thousand years of mathematical, by trial and error, set the stage for this breakthrough. But it took even longer to be fully appreciated. 5

The key idea of the 2x2 matrix solution, widely used in modern engineering, can be traced back to Brahmagupta's solution of Pell's equation, for arbitrary N. Brahmagupta's recursion, identical to that of the Pythagoreans' $N = 2$ case (Eq. 2.13), eventually led to the concept of linear algebra, defined by the simultaneous solutions of many linear equations. The recursion by the Pythagoreans (6th BCE) predated the creation of algebra by al-Khwārizmī (9th CE century) (Fig. 1.1) (Stillwell, 2010, p. 88). 10

development of algebra (Fig 1.1 p. 19) and Diophantus' book on "Arith Bombelli, Fermat and Desc

3.2.2 Taylor series

An *analytic function* is one that

1. may be expanded in a *Taylor series*

$$P(x) = \sum_{n=0}^{\infty} c_n(x - x_o)^n, \tag{3.27}$$

2. converges for $|x - x_o| < 1$, called the RoC, with coefficients c_n that are determined by the derivatives of $P(x)$, evaluated at the expansion point $x = x_o$.

3. The *Taylor series* coefficients c_n are defined by taking derivatives of $P(x)$ and evaluating them at the expansion point x_0, namely

$$c_n = \frac{1}{n!}\frac{d^n}{dx^n}P(x - x_o)\bigg|_{x=x_o}. \tag{3.28}$$

Properties: The Taylor formula is a prescription for how to uniquely define the coefficients c_n. Without the Taylor series formula, we would have no way of determining c_n. The proof of the Taylor formula is transparent, simply by taking successive derivatives of Eq. 3.27, and then evaluating the result at the expansion point. If $P(x)$ is analytic then this procedure will always work. If $P(x)$ fails to have a derivative of any order, then the function is not analytic and Eq. 3.27 is not valid for $P(x)$. For example, if $P(x)$ has a pole at x_o then it is not analytic at that point. 20

The Taylor series representation of $P(x)$ has special applications for solving differential equations because

1. it is single valued

2. all its derivatives and integrals are uniquely defined

3. it may be trivially continued into the complex plane by extending $x \in \mathbb{C}$. 25

Analytic continuation: A limitation of the Taylor series expansion is that it is not valid outside of its RoC. One method for avoiding this limitation is to move the expansion point. This is called *analytic continuation*. However, analytic continuation is a non-trivial operation because it (1) requires manipulating an infinite number of derivatives of $P(x)$, (2) at the new expansion point x_o, where (3) $P(x - x_o)$ may not have derivatives, due to possible singularities. (4) Thus one needs to know where the singularities of $P(s)$ are in the complex s plane. Due to these many problems, analytic continuation is rarely used, other than as an important theoretical concept.

Example: The trivial (corner) case is the geometric series $P(x) = 1/(1 - x)$ about the expansion point $x = 0$. The function $P(x)$ is defined everywhere, except at the singular point $x = 1$, whereas the geometric series is only valid for $|x| < 1$.

Exercise: Verify that c_o and c_1 of Eq. 3.27 follow from Eq. 3.28. **Solution:** To obtain c_0, for $n = 0$, there is no derivative (d^0/dx^0 indicates no derivative is taken), so we must simply evaluate $P(x - x_o) = c_0 + c_1(x - x_o) + \cdots$ at $x = x_o$, leaving c_o. To find c_1 we take one derivative which results in $P'(x) = c_1 + 2c_2(x - x_0)) \cdots$. Evaluating this at $x = x_0$ leaves c_1. Each time we take a derivative we reduce the degree of the series by 1, leaving the next constant term.

Exercise: Suppose we truncate the Taylor series expansion to N terms. What is the name of such functions and what are their properties? **Solution:** When an infinite series is truncated the resulting function is called an Nth degree polynomial

$$P_N(x) = \sum_{n=0}^{N} = c_0 + c_1(x - x_o) + c_2(x - x_o)^2 \cdots c_N(x - x_0)^N.$$

We can find c_o by evaluating $P_N(x)$ at the expansion point x_o, since from the above formula $P_N(x_o) = c_0$. From the Taylor formula $c_1 = \left. P'_N(x) \right|_{x_o}$.

Exercise: How many roots do $P_N(x)$ and $P'_N(x)$ have? **Solution:** According to the fundamental theorem of algebra $P_N(x)$ has N roots. $P'_N(x)$ has $N - 1$ roots. The *Gauss-Lucas theorem* states that the $N - 1$ roots of $P'_N(x)$ lie inside the convex hull of the N roots of $P_N(x)$.

   To understand the meaning of the *convex hull*, consider the following construction: If stakes are placed at each of the N roots of $P_N(x)$, and a string is then wrapped around the stakes, with all the stakes inside the string, the convex hull is then the closed set inside the string. One can then begin to imagine how the $N - 1$ roots of the derivative must evolve with each set inside the convex hull of the previous set. This concept may be recursed to $N - 2$, and below.

Exercise: Would it be possible to find the inverse Gauss-Lucas theorem that states where the roots of the integral of a polynomial might be? **Solution:** With each integral there is a new degree of freedom that must be taken into account, thus this problem is much more difficult. But since there is only 1 extra degree of freedom, it does not seem impossible. To solve this problem more constraints will be needed.

Role of the Taylor series: The Taylor series plays a major role in the mathematics of differential equations and their solution, as the coefficients of the series uniquely determine the analytic series representation via its derivatives. The implications and limitations of the power series representation are very specific: if the series fails to converge (i.e., outside the RoC), it is essentially meaningless.

   A very important fact about the RoC: It is only relevant to the series, not the function being expanded. Typically the function has a pole at the radius of the RoC, where the series fails to converge. However the function being expanded is valid everywhere, other than at the pole. It seems that this point has been poorly explained in many texts. Besides the RoC is the region of divergence RoD, which is the RoC's complement.

The Taylor series does not need to be infinite to converge to the function it represents, since it obviously works for any polynomial $P_N(x)$ of degree N. But in the finite case ($N < \infty$), the RoC is infinite, and the series *is* the function $P_N(x)$ exactly, everywhere. Of course $P_N(x)$ is called a polynomial of degree N. When $N \to \infty$, the Taylor series is only valid within the RoC, and it is (typically) the representation of the reciprocal of a polynomial.

These properties are both the curse and the blessing of the analytic function. On the positive side, analytic functions are the ideal starting point for solving differential equations, which is exactly how they were used by Newton and others. Analytic functions are "smooth" since they are infinitely differentiable, with coefficients given by Eq. 3.28. They are single valued, so there can be no ambiguity in their interpretation.

Two well-known analytic functions are the *geometric series* ($|x| < 1$)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n \tag{3.29}$$

and exponential series ($|x| < \infty$)

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n. \tag{3.30}$$

Exercise: Relate the Taylor series expressions for Eq. 3.29 to the following functions:

1.
$$F_1(x) = \int^x \frac{1}{1-x}dx \tag{3.31}$$

   Solution: $= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$.

2.
$$F_2(x) = \frac{d}{dx}\frac{1}{1-x} \tag{3.32}$$

   Solution: $1 + 2x + 3x^2 \ldots$

3.
$$F_3(x) = \ln\frac{1}{1-x} \tag{3.33}$$

   Solution: $1 + \frac{1}{2}x + \frac{1}{3}x^2$.

4.
$$F_4(x) = \frac{d}{dx}\ln\frac{1}{1-x} \tag{3.34}$$

   Solution: $1 + x + s^2 + s^3 \ldots$.

Exercise: Using symbolic manipulation (Matlab, Octave, Mathematica), expand the given function $F(s)$ in a Taylor series, and find the recurrence relation between the Taylor coefficients c_n, c_{n-1}, c_{n-2}. Assume $a \in \mathbb{C}$ and $T \in \mathbb{R}$.

1.
$$F(s) = e^{as}$$

   Solution: A Google search on "octave syms taylor" is useful to answer this question. The Matlab/Octave code is to expand this in a taylor series is

```
syms s
taylor(exp(s),s,0,'order',10)
```

Exercise: Find the coefficients of the following functions by the method of Eq. 3.28, and give the RoC.

1. $w(x) = \frac{1}{1-x\jmath}$. **Solution:** From a straightforward expansion we know the coefficients are

$$\frac{1}{1-x\jmath} = 1 + x\jmath + (x\jmath)^2 + (x\jmath)^3 \cdots = 1 + x\jmath - x^2 + -\jmath x^3 \cdots.$$

Working this out using Eq. 3.28 is more work:

$c_0 = \frac{1}{0!}w\big|_0 = 1; c_1 = \frac{1}{1!}\frac{dw}{dx}\big|_0 = -\frac{-\jmath}{(1-x\jmath)^2}\big|_{x=0} = \jmath; c_2 = \frac{1}{2!}\frac{d^2w}{dx^2}\big|_0 = \frac{1}{2!}\frac{-2}{(1-x\jmath)^3}\big|_0 = -1;$

$c_3 = \frac{1}{3!}\frac{d^3w}{dx^3}\big|_0 = \frac{-\jmath}{(1-x\jmath)^4}\big|_0 = -\jmath.$

However, if we take derivatives of the series expansion it is much easier, and one can even figure out the term for c_n:

$c_0 = 1; c_1 = \frac{d}{dx}\sum(\jmath x)^n\big|_0 = \jmath; c_2 = \frac{1}{2!}\frac{d^2}{dx^2}\sum(jx)^n\big|_0 = 2(\jmath)^2;$

$c_3 = \frac{1}{3!}\frac{d^3}{dx^3}\sum(jx)^n\big|_0 = (\jmath)^3 = -\jmath;$

$\cdots,$

$c_n = \frac{1}{n!}\jmath^n n! = \jmath^n.$

The RoC is $|x\jmath| = |x| < 1$.

2. $w(x) = e^{x\jmath}$. **Solution:** $c_n = \frac{1}{n!}\jmath^n$. The RoC is $|x| < \infty$. Functions with an ∞ RoC are called *entire*.

Brune impedances: A third special *family of functions* is formed from ratios of two polynomials $Z(s) = N(s)/D(s)$, commonly used to define an impedance $Z(s)$, denoted the *Brune impedance*. Impedance functions are a very special class of complex analytic functions because they must have a non-negative real part

$$\Re Z(s) = \Re\frac{N(s)}{D(s)} \geq 0,$$

so as to obey conservation of energy. A physical Brune impedance cannot have a negative resistance (the real part); otherwise it would act like a power source, violating conservation of energy. Most impedances used in engineering applications are in the class of Brune impedances, defined by the ratio of two polynomials, of degrees m and n

$$Z_{\text{Brune}}(s) = \frac{P_M(s)}{P_N(s)} = \frac{s^M + a_1 S^{M-1} \cdots a_0}{s^N + b_1 S^{N-1} \cdots b_0}, \tag{3.35}$$

where $M = N \pm 1$ (i.e., $N = M \pm 1$). This fraction of polynomials is sometimes known as a "Padé approximation," but more specifically this ratio is a *Brune impedance*, with poles and zeros, defined as the complex roots of the two polynomials. The key property of the Brune impedance is that the real part of the impedance is non-negative (positive or zero) in the right s half-plane

$$\Re Z(s) = \Re\left[R(\sigma,\omega) + jX(\sigma,\omega)\right] = R(\sigma,\omega) \geq 0 \qquad \text{for} \quad \Re s = \sigma \geq 0. \tag{3.36}$$

Since $s = \sigma + \omega\jmath$, the complex frequency (s) right half-plane (RHP) corresponds to $\Re s = \sigma \geq 0$). This condition defines the class of *positive-real* functions, also known as the *Brune condition*, which is frequently written in the abbreviated form

$$\Re Z(\Re s \geq 0) \geq 0.$$

As a result of this *positive-real* constraint on impedance functions, the subset of Brune impedances (those given by Eq. 3.35 satisfying Eq. 3.36) must be complex analytic in the entire right s half-plane. This is a powerful constraint that places strict limitations on the locations of both the poles and the zeros of every positive-real Brune impedance.

Exercise: Show that $Z(s) = 1/\sqrt{s}$ is positive real, but not a Brune impedance. **Solution:** Since it may not be written as the ratio of two polynomials, it is not in the Brune impedance class. By writing $Z(s) = |Z(s)e^{\phi\jmath}$ in polar coordinates, since $-\pi/4 \le \phi \le \pi/4$ when $|\angle|s < \pi/2$, $Z(s)$ satisfies the Brune condition, thus is positive real.

Determining the region of convergence (RoC): Determining the RoC for a given analytic function is quite important, and may not always be obvious. In general the RoC is a circle whose radius extends from the expansion point out to the nearest pole. Thus when the expansion point is moved, the RoC changes since the location of the pole is fixed.

Example: For the geometric series (Eq. 3.29), the expansion point is $x_o = 0$, and the RoC is $|x| < 1$, since $1/(1 - x)$ has a pole at $x = 1$. We may move the expansion point by a linear transformation, for example, by replacing x with $z + 3$. Then the series becomes $1/((z + 3) - 1) = 1/(z + 2)$, so the RoC becomes 2 because in the z plane the pole has moved to -2.

Example: A second important example is the function $1/(x^2 + 1)$, which has the same RoC as the geometric series, since it may be expressed in terms of its *residue expansion* (aka, partial fraction expansion)

$$\frac{1}{x^2 + 1} = \frac{1}{(x + 1\jmath)(x - 1\jmath)} = \frac{1}{2\jmath}\left(\frac{1}{x - 1\jmath} - \frac{1}{x + 1\jmath}\right).$$

Each term has an RoC of $|x| < |1\jmath| = 1$. The amplitude of each pole is called the *residue*, defined in Section 4.4 Eq. 4.36, p. 166. The residue for the pole at $1\jmath$ is $1/2\jmath$.

In summary, the function $1/(x^2 + 1)$ is the sum of two geometric series, with poles at $\pm 1\jmath$ which are not as obvious because the roots are complex and conjugate. Once factored, it becomes clear what is going on.

Exercise: Verify the above expression is correct, and show that the residues are $\pm 1/2\jmath$. **Solution:** Cross-multiply and cancel, leaving 1, as required. The RoC is the coefficient on the pole. Thus the residue of the pole at $x\jmath$ is $\jmath/2$.

Exercise: Find the residue of $\frac{d}{dz}z^\pi$. **Solution:** Taking the derivative gives $z^{\pi-1}$ which has a pole at $z = 0$. Applying the formula for the residue (Eq. 4.36, p. 166) we find

$$c^{-1} = \lim_{z \to 0} z z^{\pi-1} = \lim_{z \to 0} z^\pi = 0.$$

Thus the residue is zero. The exponential series converges for every finite value of $x \in \mathbb{R}$ (the RoC is the entire real line), thus the exponential is called an *entire function*.

Exercise: Find the residue of $g(z) = \frac{d}{dz}(z - 1)^{-\pi}$. **Solution:** $g(z) = -\pi/(z - 1)^{\pi+1}$, and apply the definition of the residue (Eq. 4.36).

3.2.3 Analytic functions:

Any function that has a Taylor series expansion is called an *analytic function. Within the RoC, the series expansion defines a single-valued function.* Polynomials $1/(1 - x)$ and e^x are examples of analytic functions that are real functions of their real argument x. This is not the entire story. Because analytic functions are easily manipulated term by term, they may be used to find solutions of differential equations since the derivatives of a series are uniquely determined within the RoC, due to Eq. 3.28.

Every analytic function has a corresponding differential equation, which is determined by the coefficients a_k of the analytic power series. An example is the exponential, which has the property that it is the eigen-function of the derivative operation

$$\frac{d}{dx}e^{ax} = ae^{ax},$$

which may be verified using Eq. 3.30. This relationship is a common definition of the exponential function, which is very special because it is the eigen-function of the derivative.

The complex analytic power series (i.e., complex analytic functions) may also be integrated, term by term, since

$$\int^x f(x)dx = \sum \frac{a_k}{k+1}x^{k+1}. \tag{3.37}$$

Newton took full advantage of this property of the analytic function and used the analytic series (*Taylor series*) to solve analytic problems, especially for working out integrals, allowing him to solve differen-
5   tial equations. To fully understand the theory of differential equations, one must master single-valued analytic functions and their analytic power series.

Single- vs. multi-valued functions: Polynomials and their ∞-degree extensions (analytic functions) are single valued: for each x there is a single value for $P_N(x)$. The roles of the domain and codomain may be swapped to obtain an *inverse function*, with properties that can be different from those of the
10 function. For example, $y(x) = x^2 + 1$ has the inverse $x = \pm\sqrt{y-1}$, which is double valued, and complex when $y < 1$. Periodic functions such as $y(x) = \sin(x)$ are even more "exotic" since $x(y) = \arcsin(x) = \sin^{-1}(x)$ has an infinite number of $x(y)$ values for each y. This problem was first addressed in Riemann's 1851 PhD thesis, written while he was working with Gauss.

Exercise: Let $y(x) = \sin(x)$. Then $dy/dx = \cos(x)$. Show that $dx/dy = -1/\sqrt{1-x^2}$. Hint:
15 $x(y) = \cos^{-1}(y) = \arccos(y)$. **Solution:** See the *implicit function theorem* (D'Angelo, 2017, p. 104). Add solution.

Exercise: Let $y(x) = \sin(x)$. Then $dy/dx = \cos(x\jmath)$. Show that $dx/dy = -\jmath/\sqrt{1+x^2}$. **Solution:** Add solution.

Exercise: Find the Taylor series coefficients of $y = \sin(x)$ and $x = \sin^{-1}(y)$. **Solution:** Add solution.

Complex analytic functions: When the argument of an analytic function $F(x)$ is complex, that is, $x \in \mathbb{R}$ is replaced by $s = \sigma + \omega\jmath \in \mathbb{C}$ (recall that $\mathbb{R} \subset \mathbb{C}$)

$$F(s) = \sum_{n=0}^{\infty} c_n(s - s_o)^n, \tag{3.38}$$

20 with $c_n \in \mathbb{C}$, that function is said to be a *complex analytic*.

For example, when the argument of the exponential becomes complex, it is periodic on the ω axis, since

$$e^{st} = e^{(\sigma+\omega\jmath)t} = e^{\sigma t}e^{\jmath\omega t} = e^{\sigma t}\left[\cos(\omega t) + \jmath\sin(\omega t)\right]. \tag{3.39}$$

Taking the real part gives

$$\Re\{e^{st}\} = e^{\sigma t}\frac{e^{\omega\jmath t} + e^{-\omega\jmath t}}{2} = e^{\sigma t}\cos(\omega t),$$

and $\Im\{e^{st}\} = e^{\sigma t}sin(\omega t)$. Once the argument is allowed to be complex, it becomes obvious that the exponential and circular functions are fundamentally related. This exposes the family of *entire circular functions* [i.e., $e^s, \sin(s), \cos(s), \tan(s), \cosh(s), \sinh(s)$] and their inverses [$\ln(s), \arcsin(s)$,

$\arccos(s), \arctan(s), \cosh^{-1}(s), \sinh^{-1}(s)$], first fully elucidated by Euler (c1750) (Stillwell, 2010, p. 315). Note that because a function, such as $\sin(\omega t)$, is periodic, its inverse must be multi-valued. What is needed is some systematic way to account for this multi-valued property.   Euler also studied the *Gamma function* ($\Gamma(s)$), which generalizes the factorial operation for real, and eventually by Gauss, for complex arguments ($s, \Gamma(s) \in \mathbb{C}$):

$$s! = \Gamma(s+1).$$

Given a complex analytic function of a complex variable, one must resort to the *extended complex plane*, *Riemann sheets* and *branch cuts*, as discussed in Section 3.8.1 (p. 138). The extended complex plane is a tool that extends the domain of complex analytic functions to include the point at infinity. This topic is critically important in engineering mathematics and will be discussed at length in Sections 3.8.1-3.7 (pp. 138-131).

Definition of the Taylor series of a complex analytic function: However, there is a fundamental problem: we cannot formally define the Taylor series for the coefficients c_k until we have defined the derivative with respect to the complex variable $dF(s)/ds$, with $s \in \mathbb{C}$. Thus simply substituting s for x in an analytic function leaves a major hole in one's understanding of the complex analytic function.

To gain a feeling for the nature of the problem, we make take derivatives of a function with respect to various variables. For example,

$$\frac{d}{dt}e^{st} = se^{st},$$

$$e^{\omega \jmath t}\frac{d}{d\sigma}e^{\sigma t} = \sigma e^{st},$$

and

$$e^{\sigma t}\frac{d}{d\omega \jmath}e^{\omega \jmath} = \omega \jmath e^{st}.$$

It was Cauchy (1814) (Fig. 3.1) who uncovered much deeper relationships within complex analytic functions (Sect. 3.8, p. 135) by defining differentiation and integration in the complex plane, leading to several fundamental theorems of complex calculus, including the *fundamental theorem of complex integration*, and Cauchy's formula. We shall explore this and several fundamental theorems in Sect. 4.2 (p. 148).

There seems to be some disagreement as to the status of multi-valued functions: Are they functions, or is a function strictly single valued? If so, then we are missing out on a host of interesting possibilities, including all the inverses of nearly every complex analytic function. For example, the inverse of a complex analytic function is a complex analytic function (e.g., e^s and $\log(s)$).

Impact of complex analytic mathematics on physics: It seems likely, if not obvious, that the success of Newton was his ability to describe physics by the use of mathematics. He was inventing new mathematics at the same time he was explaining new physics. The same might be said for Galileo. It seems likely that Newton was extending the successful techniques and results of Galileo's work on gravity (Galileo, 1638).[15] Galileo died on Jan 8, 1642, and Newton was born Jan 4, 1643, just short of one year later. Certainly Newton was well aware of Galileo's great success, and naturally would have been influenced by it (see p. 24).

The application of complex analytic functions to physics was dramatic, as may be seen in the six volumes on physics by Arnold Sommerfeld (1868-1951), and from the productivity of his many (36) students (e.g., Debye, Lenz, Ewald, Pauli, Guillemin, Bethe, Heisenberg, and Seebach, to name a few), notable coworkers (i.e., Leon Brillouin) and others (i.e., John Bardeen) upon whom Sommerfeld had a strong influence. Sommerfeld is famous for training many students who were awarded the Nobel Prize in Physics, yet he never won a Nobel (the prize is not awarded in mathematics). Sommerfeld brought mathematical physics (the merging of physical and experimental principles via mathematics) to a new

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[15]`http://muse.tau.ac.il/museum/galileo/galileo_low_of_fall.html`

level with the use of complex integration of analytic functions to solve otherwise difficult problems, thus following the lead of Newton who used real integration of Taylor series to solve differential equations (Brillouin, 1960, Ch. 3 by Sommerfeld, A.).

3.3 Lec 12: **Root classification by convolution**

Following the exploration of algebraic relationships by Fermat and Descartes, the first theorem was being formulated by d'Alembert. The idea behind this theorem is that every polynomial of degree N (Eq. 3.7) has at least one root. This may be written as the product of the root and a second polynomial of degree of $N-1$. By the recursive application of this concept, it is clear that every polynomial of degree N has N roots. Today this result is known as the *fundamental theorem of algebra:*

> Every polynomial equation $P(z) = 0$ has a *solution* in the complex numbers. As Descartes observed, a solution $z = a$ implies that $P(z)$ has a factor $z - a$. The quotient
>
> $$Q(z) = \frac{P(z)}{z - a} = \frac{P(z)}{a}\left[1 + \frac{z}{a} + \left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)^3 + \cdots\right] \qquad (3.10)$$
>
> is then a polynomial of one lower degree. ... We can go on to factorize $P(z)$ into n linear factors.

—Stillwell (2010, p. 285).

The ultimate expression of this theorem is given by Eq. 3.7 (p. 76), which indirectly states that an nth degree polynomial has n roots. We shall use the term *degree* when speaking of polynomials and the term *order* when speaking of differential equations. A general rule is that *order* applies to the time domain and *degree* to the frequency domain, since the Laplace transform of a differential equation, having constant coefficients, of order N, is a polynomial of degree N in Laplace frequency s.

Today this theorem is so widely accepted we fail to appreciate it. Certainly about the time you learned the quadratic formula, you were prepared to understand the concept of polynomials having roots. The simple quadratic case may be extended to a higher degree polynomial. The Octave/Matlab command `roots([1, a₂, a₁, aₒ])` provides the roots $[s_1, s_2, s_3]$ of the cubic equation, defined by the coefficient vector $[1, a_2, a_1, a_o]$. The command `poly([s₁, s₂, s₃])` returns the coefficient vector. I don't know the largest degree that can be accurately factored by Matlab/Octave, but I'm sure its well over $N = 10^3$. Today, finding the roots numerically is a solved problem.

Factorization versus convolution: The best way to gain insight into the polynomial factorization problem is through the inverse operation, multiplication of monomials. Given the roots x_k, there is a simple algorithm for computing the coefficients a_k of $P_N(x)$ for any n, no matter how large. This method is called *convolution*. Convolution is said to be a *trap-door function* since it is easy, while the inverse, factoring (*deconvolution*), is hard, and analytically intractable for degree $N \geq 5$ (Stillwell, 2010, p. 102).

3.3.1 Convolution of monomials

As outlined by Eq. 3.7, a polynomial has two descriptions, first as a series with coefficients a_n and second in terms of its roots x_r. The question is: What is the relationship between the coefficients and the roots? The simple answer is that they are related by *convolution*.

Let us start with the quadratic

$$(x + a)(x + b) = x^2 + (a + b)x + ab, \qquad (3.11)$$

where in vector notation $[-a, -b]$ are the roots and $[1, a + b, ab]$ are the coefficients.

To see how the result generalizes, we may work out the coefficients for the cubic ($N = 3$). Multiplying the following three factors gives)

$$(x-1)(x-2)(x-3) = (x^2 - 3x + 2)(x-3) = x(x^2 - 3x + 2) - 3(x^2 - 3x + 2) = x^3 - 6x^2 + 11x - 6.$$

When the roots are $[1, 2, 3]$ the coefficients of the polynomial are $[1, -6, 11, -6]$. To verify, substitute the roots into the polynomial, and show that they give zero. For example, $r_1 = 1$ is a root since $P_3(1) = 1 - 6 + 11 - 6 = 0$.

As the degree increases, the algebra becomes more difficult. Imagine trying to work out the co-efficients for $N = 100$. What is needed is a simple way of finding the coefficients from the roots. Fortunately, *convolution* keeps track of the book-keeping, by formalizing the procedure, along with Newton's deconvolution method for finding roots of polynomials (p. 79).

Convolution of two vectors: To get the coefficients by convolution, write the roots as two vectors $[1, a]$ and $[1, b]$. To find the coefficients we must convolve the vectors, indicated by $[1, a] \star [1, b]$, where $\star$ denotes convolution. Convolution is a recursive operation. The convolution of $[1, a] \star [1, b]$ is done as follows: reverse one of the two monomials, padding unused elements with zeros. Next slide one monomial against the other, forming the local *scalar product* (element-wise multiply and add):

$$
\begin{array}{llll}
a & 1 & 0 & 0 \\
0 & 0 & 1 & b \\
= & 0 & &
\end{array}
\qquad
\begin{array}{lll}
a & 1 & 0 \\
0 & 1 & b \\
= & x^2 &
\end{array}
\qquad
\begin{array}{lll}
a & 1 & 0 \\
1 & b & 0 \\
= & (a+b)x &
\end{array}
\qquad
\begin{array}{lll}
0 & a & 1 \\
1 & b & 0 \\
= & abx^0 &
\end{array}
\qquad
\begin{array}{llll}
0 & 0 & a & 1 \\
1 & b & 0 & 0 \\
= & 0 & &
\end{array} ,
$$

resulting in coefficients $[\cdots, 0, 0, 1, a + b, ab, 0, 0, \cdots]$.

By reversing one of the polynomials, and then taking successive scalar products, all the terms in the sum of the scalar product correspond to the same power of x. This explains why convolution of the coefficients gives the same answer as the product of the polynomials.

As seen by the above example, the positions of the first monomial coefficients are reversed, and then slid across the second set of coefficients, the dot-product is computed, and the result placed in the output vector. Outside the range shown, all the elements are zero. In summary,

$$[1, -1] \star [1, -2] = [1, -1 - 2, 2] = [1, -3, 2].$$

In general

$$[a, b] \star [c, d] = [ac, bc + ad, bd],$$

Convolving a third term $[1, -3]$ with $[1, -3, 2]$ gives (p. 339)

$$[1, -3] \star [1, -3, 2] = [1, -3 - 3, 9 + 2, -6] = [1, -6, 11, -6],$$

which is identical to the cubic example, found by the algebraic method.

By convolving one monomial factor at a time, the overlap is always two elements, thus it is never necessary to compute more than two multiplies and an add for each output coefficient. This greatly simplifies the operations (i.e., they are easily done in your head). Thus the final result is more likely to be correct. Comparing this to the algebraic method, convolution has the clear advantage.

Exercise:

1. What are the three nonlinear equations that one would need to solve to find the roots of a cubic?
   Solution: From our formula for the convolution of three monomials we may find the nonlinear "deconvolution" relations between the roots[19] $[-a, -b, -c]$ and the cubic's coefficients $[1, \alpha, \beta, \gamma]$

$$
\begin{aligned}
(x + a) \star (x + b) \star (x + c) &= (x + c) \star (x^2 + (a + b)x + ab) \\
&= x \cdot (x^2 + (a + b)x + ab) + c \cdot (x^2 + (a + b)x + ab) \\
&= x^3 + (a + b + c)x^2 + (ab + ac + cb)x + abc \\
&= [1, a + b + c, ab + ac + cb, abc].
\end{aligned}
$$

---

[19]By working with the negative roots we may avoid an unnecessary and messy alternating sign problem.

It follows that the nonlinear equations must be

$$\alpha = a + b + c$$
$$\beta = ab + ac + bc$$
$$\gamma = abc.$$

These may be solved by the classic cubic solution, which therefore is a *deconvolution* problem, also know as *long division of polynomials*. It follows that the following long-division of polynomials must be true:

$$\frac{x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc}{x + a} = x^2 + (b + c)x + bc$$

The product of monomial $P_1(x)$ with a polynomial $P_N(x)$ gives $P_{N+1}(x)$: This statement is another way of stating the *fundamental theorem of algebra*. Each time we convolve a monomial with a polynomial of degree N, we obtain a polynomial of degree $N + 1$. The convolution of two monomials results in a quadratic (degree 2 polynomial). The convolution of three monomials gives a cubic (degree 3). In general, the degree k of the product of two polynomials of degree n, m is the sum of the degrees ($k = n + m$). For example, if the degrees are each 5 ($n = m = 5$), then the resulting degree is 10.

While we all know this theorem from high school algebra class, it is important to explicitly identify the *fundamental theorem of algebra*.

Note that the degree of a polynomial is one less than the length of the vector of coefficients. Since the leading term of the polynomial cannot be zero, else the polynomial would not have degree N, when looking for roots, the coefficient can (and should always) be normalized to 1.

In summary, the product of two polynomials of degree m, n having m and n roots gives a polynomial of degree $m + n$. This is an analysis process of merging polynomials, by coefficient convolution. Multiplying polynomials is a merging process, into a single polynomial.

Composition of polynomials: Convolution is not the only important operation between two polynomials. Another is *composition*, which may be defined for two functions $f(z), g(z)$. Then the composition $c(z) = f(z) \circ g(z) = f(g(z))$. As a specific example, suppose $f(z) = 1 + z + z^2$ and $g(z) = e^{2z}$. With these definitions

$$f(z) \circ g(z) = 1 + e^{2z} + (e^{2z})^2 = 1 + e^{2z} + e^{4z}.$$

Note that $f(z) \circ g(z) \neq g(z) \circ f(z)$.

Exercise: Find $g(z) \circ f(z)$. **Solution:** $e^{2f(z)} = e^{2(1+z+z^2)} = e^2 e^{(1+z+z^2)} = e^3 e^z e^{z^2}$.

3.3.2 Residue expansions of rational functions

As discussed in Section 3.1.4, p. 84, there are at least seven important Matlab/Octave routines that are closely related: `conv()`, `deconv()`, `poly()`, `polyder()`, `polyval()`, `residue()`, `root()`. Several of these are complements of each other, or do a similar operation in a slightly different way. Routines `conv()`, `poly()` build polynomials from the roots, while `root()` solves for the roots given the polynomial coefficients. The operation `residue()` converts the ratio of two polynomials and expands it in a partial fraction expansion, with poles and residues.

When lines and planes are defined, the equations are said to be *linear* in the independent variables. In keeping with this definition of *linear*, we say that the equations are *non-linear* when the equations have degree greater than 1 in the independent variables. The term *bilinear* has a special meaning, in that both the domain and codomain are linearly related by lines (or planes). As an example, impedance is

defined in frequency as the ratio of the voltage over the current, but it frequently has a representation as the ratio of two polynomials $N(s)$ and $D(s)$

$$Z(s) = \frac{N(s)}{D(s)} = sL_o + R_o + \sum_{k=0}^{K} \frac{K_k}{s - s_k}. \tag{3.12}$$

Here $Z(s)$ is the impedance and V and I are the voltage and current at radian frequency ω.[20]

Such an impedance is typically specified as a *rational* or *bilinear* function, namely the ratio of two polynomials, $P_N(s) = N(s) = [a_N, a_{n-1}, \cdots, a_o]$ and $P_K(s) = D(s) = [b_K, b_{K-1}, \cdots, b_o]$ of degrees $N, K \in \mathbb{N}$, as functions of complex Laplace frequency $s = \sigma + \jmath\omega$, having simple roots. Most impedances are rational functions since they may be written as $D(s)V = N(s)I$. Since $D(s)$ and $N(s)$ are both polynomials in s, rational functions are also called *bilinear transformation* or in the mathematical literature *Möbius transformation*, which comes from a corresponding scalar differential equation, of the form

$$\sum_{k=0}^{K} b_k \frac{d^k}{dt^k} i(t) = \sum_{n=0}^{N} a_n \frac{d^n}{dt^n} v(t) \quad \leftrightarrow \quad I(\omega) \sum_{k=0}^{K} b_k s^k = V(\omega) \sum_{n=0}^{N} a_n s^n. \tag{3.13}$$

This construction is also known as the ABCD method in the engineering literature (Eq. 3.35, p. 123). This equation, as well as 3.12, follows from the Laplace transform (see Section 3.7, p. 131) of the differential equation (on left), by forming the impedance $Z(s) = V/I = A(s)/B(s)$. This form of the differential equation follows from Kirchhoff's voltage and current laws (KCL, KVL) or from Newton's laws (for the case of mechanics).

The physical properties of an impedance: Based on d'Alembert's observation that the solution to the wave equation is the sum of forward and backward traveling waves, the impedance may be rewritten in terms of forward and backward traveling waves (see p. 207)

$$Z(s) = \frac{V}{I} = \frac{V^+ + V^-}{I^+ - I^-} = r_o \frac{1 + \Gamma(s)}{1 - \Gamma(s)}, \tag{3.14}$$

where $r_o = P^+/I^+$ is called the *surge impedance* of the transmission line (e.g., wire) connected to the load impedance $Z(s)$, and $\Gamma(s) = P^-/P_+ = I^-/I^+$ is the reflection coefficient corresponding to $Z(s)$. Any impedance of this type is called a *Brune impedance* due to its special properties (discussed on p. 363) (Brune, 1931a). Like $Z(s)$, $\Gamma(s)$ is causal and complex analytic. The impedance and the reflectance function must both be complex analytic, since they are related to the bilinear transformation, which assures the mutual complex analytic properties.

Due to the bilinear transformation, the physical properties of $Z(s)$ and $\Gamma(s)$ are very different. Specifically, the real part of the load impedance must be non-negative ($\Re\{Z(\omega\jmath)\} \geq 0$), if and only if $|\Gamma(s)| \leq 1|$. In the time domain, the impedance $z(t) \leftrightarrow Z(s)$ must have a value of r_o at $t = 0$. Correspondingly, the time domain reflectance $\gamma(t) \leftrightarrow \Gamma(s)$ must be zero at $t = 0$.

This is the basis of conservation of energy, which may be traced back to the properties of the reflectance $\Gamma(s)$.

Exercise: Show that if $\Re\{Z(s)\} \geq 0$ then $|\Gamma(s)| \leq 1$. **Solution:** Taking the real part of Eq. 3.14, which must be ≥ 0, we find

$$\Re\{Z(s)\} = \frac{r_o}{2} \left[\frac{1 + \Gamma(s)}{1 - \Gamma(s)} + \frac{1 + \Gamma^*(s)}{1 - \Gamma^*(s)} \right] = r_o \frac{1 - |\Gamma(s)|^2}{|1 + \Gamma(s)|^2} \geq 0.$$

Thus $|\Gamma| \leq 1$.

---

[20]Note that the relationship between the impedance and the residues K_k is a linear one, best solved by setting up a linear system of equations in the unknown residues. Methods for doing this will be discussed in Appendix I (p. 284).

3.4 Lec 13: Introduction to Analytic Geometry

Analytic geometry came about with the merging of Euclid's geometry with algebra. The combination of Euclid's (323 BCE) geometry and al-Khwarizmi's (830 CE) algebra resulted in a totally new and powerful tool, *analytic geometry*, independently worked out by Descartes and Fermat (Stillwell, 2010). The addition of matrix algebra during the 18th century enabled analysis in more than three dimensions, which today is one of the most powerful tools used in artificial intelligence, data science and machine learning. The utility and importance of these new tools cannot be overstated. The timeline for this period is provided in Fig. 3.5.
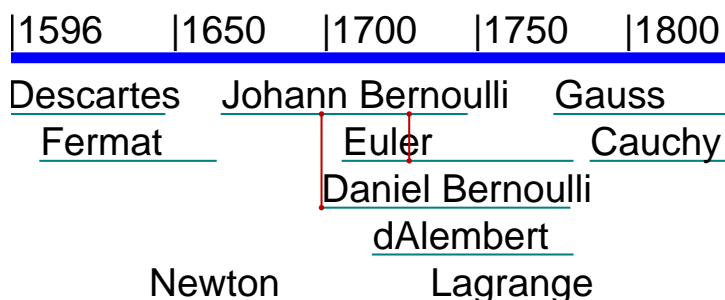
Figure 3.5:   *Expanded timeline of Fig. 3.1 (p. 73) covering the two centuries from 1596CE to 1855CE, covering the development of the modern theories of analytic geometry, calculus, differential equations and linear algebra. The vertical red lines indicate mentor-student relationships. Lagrange had a role in the development of linear algebra. Note the overlap between Newton and Johann and his son Daniel Bernoulli, and Euler. Gauss had the advantage of input from Newton, Euler, d'Alembert and Lagrange. Likely Cauchy had an influence on Gauss as well.*

There are many important relationships between Euclidean geometry and 16th century algebra. An attempt at a detailed comparison is summarized in Table 3.1. Important similarities include vectors, their Pythagorean lengths $[a, b, c]$

$$c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \tag{3.15}$$

$a = x_2 - x_1$ and $b = y_2 - y_1$, and the angles. Euclid's geometry had length and angles, but no concept of coordinates, thus of vectors. One of the main innovations of analytic geometry is that one may compute with real, and soon after, complex numbers.

There are several new concepts that come with the development of analytic geometry:

1. Composition of functions: If $y = f(x)$ and $z = g(y)$ then the composition of functions f and g is denoted $z(x) = g \circ f(x) = g(f(x))$.

2. Elimination: Given two functions $f(x, y)$ and $g(x, y)$, elimination removes either x or y. This procedure, known to the Chinese, is called *Gaussian elimination*.

3. Intersection: While one may speak of the intersection of two lines to give a point, or two planes to give a line, the term *intersection* is also an important but very different concept in set theory. This is a special case of elimination when the functions $f(x, y), g(x, y)$ are linear in their arguments.

4. Vectors: Analytic geometry provides the concept of a vector (see Appendix A), as a line with length and orientation (i.e., direction). Analytic geometry defines vectors in any number of dimensions, as ordered sets of points.

5. Analytic geometry extends the ideas of Euclidean geometry with the introduction of the scalar (dot) product of two vectors $\mathbf{f} \cdot \mathbf{g}$, and the vector (cross) product $\mathbf{f} \times \mathbf{g}$ (see Fig. 3.6, p. 109).

6. All of these exist independently of any coordinate system, thus are more general than analytic geometry.

Table 3.1: An ad-hoc comparison between Euclidean geometry and analytic geometry. I am uncertain as to the classification of the items in the third column.

Euclidean geometry: $\mathbb{R}^3$	Analytic geometry: $\mathbb{R}^n$	Uncertain
(a) Proof	(a) Numbers	(a) Cross product ($\mathbb{R}^3$)
(b) Line length	(b) Algebra	(b) Recursion
(c) Line intersection	(c) Power series	(c) Iteration $\in \mathbb{C}^2$ (e.g., Newton's method)
(d) Point	(d) Analytic functions	(d) Iteration $\in \mathbb{R}^n$
(e) Projection (e.g. scalar product)	(e) Complex analytic functions: e.g., $\sin\theta, \cos\theta, e^{\theta_j}, \log z$	
(f) Line direction	(f) Composition	
(g) Vector (sort of)	(g) Elimination	
(h) Conic section	(h) Integration	
(i) Square roots (e.g., spiral of Theodorus)	(i) Derivatives	
	(j) Calculus	
	(k) Polynomial $\in \mathbb{C}$	
	(l) Fund. thm. algebra	
	(m) Normed vector spaces	

What algebra also added to geometry was the ability to compute with complex numbers. For example, the length of a line (Eq. 3.15) was measured in Geometry with a compass: numbers played no role. Once algebra was available, the line's Euclidean length could be computed numerically, directly from the coordinates of the two ends, defined by the *3-vector*

$$\mathbf{e} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} = [x, y, z]^T,$$

which represents a point at $(x, y, z) \in \mathbb{R}^3 \subset \mathbb{C}^3$ in three dimensions, having *direction*, from the origin $(0, 0, 0)$ to (x, y, z). An alternative *matrix notation* is $\mathbf{e} = [x, y, z]^T$, a column vector of three numbers. These two notations are different ways of representing exactly the same thing.

By defining the vector, analytic geometry allows Euclidean geometry to become quantitative, beyond the physical drawing of an object (e.g., a sphere, triangle or line). With analytic geometry we have the Euclidean concept of a vector, a line having a magnitude (length) and direction, but analytic defined in terms of physical coordinates (i.e., numbers). The difference between two vectors defines a third vector, a concept already present in Euclidean geometry. For the first time, complex numbers were allowed into geometry (but rarely used until Cauchy and Riemann).

As shown in Fig. 3.6, there are two types of products: the 1) scalar $\boldsymbol{A} \cdot \boldsymbol{B}$ and 2) vector $\boldsymbol{A} \times \boldsymbol{B}$ products.

Scalar product of two vectors: When using algebra, many concepts, obvious with Euclid's geometry, may be made precise. There are many examples of how algebra extends Euclidean geometry, the most basic being the *scalar product* (aka dot product) between vectors $\boldsymbol{x}, \boldsymbol{\kappa} \in \mathbb{C}^3$

$$\mathbf{x} \cdot \boldsymbol{\kappa} = (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \cdot (\alpha\hat{\mathbf{x}} + \beta\hat{\mathbf{y}} + \gamma\hat{\mathbf{z}}), \quad \in \mathbb{C}$$
$$= \alpha x + \beta y + \gamma z.$$
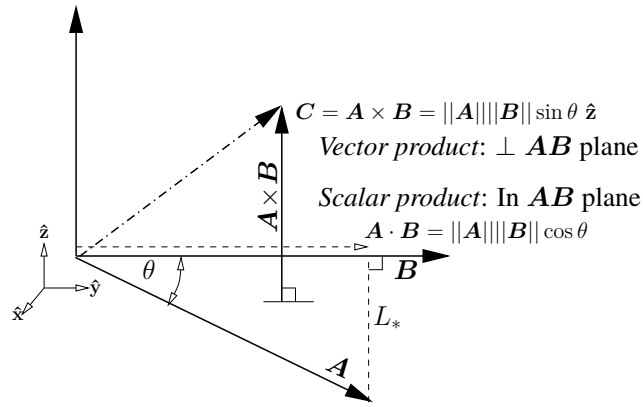
Figure 3.6: *Vectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are used to define the* scalar product $\boldsymbol{A} \cdot \boldsymbol{B}$, vector (cross) product $\boldsymbol{A} \times \boldsymbol{B}$ and triple product $\boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B})$. *The two vector products complement each other, one proportional to the* $\sin \theta$ *of the angle* θ *between them and the other to the* $\cos \theta$. *It follows that* $||\boldsymbol{A} \cdot \boldsymbol{B}||^2 + ||\boldsymbol{A} \times \boldsymbol{B}||^2 = ||\boldsymbol{A}||^2 ||\boldsymbol{B}||^2$. *The scalar product computes the projection of one vector on the other. The vector product* $\boldsymbol{A} \times \boldsymbol{B}$ *computes the area of the trapezoid formed by the two vectors, while the triple product* $\boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B})$ *defines the volume of the formed parallelepiped (i.e., prism). When all the angles are* $90°$, *the volume becomes a cuboid (see p. 211).*

In matrix notation the scalar product is written as (p. 211)

$$\mathbf{x} \cdot \boldsymbol{\kappa} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} x, y, z \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \alpha x + \beta y + \gamma z. \tag{3.16}$$

The scalar product takes the character of $\boldsymbol{\kappa}$. For example, if $\boldsymbol{\kappa}(s) \in \mathbb{C}$ is a function of complex frequency s, then the scalar product is complex. If $\zeta \in \mathbb{R}$ is real, then the scalar product is real.

Norm (length) of a vector: The *norm* of a vector

$$||\mathbf{e}|| \equiv +\sqrt{\mathbf{e} \cdot \mathbf{e}} \geq 0$$

is *defined* as the positive square root of the scalar product of the vector with itself. This is a generalization of the length, in any number of dimensions, forcing the sign of the square-root to be non-negative. The length is a concept of Euclidean geometry, and it must always be positive and real. A complex (or negative) length is not physically meaningful. More generally, the Euclidean length of a line is given as the *norm* of the difference between two real vectors $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{R}$

$$||\mathbf{e_1} - \mathbf{e_2}||^2 = (\mathbf{e}_1 - \mathbf{e}_2) \cdot (\mathbf{e}_1 - \mathbf{e}_2)$$
$$= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \geq 0.$$

From this formula we see that the norm of the difference of two vectors is simply a compact expression for the Euclidean length. A zero-length vector, such as is a point, is the result of the fact that

$$||x - x||^2 = (x - x) \cdot (x - x),$$

is zero.

Integral definition of a scalar product: Up to this point, following Euclid, we have only considered a vector to be a set of elements $\{x_n\} \in \mathbb{R}$, index over $n \in \mathbb{N}$, as defining a linear vector space with scalar product $x \cdot y$, with the scalar product defining the norm or length of the vector $||x|| = \sqrt{x \cdot x}$. Given the scalar product, the norm naturally follows.

5

At this point an obvious question presents itself: Can we extend our definition of vectors to differentiable functions (i.e., $f(t)$ and $g(t)$), indexed over $t \in \mathbb{R}$, with coefficients labeled by $t \in \mathbb{R}$, rather than by $n \in \mathbb{N}$? Clearly, if the functions are analytic, there is no obvious reason why this should be a problem, since analytic functions may be represented by a convergent series having Taylor coefficients, thus are integrable term by term.

Specifically, under certain conditions, the function $f(t)$ may be thought of as a vector, defining a normed vector space. This intuitive and somewhat obvious idea is powerful. In this case the scalar product can be defined in terms of the integral

$$f(t) \cdot g(t) = \int_t f(t)g(t)dt \tag{3.17}$$

summed over $t \in \mathbb{R}$, rather than a sum over $n \in \mathbb{N}$.

This definition of the vector scalar product allows for a significant but straightforward generalization of our vector space, which will turn out to be both useful and an important extension of the concept of a normed vector space. In this space we can define the derivative of a norm with respect to t, which is not possible for the case of the discrete case, indexed over n. The distinction introduces the concept of continuity in the index t, which does not exist for the discrete index $n \in \mathbb{N}$.

Pythagorean theorem and the Schwarz inequality: Regarding Fig. 3.6, suppose we compute the difference between vector $\boldsymbol{A} \in \mathbb{R}$ and $\alpha \boldsymbol{B} \in \mathbb{R}$ as $L = ||\boldsymbol{A} - \alpha \boldsymbol{B}|| \in \mathbb{R}$, where $\alpha \in \mathbb{R}$ is a scalar that modifies the length of $\boldsymbol{B}$. We seek the value of α, which we denote as α^*, that minimizes the length of L. From simple geometrical considerations, $L(\alpha)$ will be minimum when the difference vector is perpendicular to $\boldsymbol{B}$, as shown in the figure by the dashed line from the tip of $\boldsymbol{A} \perp \boldsymbol{B}$.

To show this algebraically we write out the expression for $L(\alpha)$ and take the derivative with respect to α, and set it to zero, which gives the formula for α^*. The argument does not change, but the algebra greatly simplifies, if we normalize $\boldsymbol{A}, \boldsymbol{B}$ to be unit vectors $\boldsymbol{a} = \boldsymbol{A}/||\boldsymbol{A}||$ and $\boldsymbol{b} = \boldsymbol{B}/||\boldsymbol{B}||$, which have norm = 1.

$$L^2 = (\boldsymbol{a} - \alpha \boldsymbol{b}) \cdot (\boldsymbol{a} - \alpha \boldsymbol{b}) = 1 - 2\alpha \boldsymbol{a} \cdot \boldsymbol{b} + \alpha^2. \tag{3.18}$$

Thus the length is shortest ($L = L_*$, as shown in Fig. 3.6) when

$$\frac{d}{d\alpha}L_*^2 = -2\boldsymbol{a} \cdot \boldsymbol{b} + 2\alpha^* = 0.$$

Solving for α^* we find $\alpha^* = \boldsymbol{a} \cdot \boldsymbol{b}$. Since $L_* > 0$ ($\boldsymbol{a} \neq \boldsymbol{b}$), Eq. 3.18 becomes

$$1 - 2|\boldsymbol{a} \cdot \boldsymbol{b}|^2 + |\boldsymbol{a} \cdot \boldsymbol{b}|^2 = 1 - |\boldsymbol{a} \cdot \boldsymbol{b}|^2 > 0.$$

In conclusion $\cos \theta \equiv |\boldsymbol{a} \cdot \boldsymbol{b}| < 1$. In terms of $\boldsymbol{A}, \boldsymbol{B}$ this is $|\boldsymbol{A} \cdot \boldsymbol{B}| < ||\boldsymbol{A}|| \, ||\boldsymbol{B}|| \cos \theta$, as shown next to $\boldsymbol{B}$ in Fig. 3.6. Thus the scalar product between two vectors is their *direction cosine*. Furthermore, since this forms a right triangle, the Pythagorean theorem must hold. The *triangle inequality* says that the lengths of the two sides must be greater than the hypotenuse. Note that $\Theta \in \mathbb{R} \notin \mathbb{C}$.

This derivation is an abbreviated version of a related discussion in Section R.1.3 (p. 114).

Vector cross ($\times$) and wedge ($\wedge$) product of two vectors: The *vector product* (aka, cross-product) $\boldsymbol{A} \times \boldsymbol{B}$ and *exterior product* (aka, wedge-product) $\boldsymbol{A} \wedge \boldsymbol{B}$ are the second and third types of vector products. As shown in Fig. 3.6,

$$\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B} = (a_1\hat{\mathbf{x}} + a_2\hat{\mathbf{y}} + a_3\hat{\mathbf{z}}) \times (b_1\hat{\mathbf{x}} + b_2\hat{\mathbf{y}} + b_3\hat{\mathbf{z}}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

is $\perp$ to the plane defined by $\boldsymbol{A}$ and $\boldsymbol{B}$. The cross product is severely limited to two input vectors $\boldsymbol{A}$ and $\boldsymbol{B} \in \mathbb{R}^2$, taken from three real dimensions (i.e., $\mathbb{R}^3$).

The exterior wedge product generalizes the cross product, since it may be defined in terms of $A, B \in \mathbb{C}^2$, taken from n dimensions ($\mathbb{C}^n$), with output in $\mathbb{C}^1$. Its output is a complex scalar with

$$A \wedge B = -B \wedge A = e^{\theta j}$$

The difference between the two is that the cross product is a vector (typically in $\mathbb{R}$) while the wedge product is a signed complex scalar $\in \mathbb{C}$.

Example: If we define $A = 3j\hat{\mathbf{x}} - 2\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$ and $B = 1\hat{\mathbf{x}} + 1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$ then the cross product is

1.
$$A \times B = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 3j & -2 & 0 \\ 1 & 1 & 0 \end{vmatrix} = (3j + 2)\hat{\mathbf{z}}.$$

Since $a_1 \in \mathbb{C}$, this example violates the common assumption that $A \in \mathbb{R}^3$.

2. The wedge product $A \wedge B$ is defined by

$$\begin{aligned} A \wedge B &= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 3j & 1 \\ -2 & 1 \end{vmatrix} \\ &= (3j\hat{\mathbf{x}} - 2\hat{\mathbf{y}}) \wedge (\hat{\mathbf{x}} + \hat{\mathbf{y}}) \\ &= 3 \cdot 0 \; \cancel{\hat{\mathbf{x}} \wedge \hat{\mathbf{x}}}^{\,0} - 2\hat{\mathbf{y}} \wedge \hat{\mathbf{x}} + 3j\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} - 2 \; \cancel{\hat{\mathbf{y}} \wedge \hat{\mathbf{y}}}^{\,0} \\ &= (3j + 2)\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \end{aligned}$$

From the above example we see that the absolute value of the wedge product $|\mathbf{a} \wedge \mathbf{b}| = ||\mathbf{a} \times \mathbf{b}||$, namely

$$|(a_2\hat{\mathbf{y}} + a_3\hat{\mathbf{z}}) \wedge (b_2\hat{\mathbf{y}} + b_3\hat{\mathbf{z}})| = ||(\mathbf{a} \times \mathbf{b})||.$$

The wedge product is especially useful because it is zero when the two vectors are co-linear, namely $\hat{\mathbf{x}} \wedge \hat{\mathbf{x}} = 0$ and $\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} = 1$. The advantage of the "wedge" product is that it is valid in $n > 3$ dimensions since it is defined for any two vectors, in any number of dimensions. Like the cross product, the magnitude of the wedge product is equal to the area of the trapezoid formed by the two vectors.

Scalar triple product: The triple of a third vector C with the vector product $A \times B \in \mathbb{R}$ is

$$C \cdot (A \times B) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \in \mathbb{R}^3,$$

which equals the volume of a parallelepiped.

Impact of Analytic Geometry: The most obvious impact of analytic geometry was its detailed analysis of the conic sections, using algebra rather than drawings via a compass and ruler. An important example is the composition of the line and circle, a venerable construction, presumably going back to before Diophantus (250CE). Once algebra was invented, the composition could be done using formulas. With this analysis came complex numbers.

The first two mathematicians to appreciate this mixture of Euclid's geometry and the new algebra were Fermat and Descartes (Stillwell, 2010, p. 111-115); soon Newton contributed to this effort by the addition of physics (e.g., calculations in acoustics, orbits of the planets, and the theory of gravity and light, significant concepts for 1687 (Stillwell, 2010, p. 115-117)).

Given these new methods, many new solutions to problems emerged. The complex roots of polynomials continued to appear, without any obvious physical meaning. Complex numbers seem to have been viewed as more of an inconvenience than a problem. Newton's solution to this dilemma was to simply ignore the imaginary cases (Stillwell, 2010, p. 119).

3.4.1 Development of Analytic Geometry

Intersection and Gaussian elimination: The first "algebra" (al-jabr) is credited to al-Khwarizmi (830 CE). Its invention advanced the theory of polynomial equations in one variable, Taylor series, and composition versus intersections of curves. The solution of the quadratic equation had been worked out thousands of years earlier, but with algebra a general solution could be defined. The Chinese had found the way to solve several equations in several unknowns, for example, finding the values of the intersection of two circles. With the invention of algebra by al-Khwarizmi, a powerful tool became available to solve the difficult problems.

Composition and Elimination   In algebra there are two contrasting operations on functions: *composition* and *elimination*.

Composition: Composition is the merging of functions, by feeding one into the other. If the two functions are f, g then their composition is indicated by $f \circ g$, meaning the function $y = f(x)$ is substituted into the function $z = g(y)$, giving $z = g(f(x))$.

Composition is not limited to linear equations, even though that is where it is most frequently applied. To compose two functions, one must substitute one equation into the other. That requires solving for that substitution variable, which is not always possible in the case of nonlinear equations. However, many tricks are available that may work around this restriction. For example, if one equation is in x^2 and the other in x^3 or $\sqrt{x}$, it may be possible to multiply the first by x or square the second. The point is that one of the variables must be isolated so that when it is substituted into the other equations, the variable is removed from the mix.

Examples: Let $y = f(x) = x^2 - 2$ and $z = g(y) = y + 1$. Then

$$g \circ f = g(f(x)) = (x^2 - 2) + 1 = x^2 - 1. \tag{3.19}$$

In general composition does not commute (i.e., $f \circ g \neq g \circ f$), as is easily demonstrated. Swapping the order of composition for our example gives

$$f \circ g = f(g(y)) = z^2 - 2 = (y + 1)^2 - 2 = y^2 + 2y - 1. \tag{3.20}$$

Intersection: Complementary to composition is *intersection* (i.e., decomposition) (Stillwell, 2010, pp. 119,149). For example, the intersection of two lines is defined as the point where they meet. This is not to be confused with finding roots. A polynomial of degree N has N roots, but the points where two polynomials intersect has nothing to do with the roots of the polynomials. The intersection is a function (equation) of lower degree, implemented with Gaussian elimination.

Intersection of two lines   Unless they are parallel, two lines meet at a point. In terms of linear algebra this may be written as two linear equations[21] (on the left), along with the intersection point $[x_1, x_2]^T$, given by the inverse of the 2x2 set of equations (on the right)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad\qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{3.21}$$

By substituting the expression for the intersection point $[\boldsymbol{x}_1, \boldsymbol{x}_2]^T$ into the original equation, we see that it satisfies the equations. Thus the equation on the right is the solution to the equation on the left.

---

[21]When writing the equation $Ax = y$ in matrix format, the two equations are $ax_1 + bx_2 = y_1$ and $dx_1 + ex_2 = y_2$ with unknowns (x_1, x_2), whereas in the original equations $ay + bx = c$ and $dy + ex = f$, they were y, x. Thus in matrix format, the names are changed. The first time you see this scrambling of variables, it can be confusing.

Note the structure of the inverse: 1) The diagonal values (a, d) are swapped, 2) the off-diagonal values (b, c) are negated and 3) the 2x2 matrix is divided by the determinant $\Delta = ad - bc$. If $\Delta = 0$, there is no solution. When the determinant is zero ($\Delta = 0$), the slopes of the two lines

$$slope = \frac{dx_2}{dx_1} = \frac{b}{a} = \frac{d}{c}$$

are equal, thus the lines are parallel. Only if the slopes differ can there be a unique solution.

Exercise: Show that the equation on the right is the solution of the equation on the left. **Solution:** By a direct substitution of the right equation into the left equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, = \frac{1}{\Delta} \begin{bmatrix} ad - bc & -ab + ab \\ cd - cd & -cb + ad \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix},$$

which gives the identity matrix.

Algebra will give the solution when geometry cannot. When the two curves fail to intersect on the real plane, the solution still exists, but is complex valued. In such cases, geometry, which only considers the real solutions, fails. For example, when the coefficients $[a, b, c, d]$ are complex, the solution exists, but the determinant can be complex. Thus algebra is much more general than geometry. Geometry fails when the solution has a complex intersection.

A system of linear $Ax = y$ equations has many interpretations, and one should not be biased by the notation. As engineers we are trained to view x as the input and y as the output, in which case then $y = Ax$ seems natural, much like the functional relation $y = f(x)$. But what does the linear relation $x = Ay$ mean, when x is the input? The obvious answer is that $y = A^{-1}x$. But when working with systems of equations, there are many uses of equations, and we need to become more flexible in our interpretation. For example $y = A^2 x$ has a useful meaning, and in fact we saw this type of relationship when working with Pell's equation (p. 63) and the Fibonacci sequence (p. 65). As another example consider

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} a_{1x} & a_{1y} \\ a_{2x} & a_{2y} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

which is reminiscent of a three-dimensional surface $\mathbf{z} = \mathbf{f}(x, y)$. We shall find that such generalizations are much more than a curiosity.

3.4.2 Lec 14: Applications of scalar products

Another important example of algebraic expressions in mathematics is Hilbert's generalization of Eq. 1.1, known as the Schwarz inequality, shown in Fig. 3.7. What is special about this generalization is that it proves that when the vertex is $90°$, the length of the leg is minimum.

Vectors may be generalized to have ∞ dimensions: $\mathbf{U}, \mathbf{V} = [v_1, v_2, \cdots, v_\infty])$. The Euclidean inner product (i.e., scalar product) between two such vectors generalizes the finite dimensional case

$$\mathbf{U} \cdot \mathbf{V} = \sum_{k=1}^{\infty} u_k v_k.$$

As with the finite case, the *norm* $||\mathbf{U}|| = \sqrt{\mathbf{U} \cdot \mathbf{U}} = \sqrt{\sum u_k^2}$ is the scalar product of the vector with itself, defining the *length* of the infinite component vector. Obviously there is an issue of convergence, if the norm for the vector is to have a finite length.

It is a somewhat arbitrary requirement that $a, b, c \in \mathbb{R}$ for the Pythagorean theorem (Eq. 1.1). This seems natural enough since the sides are lengths. But, what if they are taken from the complex numbers, as for the lossy vector wave equation, or the lengths of vectors in $\mathbb{C}^n$? Then the equation generalizes to

$$\mathbf{c} \cdot \mathbf{c} = ||\mathbf{c}||^2 = \sum_{k=1}^{n} |c_k|^2,$$

where $||\mathbf{c}||^2 = (\mathbf{c}, \mathbf{c})$ is the inner (dot) product of a vector $\mathbf{c}$ with itself, where $|c_k|$ is the magnitude the complex c_k. As before, $||\mathbf{c}|| = \sqrt{||\mathbf{c}||^2}$ is the *norm* of vector $\mathbf{c}$, akin to a length.
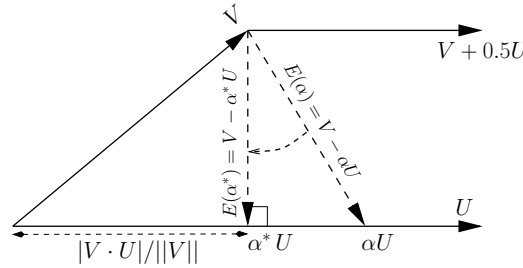
Figure 3.7: *The* Schwarz inequality *is related to the shortest distance (length of a line) between the ends of the two vectors.* $||U|| = \sqrt{U \cdot U}$ *as the scalar product of that vector with itself.*

Schwarz inequality   The *Schwarz inequality*[22] says that the magnitude of the inner product of two vectors is less than or equal to the product of their lengths

$$|U \cdot V| \le ||U|| \, ||V||.$$

This may be simplified by normalizing the vectors to have unit length ($\widehat{U} = U/||U||, \widehat{V} = V/||V||$), in which case $-1 < \widehat{U} \cdot \widehat{V} \le 1$. Another simplification is to define the scalar product in terms of the direction cosine

$$\cos\theta = |\widehat{U} \cdot \widehat{V}| \le 1.$$

A proof of the Schwarz inequality is as follows: From these definitions we may define the minimum difference between the two vectors as the perpendicular from the end of the first to the intersection with the second. As shown in Fig. 3.7, $U \perp V$ may be found by minimizing the length of the vector difference:

$$\min_{\alpha} ||V - \alpha U||^2 = ||V||^2 + 2\alpha V \cdot U + \alpha^2 ||U||^2 > 0$$

$$0 = \partial_\alpha \, (V - \alpha U) \cdot (V - \alpha U)$$

$$= V \cdot U - \alpha^* ||U||^2$$

$$\therefore \quad \alpha^* = V \cdot U/||U||^2.$$

The *Schwarz inequality* follows:

$$I_{\min} = ||V - \alpha^* U||^2 = ||V||^2 - \frac{|U \cdot V|^2}{||U||^2} > 0$$

$$0 \le |U \cdot V| \le ||U|| \, ||V||.$$

An important example of such a vector space includes the definition of the *Fourier transform*, where we may set

$$U(\omega) = e^{-\omega_0 j t} \qquad V(\omega) = e^{\omega j t} \qquad U \cdot V = \int_\omega e^{j\omega t} e^{-j\omega_0 t} \frac{d\omega}{2\pi} = \delta(\omega - \omega_0).$$

It seems that the Fourier transform is a result that follows from a minimization, unlike the Laplace transform that follows from causal system parameters. This explains the important differences between the two, in terms of their properties (unlike the LT, the FT is not complex analytic). We further explore this topic in Lecture 3.6 (p. 128).

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[22]A simplified derivation is provided in Sect. 3.4 (p. 107).

3.4.3   Lec 15: Gaussian Elimination

The method for finding the intersection of equations is based on the recursive elimination of all the variables but one. This method, known as *Gaussian elimination* (Appendix B, p. 251), works across a broad range of cases, but may be defined as a systematic algorithm when the equations are linear in the variables.[23] Rarely do we even attempt to solve problems in several variables of degree greater than 1. But Gaussian eliminations may still work in such cases (Stillwell, 2010, p. 90).

In Appendix B.2.1 (p. 257) the inverse of a 2x2 linear system of equations is derived. Even for a 2x2 case, the general solution requires a great deal of algebra. Working out a numeric example of Gaussian elimination is more instructive. For example, suppose we wish to find the intersection of the two equations

$$x - y = 3$$
$$2x + y = 2.$$

This 2x2 system of equations is so simple that you may immediately visualize the solution: By adding the two equations, y is eliminated, leaving $3x = 5$. But doing it this way takes advantage of the specific example, and we need a method for larger systems of equations. We need a generalized (algorithmic) approach. This general approach is called *Gaussian elimination*.

Start by writing the equations in *matrix* form (note this is not of the form $Ax = y$)

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}. \tag{3.22}$$

Next, eliminate the lower left term ($2x$) using a scaled version of the upper left term (x). Specifically, multiply the first equation by -2, add it to the second equation, replacing the second equation with the result. This gives

$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 - 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}. \tag{3.23}$$

Note that the top equation did not change. Once the matrix is "upper triangular" (zero below the diagonal) you have the solution. Starting from the bottom equation, $y = -4/3$. Then the upper equation gives $x - (-4/3) = 3$, or $x = 3 - 4/3 = 5/3$.

In principle, Gaussian elimination is easy, but if you make a calculation mistake along the way, it is very difficult to find your error. The method requires a lot of mental labor, with a high probability of making a mistake. Thus you do not want to apply this method every time. For example, suppose the elements are complex numbers, or polynomials in some other variable such as frequency. Once the coefficients become more complicated, the seemingly trivial problem becomes highly error prone. There is a much better way, that is easily verified, which puts all the numerics at the end, in a single step.

The above operations may be automated by finding a carefully chosen upper-diagonalization matrix U that does the same operation. For example, let

$$U = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}. \tag{3.24}$$

Multiplying Eq. 3.22 by U, we find

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \tag{3.25}$$

thus we obtain Eq. 3.23. At this point we can either back-substitute and obtain the solution, as we did above, or find a matrix L that finishes the job, by removing elements above the diagonal.

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[23]https://en.wikipedia.org/wiki/System_of_linear_equations.

Exercise: Show that $det(U) = det(UA) = 1$. **Solution:** Since

$$det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc,$$

and $det(UA) = detU \cdot det(A)$, and $det(U) = 1 - 0 = 1$, the result follows naturally. Common notation is to let $det(A) = |A|$.

In Appendix B.2.1 the inverse of a general 2x2 matrix is summarized in terms of three steps: 1) swap the diagonal elements, 2) reverse the signs of the off-diagonal elements and 3) divide by the determinant $\Delta = ab - cd$. Specifically

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{3.26}$$

There are very few things that you must memorize, but the inverse of a 2x2 is one of them. It needs to be in your tool-bag, like the quadratic formula.

While it is difficult to compute the inverse matrix from scratch (Appendix B), it takes only a few seconds to verify it (steps 1 and 2)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & -ab + ab \\ cd - cd & -bc + ad \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}. \tag{3.27}$$

5   Thus dividing by the determinant gives the 2x2 identity matrix. A good strategy, when you don't trust your memory, is to write down the inverse as best you can, and then verify.

Using the 2x2 matrix inverse on our example (Eq. 3.22), we find

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{1+2} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 \\ -6 + 2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -4/3 \end{bmatrix}. \tag{3.28}$$

If you use this method, you will rarely (never) make a mistake, and the solution is easily verified. Either you can check the numbers in the inverse, as was done in Eq. 3.27, or you can substitute the solution back into the original equation.

3.5   Lec 16: Transmission (ABCD) matrix composition method

Matrix composition: Matrix multiplication represents a composition of 2x2 matrices, because the input to the second matrix is the output of the first (this follows from the definition of composition: $f(x) \circ g(x) = f(g(x))$). Thus the ABCD matrix is also known as the *transmission* matrix method, or occasionally the *chain matrix*. The general expression for an transmission matrix $\boldsymbol{T}(s)$ is

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \mathcal{A}(s) & \mathcal{B}(s) \\ \mathcal{C}(s) & \mathcal{D}(s) \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \tag{3.35}$$

The four coefficients $\mathcal{A}(s), \mathcal{B}(s), \mathcal{C}(s), \mathcal{D}(s)$ are all complex functions of the Laplace frequency $s = \sigma + j\omega$ (p. 132). Typically they are polynomials in s, $\mathcal{C}(s) = s^2 + 1$ for example.

It is a standard convention to always define the current into the node, but since the input current (on the left) is the same as the output current on the right (I_2), hence the negative sign on I_2, to meet the convention of current into every node. When transmission matrices are cascaded, all the signs then match.

We have already used 2x2 matrix composition in representing complex numbers (p. 38), and for computing the $gcd(m, n)$ of $m, n \in \mathbb{N}$ (p. 54), Pell's equation (p. 63) and the Fibonacci sequence (p. 65). It would appear that 2x2 matricies have high utility, yet are simple to work with.

Definitions of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$:   By writing out the equations corresponding to Eq. 3.35, it is trivial to show that:

$$\mathcal{A}(s) = \left.\frac{V_1}{V_2}\right|_{I_2=0}, \quad \mathcal{B}(s) = \left.-\frac{V_1}{I_2}\right|_{V_2=0}, \quad \mathcal{C}(s) = \left.\frac{I_1}{V_2}\right|_{I_2=0}, \quad \mathcal{D}(s) = \left.-\frac{I_1}{I_2}\right|_{V_2=0}. \tag{3.36}$$

Each equation has a physical interpretation, and a corresponding name. Functions $\mathcal{A}$ and $\mathcal{C}$ are said to be *blocked*, because the output current I_2 is zero. Functions $\mathcal{B}$ and $\mathcal{D}$ are said to be *short-circuited*, because the output voltage V_2 is zero. These two terms (blocked, short-circuited) are electrical-engineering centric, arbitary, and fail to generalize to other cases, thus these terms should be avoided.

For example, in a mechanical system *blocked* would correspond to an output *isometric* (no length change) velocity of zero. In mechanics the *isometric force* is defined as the maximum applied force, conditioned on zero velocity (aka, the blocked force). Thus the *short-circuited* force (aka $\mathcal{B}$) would correspond to the force when the force is zero, which is nonsense. Thus these engineering-centric terms do not gracefully generalize, so better terminology is needed. Much of this was sorted out by Thévenin[25] c1883 (Johnson, 2003) and (Kennelly, 1893).

$\mathcal{A}, \mathcal{D}$ are called voltage (force) and current (velocity) *transfer functions*, since they are ratios of voltages and currents, whereas $\mathcal{B}, \mathcal{C}$ are known as the *transfer impedance* and *transfer admittance*. For example, the unloaded (blocked) ($I_2 = 0$) output voltage $V_2 = I_1/\mathcal{C}$, which would correspond to the isometric force in mechanics. In this way each term expresses an output (port 2) in terms of an input (port 1), for a given load condition.

Exercise: Explain why $\mathcal{C}$ is given as above. **Solution:** Writing out the lower equation gives $I_1 = \mathcal{C}V_2 - \mathcal{D}I_2$. Setting $I_2 = 0$, we obtain the equation for $\mathcal{C} = I_1/V_2|_{I_2=0}$.

Exercise: Can $\mathcal{C} = 0$? **Solution:** Yes, if $I_2 = 0$ and $I_1 = I_2$, $\mathcal{C} = 0$. For $\mathcal{C} \neq 0$ there needs to be a finite "shunt" impedance across V_1, so that $I_1 \neq I_2 = 0$.

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[25]https://en.wikipedia.org/wiki/Leon_Charles_Thevenin

3.5.1   Thévenin parameters of a source:

An important concept in circuit theory is that of the *Thévenin parameters*, the *open-circuit voltage* and the *short-circuit current*, the ratio of which defines the *Thévenin impedance* (Johnson, 2003). The open-circuit voltage is defined as the voltage V_2 when the load current $I_2 = 0$, which was shown in the previous exercise to be $V_2/I_1|_{I_2=0} = 1/\mathcal{C}$.

Thévenin Voltage: The definition Thévenin voltage $V_{\text{Thév}}$ is the same as that for $\mathcal{A}$, thus

$$\left.\frac{V_{\text{Thév}}}{V_1}\right|_{I_2=0} = \frac{1}{\mathcal{A}}. \tag{3.37}$$

Thévenin impedance   The Thévenin impedance is the impedance looking into port 2 with $V_1 = 0$, thus

$$Z_{\text{Thév}} = \left.\frac{V_2}{I_2}\right|_{V_1=0}.$$

From the upper equation of Eq. 3.35, with $V_1 = 0$, we obtain $\mathcal{A}V_2 = \mathcal{B}I_2$, thus

$$Z_{\text{Thév}} = \frac{\mathcal{B}}{\mathcal{A}}.$$

Properties of the transmission matrix: The transmission matrix is always constructed from the product of elemental matrices of the form

$$\begin{bmatrix} 1 & Z(s) \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ Y(s) & 1 \end{bmatrix}.$$

Thus for the case of reciprocal systems (P6, p. 134),

$$\Delta_T = \det \begin{bmatrix} \mathcal{A}(s) & \mathcal{B}(s) \\ \mathcal{C}(s) & \mathcal{D}(s) \end{bmatrix} = 1,$$

since the determinant of the product of each elemental matrix is 1, the determinant their product is 1. An anti-reciprocal system may be synthesized by the use of a gyrator, and for such cases $\Delta_T = -1$.

The eigenvalues and vectors equations for a T matrix are summarized on p. 183 and discussed in Appendix D (p. 263).

3.5.2   Lec 17: The impedance matrix

With a bit of algebra, one may find the impedance matrix in terms of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ (Van Valkenburg, 1964a, p. 310)

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{\mathcal{C}} \begin{bmatrix} \mathcal{A} & \Delta_T \\ 1 & \mathcal{D} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{3.38}$$

Note that if $\mathcal{C} = 0$, the impedance matrix does not exist.

Definitions of $z_{11}(s), z_{12}(s), z_{21}(s), z_{22}(s)$**:**   The definitions of the matrix elements are easily read off of the equation, as

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0}, \quad z_{12} = -\left.\frac{V_1}{I_2}\right|_{I_1=0}, \quad z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}, \quad z_{22} = -\left.\frac{V_2}{I_2}\right|_{I_1=0}. \tag{3.39}$$

These definitions follow trivially from Eq. 3.38 and each element has a physical interpretation. For example, the unloaded ($I_2 = 0$, aka blocked or isometric) input impedance is $z_{11}(s) = \mathcal{A}(s)/\mathcal{C}(s)$ while the unloaded transfer impedance is $z_{21}(s) = 1/\mathcal{C}(s)$. For reciprocal systems (P6, p. 134) $z_{12} = z_{21}$ since $\Delta_T = 1$. For anti-reciprocal systems, such as dynamic (aka, magnetic) loudspeakers and microphones (Kim and Allen, 2013)), $\Delta_T = -1$, thus $z_{21} = -z_{12} = 1/\mathcal{C}$. Finally z_{22} is the impedance looking into port 2 with port 1 open/blocked ($I_1 = 0$).

3.5.3   Lec 18: Network power relations

Impedance is a very general concept, closely tied to the definition of power $\mathcal{P}(t)$ (and energy). Power is defined as the product of the effort (force) and the flow (current). As described in Fig. 3.2, these concepts are very general, applying to mechanics, electrical circuits, acoustics, thermal circuits, or any other case where conservation of energy applies. Two basic variables are defined, *generalized force* and *generalized flow*, also called *conjugate variables*. The product of the conjugate variables is the power, and the ratio is the impedance. For example, for the case of voltage and current,

$$\mathcal{P}(t) \equiv \int^t v(t)i(t)dt, \qquad\qquad Z(s) \equiv \frac{V(\omega)}{I(\omega)}.$$

3.5.4   Power vs. power series, linear vs. nonlinear

Another place where equations of second degree appear in physical applications is in energy and power calculations. The electrical power is given by the product of the voltage $v(t)$ and current $i(t)$ (or in mechanics as the force times the velocity). For example if we define $\mathcal{P} = v(t)i(t)$ to be the power $\mathcal{P}$ [watts], then the *total energy* [joules] at time t is (Van Valkenburg, 1964a, Chapter 14)

$$\mathcal{E}(t) = \int_0^t v(t)i(t)dt.$$

From this observe that the power is the rate of change of the total energy

$$\mathcal{P}(t) = \frac{d}{dt}\mathcal{E}(t),$$

reminiscent of the fundamental theorem of calculus [Eq. 4.7, (p. 148)].

Ohm's law and impedance: The ratio of voltage over the current is called the *impedance* which has units of [ohms]. For example given a resistor of $R = 10$ [ohms],

$$v(t) = R\,i(t).$$

Namely, 1 [amp] flowing through the resistor would give 10 [volts] across it. Merging the linear relation due to Ohm's law with the definition of power shows that the instantaneous power in a resistor is quadratic in voltage and current

$$\mathcal{P}(t) = v(t)^2/R = i(t)^2 R. \tag{3.40}$$

Note that Ohm's law is linear in its relation between voltage and current whereas the power and energy are *nonlinear*. ⁵

   Ohm's law generalizes in a very important way, allowing the impedance (e.g., resistance) to be a linear complex analytic function of complex frequency $s = \sigma + \omega\jmath$ (Kennelly, 1893; Brune, 1931a). Impedance is a fundamental concept in many fields of engineering. For example:[26] Newton's second law $F = ma$ obeys Ohm's law, with mechanical impedance $Z(s) = sm$. Hooke's law $F = kx$ for ¹⁰ a spring is described by a mechanical impedance $Z(s) = k/s$. In mechanics a "resistor" is called a *dash-pot* and its impedance is a positive and real constant.[27]

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[26]In acoustics the pressure is a potential, like voltage. The force per unit area is given by $f = -\nabla p$, thus $F = -\int \nabla p\, dS$. Velocity is analogous to a current. In terms of the velocity potential, the velocity per unit area is $v = -\nabla\phi$.

[27]https://en.wikipedia.org/wiki/Impedance_analogy

Kirchhoff's laws KCL, KVL: The laws of electricity and mechanics may be written down using Kirchhoff's current and voltage laws (KCL, KVL), which lead to linear systems of equations in the currents and voltages (velocities and forces) of the system under study, with complex coefficients having positive real parts.

5   Points of major confusion are a number of terms that are misused, and overused, in the fields of mathematics, physics and engineering. Some of the most obviously abused terms are *linear/nonlinear*, *energy*, *power*, *power series*. These have multiple meanings, which can, and are, fundamentally in conflict.

Transfer functions (transfer matrix):   The only method that seems to work, to sort this out, is to cite the relevant physical application, in specific contexts. The most common standard reference is a physical system that has an input $x(t)$ and an output $y(t)$. If the system is linear, then it may be represented by its *impulse response* $h(t)$. In such cases the system equation is

$$y(t) = h(t) \star x(t) \leftrightarrow Y(\omega) = H(s)|_{\sigma=0} X(\omega);$$

namely, the convolution of the input with the impulse response gives the output. From Fourier analysis 10   this relation may be written in the real frequency domain as a product of the Laplace transform of the impulse response, evaluated on the $\omega_\jmath$ axis and the Fourier transform of the input $X(\omega) \leftrightarrow x(t)$ and output $Y(\omega) \leftrightarrow y(t)$.

If the system is nonlinear, then the output is not given by a convolution, and the Fourier and Laplace transforms have no obvious meaning.

15   The question that must be addressed is why the power is nonlinear, whereas a power series of $H(s)$ is linear: Both have powers of the underlying variables. This is naturally confusing and rarely, if ever, address. The quick answer is that powers of the Laplace frequency s correspond to derivatives, which are linear, whereas the product of the voltage $v(t)$ and current $i(t)$ is nonlinear. The important and interesting question will be addressed in Section 3.7.1 (p. 133), in terms of the system postulates of 20   physical systems.

Whats going on?   The domain variables must be separated from the codomain variables. In our example, the voltage and current are multiplied together, resulting in a nonlinear output, the power. If the frequency is squared, this is describing the degree of a polynomial. This is not nonlinear because it does not impact the signal output; rather it characterizes the Laplace transform of the system response.

Table 3.2: *The generalized impedance is defined as the ratio of a force over a flow, a concept that also holds in mechanics and acoustics. In mechanics, the 'force' is equal to the mechanical force on an element (e.g. a mass, dash-pot, or spring), and the 'flow' is the velocity. In acoustics, the 'force' is pressure, and the 'flow' is the volume velocity or particle velocity of air molecules.*

Case	Force	Flow	Impedance	units *ohms* [Ω]
Electrical	voltage (V)	current (I)	$Z = V/I$	[Ω]
Mechanics	force (F)	velocity (U)	$Z = F/U$	mechanical [Ω]
Acoustics	pressure (P)	particle velocity (V)	$Z = P/V$	specific [Ω]
Acoustics	mean pressure ($\mathcal{P}$)	volume velocity ($\mathcal{V}$)	$Z = \mathcal{P}/\mathcal{V}$	acoustic [Ω]
Thermal	temperature (T)	heat-flux (J)	$Z = T/J$	thermal [Ω]

25   **Ohm's law:**   In general, impedance is defined as the ratio of a force over a flow. For electrical circuits (Table 3.2), the voltage is the 'force' and the current is the 'flow.' Ohm's law states that the voltage across and the current through a circuit element are linearly related by the *impedance* of that element (which is typically a complex function of the complex Laplace frequency $s = \sigma + \omega_\jmath$). For resistors, the voltage over the current is called the *resistance*, and is a constant (e.g. the simplest case, $V/I = R$). 30   For inductors and capacitors, the impedance depends on the Laplace frequency s [e.g. $V/I = Z(s)$]).

As discussed in Fig. 3.2, the impedance concept also holds for mechanics and acoustics. In mechanics, the 'force' is equal to the mechanical force on an element (e.g. a mass, dash-pot, or spring), and the 'flow' is the velocity. In acoustics, the 'force' is pressure, and the 'flow' is the volume velocity or particle velocity of air molecules.

In this section we shall derive the method of linear composition of systems, also known as the *ABCD transmission matrix method*, or in the mathematical literature as the Möbius (bilinear) transformation. Using the method of *matrix composition*, a linear system of 2x2 matrices can represent a large important family of networks. By the application of Ohm's law to the circuit shown in Fig. 3.2, we can model a cascade of such cells.
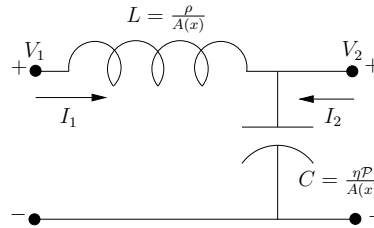
Figure 3.2: *A single LC cell of the LC transmission line (see Fig. Q.1, p. 328). Every cell of any transmission line may be modeled by the ABCD method, as the product of two matrices. For the example shown here, the inductance L of the coil and the capacitance C of capacitor are in units of [henry/m] and [farad/m], thus they depend on length Δ_x [m] that the cell represents. Note the flows are always defined as into the $+$ node.*

Example of the use of the ABCD matrix composition: In Fig. 3.2 we see the network is composed of a series inductor (mass) having an impedance $Z_l = sL$, and a shunt capacitor (compliance) having an impedance $Z_c = 1/sC$, where $s \in \mathbb{C}$. As determined by Ohm's law, each impedance is described by a linear relation between the current and the voltage. For the inductive impedance, applying Ohm's law, we find

$$V_1 - V_2 = Z_l I_1,$$

where Z_l is the complex impedance of the inductor. For the capacitive impedance, applying Ohm's law we find

$$V_2 = (I_1 + I_2)Z_c,$$

where Z_c is the complex impedance of the capacitor.

Each of these linear impedance relations may be written in matrix form. The series inductor equation gives (note $I_1 = -I_2$)

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & Z_l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}, \tag{3.41}$$

while the shunt capacitor equation yields (note $V_1 = V_2$)

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} V_3 \\ -I_3 \end{bmatrix}, \tag{3.42}$$

where $Y_c = 1/Z_c$ is called the *admittance*.

When the second matrix equation for the shunt admittance (Eq. 3.42) is substituted into the series impedance equation (Eq. 3.41), we find the ABCD matrix ($\boldsymbol{T}_1 \circ \boldsymbol{T}_2$) for the cell is simply the product of two matrices

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & Z_l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} V_3 \\ -I_3 \end{bmatrix} = \begin{bmatrix} 1 + Z_l Y_c & Z_l \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} V_3 \\ -I_3 \end{bmatrix}. \tag{3.43}$$

Note that the determinant of the matrix $\Delta = \mathcal{AD} - \mathcal{BC} = 1$. This is not an accident since the determinant of the two matricies are each 1, thus the determinant of their product is 1. Every cascade of series and shunt elements will always have $\Delta = 1$.

Thus for Fig. 3.2, Eq. 3.43 has $\mathcal{A}(s) = 1 + s^2 LC$, $\mathcal{B}(s) = sL$, $\mathcal{C}(s) = sC$ and $\mathcal{D} = 1$. This equation characterizes every possible relation between the input and output voltage and current of the cell. For example, the ratio of the output over input voltage, with the output unloaded, is

$$\frac{V_3}{V_1}\bigg|_{I_3=0} = \frac{1}{\mathcal{A}(s)} = \frac{1}{1 + Z_l Y_c} = \frac{1}{1 + Z_l/Z_c} = \frac{Z_c}{Z_c + Z_l}.$$

This relation is known as the *voltage divider relation*.

To derive the formula for the *current divider equation*, use the lower equation and set $V_3 = 0$, giving

$$\frac{-I_3}{I_1}\bigg|_{V_2=0} = \frac{1}{\mathcal{D}} = 1.$$

3.6   Lec 19: Signals: Fourier transforms

Two basic transformations in engineering mathematics are the Fourier and the Laplace transforms, which deal with time–frequency analysis. the best books on this topic is Papoulis (1962).

Notation: The Fourier transform takes a time domain signal $f(t) \in \mathbb{R}$ and transforms it to the frequency domain $\omega \in \mathbb{R}$, where it is complex ($F(\omega) \in \mathbb{C}$). For the Fourier transform, both the time $-\infty < t < \infty$ and frequency $\infty < \omega < \infty$ are strictly real. The relationship between $f(t)$ and its transform $F(\omega)$ is indicated by the double arrow symbol

$$f(t) \leftrightarrow F(\omega).$$

Since the FT obeys superposition, it is possible to define a FT of a complex time function $f(t) \in \mathbb{C}, t \in \mathbb{R}$. This is useful in certain engineering applications (i.e., Hilbert envelope, Hilbert transforms). It is accepted in the engineering and physics literature to use the case of the variable to indicate the type of argument. A time function is $f(t)$, where t has units of seconds [s] and is in lower case, whereas its Fourier transform, a function of frequency, having units of either hertz [Hz] or radians [2πHz], is written using upper case $F(\omega)$. This helps the reader parse the type of variable under consideration. This is a helpful piece of notation, but not entirely in agreement with notation used in mathematics.

Definition of the Fourier transform: The forward transform takes $f(t)$ to $F(\omega)$ while the inverse transform takes $F(\omega)$ to $\widetilde{f}(t)$. The tilde symbol indicates that in general the recovered inverse transform signal can be slightly different from $f(t)$. We give examples of this in Table 3.3.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-\jmath\omega t}dt \qquad\qquad \widetilde{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{\jmath\omega t}d\omega \qquad (3.44)$$

$$F(\omega) \leftrightarrow f(t) \qquad\qquad\qquad \widetilde{f}(t) \leftrightarrow F(\omega). \qquad\qquad (3.45)$$

Properties of Fourier Transforms:

1. Both time t and frequency ω are real.

2. For the forward transform (time to frequency), the sign of the exponential is negative.

3. The limits on the integrals in both the forward and reverse FTs are $[-\infty, \infty]$.

4. When taking the inverse Fourier transform, the scale factor of $1/2\pi$ is required to cancel the 2π in the differential $d\omega$.

Table 3.3: *Basic (Level I) Fourier transforms. Note $a > 0 \in \mathbb{R}$ has units [rad/s]. To flag this necessary condition, we use $|a|$ to assure this condition will be met. The other constant $T_o \in \mathbb{R}$ [s] has no restrictions, other than being real. Complex constants may not appear as the argument to a delta function, since complex numbers do not have the order property.*

$f(t) \leftrightarrow F(\omega)$	Name
$\widetilde{\delta}(t) \leftrightarrow 1(\omega) \equiv 1 \; \forall \; \omega$	Dirac
$1(t) \equiv 1 \, \forall \, t \leftrightarrow 2\pi \widetilde{\delta}(\omega)$	Dirac
$\mathrm{sgn}(t) = \dfrac{t}{|t|} \leftrightarrow \dfrac{2}{\jmath\omega}$	
$\widetilde{u}(t) = \dfrac{1(t) + \mathrm{sgn}(t)}{2} \leftrightarrow \pi\widetilde{\delta}(\omega) + \dfrac{1}{\jmath\omega} \equiv \widetilde{U}(\omega)$	step
$\widetilde{\delta}(t - T_o) \leftrightarrow e^{-\jmath\omega T_o}$	delay
$\widetilde{\delta}(t - T_o) \star f(t) \leftrightarrow F(\omega)e^{-\jmath\omega T_o}$	delay
$\widetilde{u}(t)e^{-|a|t} \leftrightarrow \dfrac{1}{\jmath\omega + |a|}$	exp
$\mathrm{rec}(t) = \dfrac{1}{T_o}\left[\widetilde{u}(t) - \widetilde{u}(t - T_o)\right] \leftrightarrow \dfrac{1}{T_o}\left(1 - e^{-\jmath\omega T_o}\right)$	pulse
$\widetilde{u}(t) \star \widetilde{u}(t) \leftrightarrow \widetilde{\delta}^2(\omega) \quad$ Not defined	NaN

5. The Fourier step function may be defined by the use of superposition of 1 and $\mathrm{sgn}(t) = t/|t|$ as

$$\widetilde{u}(t) \equiv \frac{1 + \mathrm{sgn}(t)}{2} = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases}.$$

Taking the FT of a delayed step function

$$\widetilde{u}(t - T_o) \leftrightarrow \frac{1}{2}\int_{-\infty}^{\infty}\left[1 - \mathrm{sgn}(t - T_o)\right]e^{-j\omega t}dt = \pi\tilde{\delta}(\omega) + \frac{e^{-j\omega T_o}}{j\omega}.$$

Thus the FT of the step function has the term $\pi\delta(\omega)$ due to the 1 in the definition of the Fourier step. This term introduces a serious flaw with the FT of the step function: While it appears to be causal, it is not.

6. The convolution $\widetilde{u}(t) \star \widetilde{u}(t)$ is not defined because both $1 \star 1$ and $\widetilde{\delta}^2(\omega)$ do not exist (and cannot be defined). ₅

7. The inverse FT has convergence problems whenever there is a discontinuity in the time response. This we indicate with a hat over the reconstructed time response. The error between the target time function and the reconstructed is zero in the root-mean sense, but not point-wise.

   Specifically, at the discontinuity point for the Fourier step function ($t = 0$), $\widetilde{u}(t) \neq u(t)$, yet $\int |\widetilde{u}(t) - u(t)|^2 dt = 0$. At the point of the discontinuity the reconstructed function displays *Gibbs* ₁₀ *ringing* (it oscillates around the step, hence does not converge at the jump).[28] The $\mathcal{LT}$ does not exhibit Gibbs ringing.

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[28]https://en.wikipedia.org/wiki/Gibbs_phenomenon

8. The FT is not always analytic in ω, as in this example of the step function. The step function cannot be expanded in a Taylor series about $\omega = 0$ because $\widetilde{\delta}(\omega)$ is not analytic in ω.

9. The Fourier δ function is denoted $\widetilde{\delta}(t)$, to differentiate it from the Laplace delta function $\delta(t)$. They differ because the step functions differ, due to the convergence problem described above.

10. One may define

$$\widetilde{u}(t) = \int_{-\infty}^{t} \widetilde{\delta}(t)dt,$$

and define the somewhat questionable notation

$$\widetilde{\delta}(t) = \frac{d}{dt}\widetilde{u}(t),$$

since the Fourier step function is not analytic.

11. The $\text{rec}(t)$ function is defined as

$$\text{rec}(t) = \frac{\widetilde{u}(t) - \widetilde{u}(t - T_o)}{T_o} = \begin{cases} 0 & t < 0 \\ 1/T_o & 0 < t < T_o \\ 0 & t > T_0 \end{cases}.$$

It follows that $\widetilde{\delta}(t) = \lim_{T_o \to 0}$. Like $\widetilde{\delta}(t)$, the $\text{rec}(t)$ has unit area.

Table 3.4: *Summary of key properties of FTs.*

FT Properties	
$\frac{d}{dt}v(t) \leftrightarrow \jmath\omega V(\omega)$	deriv
$f(t) \star g(t) \leftrightarrow F(\omega)G(\omega)$	conv
$f(t)g(t) \leftrightarrow \frac{1}{2\pi}F(\omega) \star G(\omega)$	conv
$f(at) \leftrightarrow \frac{1}{a}F\left(\frac{\omega}{a}\right)$	scaling

Periodic signals: Besides these two basic types of time–frequency transforms, there are several variants that depend on the symmetry in time and frequency. For example, when the time signal is sampled (discrete in time), the frequency response becomes periodic, leading to the *discrete-time Fourier transform* (DTFT). When a time response is periodic, the frequency response is sampled (discrete in frequency), leading to the *Fourier series*. These two symmetries may be simply characterized as *periodic in time* $\Rightarrow$ discrete in frequency, and *periodic in frequency* $\Rightarrow$ discrete in time. When a function is discrete both in time and frequency, it is necessarily periodic in time and frequency, leading to the *discrete Fourier transform* (DFT). The DFT is typically computed with an algorithm called the *fast Fourier transform* (FFT), which can dramatically speed up the calculation when the data is a power of 2 in length.

An important symmetry occurs given functions that are causal (in time) *and* periodic (in frequency). The best known example is the class of signals that have *z transforms*, which are causal (one-sided in

time) discrete-time signals. The harmonic series (Eq. 3.29, p. 91) is the z-transform of the discrete-time step function and is thus, due to symmetry, analytic within the RoC in the complex frequency (z) domain.

The double brackets on $f((t))_{T_o}$ indicate that $f(t)$ is periodic in t with period T_o, i.e., $f(t) = f(t + kT_o)$ for all $k \in \mathbb{N}$. Averaging over one period and dividing by the T_o gives the average value of $f(t)$.

Exercise: Consider the $\mathcal{FT}$ as a scalar (dot) product (Eq. 3.16, p. 109) between "vectors" $f((t))_{T_o}$ and $e^{-\jmath\omega_k t}$

$$F(\omega_k) = f((t))_{T_o} \cdot e^{-\jmath\omega_k t}$$
$$\equiv \frac{1}{T_o} \int_0^{T_o} f(t) e^{-\jmath\omega_k t} dt,$$

where $\omega_0 = 2\pi/T_o$, $f(t)$ has period T_o, i.e., $f(t) = f(t + nT_o) = e^{\jmath\omega_n t}$ with $n \in \mathbb{N}$, and $\omega_k = k\omega_o$.

What is the value of the scalar product? **Solution:** Evaluating the scalar product we find

$$e^{\jmath\omega_n t} \cdot e^{-\jmath\omega_k t} = \frac{1}{T_o} \int_0^{T_o} e^{\jmath\omega_n t} e^{-\jmath\omega_k t} dt$$
$$= \frac{1}{T_o} \int_0^{T_o} e^{2\pi\jmath(n-k)t/T_o} dt = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}.$$

The two signals (vectors) are orthogonal.

Exercise: Consider the discrete time $\mathcal{FT}$ (DTFT) as a scalar (dot) product (Eq. 3.16, between "vectors" $f_n = f(t)|_{t_n}$ and $e^{-\jmath\omega t_n}$ where $t_n = nT_s$ and $T_s = 1/2F_{\max}$ is the *sample period*. The scalar product over $n \in \mathbb{Z}$ is

$$F((\omega))_{2\pi} = f_n \cdot e^{-\jmath\omega t_n}$$
$$\equiv \sum_{n=-\infty}^{\infty} f_n e^{-\jmath\omega t_n},$$

where $\omega_0 = 2\pi/T_o$ and $\omega_k = k\omega_o$. If $f_n = e^{\jmath\omega_n t}$ what is the value of the scalar product? **Solution:** Evaluating the scalar product we find

$$e^{\jmath\omega_n t} \cdot e^{-\jmath\omega_k t} = \frac{1}{T_o} \int_0^{T_o} e^{\jmath\omega_n t} e^{-\jmath\omega_k t} dt. = \frac{1}{T_o} \int_0^{T_o} e^{2\pi\jmath(n-k)t/T_o} dt = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}.$$

The two signals (vectors) are orthogonal.

Add Kernel methods
applications
Poisson Sum Formula

3.7 Lec 20: Systems: Laplace transforms

The *Laplace transform* takes real signals $f(t) \in \mathbb{R}$, as a function of real time $t \in \mathbb{R}$, that are *causal*, i.e., strictly zero for negative time ($f(t) = 0$ for $t < 0$), and transforms them into complex functions $F(s) \in \mathbb{C}$ of complex frequency $s = \sigma + \omega\jmath$. As for the Fourier transform, there is the notation $f(t) \leftrightarrow F(s)$.

When a signal is zero for negative time $f(t < 0) = 0$, it is said to be *causal*, and the resulting transform $F(s)$ is then complex analytic over significant regions of the s plane. For a function of time to be causal, time *must* be real ($t \in \mathbb{R}$), since if it were complex, it would lose the order property (thus it could not be causal). It is helpful to emphasize the causal nature of $f(t)u(t)$ by indicating the causal nature, using the Heaviside step function $u(t)$.

Any restriction on a function (e.g., real, causal, periodic, positive real part, etc.) is called a *symmetry property*. There are many forms of symmetry (Section 3.7.1, p. 133). The concept of symmetry is very general and widely used in both mathematics and physics, where it is more generally known as *group theory*. We shall restrict ourselves to only a few very basic symmetries.

Definition of the Laplace transform: The forward and inverse Laplace transforms are

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt \qquad\qquad f(t) = \frac{1}{2\pi\jmath} \int_{\sigma_o - \infty\jmath}^{\sigma_o + \infty\jmath} F(s)e^{st}ds \qquad\qquad (3.46)$$

$$F(s) \leftrightarrow f(t) \qquad\qquad\qquad f(t) \leftrightarrow F(s) \qquad\qquad\qquad (3.47)$$

⁵ Here $s = \sigma + j\omega \in \mathbb{C}$ [2πHz] is the complex Laplace frequency in radians and $t \in \mathbb{R}$ [s] is the time in seconds.

When dealing with engineering problems it is convenient to separate the *signals* we use from the *systems* that process them. We do this by treating signals, such as a music signal, differently from a system, such as a filter. In general signals may start and end at any time. The concept of causality has ¹⁰ no mathematical meaning in signal space. Systems, on the other hand, obey very rigid rules (to assure that they remain physical). These physical restrictions are described in terms of the *network postulates*, which are discussed in Sect. 3.7.1 (p. 133).   There is a question as to why postulates are needed, and which ones are the best choices.[29] There may be no answers to these questions, but having a set of postulates is a useful way of thinking about physics.[30]

As discussed in Section 4.5.1 (p. 178), we must use the Cauchy residue theorem (CRT), requiring closure of the contour $\mathcal{C}$ at $\omega\jmath \to \pm j\infty$

$$\oint_{\mathcal{C}} = \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} + \int_{\subset_{\infty}},$$

¹⁵ where the path represented by '$\subset_{\infty}$' is a semicircle of infinite radius. For a causal, 'stable' (e.g. doesn't "blow up" in time) signal, all of the poles of $F(s)$ *must* be inside of the Laplace contour, in the left half s-plane.
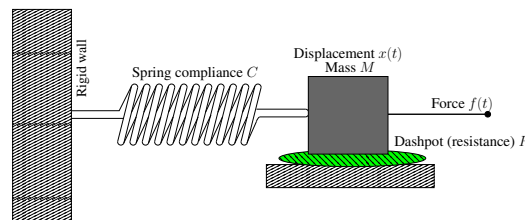
Figure 3.3: *Three-element mechanical resonant circuit consisting of a spring, mass and dash-pot (e.g., viscous fluid).*

Hooke's law for a spring states that the force $f(t)$ is proportional to the displacement $x(t)$, i.e., $f(t) = Kx(t)$. The formula for a dash-pot is $f(t) = Rv(t)$, and Newton's famous formula for mass is ²⁰ $f(t) = d[Mv(t)]/dt$, which for constant M is $f(t) = Mdv/dt$.

The equation of motion for the mechanical oscillator in Fig. 3.3 is given by Newton's second law; the sum of the forces must balance to zero

$$M\frac{d^2}{dt^2}x(t) + R\frac{d}{dt}x(t) + Kx(t) = f(t). \qquad\qquad (3.48)$$

These three constants, the mass M, resistance R and stiffness K ($\in \mathbb{R} \geq 0$) are real and non-negative. The dynamical variables are the driving force $f(t) \leftrightarrow F(s)$, the position of the mass $x(t) \leftrightarrow X(s)$ and its velocity $v(t) \leftrightarrow V(s)$, with $v(t) = dx(t)/dt \leftrightarrow V(s) = sX(s)$.

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[29]https://www.youtube.com/watch?v=JXAfEBbaz_4
[30]https://www.youtube.com/watch?v=YaUlqXRPMmY

Newton's second law (c1650) is the mechanical equivalent of Kirchhoff's (c1850) voltage law (KVL), which states that the sum of the voltages around a loop must be zero. The gradient of the voltage results in a force on a charge (i.e., $F = qE$).

Equation 3.48 may be re-expressed in terms of impedances, the ratio of the force to velocity, once it is transformed into the Laplace frequency domain.

Example: The divergent series

$$e^t u(t) = \sum_0^\infty \frac{1}{n!} t^n \leftrightarrow \frac{1}{s-1}$$

is a valid description of $e^t u(t)$, with an unstable pole at $s = 1$. For values of $|x - x_o| < 1$ ($x \in \mathbb{R}$), the *analytic function* $P(x)$ is said to have a *region of convergence* (RoC). For cases where the argument is complex ($s \in \mathbb{C}$), this is called the *radius of convergence* (RoC). We call the region $|s - s_o| > 1$ the *region of divergence* (RoD), and $|s - s_o| = 0$, the *singular circle*. Typically the underlying function $P(s)$, defined by the series, has a pole on the singular circle for $s \in \mathbb{C}$. One may isolate such poles by moving the expansion point s_o until the RoC approaches zero. Outside the circle the Taylor series is divergent and inside it is convergent.

There seems to be a conflict with the time response $f(t) = e^{at} u(t)$, which has a divergent series (unstable pole). I'm not sure how to explain this conflict, other than to point out that $t \in \mathbb{R}$: thus, the series expansion of the diverging exponential is real-analytic, not complex analytic. *First* $f(t)$ has a Laplace transform with a pole at $s = 1$, in agreement with its unstable nature. *Second*, every analytic function must be single valued. This follows from the fact that each term in Eq. 3.27 is single valued. *Third*, analytic functions are "smooth" since they may be differentiated an infinite number of times and the series still converges. There can be no jumps or kinks in such functions.

The key idea that every impedance must be complex analytic and ≥ 0 for $\sigma > 0$ was first proposed by Otto Brune in his PhD at MIT, supervised by Ernst A. Guillemin, an MIT electrical engineering professor who played an important role in the development of circuit theory and was a student of Arnold Sommerfeld.[31] Other MIT advisers were Norbert Wiener and Vannevar Bush. Brune's primary, but non-MIT, advisor was W. Cauer, also perhaps a student of Sommerfeld, who was well trained in 19th century German mathematics (Brune, 1931b).

Summary: While the definitions of the FT ($\mathcal{FT}$) and LT ($\mathcal{LT}$) transforms may appear similar, they are not. The key difference is that the time response of the Laplace transform is causal, leading to a complex analytic frequency response. While the frequency response of the Fourier transform is complex, it is not complex analytic since the frequency ω is real. Fourier transforms do not have poles. These are not superficial differences.

The concept of symmetry is helpful in understanding the many different types of time-frequency transforms. Two fundamental types of symmetry are causality and periodicity.

The Fourier transform $\mathcal{FT}$ characterizes the steady-state response while the Laplace transform $\mathcal{LT}$ characterizes both the transient and steady-state response. Given a causal system response $H(s) \leftrightarrow h(t)$ with input $x(t)$, the output is

$$y(t) = h(t) \star x(t) \leftrightarrow Y(\omega) = H(s)\Big|_{s=\jmath\omega} X(\omega).$$

3.7.1 Lec 21: **Ten system postulates**

Solutions of differential equations, such as the wave equation, are conveniently described in terms of mathematical properties, which we present here in terms of 10 *network postulates*:

---

[31] It must be noted that University of Illinois Prof. 'Mac' Van Valkenburg was arguably more influential in circuit theory during the same period. Mac's books are certainly more accessible, but perhaps less widely cited.

(P1) *Causality* (non-causal/acausal): Causal systems respond when acted upon. Virtually all physical systems obey causality. An example of a causal system is an integrator, which has a response of a step function. Filters are also examples of causal systems. Signals represent acausal responses. They do not have a clear beginning or end, such as the sound of the wind or traffic noise. A causal linear system is typically complex analytic. A nonlinear system can be causal, but not complex analytic.

(P2) *Linearity* (nonlinear): Linear systems obey superposition. Two signals $x(t)$ and $y(t)$ are the inputs to a linear system, producing outputs $x'(t)$ and $y'(t)$. When the inputs are presented together as $ax(t) + by(t)$ with weights $a, b \in \mathbb{C}$, the output will be $ax'(t) + by'(t)$. If either a or b is zero, the corresponding signal is removed from the output.

Nonlinear systems mix the two inputs, thereby producing signals not present in the input. For example, if the inputs to a nonlinear system are two sine waves, the output will contain distortion components, having frequencies not present at the input. An example of a nonlinear system is one that multiplies the two inputs. A second is a diode, which rectifies a signal, letting current flow only in one direction. Most physical systems have some degree of nonlinear response, but this is not always desired. Other systems are designed to be nonlinear, such as the diode example.

(P3) *Passive* (active): An active system has a power source, such as a battery, while a passive system has no power source. While you may consider a transistor amplifier to be active, it is only so when connected to a power source.

(P4) *Real* (complex) time response: Typically systems are "real in, real out." They do not naturally have complex responses (real and imaginary parts). While a Fourier transform takes real inputs and produces complex outputs, this is not an example of a complex time response. P4 is a characterization of the input signal, not its Fourier transform.

(P5) *Time-invariant* (time varying): For a system to be a time varying system the output must depend on when the input signal starts or stops. If the output, relative to the input, is independent of the starting time, then the system is said to be time-invariant (static).

(P6) *Reciprocal* (non- or anti-reciprocal): In many ways this is the most difficult propriety to characterize and thus to understand. It is best characterized by the ABCD matrix (Section 3.5, p. 123). If $B = C$, the system is said to be *reciprocal*. If $B = -C$, it is said to be *anti-reciprocal*. The impedance matrix is reciprocal while a loudspeaker is anti-reciprocal and modeled by the gyrator rather than a transformer. Non-reciprocal systems are modeled by gyrators, which may be thought of as transformers which swap the force and flow variables. For example, the input impedance of a gyrator, terminated by an inductor, is a capacitor. For an expanded discussion on reciprocity, see Sect. K, p. 289.

Add fig T network: Za, Zb, Zc, to explain P6

(P7) *Reversibility* (non-reversible): If swapping the input and output of a system leaves the system invariant, it is said to be reversible. When $A = D$, the system is reversible. Note the similarity and differences between reversible and reciprocal.

(P8) *Space-invariant* (space-variant): If a system operates independently as a function of where it physically is in space, then it is space-invariant. When the parameters that characterize the system depend on position, it is space-variant.

(P9) *Deterministic* (random): Given the wave equation, along with the boundary conditions, the system's solution may be deterministic, or not, depending on its extent. Consider a radar or sonar wave propagating out into uncharted territory. When the wave hits an object, the reflection can return waves that are not predicted, due to unknown objects. This is an example where the boundary condition is not known in advance.

(P10) *Quasi-static* ($ka < 1$) Quasi-statics follows the Nyquist sampling theorem, for systems that have dimensions that are small compared to the local wavelength. This assumption fails when the frequency is raised (the wavelength becomes short). Thus this is also known as the *long-wavelength* approximation. Quasi-statics is typically stated as $ka < 1$, where $k = 2\pi/\lambda = \omega/c_o$ and a is the smallest dimension of the system. see p. 235 for a detailed discussion of the role of quasi-statics in acoustic horn wave propagation.

Postulate (P10) is closely related to the Feynman lecture *The "underlying unity" of nature* where Feynman asks (Feynman, 1970c, Ch. 12-7): Why do we need to treat the fields as smooth? His answer is related to the wavelength of the probing signal relative to the dimensions of the object being probed. This raises the fundamental question: Are Maxwell's equations a band-limited approximation to reality? Today we have no answers to this question.

Here is one helpful discussion:[32]

> The Lorentz force formula and Maxwell's equations are two distinct physical laws, yet the two methods yield the same results.
>
> Why the two results coincide was not known. In other words, the flux rule consists of two physically different laws in classical theories. Interestingly, this problem was also a motivation behind the development of the theory of relativity by Albert Einstein. In 1905, Einstein wrote in the opening paragraph of his first paper on relativity theory, "It is known that Maxwell's electrodynamics – as usually understood at the present time – when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena." But Einstein's argument moved away from this problem and formulated special theory of relativity, thus the problem was not solved.
>
> Richard Feynman once described this situation in his famous lecture (The Feynman Lectures on Physics, Vol. II, 1964), "we know of no other place in physics where such a simple and accurate general principle requires for its real understanding an analysis in terms of two different phenomena. Usually such a beautiful generalization is found to stem from a single deep underlying principle. $\cdots$ We have to understand the "rule" as the combined effects of two quite separate phenomena."

Summary discussion of the 10 network postulates: Each postulate has at least two categories. For example, (P1) is either causal, non-causal or acausal while (P2) is either linear or non-linear. (P6) and (P9) only apply to 2-port *algebraic networks* (those having an input and an output). The others apply to both 2– and 1–port networks (e.g., an impedance is a 1-port). An interesting example is the anti-reciprocal transmission matrix of a loudspeaker (Allen, 2019, Lec21).

Related forms of these postulates may be found in the network theory literature (Van Valkenburg, 1964a,b; Ramo *et al.*, 1965). Postulates (P1-P6) were introduced by Carlin and Giordano (1964) and (P7-P9) were added by Kim *et al.* (2016). While linearity (P2), passivity (P3), realness (P4) and time-invariance (P5) are independent, causality (P1) is a consequence of linearity (P2) and passivity (P3) (Carlin and Giordano, 1964, p. 5).

3.8   Lec 22: Complex analytic mappings (domain-coloring)

One of the most difficult aspects of complex functions of a complex variable is visualizing the mappings from the $z = x + y\jmath$ to $w(z) = u + v\jmath$ planes. For example, $w(z) = \sin(x)$ is trivial when $z = x + y\jmath$ is real (i.e., $y = 0$), because $\sin(x)$ is real. Likewise for the case where $x = 0$,

$$\sin(y\jmath) = \frac{e^{-y} - e^{y}}{2\jmath} = -\jmath \sinh(y)$$

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[32] https://www.sciencedaily.com/releases/2017/09/170926085958.htm

is pure imaginary. However, the more general case

$$w(z) = \sin(z) \in \mathbb{C}$$

is not easily visualized. And when $u(x, y)$ and $v(x, y)$ are less well known functions, $w(z)$ can be even more difficult to visualize. For example, if $w(z) = J_0(z)$ then $u(x, y), v(x, y)$ are the real and imaginary parts of the Bessel function.

A software solution: Fortunately with computer software today, this problem can be solved by adding color to the chart. An Octave/Matlab script[33] `zviz.m` has been used to make the charts shown in Fig. 3.4. Such charts are known as *domain-coloring*.[34]

Rather than plotting $u(x, y)$ and $v(x, y)$ separately, domain-coloring allows us to display the entire function on one color chart (i.e., colorized plot). For this visualization we see the complex polar form of $w(s) = |w|e^{j\angle w}$, rather than the 2x2 (four-dimensional) Cartesian graph $w(x+yj) = u(x, y)+v(x, y)j$. On the left is the reference condition, the identity mapping ($w = s$), and on the right the origin has been shifted to the right and up by $\sqrt{2}$.

Mathematicians typically use abstract (i.e., non–physical) notation $w(z)$, where $w = u + vi$ and $z = x + yi$. Engineers prefer to think in terms of a physical complex impedance $Z(s) = R(s) + jX(s)$,
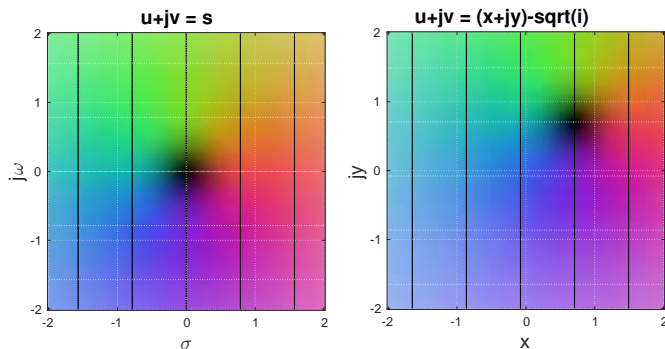
Figure 3.4: *Left: Domain-colorized map showing the complex mapping from the $s = \sigma + \omega j$ plane to the $w(s) = u(\sigma, \omega) + v(\sigma, \omega)j$ plane. This mapping may be visualized by the use of intensity (light/dark) to indicate magnitude, and color (hue) to indicate angle (phase) of the mapping. Right: This shows the $w(z) = z - \sqrt{j}$ plane, shifted to the right and up by $\sqrt{2}/2 = 0.707$. The white and black lines are the iso-real and iso-imaginary contours of the mapping.*

having *resistance* $R(s)$ and *reactance* $X(s)$ [ohms], as a function of the complex Laplace radian frequency $s = \sigma + \omega j$ [rad], as used, for example, with the Laplace transform (Sect. 3.7, p. 131).

In Fig. 3.4 we use both notations, with $Z(s) = s$ on the left and $w(z) = z - \sqrt{j}$ on the right, where we show this color code as a 2x2 dimensional domain-coloring graph. Intensity (dark to light) represents the magnitude of the function, while hue (color) represents the phase, where (see Fig. 3.4) red is $0°$, sea-green is $90°$, blue-green is $135°$, blue is $180°$, and violet is $-90°$ (or $270°$).[35]

The function $w = s = |s|e^{j\theta}$ has a dark spot (a zero) at $s = 0$, and becomes brighter away from the origin. On the right is $w(z) = z - \sqrt{j}$, which shifts the zero to $z = \sqrt{j}$. Thus domain-coloring gives the full 2x2 complex analytic function mapping $w(x, y) = u(x, y) + v(x, y)j$, in colorized polar coordinates.

Visualizing complex functions: The mapping from $z = x + iy$ to $w(z) = u(x, y) + iv(x, y)$ is difficult to visualize because for each point in the domain z, we would like to represent both the magnitude and phase (or real and imaginary parts) of $w(z)$. A good way to visualize these mappings is to use color (hue) to represent the phase and intensity (dark to light) to represent the magnitude. The Octave/Matlab program `zviz.m` has been provided to d

To use the program in Matlab/Octave, use the syntax `zviz <function of z>` (for example, type `zviz z.^2`). Note the period between z and `^2`. This will render a domain-coloring (aka colorized) version of the function. Examples you can render with

Figure 3.5: *Plot of $\sin(0.5\pi(z - i))$.*

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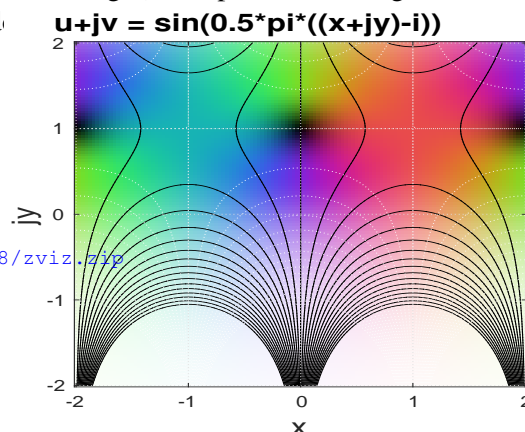
[33] http://jontalle.web.engr.illinois.edu/uploads/298/zviz.zip
[34] https://en.wikipedia.org/wiki/Domain_coloring
[35] Colors vary depending on both the display medium and the eye.

`zviz` are given in the comments at the top of the
`zviz.m` program. A good example for testing is
`zviz z-sqrt(j)`, which will show a dark spot (a
zero) at $(1 + 1\jmath)/\sqrt{2} = 0.707(1 + 1\jmath)$.

Example: Figure 3.5 shows a colorized plot of ⁵
$w(z) = \sin(\pi(z - i)/2)$ resulting from the Matlab/Oc-
tave command `zviz sin(pi*(z-i)/2)`. The abscissa (horizontal axis) is the real x axis and the
ordinate (vertical axis) is the complex iy axis. The graph is offset along the ordinate axis by $1i$, since the
argument $z - i$ causes a shift of the sine function by 1 in the positive imaginary direction. The visible
zeros of $w(z)$ appear as dark regions at $(-2, 1), (0, 1), (2, 1)$. As a function of x, $w(x + 1\jmath)$ oscillates ¹⁰
between red (phase is zero degrees), meaning the function is positive and real, and sea-green (phase is
$180°$), meaning the function is negative and real.

Along the vertical axis, the function is either a $\cosh(y)$ or $\sinh(y)$, depending on x. The intensity
becomes lighter as $|w|$ increases, and white as $w \to \infty$. The intensity becomes darker as $|w|$ decreases,
and black as $w \to 0$. ¹⁵

u+jv = exp((x+jy))   **u+jv = log((x+jy))**
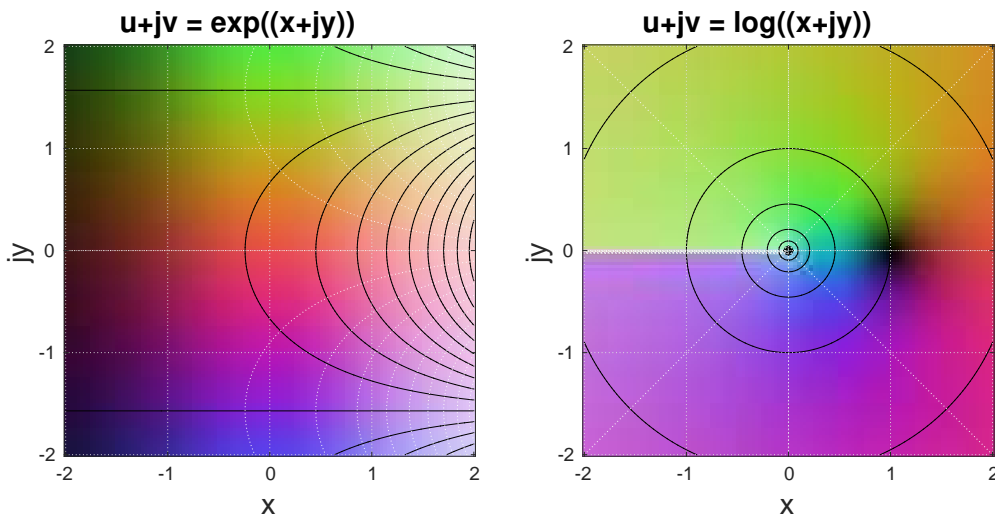
Figure 3.6: *This domain-color map allows one to visualize complex mappings by the use of intensity (light/dark) to indicate
magnitude, and color (hue) to indicate angle (phase). The white and black lines are the iso-real and iso-imaginary contours
of the mapping. Left: This figure shows the domain-color map for the complex mapping from the $z = x + y\jmath$ plane to
the $w(z) = u + v\jmath = e^{x+y\jmath}$ plane, which goes to zero as $x \to -\infty$, causing the domain-color map to become dark for
$x < -2$. The white and black lines are always perpendicular because e^z is complex analytic everywhere. Right: This shows
the principal value of the inverse function $u(x, y) + v((x, y)\jmath = \log(x + y\jmath)$, which has a zero (dark) at $z = 1$, since there
$\log(1) = 0$ (the imaginary part is zero). Note the branch cut from $x = [0, -\infty\jmath)$. On branches other than the one shown, there
are no zeros since the phase ($\angle z = 2\pi n \in \mathbb{Z}$) is not zero. n is called the branch index.*

Example: Two additional examples are given in Fig. 3.6 using the notation $w(z) = u(x, y) + v(x, y)\jmath$
showing the two complex mappings $w = e^s$ (left) and its inverse $s = \ln(w)$. The exponential is
relatively easy to understand because $w(s) = |e^\sigma e^{\omega\jmath}| = e^\sigma$.

The red region is where $\omega \approx 0$ in which case $w \approx e^\sigma$. As σ becomes large and negative, $w \to 0$, thus
the entire field becomes dark on the left. The field is becoming light on the right where $w = e^\sigma \to \infty$. ²⁰
If we let $\sigma = 0$ and look along the ω axis, we see that the function is changing phase, sea-green ($90°$) at
the top and violet ($-90°$) at the bottom.

In the right panel note the zero for $\ln(w) = \ln|w| + \omega\jmath$ at $w = 1$. The root of the log function is
$\log(w_r) = 0, w_r = 1, \phi = 0$, since $\log(1) = 0$. More generally, the log of $w = |w|e^{\phi\jmath}$ is $s = \ln|w| + \phi\jmath$.
Thus $s(w)$ can be zero only when the angle of w is zero. ²⁵

The $\ln(w)$ function has a *branch cut* along the $\phi = 180°$ axis. As one crosses over the cut, the phase
goes above $180°$, and the plane changes to the next *sheet* of the log function. The only sheet with a

zero is the principle value, as shown. For all others, the log function is either increasing or decreasing monotonically, and there is no zero, as seen for sheet 0 (the one showing in Fig. 3.6).

3.8.1   Riemann Sphere: 3^d extension of chord and tangent method

Once algebra was formulated by c830 CE, mathematicians were able to expand beyond the limits set
5 by geometry on the real plane, and the verbose descriptions of each problem in prose (Stillwell, 2010, p. 93). The geometry of Euclid's *Elements* had paved the way, but after 2000 years, the addition of the language of algebra changed everything. The analytic function was a key development, heavily used by both Newton and Euler. Also the investigations of Cauchy made important headway with his work on complex variables. Of special note was integration and differentiation in the complex plane of complex
10 analytic functions, which is the topic of stream 3.

It was Riemann, working with Gauss in the final years of Gauss's life, who made the breakthrough, with the concept of the *extended complex plane*.[36] This concept was based on the composition of a line with the sphere, similar to the derivation of Euclid's formula for Pythagorean triplets (Fig. 3.4, p. 101). While the importance of the extended complex plane was unforeseen, it changed analytic mathematics
15 forever, along with the physics it supported. It unified and thus simplified many important integrals, to the extreme. The basic idea is captured by the *fundamental theorem of complex integral calculus* (Table 4.1, p. 149).
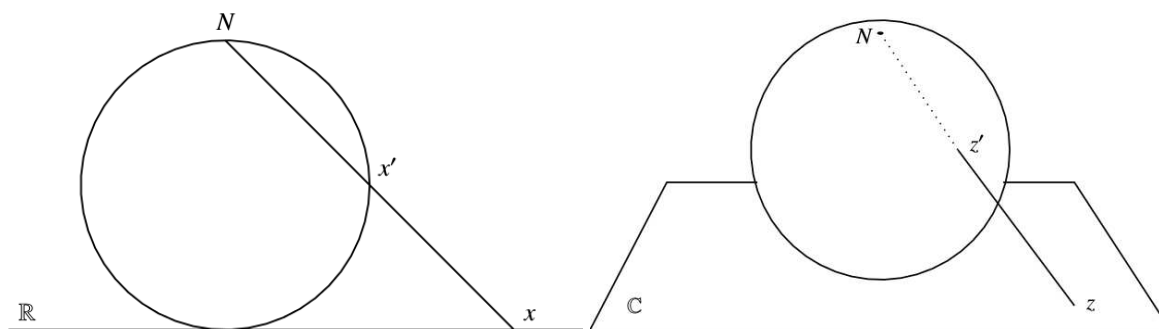
Figure 3.7:   *The left panel shows how the real line may be composed with the circle. Each real x value maps to a corresponding point x' on the unit circle. The point $x \to \infty$ maps to the north pole N. This simple idea may be extended with the composition of the complex plane with the unit sphere, thus mapping the plane onto the sphere. As with the circle, the point on the complex plane $z \to \infty$ maps onto the north pole N. This construction is important because while the plane is open (does not include $z \to \infty$), the sphere is analytic at the north pole. Thus the sphere defines the* closed *extended plane. Figure adapted from Stillwell (2010, pp. 299-300).*

The idea is outlined in Fig. 3.7. On the left is a circle and a line. The difference between this case and the derivation of the Pythagorean triplets is that the line starts at the north pole, and ends on the
20 real $x \in \mathbb{R}$ axis at point x. At point x', the line cuts through the circle. Thus the mapping from x to x' takes every point on $\mathbb{R}$ to a point on the circle. For example, the point $x = 0$ maps to the south pole (not indicated). To express x' in terms of x one must compose the line and the circle, similar to the composition used in Fig. 3.4 (p. 101). The points on the circle, indicated here by x', require a traditional polar coordinate system, having a unit radius and an angle defined between the radius and a vertical line
25 passing through the north pole. When $x \to \infty$ the point $x' \to N$, known as the *point at infinity*. But this idea goes much further, as shown on the right half of Fig. 3.7.

Here the real tangent line is replaced by a tangent complex plane $z \in \mathbb{C}$, and the complex puncture point $z' \in \mathbb{C}$, in this case on the complex sphere, called the *extended complex plane.* This is a natural extension of the chord/tangent method on the left, but with significant consequences. The main differ-
30 ence between the complex plane z and the extended complex plane, other than the coordinate system, is

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[36]"Gauss did lecture to Riemann but he was only giving elementary courses and there is no evidence that at this time he recognized Riemann's genius." Then "In 1849 he [Riemann] returned to Göttingen and his Ph.D. thesis, supervised by Gauss, was submitted in 1851." http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Riemann.html

what happens at the north pole. The point at $|z| = \infty$ is not defined on the plane, whereas on the sphere, the point at the north pole is simply another point, like every other point on the sphere.

Open vs. closed sets: Mathematically the plane is said to be an *open set*, since the limit $z \to \infty$ is not defined, whereas on the sphere, the point z' is a member of a *closed set*, since the north pole *is* defined. The distinction between an open and closed set is important, because the closed set allows the function to be complex analytic at the north pole, which it cannot be on the plane (since the point at infinity is not defined).

The z plane may be replaced with another plane, say the $w = F(z) \in \mathbb{C}$ plane, where w is some function F of $z \in \mathbb{C}$. For the moment we shall limit ourselves to *complex analytic functions* of z, namely $w = F(z) = u(x, y) + v(x, y)\jmath = \sum_{n=0}^{\infty} c_n z^n$.

In summary, given a point $z = x + y\jmath$ on the open complex plane, we map it to $w = F(z) \in \mathbb{C}$, the complex $w = u + v\jmath$ plane, and from there to the closed extended complex plane $w'(z)$. The point of doing this is that it allows us to allow the function $w'(z)$ to be analytic at the north pole, meaning it can have a convergent Taylor series at the point at infinity $z \to \infty$. Since we have not yet defined $dw(z)/dz$, the concept of a complex Taylor series remains undefined.

3.8.2 Bilinear transformation

In mathematics the *bilinear transformation* has special importance because it is linear in its action on both the input and output variables. Since we are engineers we shall stick with the engineering terminology. But if you wish to read about this on the internet, be sure to also search for the mathematical term, Möbius transformation.

When a point on the complex plane $z = x + y\jmath$ is composed with the bilinear transform ($a, b, c, d \in \mathbb{C}$), the result is $w(z) = u(x, y) + v(x, y)\jmath$

$$ w = \frac{az + b}{cz + d}. \tag{3.49} $$

The transformation from $z \to w$ is a cascade of four independent compositions:

1. translation ($w = z + b$: $a = 1, b \in \mathbb{C}, c = 0, d = 1$),

2. scaling ($w = |a|z$: $a \in \mathbb{R}, b = 0, c = 0, d = 1$),

3. rotation ($w = \frac{a}{|a|}z$: $a \in \mathbb{C}, b = 0, c = 0, d = |a|$) and

4. inversion ($w = \frac{1}{z}$: $a = 0, b = 1, c = 1, d = 0$).

Each of these transformations is a special case of Eq. 3.49, with the inversion the most complicated. I highly recommend a video showing the effect of the bilinear (Möbius) transformation on the plane.[37]

The bilinear transformation is the most general way to move the expansion point in a complex analytic expansion. For example, starting from the harmonic series, the bilinear transform gives

$$ \frac{1}{1 - w} = \frac{1}{1 - \frac{az+b}{cz+d}} = \frac{cz + d}{(c - a)z + (d - b)}. $$

The RoC is transformed from $|w| < 1$ to $|(az - b)/(cz - d)| < 1$. An interesting application might be in moving the expansion point until it is on top of the nearest pole, so that the RoC goes to zero. This might be a useful way of finding a pole, for example.

When the extended plane (Riemann sphere) is analytic at $z = \infty$, one may take the derivatives there, defining the Taylor series with the expansion point at ∞. When the bilinear transformation rotates the Riemann sphere, the point at infinity is translated to a finite point on the complex plane, revealing the

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[37]https://www.youtube.com/watch?v=0z1fIsUNhO4

analytic nature at infinity. A second way to transform the point at infinity is by the bilinear transformation $\zeta = 1/z$, mapping a zero (or pole) at $z = \infty$ to a pole (or zero) at $\zeta = 0$. Thus this construction of the Riemann sphere and the Möbious (bilinear) transformation allows us to understand the point at infinity, and treat it like any other point. If you felt that you never understood the meaning of the point at ∞ (likely), this should help.

Chapter 4

Stream 3a: Scalar (i.e., Ordinary) Differential Equations

Stream 3 is ∞, a concept which typically means unbounded (immeasurably large), but in the case of calculus, ∞ means *infinitesimal* (immeasurably small), since taking a limit requires small numbers. Taking a limit means you may never reach the target, a concept that the Greeks called *Zeno's paradox* (Stillwell, 2010, p. 76).

When speaking of the class of *ordinary* (versus *vector*) differential equations, the term *scalar* is preferable, since the term "ordinary" is vague, if not a meaningless label. There is a special subset of fundamental theorems for scalar calculus, all of which are about integration, as summarized in Table 4.1 (p. 149), starting with Leibniz's theorem. These will be discussed below, and more extensively in Lec. 4.2 (p. 148) and Sections 4 (p. 145) and 5 (p. 145).

Chronological history post 16th century

16th Bombelli *1526–1572*

17th Galileo *1564–1642*, Kepler *1571–1630*, Newton *1642–1727* Principia *1687*; *Mersenne; Huygens; Pascal; Fermat, Descartes (analytic geometry); and the three Bernoullis (Jakob, Johann & son Daniel)*

18th Euler *1707–1783* (Student of Johann Bernoulli*), d'Alembert *1717–1783*, Lagrange *1736–1833*, Laplace *1749-1827*, Fourier *1768-1830*, Gauss *1777–1855* Cauchy *1789–1857*,

19th Helmholtz *1821–1894*, Kelvin *1824–1907*, Helmholtz *1821-1894*, Kirchhoff *1824-1887*, Riemann *1826–1866*, Maxwell *1831–1879*, Rayleigh *1842–1919* Heaviside *1850–1925*,

20th Hilbert *1862-1942*, Einstein *1879-1955*

Following the integral theorems on scalar calculus are those on vector calculus, without which there could be no understanding of Maxwell's equations. Of these, the *fundamental theorem of complex calculus* (aka, Helmholtz decomposition), Gauss's law and Stokes's theorem form the cornerstone of modern vector field analysis. These theorems allow one to connect the differential (point) and macroscopic (integral) relationships. For example, Maxwell's equations may be written as either vector differential equations, as shown by Heaviside (along with Gibbs and Hertz),[1] or in integral form. It is helpful to place these two forms side-by-side, to fully appreciate their significance. To understand the differential (microscopic) view, one must understand the integral (macroscopic) view. These are presented in Section 5.7 (p. 223) and Fig. 5.7 (p. 226).

4.1   The beginning of modern mathematics

As outlined in Fig. 4.1, mathematics as we know it today began in the 16th to 18th centuries, arguably starting with Galileo, Descartes, Fermat, Newton, the Bernoulli family, and Euler. Galileo

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[1] https://en.wikipedia.org/wiki/History_of_Maxwell\%27s_equations

TimeLine

|1525 |1600 |1700 |1800 |1875 |1925

Bombelli   Newton   Maxwell
            Descartes Johann Bernoulli Gauss   Hilbert
            Mersenne   Euler   Riemann
                Huygens   Daniel Bernoulli   Rayleigh
        Galileo   Jacob Bernoulli   Cauchy   Einstein
            Fermat   d'Alembert   Helmholtz
                                Mozart   US Civil War
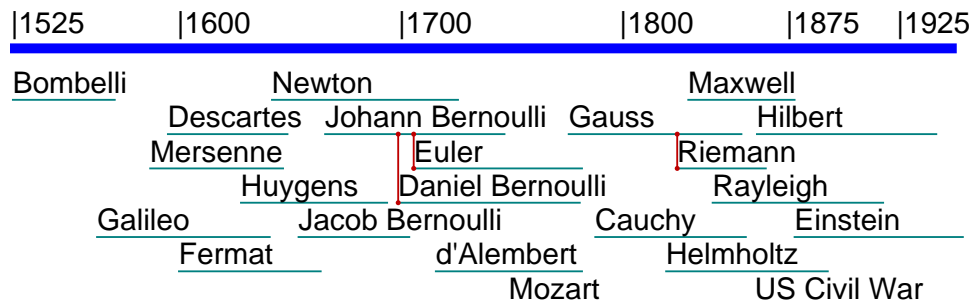
Figure 4.1: *Final overview of the timeline for the four centuries from the 16th and 20th CE covering Bombelli to Einstein. Mozart and the US Civil War are indicated along the bottom, for orientation.*

was formidable, due to his fame, fortune, and his "successful" stance against the powerful Catholic establishment. His creativity in scientific circles was certainly well known due to his many skills and accomplishments. Descartes and Fermat were at the forefront of merging algebra and geometry. While Fermat kept meticulous notebooks, he did not publish, and tended to be secretive. Thus Descartes's contributions were more widely acknowledged, but not necessarily deeper.

Regarding the development of calculus, much was yet to be done by Newton and Leibniz, using term-by-term integration of functions based on Taylor series representation. This was a powerful technique, but as stated earlier, incomplete because the Taylor series can only represent single-valued functions, within the RoC. But more importantly, Newton (and others) failed to recognize the powerful generalization to complex analytic functions. The first major breakthrough was Newton's publication of *Principia* (1687), and the second was Riemann (1851), advised by Gauss.

Following Newton's lead, the secretive and introverted behavior of the typical mathematician dramatically changed with the Bernoulli family (Fig. 4.1). The oldest brother Jacob taught his much younger brother Johann, who then taught his son Daniel. But Johann's star pupil was Euler. Euler first mastered all the tools and then published, with a prolificacy previously unknown.

Euler and the circular functions: The first major task was to understand the family of analytic circular functions, $e^x, \sin(x), \cos(x)$, and $\log(x)$, a task begun by the Bernoulli family, but mastered by Euler. Euler sought relations between these many functions, some of which may not be thought of as being related, such as the log and sin functions. The connection that may "easily" be made is through their complex Taylor series representation (Eq. 3.28, p. 89). By the manipulation of the analytic series representations, the relationship between e^x, and the $\sin(x)$ and $\cos(x)$, was precisely captured with the relation

$$e^{j\omega} = \cos(\omega) + j\sin(\omega),$$
(4.1)

and its analytic inverse (Greenberg, 1988, p. 1135)

$$\tan^{-1}(z) = \frac{1}{2j} \ln\left(\frac{1j-z}{1j+z}\right).$$
(4.2)

The phase is $\phi(\omega) = \angle z$, from which one may compute the *group delay*

$$\tau_g(\omega) = -\frac{1}{2\pi}\frac{\partial\phi(\omega)}{\partial\omega}.$$

Exercise: Starting from Eq. 4.1, derive Eq. 4.2. **Solution:** Solution: Let $z(\omega) = \tan \omega$, then

$$z(\omega) = \frac{\sin \omega}{\cos \omega} = \tan(\omega) = -\jmath \frac{e^{\omega \jmath} - e^{-\omega \jmath}}{e^{\omega \jmath} + e^{-\omega \jmath}} = -\jmath \frac{e^{2\omega \jmath} - 1}{e^{2\omega \jmath} + 1}. \tag{4.3}$$

Solving this for $e^{2\omega \jmath}$ gives

$$e^{2\omega \jmath} = \frac{1 + z\jmath}{1 - z\jmath}. \tag{4.4}$$

Taking the square root results in the analytic relationship between $e^{\omega \jmath}$ and $\jmath \tan \omega$

$$e^{\omega \jmath} = \pm \sqrt{\frac{1 + z\jmath}{1 - z\jmath}}.$$

Taking the $\log$, and using the definition of $\omega(z) = \tan^{-1}(z)$, we obtain Eq. 4.2.

These equations are the basis of transmission lines (TL) and the Smith chart. Here $z(\omega)$ is the TL's input impedance and Eq. 4.4 is the *reflectance*.
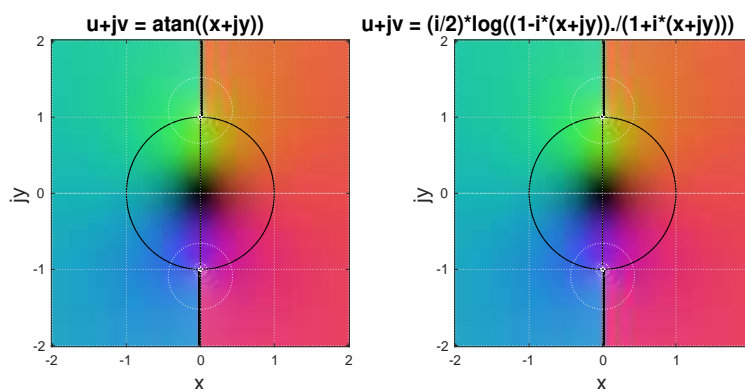
Figure 4.2: *Colorized plots of $\omega(z) = tan^{-1}(z)$ and $\omega(z) = \frac{i}{2} \ln(1 - iz)/(1 + iz)$, verifying they are the same complex analytic function.*

While many high school students memorize Euler's relation, it seems unlikely they appreciate the significance of *complex analytic functions* (Eq. 3.39, p. 94).

A brief history of complex analytic functions: Newton famously ignored imaginary numbers, and called them imaginary in a disparaging (pejorative) way. Given Newton's prominence, his view certainly must have keenly attenuated interest in complex algebra, even though it had been previously quantified by Bombelli in 1525, likely based on his serendipitous finding of Diophantus's book *Arithmetic* in the Vatican library.

Euler derived his relationships using real power series (i.e., real analytic functions). While Euler was fluent with $\jmath = \sqrt{-1}$, he did not consider functions to be complex analytic. That concept was first explored by Cauchy almost a century later. The missing link to the concept of complex analytic is the definition of the derivative with respect to the complex argument

$$F'(s) = \frac{dF(s)}{ds}, \tag{4.5}$$

where $s = \sigma + \omega \jmath$, without which the complex analytic Taylor coefficients may not be defined.

Euler did not appreciate the role of complex analytic functions because these were first discovered well after his death (1785) by Augustin-Louis Cauchy (1789–1857), and extended by Riemann in 1851 (p. 138).

4.2 Lec 23: **Fundamental theorems of scalar calculus**

History of the fundamental theorem of scalar calculus: It some sense, the story of calculus begins with the *fundamental theorem of calculus* (FTC), also known generically as *Leibniz's formula*. The simplest integral is the length of a line $L = \int_0^L dx$. If we label a point on a line as $x = 0$ and wish to measure the distance to any other point x, we form the line integral between the two points. If the line is straight, this integral is simply the Euclidean length given by the difference between the two ends (Eq. 3.4, p. 109).

If $F(\chi) \in \mathbb{R}$ describes a height above the line $\chi \in \mathbb{R}$, then $f(x)$

$$f(x) - f(0) = \int_{x=0}^x F(\chi)d\chi \tag{4.6}$$

may be viewed as the *anti-derivative* of $F(\chi)$. Here χ is a dummy variable of integration. Thus the area under $F(\chi)$ only depends on the difference of the area evaluated at the end points. It makes intuitive sense to view $f(x)$ as the anti-derivative of $F(\chi)$.

This property of the area as an integral over an interval, only depending on the end points, has important consequences in physics in terms of conservation of energy, allowing for important generalizations. For example, as long as $\chi \in \mathbb{R}$, one may let $F(\chi) \in \mathbb{C}$ with no loss of generality, due to the linear propriety (P1, p. 133) of the integral.

If $f(x)$ is analytic (Eq. 3.27, p. 89), then

$$F(x) = \frac{d}{dx}f(x) \tag{4.7}$$

is an exact real differential. It follows that $F(x)$ is analytic. This is known as the *fundamental theorem of (real) calculus* (FTC). Thus Eq. 4.7 may be viewed as an *exact real differential*. This is easily shown by evaluating

$$\frac{d}{dx}f(x) = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta} = F(x),$$

starting from the *anti-derivative* Eq. 4.6. If $f(x)$ is not analytic then the limit may not exist, so this is a necessary condition.

There are many important variations on this very basic theorem (i.e., Sect. 4, p. 145). For example, the limits could depend on time. Also when taking Fourier transforms, the integrand depends on both time $t \in \mathbb{R}$ and frequency $\omega \in \mathbb{R}$ via a complex exponential "kernel" function $e^{\pm j\omega t} \in \mathbb{C}$, which is analytic in both t and ω.

4.2.1   The fundamental theorems of complex calculus:

The *fundamental theorem of complex calculus* (FTCC) states (Greenberg, 1988, p. 1197) that for any *complex analytic function* $F(s) \in \mathbb{C}$ with $s = \sigma + \omega j \in \mathbb{C}$,

$$f(s) - f(s_o) = \int_{s_o}^s F(\zeta)d\zeta. \tag{4.8}$$

Equations 4.6 and 4.8 differ because the path of the integral is complex. Thus the line integral is over $s \in \mathbb{C}$ rather than a real integral over $\chi \in \mathbb{R}$. The *fundamental theorem of complex calculus* (FTCC) states that the integral only depends on the end points, since

$$F(s) = \frac{d}{ds}f(s). \tag{4.9}$$

Comparing exact differentials Eq. 4.5 (FTCC) and Eq. 4.7 (FTC), we see that $f(s) \in \mathbb{C}$ must be *complex analytic*, and have a Taylor series in powers in $s \in \mathbb{C}$. It follows that $F(s)$ is also complex analytic.

Complex analytic functions: The definition of a *complex analytic function* $F(s)$ of $s \in \mathbb{C}$ is that the function may be expanded in a Taylor series (Eq. 3.38, p. 94) about an *expansion point* $s_o \in \mathbb{C}$. This definition follows the same logic as the FTC. Thus we need a definition for the coefficients $c_n \in \mathbb{C}$, which most naturally follow from Taylor's formula

$$c_n = \frac{1}{n!} \frac{d^n}{ds^n} F(s) \bigg|_{s=s_o}. \tag{4.10}$$

The requirement that $F(s)$ have a Taylor series naturally follows by taking derivatives with respect to s at s_o. *The problem is that both integration and differentiation of functions of complex Laplace frequency $s = \sigma + \omega \jmath$ have not yet been defined.*

Thus the question is: What does it mean to take the derivative of a function $F(s) \in \mathbb{C}$, $s = \sigma + \omega \jmath \in \mathbb{C}$, with respect to s, where s defines a plane rather than a real line? We learned how to form the derivative on the real line. Can the same derivative concept be extended to the complex plane?

The answer is affirmative. The question may be resolved by applying the rules of the real derivative when defining the derivative in the complex plane. However, for the complex case, there is an issue regarding direction. Given any analytic function $F(s)$, is the partial derivative with respect to σ different from the partial derivative with respect to $\omega \jmath$? For complex analytic functions, the FTCC states that the integral is independent of the path in the s plane. *Based on the chain rule, the derivative must also be independent of direction at s_o.* This directly follows from the FTCC. *If the integral of a function of a complex variable is to be independent of the path, the derivative of a function with respect to a complex variable must be independent of the direction.* This follows from Taylor's formula, Eq. 4.10, for the coefficients of the complex analytic formula.

The Cauchy-Riemann conditions: The FTC defines the area as an integral over a real differential $(dx \in \mathbb{R})$, while the FTCC relates an integral over a complex function $F(s) \in \mathbb{C}$, along a complex interval (i.e., path) $(ds \in \mathbb{C})$. For the FTC the area under the curve only depends on the end points of the anti-derivative $f(x)$. But what is the meaning of an "area" along a complex path? The *Cauchy-Riemann conditions* provide the answer.

Table 4.1: *Summary of the fundamental theorems of integral calculus, each of which deals with integration. There are at least two main theorems related to scalar calculus, and three more for vector calculus.*

Name	Mapping	p.	Description
Leibniz (FTC)	$\mathbb{R}^1 \to \mathbb{R}^0$	148	Area under a real curve
Cauchy (FTCC)	$\mathbb{C}^1 \to \mathbb{R}^0$	148	Area under a complex curve
Cauchy's theorem	$\mathbb{C}^1 \to \mathbb{C}^0$	165	Close integral over analytic region is zero
Cauchy's integral formula	$\mathbb{C}^1 \to \mathbb{C}^0$	166	Fundamental theorem of complex integral calculus
residue theorem	$\mathbb{C}^1 \to \mathbb{C}^0$	166	Residue integration

4.2.2 Lec 24: Cauchy-Riemann conditions

For the integral of $Z(s) = R(\sigma, \omega) + X(\sigma, \omega)\jmath$ to be independent of the path, the derivative of $Z(s)$ must be independent of the direction of the derivative. As we show next, this leads to a pair of equations known as the *Cauchy-Riemann* conditions. This is an important generalization of Eq. 1.1 (p. 21), which goes from real integration $(x \in \mathbb{R})$ to complex integration $(s \in \mathbb{C})$ based on lengths, thus on area.

To define

$$\frac{d}{ds} Z(s) = \frac{d}{ds} [R(\sigma, \omega) + \jmath X(\sigma, \omega)],$$

take partial derivatives of $Z(s)$ with respect to σ and $\jmath \omega$, and equate them:

$$\frac{\partial Z}{\partial \sigma} = \frac{\partial R}{\partial \sigma} + \jmath \frac{\partial X}{\partial \sigma} \qquad \equiv \qquad \frac{\partial Z}{\partial \jmath \omega} = \frac{\partial R}{\partial \jmath \omega} + \jmath \frac{\partial X}{\partial \jmath \omega}.$$

This says that a horizontal derivative, with respect to σ, is equivalent to a vertical derivative, with respect to $\jmath\omega$. Taking the real and imaginary parts gives the two equations

$$\text{CR-1:} \quad \frac{\partial R(\sigma,\omega)}{\partial \sigma} = \jmath\frac{\partial X(\sigma,\omega)}{\partial \omega \jmath} \quad \text{and} \quad \text{CR-2:} \quad \frac{\partial R(\sigma,\omega)}{\partial \omega \jmath} = -\jmath\frac{\partial X(\sigma,\omega)}{\partial \sigma}, \tag{4.11}$$

known as the *Cauchy-Riemann (CR) conditions*. The $\jmath$ cancels in CR-1, but introduces a $\jmath^2 = -1$ in CR-2. They may also be written in polar coordinates ($s = re^{\theta\jmath}$) as

$$\frac{\partial R}{\partial r} = \frac{1}{r}\frac{\partial X}{\partial \theta} \quad \text{and} \quad \frac{\partial X}{\partial r} = -\frac{1}{r}\frac{\partial R}{\partial \theta}.$$

If you are wondering what would happen if we took a derivative at 45 degrees, then we only need to multiply the function by $e^{\jmath\pi/4}$. But doing so will not change the derivative. Thus we may take the derivative in any direction by multiplying by $e^{\theta\jmath}$, and the CR conditions will not change.

The CR conditions are necessary conditions that the integral of $Z(s)$, and thus its derivative, be independent of the path, expressed in terms of conditions on the real and imaginary parts of Z. This is a very strong condition on $Z(s)$, which follows assuming that $Z(s)$ may be written as a Taylor series in s:

$$Z(s) = Z_o + Z_1 s + \frac{1}{2}Z_2 s^2 + \cdots, \tag{4.12}$$

where $Z_n \in \mathbb{C}$ are complex constants given by the Taylor series formula (Eq. 4.10, p. 149). As with the real Taylor series, there is the convergence condition, that $|s| < 1$, called the *radius of convergence* (RoC). This is an important generalization of the region of convergence (ROC) for real $s = x$.

Every function that may be expressed as a *Taylor series* in $s - s_o$ about point $s_o \in \mathbb{C}$ is said to be *complex analytic* at s_o. This series, *which must be single valued*, is said to converge within a *radius of convergence* (RoC). This highly restrictive condition has significant physical consequences. For example, every impedance function $Z(s)$ obeys the CR conditions over large regions of the s plane, including the entire right half-plane (RHP) ($\sigma > 0$). This condition is summarized by the Brune condition $\Re\{Z(\sigma > 0)\} \geq 0$ (Section 4.3, Eq. 4.26, p. 158). When the CR condition is generalized to volume integrals, it is called *Green's theorem*, used heavily in the solution of boundary value problems in engineering and physics (Kusse and Westwig, 2010). Sections 4 (p. 145) and 5 (p. 187) depend heavily on these concepts.

We may merge these equations into a pair of second-order equations by taking a second round of partials. Specifically, eliminating the real part $R(\sigma,\omega)$ of Eq. 4.11 gives

$$\frac{\partial^2 R(\sigma,\omega)}{\partial\sigma\partial\omega} = \frac{\partial^2 X(\sigma,\omega)}{\partial^2\omega} = -\frac{\partial^2 X(\sigma,\omega)}{\partial^2\sigma}, \tag{4.13}$$

which may be compactly written as $\nabla^2 X(\sigma,\omega) = 0$. Eliminating the imaginary part gives

$$\frac{\partial^2 X(\sigma,\omega)}{\partial\omega\partial\sigma} = \frac{\partial^2 R(\sigma,\omega)}{\partial^2\sigma} = -\frac{\partial^2 R(\sigma,\omega)}{\partial^2\omega}, \tag{4.14}$$

which may be written as $\nabla^2 R(\sigma,\omega) = 0$.

In summary, for a function $Z(s)$ to be complex analytic, the derivative dZ/ds must be independent of direction (path), which requires that the real and imaginary parts of the function obey Laplace's equation, i.e.,

$$\text{CR-3:} \; \nabla^2 R(\sigma,\omega) = 0 \quad \text{and} \quad \text{CR-4:} \; \nabla^2 X(\sigma,\omega) = 0. \tag{4.15}$$

The CR equations are easier to work with because they are first-order, but the physical intuition is best understood by noting two facts (1) the derivative of a complex analytic function is independent of its direction, and (2) the real and imaginary parts of the function both obey Laplace's equation. Such relationships are known as *harmonic functions*.[2]

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[2]When the function is the ratio of two polynomials, as in the cases of the Brune impedance, they are also related to Möbius transformations, also know as *bi-harmonic* operators.

As we shall see in the next few lectures, complex analytic functions must be smooth since every analytic function may be differentiated an infinite number of times, within the RoC. The magnitude must attain its maximum *and* minimum on the boundary. For example, when you stretch a rubber sheet over a jagged frame, the height of the rubber sheet obeys Laplace's equation. Nowhere can the height of the sheet rise above or below its value at the boundary. 5

Harmonic functions define *conservative fields*, which means that energy (like a volume or area) is conserved. The work done in moving a mass from a to b in such a field is conserved. If you return the mass from b back to a, the energy is retrieved, and zero net work has been done.

4.3   Lec 25: Complex Analytic functions and Brune Impedance

It is rarely stated that the variable that we are integrating over, either x (space) or t (time), is real $(x, t \in R)$, since that fact is implicit, due to the physical nature of the formulation of the integral. But this intuition must be refined once complex numbers are included with $s \in \mathbb{C}$, where $s = \sigma + \omega\jmath$.

That time and space are real variables is more than an assumption: it is a requirement that follows from the real *order property*. Real numbers have order. For example, if $t = 0$ is now (the present), then $t < 0$ is the past and $t > 0$ is the future. The order property of time and space allows one to order these along a real axis. To have time travel, time and space would need to be complex (they are not), since if the space axis were complex, as in frequency s, the order property would be invalid. It follows that if we require order, time and space must be real $(t, x \in \mathbb{R})$. Interestingly, it was shown by d'Alembert (1747) that time and space are related by the pure delay due to the wave speed c_o. To obtain a solution to the governing wave equation, which d'Alembert first proposed for sound waves, $x, t \in \mathbb{R}^3$ may be combined as functions of

$$\zeta_{\pm} = t \pm x/c_o,$$

where c [m/s] is the phase velocity of the waves. The d'Alembert solution to the wave equation, describing waves on a string under tension, is

$$u(x, t) = f(t - x/c_o) + g(t + x/c_o), \tag{4.22}$$

which describes the transverse velocity (or displacement) of two independent waves $f(\zeta_-)$, $g(\zeta_+)$ on the string, which represent forward and backward traveling waves.[3] For example, starting with a string at rest, if one displaces the left end, at $x = 0$, by a step function $u(t)$, then that step displacement will propagate to the right as $u(t - x/c_o)$, arriving at location x_o [m], at time x_o/c_o [s]. Before this time, the string will not move to the right of the wave-front, at x_o [m], and after t_o [s] it will have displacement 1. Since the wave equation obeys superposition (postulate P1, p. 133), it follows that the "plane-wave" eigen-functions of the wave equation for $\mathbf{x}, \mathbf{k} \in \mathbb{R}^3$ are given by

$$\psi_{\pm}(\mathbf{x}, t) = \delta(t \mp \mathbf{k} \cdot \mathbf{x}) \leftrightarrow e^{st \pm \jmath\mathbf{k}\cdot\mathbf{x}}, \tag{4.23}$$

where $|\mathbf{k}| = 2\pi/|\lambda| = \omega/c_o$ is the *wave number*, $|\lambda|$ is the wavelength, and $s = \sigma + \omega\jmath$, the Laplace frequency.

When propagation losses are considered, we must replace the *wave number* $\jmath\mathbf{k}$ with a complex analytic vector wave number $\boldsymbol{\kappa}(s) = \mathbf{k}_r(s) + \jmath\mathbf{k}(\mathbf{s})$, which is denoted as either the *complex propagation function*, *propagation function*, or the *dispersion relation*. The vector complex propagation function is a key generalization of the scalar wave number $k = 2\pi/\lambda$.

Forms of loss include viscosity and radiation. Important examples include acoustic plane waves, electromagnetic wave propagation, antenna theory, and the more complex case of 3D electron wave propagating in crystals (i.e., silicon), where electrons and EM waves are in a state of equilibrium.

Electrons and photons are simply different EM states, where $\boldsymbol{\kappa}(\boldsymbol{x}, s)$ describes the crystal's dispersion relations as functions of both frequency and direction, famously known as *Brillouin zones*. Dispersion is a property of the medium such that the wave velocity is a function of frequency and direction, as in silicon.[4] Informed discussions on the history of this topic may be found in Brillouin (1953).

4.3.1   Complex impedance functions

Conservation of energy (or power) is a cornerstone of modern physics. It may first have been under consideration by Galileo Galilei (1564-1642) and Marin Mersenne (1588-1648). Today the question is not whether it is true, but why. Specifically, what is the physics behind conservation of energy?

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[3] d'Alembert's solution is valid for functions that are not differentiable, such as $\delta(t - c_o x)$.

[4] In case you missed it, I'm suggesting is that photons (propagating waves) and electrons (evanescent waves) are different wave "states." The difference is the medium, which determines the dispersion relation (Papasimakis *et al.*, 2018).

Surprisingly, the answer is straightforward, based on its definition and the properties of impedance. Recall that the power is the product of the force and flow, and impedance is their ratio.

The power is given by the product of two variables, sometimes called *conjugate* variables, the force and the flow. In electrical terms, these are voltage (force) ($v(t) \leftrightarrow V(\omega)$) and current (flow) ($i(t) \leftrightarrow I(\omega)$); thus, the electrical power at any instant of time is

$$\mathcal{P}(t) = v(t)i(t). \tag{4.24}$$

The total energy $E(t)$ is the integral of the power, since $\mathcal{P}(t) = dE/dt$. Thus if we start with all the elements at rest (no currents or voltages), then the energy as a function of time is always positive

$$E(t) = \int_0^t \mathcal{P}(t)dt \geq 0, \tag{4.25}$$

and is simply the total energy applied to the network (Van Valkenburg, 1964a, p. 376). Since the voltage and current are related by either an impedance or an admittance, conservation of energy depends on the property of impedance. From Ohm's law and P1 (every impedance is causal)

$$v(t) = z(t) \star i(t) = \int_{\tau=0}^t z(\tau)i(t-\tau)d\tau \leftrightarrow V(s) = Z(s)I(s).$$

Needs work!

Example: Let $i(t) = \delta(t)$. Then $|w|^2(\tau) = i(t) \star i(t) = \delta(\tau)$. Thus

$$I_{xx}(t) = \int_{\tau=0}^t z(\tau)|w|^2(\tau)d\tau = \int_{\tau=0}^t z(\tau)\delta(\tau)d\tau = \int_0^t z(\tau)d\tau.$$

The Brune impedance always has the form $z(t) = r_o\delta(t) + \zeta(t)$. The surge impedance is defined as

$$r_o = \int_{0^-}^\infty z(t)dt.$$

The integral of the reactive part ($\zeta(t)$) is always zero, since the reactive part cannot not store energy.

Perhaps easier to visualize is when working in the frequency domain where the total energy, equal to the integral of the real part of the power, is

$$\frac{1}{s}\Re VI = \frac{1}{2s}(V^*I + VI^*) = \frac{1}{2s}(Z^*I^*I + ZII^*) = \frac{1}{s}\Re Z(s)|I|^2 \geq 0.$$

Formally this is related to a positive definite operator where the positive resistance forces the definiteness, which is sandwiched between the current.   5

In conclusion, conservation of energy is totally dependent on the properties of the impedance. Thus one of the most important and obvious applications of complex functions of a complex variable is the impedance function. This seems to be the ultimate example of the FTC, applied to $z(t)$, in the name of conservation of energy.

Poles and zeros of PR functions must be first degree: We conjecture that this proof also requires   10
that the poles and the zeros of the impedance function be simple (only first degree). Second degree poles would have a reactive "secular" response of the form $h(t) = t\sin(\omega_k t + \phi)u(t)$, and these terms would not average to zero, depending on the phase, as is required of an impedance. As a result, only single degree poles would be possible.[5] Furthermore, when the impedance is the ratio of two polynomials, where the lower degree polynomial is the derivative of the higher degree one, then the poles and zeros   15
must alternate. This is a well-known property of the Brune impedance that has never been adequately

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[5]Secular terms result from second degree poles since $u(t) \star u(t) = tu(t) \leftrightarrow 1/s^2$.

explained except for very special cases (Van Valkenburg, 1964b, p. 104-107). I suspect that no one has ever reported an impedance having second degree poles and zeros as that would be rare impedance. Network analysis books never report 2nd degree poles and zeros in their impedance functions. Nor has there ever been any guidance as to where the poles and zeros might lie in the *left-hand s plane*. Understanding the exact relationships between pairs of poles and zeros, to assure that the real part of the impedance is real, would resolve a longstanding unsolved problem (Van Valkenburg, 1964b).

Every impedance must obey conservation of energy (P3):   The impedance function $Z(s)$ has resistance R and reactance X as a function of complex frequency $s = \sigma + \jmath\omega$. From the causality postulate (P1) of Sections 3.7.1 and K (p. 289), $z(t < 0) = 0$. Every impedance is defined by a Laplace transform pair

$$z(t) \leftrightarrow Z(s) = R(\sigma, \omega) + \jmath X(\sigma, \omega),$$

with $R, X \in \mathbb{R}$.

According to Postulate P3 (Sect. 3.7.1, p. 133), a system is passive if it does not contain a power source. Drawing power from an impedance violates conservation of energy. This propriety is also called *positive-real*, which was defined by Brune (1931b,a)

$$\Re\{Z(s \geq 0)\} \geq 0. \tag{4.26}$$

'

Positive-real systems cannot draw more power than is stored in the impedance.[6] The region $\sigma \leq 0$ is called the *left half s plane* (LHP), and the complementary region $\sigma > 0$ is called the *right half s plane* (RHP). According to the Brune condition the real part of every impedance must be non-negative in the RHP.

It is easy to construct examples of second-order poles or zeros in the RHP, such that P3 is violated. Thus P3 implies that the impedance may *not* have more than simple (first-order) poles and zeros, strictly in the LHP. But there is yet more: These poles and zero in the LHP must have order, to meet the minimum phase condition. This minimum phase condition is easily stated

$$\angle Z(s) < \angle s,$$

but difficult to prove.[7] There seems to be no proof that second-order poles and zeros (e.g., second-order roots) are not allowed. However such roots must violate a requirement that the poles and zeros must alternate on the $\sigma = 0$ axis, which follows from P3. In the complex plane the concept of 'alternate' is not defined (complex numbers cannot be ordered). What *has* been proved (i.e., Foster's reactance theorem[8]) is that if the poles are on the real or imaginary axis, they must alternate, leading to simple poles and zeros (Van Valkenburg, 1964a). The restriction on poles is sufficient, but not necessary, as $Z(s) = 1/\sqrt{s}$ is a *physical realizable* (PR) impedance, but is less than a first-degree pole (Kim and Allen, 2013). The corresponding condition in the LHP, and its proof, remains elusive (Van Valkenburg, 1964a).

As an example, a series resistor R_o and capacitor C_o have an impedance given by (Table I.2, p. 282)

$$Z(s) = R_o + 1/sC_o \leftrightarrow R_o\delta(t) + \frac{1}{C_o}u(t) = z(t), \tag{4.27}$$

with constants $R_o, C_o \in \mathbb{R} > 0$. In mechanics an impedance composed of a dash-pot (damper) and a spring has the same form. A resonant system has an inductor, resistor and a capacitor, with an impedance given by (Table I.1, p. 281)

$$Z(s) = \frac{sC_o}{1 + sC_oR_o + s^2C_oM_o} \leftrightarrow C_o\frac{d}{dt}\left(c_+e^{s_+t} + c_-e^{s_-t}\right) = z(t), \tag{4.28}$$

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[6]Does this condition hold for the LHP $\sigma < 0$? It does for Eq. 4.28.

[7]As best I know, this is an open problem in network theory (Brune, 1931a; Van Valkenburg, 1964a).

[8]https://en.wikipedia.org/wiki/Foster\%27s_reactance_theorem

which is a second degree polynomial with two complex resonant frequencies $s_\pm$. When $R_o > 0$ these roots are in the left half s plane, with $z(t) \leftrightarrow Z(s)$.

Systems (networks) containing many elements, and transmission lines, can be much more complicated, yet still have a simple frequency domain representation. This is the key to understanding how these physical systems work, as will be described below.

Complex analytic functions: To solve a differential equation, or integrate a function, Newton used the Taylor series to integrate one term at a time. However, he only used real functions of a real variable, due to the fundamental lack of appreciation of the complex analytic function. This same method is how one finds solutions to scalar differential equations today, but using an approach that makes the solution method less obvious. Rather than working directly with the Taylor series, today we use the complex exponential, since the complex exponential is an eigen-function of the derivative

$$\frac{d}{dt} e^{st} = s e^{st}.$$

Since e^{st} may be expressed as a Taylor series, having coefficients $c_n = 1/n!$, in some real sense the modern approach is a compact way of doing what Newton did. Thus every linear constant coefficient differential equation in time may be simply transformed into a polynomial in complex Laplace frequency s, by looking for solutions of the form $A(s)e^{st}$, transforming the differential equation into a polynomial $A(s)$ in complex frequency. For example,

$$\frac{d}{dt} f(t) + a f(t) \leftrightarrow (s + a) F(s).$$

The root of $A(s_r) = s_r + a = 0$ is the eigenvalue of the differential equation. A powerful tool for understanding the solutions of differential equations, both scalar and vector, is to work in the Laplace frequency domain. The Taylor series has been replaced by e^{st}, transforming Newton's real Taylor series into the complex exponential eigen-function. In some sense, these are the same method, since

$$e^{st} = \sum_{n=0}^{\infty} \frac{(st)^n}{n!}. \tag{4.29}$$

Taking the derivative with respect to time gives

$$\frac{d}{dt} e^{st} = s e^{st} = s \sum_{n=0}^{\infty} \frac{(st)^n}{n!}, \tag{4.30}$$

which is also complex analytic. Thus if the series for $F(s)$ is valid (i.e., it converges), then its derivative is also valid. This was a very powerful concept, exploited by Newton for real functions of a real variable, and later by Cauchy and Riemann for complex functions of a complex variable. The key question here is: Where does the series fail to converge? If it does not converge, the representation fails. This is the main message behind the FTCC (Eq. 4.8).

The FTCC (Eq. 4.5) is formally the same as the FTC (Eq. 4.7) (Leibniz formula), the key (and significant) difference being that the argument of the integrand $s \in \mathbb{C}$. Thus this integration is a line integral in the complex plane. One would naturally assume that the value of the integral depends on the path of integration.

But, according to FTCC, it does not. In fact they are clearly distinguishable from the FTC. And the reasoning is the same. If $F(s) = df(s)/ds$ is complex analytic (i.e., has a power series $f(s) = \sum_k c_k s^k$, with $f(s), c_k, s \in \mathbb{C}$), then it may be integrated, and *the integral does not depend on the path*. At first blush, this is sort of amazing. The key is that $F(s)$ and $f(s)$ must be complex analytic, which means they are differentiable. This all follows from the Taylor series formula Eq. 4.10 (p. 149) for the coefficients of the complex analytic series. For Eq. 4.8 to hold, the derivatives must be independent of the direction, as discussed in Section 4.2.2. The concept of a complex analytic function therefore

has eminent consequences, in the form of several key theorems on complex integration discovered by Cauchy (c1820).

The use of the complex Taylor series generalizes the functions it describes, with unpredictable consequences, as nicely shown by the domain coloring diagrams presented in Section 3.8 (p. 135). Cauchy's tools of complex integration were first exploited in physics by Sommerfeld (1952) to explain the onset transients in waves, as explained in detail in Brillouin (1960, Chap. 3).

Up to 1910, when Sommerfeld first published his results using complex analytic signals and saddle point integration in the complex plane, there was a poor understanding of the implications of the causal wave-front. It would be reasonable to say that his insights changed our understanding of wave propagation, for both light and sound. Sadly this insight has not been fully appreciated, even to this day. If you question my summary, please read Brillouin (1960, Chap. 1).

The full power of the complex analytic function was first appreciated by Bernard Riemann (1826-1866) in his University of Göttingen PhD thesis of 1851, under the tutelage of Carl Friedrich Gauss (1777-1855), and drawing heavily on the work of Cauchy.

The key definition of a complex analytic function is that it has a Taylor series representation over a region of the complex frequency plane $s = \sigma + j\omega$, that converges in a *region of convergence* (RoC) about the expansion point, with a radius determined by the nearest pole of the function. A further surprising feature of all analytic functions is that within the RoC, the inverse of that function also has a complex analytic expansion. Thus given $w(s)$, one may also determine $s(w)$ to any desired accuracy, critically depending on the RoC.

4.3.2 Lec 26: Multi-valued functions

In the field of mathematics there seems to have been a tug-of-war regarding the basic definition of the concept of a function. The accepted definition today seems to be a single-valued mapping from the domain to the codomain (or range). This makes the discussion of multi-valued functions somewhat tedious. In 1851 Riemann (working with Gauss) seems to have resolved this problem for the natural set of multi-valued functions by introducing the concept of single-valued *sheets*, delineated by *branch-cuts*.

Two simple yet important examples of multi-valued functions are the circle $z^2 = x^2 + y^2$ and $w = \log(z)$. For example, assuming z is the radius of the circle, solving for $y(x)$ gives the double-valued function

$$y(x) = \pm\sqrt{z^2 - x^2}.$$

If we accept the modern definition of a function as the mapping from one set to a second, then $y(x)$ is not a function, or even two functions. For example, what if $x > z$? Or worse, what if $z = 2\jmath$ with $|x| < 1$? Riemann's construction, using branch cuts for multivalued function, resolves all these difficulties (as best I know).

To proceed, we need definitions and classifications of the various types of complex singularities:

1. Poles of degree 1 are called *simple poles*. Their amplitude is called the *residue* (e.g. α/s has residue α). Simple poles are special (Eq. 4.35, p. 166)[9] as they play a key role in mathematical physics, since their inverse Laplace transform defines a causal eigen-function.

2. When the numerator and denominator of a *rational function* (i.e., ratio of two polynomials) have a common root (i.e., factor), that root is said to be *removable*.

3. A singularity that is not (1) removable, (2) a pole or (3) a branch point is called *essential*.

4. A complex analytic function (except for isolated poles) is called *meromorphic*.[10] Meromorphic functions can have any number of poles, even an infinite number. The poles need not be simple.

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[9]https://en.wikipedia.org/wiki/Pole_(complex_analysis)
[10]https://en.wikipedia.org/wiki/Meromorphic_function

5. A complex analytic function (except for isolated poles), is called *meromorphic*.[11] Meromorphic functions can have any number of poles, even an infinite number. The poles need not be simple.

6. When the first derivative of a function $Z(s)$ has a simple pole at a, then a is said to be a *branch point* of $Z(s)$. An important example is the logarithmic derivative

$$d \ln(s-a)^\alpha/ds = \alpha/(s-a), \ \alpha \in \mathbb{I}.$$

However, the converse does not necessarily hold.

7. I am not clear about the interesting case of an *irrational pole* ($\alpha \in \mathbb{I}$). In some cases this may be simplified via the logarithmic derivative operation, as mentioned above.

5

More complex topologies are being researched today, and progress is expected to accelerate due to modern computing technology.[12] It is helpful to identify the physical meaning of these more complex surfaces, to guide us in their interpretation and possible applications.[13]

Branch cuts: Up to this point we have only considered poles of degree $\alpha \in \mathbb{N}$ of the form $1/s^\alpha$.
The concept of a *branch cut* allows one to manipulate (and visualize) multi-valued functions, for which

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$\alpha \in \mathbb{F}$. This is done by breaking each region into single-valued *sheets*, as shown in Fig. 4.3 (right). The branch cut is a curve $\in \mathbb{C}$ that separates the various single-valued sheets of a multi-valued function. The concepts of branch cuts, sheets and the *extended plane* were first devised by Riemann, working with Gauss (1777-1855), as first described in his thesis of 1851. It was these three mathematical and geometrical constructions that provided the deep insight into complex analytic functions, greatly extending the

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important earlier work of Cauchy (1789-1857) on the calculus of complex analytic functions. For alternative helpful discussion of Riemann sheets and branch cuts, see Boas (1987, Section 29, pp. 221-225) and (Kusse and Westwig, 2010).

Multi-valued functions: The best way to explore the complex mapping from the complex planes
$s \to w(s)$ is to master the single valued function $s = w^2(s)$ and its double-valued inverse $w(s) = \sqrt{s}$,

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the square root function (Fig. 4.3). Single valued functions such as $W(s) = s^2$ are relatively straightforward. Multivalued functions require the concept of a *branch cut*, defined in the image ($w(s)$) plane, which is a technique to render the multiple values as single valued on each of several *sheets*, defined by the sheet index in the domain (s) plane, and delineated by a branch cut (in the w plane). The sheets are labeled in the domain (s) plane by a *sheet index* $k \in \mathbb{Z}$ while branch points and cuts are defined in the

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image (w) plane (a.k.a., the range). *It is important to understand that the path of every branch cut is not unique, and may be moved.* However branch points are unique thus not movable.

The multi-valued nature of $w(s) = \sqrt{s}$ is best understood by working with the function in polar coordinates. Let

$$s_k = re^{\theta \jmath}e^{2\pi k \jmath}, \tag{4.31}$$

where $r = |s|, \theta = \angle s, \in \mathbb{R}$ and $k \in \mathbb{Z}$ is the *sheet-index*. Furthermore let

$$w = \rho e^{\psi \jmath} = \sqrt{r}e^{\jmath\theta/2}e^{\jmath\pi k}, \quad \text{(4.32)}$$

where $\rho = |w|, \psi \angle w, \in \mathbb{R}$. The generic Cartesian coordinates are $s = \sigma + \omega \jmath$ and $w(s) = u(\sigma,\omega) + v(\sigma,\omega)\jmath$. For single valued functions such as $w(s) = s^2$ (**Left**) there is no branch

30
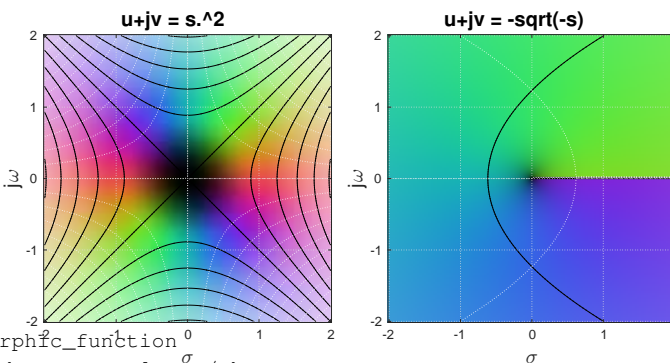
Figure 4.3: *Here we use Cartesian coordinates in the domain $s = \sigma + \omega\jmath$ and polar coordinates for the range $w(\sigma,\omega) = |w|e^{\psi\jmath}$. The color intensity indicates the magnitude $|s|$, with black being $|s| = 0|$ and bright (eventually white), indicating $|s| \to \infty$.* **Left:** *Mapping: $w(s) = s^2$.* **Right:** *Mapping of the principal branch from $-s$ to $w(s) = -\sqrt{-s}$ (i.e., the rotated inverse of s^2). This sheet was generated by rotating w by 180°. The branch cut is on the $\psi = 0$ axis, with branch points at $|w| = 0$ and ∞.*

cut since $\psi = 2\theta$. Note how the red
color ($\theta = 0°$) appears twice in this
mapping. For multi-valued functions a
branch-cut is required, typically along
the negative $v(\sigma, \omega)$ axis (i.e, $\psi = \pi$)
but may be freely distorted, as seen by
comparing the right panel of Fig. 4.3
with the right panel of Fig. 4.5.
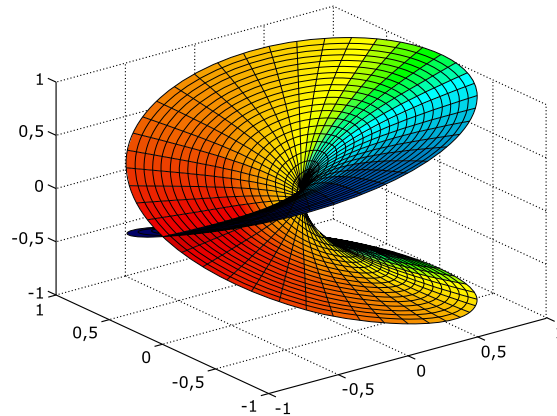
Figure 4.4:   *Here we show the mapping for the square root function $z = \pm\sqrt{x}$ which has two single-valued sheets of the x plane corresponding to the two signs of the square root. The branch cut runs along the real axis, from $z = 0$ to ∞. A more comprehensive explanation is presented in the text. Figure from: https://en.wikipedia.org/wiki/Riemann_surface*

Hue: By studying the output of `zviz.m` (p. 136) we may understand domain-coloring. The domain angles $\angle s$ go from $-\pi < \theta < \pi$, with $\theta = 0$ being red and $\pm\pi$ being green (between yellow and blue). The angle to hue map is shown in the left panel of Fig. 4.5. For the $w(s) = s^2$ the $\angle s$ is expanded by 2, since $\psi = 2\theta$. For $w(s) = \sqrt{s}$ the $\angle s$ is compressed by a factor of 2, since $\psi = \theta/2$. Thus the principle angle $k = 0$, $-180° < \theta < 180°$) maps to half the w plane ($-90° < \psi < 90°$) from purple to yellow, while the $k = 1$ branch maps to $90° < \psi < 270°$. Note how the panel on the right of Fig. 4.3 matches the right half of s (purple = -90°, yellow/green = +90°) while the middle panel above comes from the left side of s (green to purple). The center panel is green at -180°, and purple at +180°, which matches the left panel at $\pm 180°$, respectively (i.e., $e^{\pi j}\sqrt{s}$).

The principle (i.e., first) Riemann sheet of $\sqrt{s}$, corresponding to $-\pi < \theta < \pi$ (i.e., $k = 0$), is shown in the center of Fig. 4.5. This differs from the neighboring sheet ($k = 1$), shown on the right. Thus $w(s) = \sqrt{s}$ is a multi-valued function of s, having two single-valued sheets.

The location of the branch cut may be moved by rotating the z coordinate system. For example, $w(z) = \pm j\sqrt{z}$ and $w(z) = \pm\sqrt{-z}$ have different branch cuts, as may be easily verified using the Matlab/Octave commands `j*zviz(z)` and `zviz(-z)`, as shown in Fig. 4.5. Since the cut may be moved, every function is analytic on the branch cut, if a Taylor series is formed on the branch cut, it will describe the function on the two different sheets. Thus the complex analytic series (i.e., the Taylor formula, Eq. 4.10) does not depend on the location of a branch cut, as it only describes the function uniquely (as a single-valued function), valid in its local region of convergence.

The second sheet ($k = 1$) (Fig. 4.5) picks up at $\theta = \pi$ and continues on to $\pi + 2\pi = 3\pi$. The first sheet maps the angle of w (i.e., $\phi = \angle w = \theta/2$) from $-\pi/2 < \phi < \pi/2$ ($w = \sqrt{r}e^{j\theta/2}$). This corresponds to $u = \Re\{w(s)\} > 0$. The second sheet maps $\pi/2 < \psi < 3\pi/2$ (i.e., 90° to 270°), which is $\Re\{w\} = u < 0$. In summary, twice around the s plane is once around the $w(s)$ plane, because the angle is half due to the $\sqrt{s}$. This then describes the multi-valued nature of the square root function.
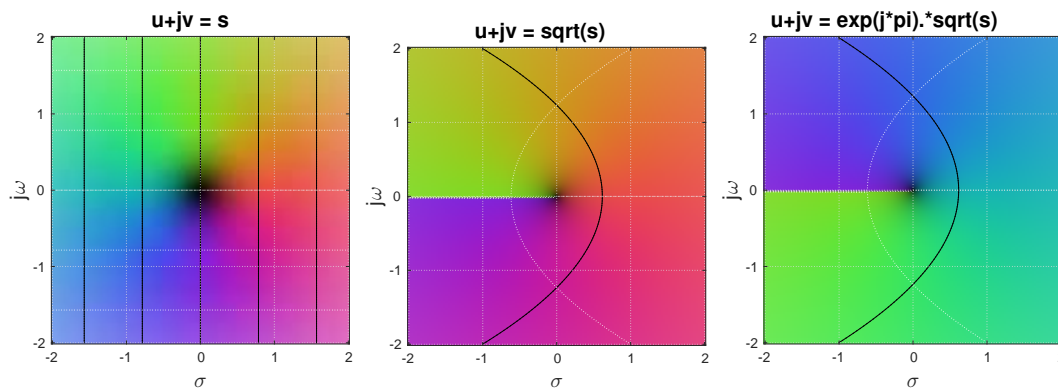
Figure 4.5: *Colorized plots of $s_k = |s|e^{\theta_J}e^{2\pi k_J}$ and $w_k(s) = \sqrt{s_k} = \sqrt{|s|}e^{\theta_J/2}e^{\pi k_J}$, as defined in polar coordinates by Eqs. 4.31 and 4.32.* **Left:** *Color map hue reference plane $s = |s|e^{\theta_J}e^{2\pi k_J}$.* **Center:** *Sheet index $k = 0$, $w(s) = \sqrt{|s|}e^{\theta/2}$ for $-\pi < \theta < +\pi$ and $\psi = \angle w(s) = \theta/2$ between $\pm\pi/2$ [rad].* **Right:** *Sheet index $k = 1$, $w(s) = e^{\theta/2}e^{J\pi}\sqrt{|s|}$ for $\pi < \theta < 3\pi$ and $\pi/2 < \psi < 3\pi/2$ [rad]. The branch cut is at $\theta = \angle s = \pm 180°$ (π [rad]) where the hue of w changes abruptly from green to purple (center) and from blue back to green (right). Note how the hue matches between the center and right panels at the branch cut: in the center panel purple runs along -180° and along +180° on the right. Likewise, green runs along +180° in the center and along -180° on the right. Thus $w(s) = \sqrt{s}$ is analytic on the branch cut connecting the two sheets ($k = 0 \rightarrow k = 1$).*

This concept of analytic inverses becomes important only when the function is multi-valued. For example, if $w(s) = s^2$ then $s(w) = \pm\sqrt{w}$ is multivalued. Riemann dealt with such extensions with the concept of a *branch-cut* with multiple *sheets*, labeled by a sheet number. Each sheet describes an analytic function (Taylor series) that converges within some RoC, having a radius out to the nearest pole. Thus Riemann's branch cuts and sheets explicitly deal with the need to define unique single-valued inverses of multi-valued functions. Since the square root function has two overlapping regions, corresponding to the $\pm$ due to the radical, there must be two connected regions, sort of like mathematical Siamese twins: distinct, yet the same.

Figure 4.3 shows the single-valued function $w(z) = s^2$ (left), and Fig. 4.5 (right) its inverse, the double-valued mapping of $s(w) = \pm\sqrt{w}$.

Branch cuts emanate and terminate at *branch points*, defined as singularities (poles), that can even have fractional degree, as for example $1/\sqrt{s}$, and terminate at one of the matching roots, which includes the possibility of ∞.[14] For example, suppose that in the neighborhood of the pole, at s_o the function is

$$f(s) = \frac{w(s)}{(s - s_o)^k},$$

where $w, s, s_o \in \mathbb{C}$ and $k \in \mathbb{Q}$. When $k = 1$, $s_o = \sigma_o + \omega_o J$ is a *first degree "simple pole"*, having degree 1 in the s plane, with *residue* $w(s_o)$.[15] Typically the order and degree are positive integers, but fractional degrees and orders are not uncommon in modern engineering applications (Kirchhoff, 1868; Lighthill, 1978). Here we shall allow both the degree and order to be fractional ($\in \mathbb{F}$). When $k \in \mathbb{F} \subset \mathbb{R}$ ($k = n/m$ is a real reduced fraction. Namely when `GCD(n,m)=1`, $n \perp m$). This defines the *degree* of a fractional pole. In such cases there must be two sets of branch cuts of degree n and m. For example, if $k = 1/2$, the singularity (branch cut) is of degree 1/2, and there are two Riemann sheets, as shown in Fig. 4.3 (p. 161). The analytic inverse of this function is $s = w^2$, as shown in the same figure.

Fractional-order Bessel function: An important example is the Bessel function

$$\delta(t) + \frac{1}{t}J_1(t)u(t) \leftrightarrow \sqrt{s^2 + 1},$$

---

[14]This presumes that poles appear in pairs, one of which may be at ∞.

[15]We shall refer to the *order* of a derivative, or differential equation, and the *degree* of a polynomial, as commonly used in engineering applications (`https://en.wikipedia.org/wiki/Differential_equation#Equation_order`).

which is related to the solution to the wave equation in two-dimensional cylindrical coordinates (Table I.2, p. 282). Bessel functions are the solutions (i.e., eigen-functions) of guided acoustic waves in round pipes, or surface waves on the earth (seismic waves), or waves on the surface of a pond (Table 5.2, p. 207).
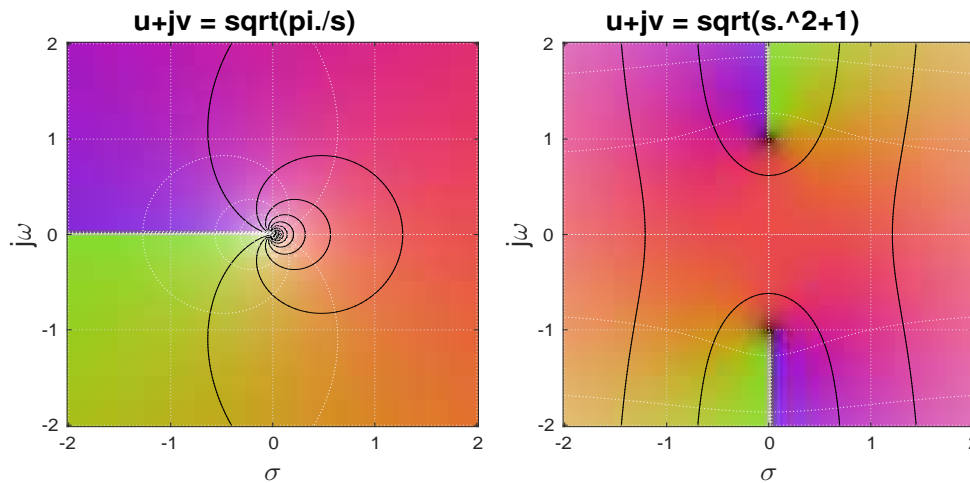
Figure 4.6: *Colorized plots of two $\mathcal{LT}$ pairs: Left: $\sqrt{\pi/s} \leftrightarrow u(t)/\sqrt{t}$. Right: $\sqrt{s^2+1} \leftrightarrow \delta(t) + \frac{1}{t}J_1(t)u(t)$.*

It is important to understand that the function is analytic on the branch cut, but not at the branch point. One is free to move the branch cut (at will). It does not need to be on a line: it could be cut in almost any connected manner, such as a spiral. The only rule is that it must start and stop at the matching branch points, or at ∞, which must have the same degree.

There are a limited number of possibilities for the degree, $k \in \mathbb{Z}$ or $\in \mathbb{F}$. If the degree is drawn from $\mathbb{R} \notin \mathbb{F}$, the pole cannot not have a residue. According to the definition of the residue, $k \in \mathbb{F}$ will not give a residue. But there remains open the possibility of generalizing the concept of the Riemann integral theorem, to include $k \in \mathbb{F}$. One way to do this is to use the *logarithmic derivative* which renders fractional poles to simple poles with fractional residues.

If the singularity has an irrational degree ($k \in \mathbb{I}$), the branch cut has the same "irrational degree." Accordingly there would be an infinite number of Riemann sheets, as in the case of the log function. An example is $k = \pi$, for which

$$F(s) = \frac{1}{s^\pi} = e^{-\log(s^\pi)} = e^{-\pi \log(s)} = e^{-\pi \log(\rho)} e^{-\pi \theta \jmath},$$

where the domain is expressed in polar coordinates $s = \rho e^{\theta \jmath}$. When $k \in \mathbb{F}$ it may be maximally close (e.g., $k = \pi_{152}/\pi_{153} = \pi_{152}/(\pi_{152} + 2) = 881/883 \approx 0.99883$, or its reciprocal ≈ 1.0023).[16] The branch cut could be very subtle (it could even go unnoticed), but it would have a significant impact on the function, and on its inverse Laplace transform.

Example: Find poles, zeros and residues of $F(s)$.

1.

$$F(s) = \frac{d}{ds} \ln \frac{s+e}{s+\pi},$$

Solution:

$$F(s) = \frac{d}{ds} \left[\ln(s+e) - \ln(s+\pi)\right] = \left(\frac{1}{s+e} - \frac{1}{s+\pi}\right)$$

The the poles are at $s_1 = -e$ and $s_2 = -\pi$ with respective residues of ± 1.

---

[16]Since there are no even primes other than $\pi_1 = 2$, the minimum difference is 2. Out of 10^6 primes, only 5 have a minimum difference of more than $e80$, with these 5 having a uniform distribution on a log scale. verify 80

2.
$$F(s) = \frac{d}{ds} \ln \frac{(s+3)^e}{(s+\jmath)^{-\pi}},$$

Solution: There is a very important take-home message here regarding the utility of the logarithmic derivative.

3.
$$F(s) = e^{\pi \ln s},$$

Solution: Add solution.

4.
$$F(s) = \pi^{-s}.$$

Solution: Add solution.

Multivalued functions: The two basic functions we review, to answer the questions about multivalued functions and branch cuts, are $w(s) = \sqrt{s}$ and $w(s) = \log(s)$ along with their inverse functions $w(s) = s^2$ and $w(s) = e^s$. For uniformity we shall refer to the *complex abscissa* ($s = \sigma + \omega\jmath$) and the *complex ordinate* $w(s) = u + v\jmath$. When the complex abscissa and domain are swapped, by taking the inverse of a function, multi-valued functions are a common consequence. For example, $f(t) = \sin(t)$ is single valued, and analytic in t, thus has a Taylor series. The inverse function $t(f)$ is multivalued. [10]

The modern terminology is the *domain* and *range*, or alternatively the *co-domain*.[17]

Log function: Next we discuss the multi-valued nature of the log function. In this case there are an infinite number of Riemann sheets, not well captured by Fig. 3.6 (p. 137), which only displays the principal sheet. However, if we look at the formula for the log function, the nature is easily discerned. The abscissa s may be defined as multi-valued since

$$s_k = re^{2\pi k\jmath}e^{\theta\jmath}.$$

Here we have extended the angle of s by $2\pi k$, where k is the sheet index $\in \mathbb{Z}$. Taking the log

$$\log(s) = \log(r) + (\theta + 2\pi k)\jmath.$$

When $k = 0$ we have the *principal value* sheet, which is zero when $s = 1$. For any other value of k, $w(s) \neq 0$, even when $r = 1$, since the angle is not zero, except for the $k = 0$ sheet.

4.4 Lec 27: Three Cauchy integral theorems

4.4.1 Cauchy's theorems for integration in the complex plane [15]

There are three basic definitions related to Cauchy's integral formula. They are closely related, and can greatly simplify integration in the complex plane. The choice of names seems unfortunate, if not totally confusing.

1. **Cauchy's (Integral) Theorem (CIT):**

$$\oint_{\mathcal{C}} F(s)ds = 0, \tag{4.33}$$

if and only if $F(s)$ is complex analytic inside of a simple closed curve $\mathcal{C}$ (Boas, 1987, p. 45); (Stillwell, 2010, 319). The FTCC (Eq. 4.8) says that the integral only depends on the end points [20] if $F(s)$ is complex analytic. By closing the path (contour $\mathcal{C}$) the end points are the same, thus the integral must be zero, as long as $F(s)$ is complex analytic.

---

[17]The best way to create confusion is to rename something. The confusion grows geometrically with each renaming. I suspect that everyone who cares knows the terms abscissa and ordinate, and some fraction know the equivalent terms *domain* and *range*.

2. **Cauchy's Integral Formula (CIF):**

$$\frac{1}{2\pi j}\oint_{\mathcal{B}}\frac{F(s)}{s-s_o}ds = \begin{cases} F(s_o), s_o \in \mathbb{C} < \mathcal{B} \text{ (inside)} \\ 0, s_o \notin \mathbb{C} > \mathcal{B} \text{ (outside).} \end{cases} \quad (4.34)$$

Here $F(s)$ is required to be analytic everywhere within (and on) the boundary $\mathcal{B}$ of integration (Greenberg, 1988, p. 1200); (Boas, 1987, p. 51); (Stillwell, 2010, p. 220). When the point $s_o \in \mathbb{C}$ is within the boundary the value $F(s_o) \in \mathbb{C}$ is the *residue* of the pole s_o of $F(s)/(s-s_o)$. When the point s_o lies outside the boundary, the integral is zero.

3. **The (Cauchy) Residue Theorem (CRT):** (Greenberg, 1988, p. 1241), (Boas, 1987, p. 73)

$$\oint_{\mathcal{C}}F(s)ds = 2\pi j\sum_{k=1}^{K}c_k = \sum_{k=1}^{K}\oint\frac{F(s)}{s-s_k}ds, \quad (4.35)$$

where the *residues* $c_k \in \mathbb{C}$ corresponds to the kth pole of $f(s)$ enclosed by the contour $\mathcal{C}$. By the use of Cauchy's integral formula, the right-most form of the residue theorem is equivalent to the CIF.[18]

How to calculate the residue: The case of first degree poles has special significance, because the Brune impedance only allows simple poles and zeros, increasing its utility. The residues for simple poles are $F(s_k)$, which is complex analytic in the neighborhood of the pole, but not at the pole.

Consider the function $f(s) = F(s)/(s-s_k)$, where we have factored $f(s)$ to isolate the first-order pole at $s = s_k$, with $F(s)$ analytic at s_k. Then the residue of the poles at $c_k = F(s_k)$. This coefficient is computed by removing the singularity, by placing a zero at the pole frequency, and taking the limit as $s \to s_k$, namely

$$c_k = \lim_{s\to s_k}[(s-s_k)F(s)] \quad (4.36)$$

(Greenberg, 1988, p. 1242), (Boas, 1987, p. 72).

When the pole is an Nth degree, the procedure is much more complicated, and requires taking $N-1$ order derivatives of $f(s)$, followed by the limit process (Greenberg, 1988, p. 1242). Higher degree poles are rarely encountered; thus, it is good to know that this formula exists, but perhaps it is not worth the effort to memorize it.

Summary and examples: These three theorems, all attributed to Cauchy, collectively are related to the fundamental theorems of calculus. Because the names of the three theorems are so similar, they are easily confused.

1. In general it makes no sense (nor is there any need) to integrate *through* a pole, thus the poles (or other singularities) must not lie on $\mathcal{C}$.

2. The Cauchy integral theorem (Eq. 4.33) follows trivially from the fundamental theorem of complex calculus (Eq. 4.8, p. 148), since if the integral is independent of the path, and the path returns to the starting point, the closed integral must be zero. Thus Eq. 4.33 holds when $F(s)$ is complex analytic within C.

3. Since the real and imaginary parts of every complex analytic function obey Laplace's equation (Eq. 4.15, p. 150), it follows that every closed integral over a Laplace field, i.e., one defined by Laplace's equation, must be zero. *In fact this is the property of a conservative system,* corresponding to many physical systems. If a closed box has fixed potentials on the walls, with any distribution whatsoever, and a point charge (i.e., an electron) is placed in the box, then a force equal to $F = qE$ is required to move that charge, and thus work is done. However, if the point is returned to its starting location, the net work done is zero.

---

[18]This theorem is the same as a 2D version of Stokes's theorem (Boas, 1987).

4. Work is done in charging a capacitor, and energy is stored. However, when the capacitor is discharged, all of the energy is returned to the load.

5. Soap bubbles and rubber sheets on a wire frame obey Laplace's equation.

6. These are all cases where the fields are Laplacian, thus closed line integrals must be zero. Laplacian fields are commonly observed because they are so basic.

5

7. We have presented the impedance as the primary example of a complex analytic function. Physically, every impedance has an associated stored energy, and every system having stored energy has an associated impedance. This impedance is usually defined in the frequency s domain, as a force *over* a flow (i.e., voltage over current). The power $\mathcal{P}(t)$ is defined as the force *times* the flow and the energy $\mathcal{E}(t)$ as the time integral of the power

$$\mathcal{E}(t) = \int_{-\infty}^{t} \mathcal{P}(t)dt, \tag{4.37}$$

which is similar to Eq. 4.6 (p. 148) [see Section 3.5.3, Eq. 3.40 (p. 125)]. In summary, impedance and power and energy are all fundamentally related.

4.4.3 Lec 28: Cauchy Integral Formula & Residue Theorem

The Cauchy integral formula (Eq. 4.34) is an important extension of the Cauchy integral theorem (Eq. 4.33) in that a pole has been explicitly injected into the integrand at $s = s_o$. If the pole location is outside of the curve $\mathcal{C}$, the result of the integral is zero, in keeping with Eq. 4.33. When the pole is inside of $\mathcal{C}$, the integrand is no longer complex analytic at the enclosed pole. When this pole is simple, the residue theorem applies. By a manipulation of the contour in Eq. 4.34, the pole can be isolated with a circle around the pole, and then taking the limit, the radius may be taken to zero, in the limit, isolating the pole.

For the related *Cauchy residue theorem* (Eq. 4.35) the same result holds, except it is assumed that there are K simple poles in the function $F(s)$. This requires the repeated application of Eq. 4.34, K times, so it represents a minor extension of Eq. 4.34. The function $F(s)$ may be written as $f(s)/P_K(s)$, where $f(s)$ is analytic in $\mathcal{C}$ and $P_K(s)$ is a polynomial of degree K, with all of its roots $s_k \in \mathcal{C}$.

Non-integral degree singularities: The key point is that this theorem applies when $n \in \mathbb{I}$, including fractionals $n \in \mathbb{F}$, but for these cases the residue is always zero, since by definition, the residue is the amplitude of the $1/s$ term (Boas, 1987, p. 73).

Examples:

1. When $n \in \mathbb{F}$ (e.g., $n = 2/3$), the residue is, by definition, zero.

2. The function $1/\sqrt{s}$ has a zero residue (apply the definition of the residue Eq. 4.36).

3. When $n \neq 1 \in \mathbb{I}$, the residue is, by definition, zero.

4. When $n = 1$, the residue is given by Eq. 4.36.

5. This method is necessary when computing the inverse Laplace transform.

4.5   Lec 29: Inverse Laplace transform ($t < 0$): Cauchy residue theorem

The inverse Laplace transform Eq. 3.47 transforms a function of complex frequency $F(s)$ and returns a causal function of time $f(t)$

$$f(t) \leftrightarrow F(s),$$

where $f(t) = 0$ for $t < 0$. Examples are provided in Table I.2 (p. 282). We next discuss the details of finding the inverse transform by use of the Cauchy residue theorem, and how the causal requirement $f(t < 0) = 0$ comes about.

The integrand of the inverse transform is $F(s)e^{st}$ and the limits of integration are $-\sigma_o \mp \omega \jmath$. To find the inverse we must close the curve, at infinity, and show that the integral at $\omega \jmath \to \infty$. There are two ways to close these limits – to the right $\sigma > 0$ (RHP), and to the left $\sigma < 0$ (LHP) – but there needs to be some logical reason for this choice. That logic is the sign of t. For the integral to converge, the term e^{st} must go to zero as $\omega \to \infty$. In terms of the real and imaginary parts of $s = \sigma + \omega \jmath$, the exponential may be rewritten as $e^{\sigma t}e^{\jmath \omega t}$. Note that both t and ω go to ∞. So it is the interaction between these two limits that determines how we pick the closure, RHP vs. LHP.

Case for causality ($t < 0$): Let us first consider negative time, including $t \to -\infty$. If we were to close $\mathcal{C}$ in the left half-plane ($\sigma < 0$), then the product σt is positive ($\sigma < 0, t < 0$, thus $\sigma t > 0$). In this case as $\omega \to \infty$, the closure integral $|s| \to \infty$ will diverge. Thus we may not close in the LHP for negative time. If we close in the RHP $\sigma > 0$ then the product $\sigma t < 0$ and e^{st} will go to zero as $\omega \to \infty$. This then justifies closing the contour, allowing for the use of the Cauchy theorems.

If $F(s)$ is analytic in the RHP, the FTCC applies, and the resulting $f(t)$ must be zero, and the inverse Laplace transform must be causal. This argument holds for any $F(s)$ that is analytic in the RHP ($\sigma > 0$).

Case of unstable poles: An important but subtle point arises: If $F(s)$ has a pole in the RHP, then the above argument still applies if we pick σ_o to be to the right of the RHP pole. This means that the inverse transform may still be applied to unstable poles (those in the RHP). This explains the need for the σ_o in the limits. If $F(s)$ has no RHP poles, then $\sigma_o = 0$ is adequate, and this factor may be ignored.

5   **Case for zero time** ($t = 0$):   When time is zero, the integral does not, in general, converge, leaving $f(t)$ undefined. This is most clear in the case of the step function $u(t) \leftrightarrow 1/s$, where the integral may not be closed, because the convergence factor $e^{st} = 1$ is lost for $t = 0$.

   The fact that $u(t)$ does not exist at $t = 0$ explains the Gibbs phenomenon in the inverse Fourier transform. At times where a jump occurs, the derivative of the function does not exist, and thus the time

10 response function is not analytic. The Fourier expansion cannot converge at places where the function is not analytic. A low-pass filter may be used to smooth the function, but at the cost of temporal resolution. Forcing the function to be analytic at the discontinuity, by smoothing the jumps, is an important computational method.

4.5.1   Lec 30: **Inverse Laplace transform** ($t > 0$)

15 **Case of $t > 0$:**   Next we investigate the convergence of the integral for positive time $t > 0$. In this case we must close the integral in the LHP ($\sigma < 0$) for convergence, so that $st < 0$ ($\sigma \le 0$ and $t > 0$). When there are poles on the $\omega \jmath = 0$ axis, $\sigma_o > 0$ assures convergence by keeping the on-axis poles inside the contour. At this point the Cauchy residue theorem (Eq. 4.35) is relevant. If we restrict ourselves to simple poles (as required for a Brune impedance), the residue theorem may be directly applied.

   The most simple example is the step function, for which $F(s) = 1/s$ and thus

$$ u(t) = \oint_{\text{LHP}} \frac{e^{st}}{s} \frac{ds}{2\pi\jmath} \leftrightarrow \frac{1}{s}, $$

20 which is a direct application of the Cauchy residue theorem, Eq. 4.35 (p. 166). The forward transform of $u(t)$ is straightforward, as discussed in Section 3.7 (p. 131). This is true of most if not all of the elementary forward Laplace transforms. In these cases, causality is built into the integral by the limits, so it is not a result, as it must be in the inverse transform. An interesting problem is to prove that $u(t)$ is not defined at $t = 0$.
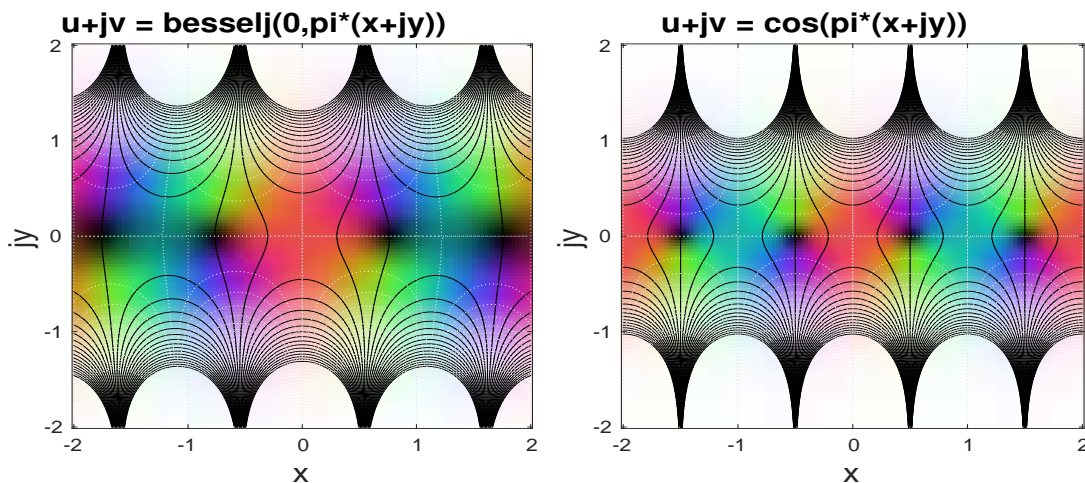
Figure 4.9: *Left: Colorized plot of $w(z) = J_o(\pi z)$. The first zero is at 2.405, and thus appears at $0.7655 = 2.405/\pi$, somewhat larger than the root of $\cos(\pi/2)$. Right: Note the similarity to $w(z) = \sin(z)$. The $\mathcal{LT}$'s have similar characteristics, as documented in Table I.2 (p. 282).*

The inverse Laplace transform of $F(s) = 1/(s + 1)$ has a residue of 1 at $s = -1$, thus that is the

only contribution to the integral. More demanding cases are Laplace transform pairs

$$\frac{1}{\sqrt{t}}u(t) \leftrightarrow \sqrt{\frac{\pi}{s}} \quad \text{and} \quad J_o(t)u(t) \leftrightarrow \frac{1}{\sqrt{s^2 + 1}},$$

and more on p. 282. Many of these are easily proved in the forward direction, but are much more difficult in the inverse direction, due to the properties at $t = 0$, unless of course the residue theorem (Eq. 4.35, p. 166) is invoked. The last $\mathcal{LT}$-pair helps us understand the basic nature of the Bessel functions $J_0(z)$ and $H_0^{(1)}(z^2)$, with a branch cut along the negative axis (Allen, 2019).

The form $\sqrt{s}$ is called a semi-inductor (Kim and Allen, 2013), also know as the *skin effect* in EM   5
theory. The form $1/\sqrt{s}$ is a semi-capacitor.

Some open questions: Without the use of the CRT (Eq. 4.35) it is difficult to see how to evaluate the inverse Laplace transform of $1/s$ directly. For example, how does one show that the above integral is zero for negative time (or that it is 1 for positive time)? The CRT neatly solves this difficult problem by the convergence of the integral for negative and positive time. Clearly the continuity of the integral at   10
$\omega \to \infty$ plays an important role. Perhaps the Riemann sphere plays a role in this that has not yet been explored.

4.5.2   Lec 31: Properties of the LT (e.g., Linearity, convolution, time-shift, modulation, etc.)

As shown in the table of Laplace transforms, there are integral (i.e., integration, not integer) relationships, or properties, that are helpful to identify. The first of these is a definition, not a property: Add notation to Appdx A.

$$f(t) \leftrightarrow F(s).$$

When taking the LT, the time response is given in lower case (e.g., $f(t)$) and the frequency domain   15
transform is denoted in upper case (e.g., $F(s)$). It is required, but not always explicitly specified, that $f(t < 0) = 0$, that is, the time function must be *causal* (P1: Section 3.7.1).

Linearity: A *key property* so basic that it is almost forgotten, is the linearity property of the LT, summarized in P2 of Section 3.7.1 (p. 134).

Convolution property: One of the most basic and useful properties is that the product of two LTs in frequency results in convolution in time

$$f(t) \star g(t) = \int_0^t f(\tau)g(t - \tau)d\tau \leftrightarrow F(s)G(s),$$

where we use the $\star$ operator as a shorthand for the convolution of two time functions.   20

A key application of convolution is filtering, which takes many forms. The most basic filter is the *moving average*, the moving sum of data samples, normalized by the number of samples. Such a filter has very poor performance. It also introduces a delay of half the length of the average, which may or may not constitute a problem, depending on the application. Another important example is a low-pass filter that removes high-frequency noise, or a notch filter that removes line-noise (i.e., 60 [Hz] in the US,   25
and its 2nd and 3d harmonics, 120 and 180 [Hz]). Such noise is typically a result of poor grounding and ground loops. It is better to solve the problem at its root than to remove it with a notch filter. Still, filters are very important in engineering.

By taking the LT of the convolution we can derive this relationship:

$$\int_0^\infty [f(t) \star g(t)]e^{-st}dt = \int_{t=0}^\infty \left[\int_0^t f(\tau)g(t-\tau)d\tau \right] e^{-st}dt$$

$$= \int_0^t f(\tau) \left(\int_{t=0}^\infty g(t-\tau)e^{-st}dt \right) d\tau$$

$$= \int_0^t f(\tau) \left(e^{-s\tau} \int_{t'=0}^\infty g(t')e^{-st'}dt' \right) d\tau$$

$$= G(s) \int_0^t f(\tau)e^{-s\tau}d\tau$$

$$= G(s)F(s).$$

We first encountered this relationship in Section **??** (p. **??**) in the context of multiplying polynomials, which was the same as convolving their coefficients. Hopefully the parallel is obvious. In the case of polynomials, the convolution was discrete in the coefficients, and here it is continuous in time. But the relationships are the same.

Time-shift property: When a function is time-shifted by time T_o, the LT is modified by e^{sT_o}, leading to the property

$$f(t - T_o) \leftrightarrow e^{-sT_o} F(s).$$

5   This is easily shown by applying the definition of the LT to a delayed time function.

Time derivative: The key to the eigen-function analysis provided by the LT is the transformation of a time derivative on a time function, that is,

$$\frac{d}{dt}f(t) \leftrightarrow sF(s).$$

Here s is the eigenvalue corresponding to the time derivative of e^{st}. Given the definition of the derivative of e^{st} with respect to time, this definition seems trivial. Yet that definition was not obvious to Euler. It needed to be extended to the space of complex analytic function e^{st}, which did not happen until at least Riemann (1851).

10   *Given a differential equation of order K, the LT results in a polynomial in s, of degree K.* It follows that this LT property is the corner-stone of why the LT is so important to scalar differential equations, as it was to the early analysis of Pell's equation and the Fibonacci sequence, as presented in earlier chapters. This property was first uncovered by Euler. It is not clear if he fully appreciated its significance, but by the time of his death, it certainly would have been clear to him. Who first coined the terms *eigenvalue*

15   and *eigen-function*? The word *eigen* is a German word meaning *of one*.

Initial and final value theorems: There are much more subtle relations between $f(t)$ and $F(s)$ that characterize $f(0^+)$ and $f(t \to \infty)$. While these properties can be very important in certain applications, they are beyond the scope of the present treatment. These relate to so-called *initial value theorems*. If the system under investigation has potential energy at $t = 0$, then the voltage (velocity) need not be zero

20   for negative time. An example is a charged capacitor or a moving mass. These are important situations, but better explored in a more in-depth treatment.

4.5.3   Solving differential equations: Method of Frobenius

Many differential equations may be solved by assuming a power series (i.e., Taylor series) solution of the form

$$y(x) = x^r \sum_{n=0}^\infty c_n x^n \tag{4.49}$$

with coefficients $c_n \in \mathbb{C}$. The *method of Frobenius* is quite general (Greenberg, 1988, p. 193).

Example: When a solution of this form is substituted into the differential equation, a recursion relation in the coefficients results. For example, if the equation is

$$y''(x) = \lambda^2 y(x)$$

the recursion is $c_n = c_{n-1}/n$. The resulting equation is

$$y(x) = e^{\lambda x} = x^0 \sum_{n=0}^{\infty} \frac{1}{n!} x^n,$$

namely $c_n = 1/n!$, thus $nc_n = 1/(n-1)! = c_{n-1}$.

Exercise: Find the recursion relation for Bessel's equation of order ν

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0.$$

Solution: If we assume a complex analytic solution of the form Eq. 4.49, we find the Bessel recursion relation for coefficients c_k to be

$$c_k = -\frac{1}{k(k + 2\nu)} c_{k-2}$$

(Greenberg, 1988, p. 231).

Appendix A

Notation

A.1 Number systems

The notation used in this book is defined in this appendix so that it may be quickly accessed.[1] Where the definition is sketchy, page numbers are provided where these concepts are fully explained, along with many other important and useful definitions. For example a discussion of $\mathbb{N}$ may be found on page 36. Math symbols such as $\mathbb{N}$ may be found at the top of the index, since they are difficult to alphabetize.

A.1.1 Units

Strangely, or not, classical mathematics, as taught today in schools, does not seem to acknowledge the concept of physical units. Units seem to have been abstracted away. This makes mathematics distinct from physics, where almost everything has units. Presumably this makes mathematics more general (i.e., abstract). But for the engineering mind, this is not ideal, or worse, as it necessarily means that important physical meaning, by design, has been surgically removed. We shall use SI units whenever possible, which means this book is not not a typical book on mathematics. Spatial coordinates are quoted in meters [m], and time in seconds [s]. Angles in degrees have no units, whereas radians have units of inverse-seconds [s^{-1}]. For a complete list of SI units, refer to https://physics.nist.gov/cuu/pdf/sp811.pdf.

A.1.2 Symbols and functions

For more on basic math notation, see to Graham *et al.* (1994).

We use $\ln$ as the log function base e, $\log$ as base 2, and π_k to indicate the kth prime (e.g., $\pi_1 = 2, \pi_2 = 3$).

When working with Fourier $\mathcal{FT}$ and Laplace $\mathcal{LT}$ transforms, lower case symbols are in the time domain while upper case indicates the frequency domain, as $f(t) \leftrightarrow F(\omega)$. An important exception is Maxwell's equations because they are so widely used as upper-case bold letters (e.g., $\boldsymbol{E}(\boldsymbol{x}, \omega)$). It seems logical to change this to conform to lower case, with $\boldsymbol{e}(\boldsymbol{x}, t) \leftrightarrow \boldsymbol{E}(\boldsymbol{x}, \omega)$ as the preferred notation.

A.1.3 Common mathematical symbols:

There are many pre-defined symbols in mathematics, too many to summarize here. We shall only use a small subset, defined here.

- A set is a collection of objects that have a common property. A set is defined using braces. For example, if set $P = \{a, b, c\}$ such that $a^2 + b^2 = c^2$, then members of P obey the Pythagorean theorem. Thus we could say that $\{1, 1, \sqrt{2}\} \in P$.

---

[1] https://en.wikipedia.org/wiki/List_of_mathematical_symbols_by_subject#Definition_symbols

- Number sets: $\mathbb{N}, \mathbb{P}, \mathbb{Z}, \mathbb{Q}, \mathbb{F}, \mathbb{I}, \mathbb{R}, \mathbb{C}$ are briefly discussed below, and in greater detail in Section 2.1 (pp. 36–37).

- One can define sets of sets and subsets of sets, and this is prone (in my experience) to error. For example, what is the difference between the number 0 and the null set $\varnothing = \{0\}$? Is $0 \in \varnothing$? Ask a mathematician. This seems a lackluster construction in the world of engineering.

- Define a vector as an n-tuple.

- The symbol $\perp$ is used in many different ways to indicate two things are perpendicular, orthogonal, or in disjoint sets. In set theory $A \perp B$ is equivalent to $A \cap B = \varnothing$. If two vectors $\boldsymbol{E}, \boldsymbol{H}$ are perpendicular $\boldsymbol{E} \perp \boldsymbol{H}$, then their inner product $\boldsymbol{E} \cdot \boldsymbol{H} = 0$ is zero. One must infer the meaning of $\perp$ from its context.

Table A.1: *List of all upper and lower case Greek letters used in the text.*

A	**A**	α	$\boldsymbol{\alpha}$	alpha	N	**N**	ν	$\boldsymbol{\nu}$	nu
B	**B**	β	$\boldsymbol{\beta}$	beta	Ξ	$\boldsymbol{\Xi}$	ξ	$\boldsymbol{\xi}$	xi
Γ	$\boldsymbol{\Gamma}$	γ	$\boldsymbol{\gamma}$	gamma	O	**O**	o	**o**	omicron
Δ	$\boldsymbol{\Delta}$	δ	$\boldsymbol{\delta}$	delta	Π	$\boldsymbol{\Pi}$	π	$\boldsymbol{\pi}$	pi
E	**E**	ε	$\boldsymbol{\varepsilon}$	epsilon	P	**P**	ρ	$\boldsymbol{\rho}$	rho
Z	**Z**	ζ	$\boldsymbol{\zeta}$	zeta	Σ	$\boldsymbol{\Sigma}$	σ	$\boldsymbol{\sigma}$	sigma
H	**H**	η	$\boldsymbol{\eta}$	eta	T	**T**	τ	$\boldsymbol{\tau}$	tau
Θ	$\boldsymbol{\Theta}$	θ	$\boldsymbol{\theta}$	theta	Υ	$\boldsymbol{\Upsilon}$	υ	$\boldsymbol{\upsilon}$	upsilon
I	**I**	ι	$\boldsymbol{\iota}$	iota	Φ	$\boldsymbol{\Phi}$	ϕ	$\boldsymbol{\phi}$	phi
K	**K**	κ	$\boldsymbol{\kappa}$	kappa	X	**X**	χ	$\boldsymbol{\chi}$	chi
Λ	$\boldsymbol{\Lambda}$	λ	$\boldsymbol{\lambda}$	lambda	Ψ	$\boldsymbol{\Psi}$	ψ	$\boldsymbol{\psi}$	psi
M	**M**	μ	$\boldsymbol{\mu}$	mu	Ω	$\boldsymbol{\Omega}$	ω	$\boldsymbol{\omega}$	omega

A.1.4   Greek letters

Frequently Greek letters, as provided in Fig. A.1, are associated in engineering and physics with a specific physical meaning. For example, ω [rad] is the radian frequency $2\pi f$, ρ [kgm/m^3] is commonly the density. ϕ, ψ are commonly used to indicate angles of a triangle, and $\zeta(s)$ is the Riemann zeta function. Many of these are so well established it makes no sense to define new terms, so we will adopt these common terms (and define them).

Likely you do not know all of these Greek letters, commonly used in mathematics. Some of them are pronounced in strange ways. The symbol ξ is pronounced "see," ζ is "zeta," β is "beta," and χ is "kie" (rhymes with pie and sky). I will assume you know how to pronounce the others, which are more phonetic in English. One advantage of learning LaTeX is that all of these math symbols are built in, and thus more easily learned, once you have adopted this powerful open-source math-oriented word-processing system (e.g., used to write this book).

A.1.5   Double-Bold notation

Table A.2 indicates the symbol followed by a page number indication where it is discussed, and the Genus (class) of the number type. For example, $\mathbb{N} > 0$ indicates the infinite set of *counting numbers* $\{1, 2, 3, \cdots\}$, not including zero. Starting from any counting number, you get the next one by adding 1. Counting numbers are also known as the *cardinal numbers*.

We say that a number is in the set with the notation $3 \in \mathbb{N} \subset \mathbb{R}$, which is read as "3 is in the set of counting numbers, which in turn is in the set of real numbers," or in vernacular language "3 is a real counting number."

Table A.2: Double-bold notation for the types of numbers. (#) is a page number. Symbol with an exponent denote the dimensionality. Thus $\mathbb{R}^2$ represents the real plane. An exponent of 0 denotes point, e.g., $\jmath \in \mathbb{C}^0$.

Symbol (p. #)	Genus	Examples	Counter Examples
$\mathbb{N}$ (36)	Counting	1,2,17,3, 10^{20}	-5, 0, π, -10, 5j
$\mathbb{P}$ (36)	Prime	2,17,3, 10^{20}	0, 1, 4, 3^2, 12, -5
$\mathbb{Z}$ (36)	Integer	-1, 0, 17, 5j, -10^{20}	1/2,π, $\sqrt{5}$
$\mathbb{Q}$ (37)	Rational	2/1, 3/2, 1.5, 1.14	$\sqrt{2}$, $3^{-1/3}$, π
$\mathbb{F}$ (37)	Fractional	1/2, 7/22	$1/\sqrt{2}$
$\mathbb{I}$ (37)	Irrational	$\sqrt{2}$, $3^{-1/3}$, π, e	Vectors
$\mathbb{R}$ (37)	Reals	$\sqrt{2}$, $3^{-1/3}$, π	$2\pi j$
$\mathbb{C}$ (37)	Complex	1, $\sqrt{2}j$, $3^{-j/3}$, π^j	Vectors
$\mathbb{G}$	Gaussian integers	$3 - 2\jmath \in \mathbb{Z} \cup \mathbb{C}$	complex integers

Prime numbers ($\mathbb{P} \subset \mathbb{N}$) are taken from the counting numbers, but do not include 1.

The signed integers $\mathbb{Z}$ include 0 and negative integers. Rational numbers $\mathbb{Q}$ are historically defined to include $\mathbb{Z}$, a somewhat inconvenient definition, since the more interesting class are the *fractionals* $\mathbb{F}$, a subset of rationals $\mathbb{F} \in \mathbb{Q}$ that exclude the integers (i.e., $\mathbb{F} \perp \mathbb{Z}$). This is a useful definition because the rationals $\mathbb{Q} = \mathbb{Z} \cup \mathbb{F}$ are formed from the union of integers and fractionals.

The rationals may be defined, using set notation (a very sloppy notation with an incomprehensible syntax), as

$$\mathbb{Q} = \{p/q : q \neq 0 \ \& \ p, q \in \mathbb{Z}\},$$

which may be read as "the set '$\{\cdots\}$' of all p/q such that ':' $q \neq 0$, 'and' $p, q \subset \mathbb{Z}$. The translation of the symbols is in single ('$\cdots$') quotes.

Irrational numbers $\mathbb{I}$ are very special: They are formed by taking a limit of fractionals, as the numerator and denominator $\to \infty$, and approach a limit point. It follows that irrational numbers must be approximated by fractionals.

The reals ($\mathbb{R}$) include complex numbers ($\mathbb{C}$) having a zero imaginary part (i.e., $\mathbb{R} \subset \mathbb{C}$).

The *size* of a set is denoted by taking the absolute value (e.g., $|\mathbb{N}|$). Normally in mathematics this symbol indicates the cardinality, so we are defining it differently from the standard notation.

Classification of numbers: From the above definitions there exists a natural hierarchical structure of numbers:

$$\mathbb{P} \in \mathbb{N}, \quad \mathbb{Z} : \{\mathbb{N}, 0, -\mathbb{N}\}, \quad \mathbb{F} \perp \mathbb{Z}, \quad \mathbb{Q} : \mathbb{Z} \cup \mathbb{F}, \quad \mathbb{R} : \mathbb{Q} \cup \mathbb{I} \subset \mathbb{C}$$

1. The primes are a subset of the counting numbers: $\mathbb{P} \subset \mathbb{N}$.

2. The signed integers $\mathbb{Z}$ are composed of $\pm\mathbb{N}$ and 0, thus $\mathbb{N} \subset \mathbb{Z}$.

3. The fractionals $\mathbb{F}$ do not include of the signed integers $\mathbb{Z}$.

4. The rationals $\mathbb{Q} = \mathbb{Z} \cup \mathbb{F}$ are the union of the signed integers and fractionals.

5. Irrational numbers $\mathbb{I}$ have the special properties $\mathbb{I} \perp \mathbb{Q}$.

6. The reals $\mathbb{R} : \mathbb{Q}, \mathbb{I}$ are the union of rationals and *irrationals* $\mathbb{I}$.

7. Reals $\mathbb{R}$ may be defined as a subset of those complex numbers $\mathbb{C}$ having zero imaginary part.

A.2   Rounding schemes

In Matlab/Octave there are five different rounding schemes (i.e., mappings): `round(x)`, `fix(x)`, `floor(x)`, `ceil(x)`, `roundb(x)`, with $x \in \mathbb{R}$. Function `round(π)=3` (e.g. $3 = \lfloor \pi \rfloor$) rounds to the nearest integer, whereas `floor(π)` (i.e., $3 = \lfloor \pi \rfloor$) rounds down and `ceil(π)` rounds up (i.e.,
5    $4 = \lceil \pi \rceil$).

Note `round()` $\equiv \lfloor \pi \rceil$ introduces negative remainders, when ever a number rounds up. The expressions in brackets is the notation for the continued fraction expansion (p. 57).

A.3   Periodic functions

Fourier series tells us that periodic functions are discrete in frequency, with frequencies given by nT_x,
10   where T_s is the sample period ($T_s = 1/2F_{max}$ and $F_{min} = F_{max}/\text{NFT}$).

This concept is captured by the *Fourier series*, which is a frequency expansion of a periodic function. This concept is quite general. Periodic in frequency implies discrete in time. Periodic and discrete in time requires periodic and discrete in frequency (the case of the DFT). The modulo function $m \bmod n$ is periodic with period n.

A periodic function may be conveniently indicated using double-parentheses notation. This is sometimes known as modular arithmetic. For example,

$$f((t))_T = f(t) = f(t \pm kT)$$

15   is periodic on $t, T \in \mathbb{R}$ with a period of T and $k \in \mathbb{Z}$. This notation is useful when dealing with Fourier series of periodic functions.

When a discrete valued (e.g., time $t \in \mathbb{N}$) sequence is periodic with period $N \in \mathbb{Z}$, we use square brackets

$$f[[n]]_N = f[n] = f[n \pm kN],$$

with $k \in \mathbb{Z}$. This notation will be used with discrete-time signals that are periodic, such as the case of the DFT.

A.4   Differential equations vs. polynomials

A polynomial has *degree* N defined by the largest power. A quadratic equation is degree 2, and a cubic has degree 3. We shall indicate a polynomial by the notation

$$P_N(z) = z^N + a_{N-1}z^{N-1} \cdots a_0.$$

20   It is a good practice to normalize the polynomial so that $a_N = 1$. This will not change the roots, defined by Eq. 3.7 (p. 76). The coefficient on z^{N-1} is always the sum of the roots z_n ($a_{N-1} = \sum_n^N z_n$), and a_0 is always their product ($a_0 = \prod_n^N z_n$).

Differential equations have *order* (polynomials have degree). If a second-order differential equation is Laplace transformed (Lec. 3.7, p. 131), one is left with a degree 2 polynomial.

Example:

$$\frac{d^2}{dt^2}y(t) + b\frac{d}{dt}y(t) + cy(t) = \alpha\left(\frac{d}{dt}x(t) + \beta x(t)\right) \leftrightarrow \tag{A.1}$$

$$(s^2 + bs + c)Y(s) = \alpha(s + \beta)X(s). \tag{A.2}$$

$$\frac{Y(s)}{X(s)} = \alpha\frac{s + \beta}{s^2 + bs + c} \equiv H(s) \leftrightarrow h(t). \tag{A.3}$$

Using the same argument as for polynomials, the lead coefficient must always be 1. The coefficient $\alpha \in \mathbb{R}$ is called the *gain*. The complex variable s is the *Laplace frequency*.

Discuss half-derivatives?

The ratio of the output $Y(s)$ over the input $X(s)$ is called the system *transfer function $H(s)$*. When $H(s)$ is the ratio of two polynomials in s, the transfer function is said to be *bilinear*, since it is linear in both the input and output. The roots of the numerator are called the *zeros* and those of the denominator, the *poles*. The inverse Laplace transform of the transfer function is called the system *impulse response*, which describes the system's output signal $y(t)$ for any given input signal $x(t)$, via convolution (i.e., $y(t) = h(t) \star x(t)$).

5

Appendix D

Symbolic analysis of $TE = E\Lambda$

D.1 General case

Here we assume

$$T = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$$

with $\Delta_T = 1$.

The eigenvectors $e_\pm$ of T are

$$e_\pm = \begin{pmatrix} \frac{1}{2\mathcal{C}} \left[(\mathcal{A} - \mathcal{D}) \mp \sqrt{(\mathcal{A} - \mathcal{D})^2 + 4\mathcal{B}\mathcal{C}} \right] \\ 1 \end{pmatrix} \tag{D.1}$$

and eigenvalues are

$$\lambda_\pm = \frac{1}{2} \begin{pmatrix} (\mathcal{A} + \mathcal{D}) - \sqrt{(\mathcal{A} - \mathcal{D})^2 + 4\mathcal{B}\mathcal{C}} \\ (\mathcal{A} + \mathcal{D}) + \sqrt{(\mathcal{A} - \mathcal{D})^2 + 4\mathcal{B}\mathcal{C}} \end{pmatrix}. \tag{D.2}$$

D.1.1 Case of the Transmission matrix:

For the case of the ABCD matrix the eigenvalues depend on reciprocity, since $\Delta_T = 1$ if $T(s)$ is reciprocal, and $\Delta_T = -1$ if it is anti-reciprocal. Thus it is helpful to display the eigenfunctions and values in terms of Δ_T so this distinction is explicit.

The term under the radical (i.e., the discriminant) may be rewritten in terms of the determinant of T (i.e., $\Delta_T = \mathcal{A}\mathcal{D} - \mathcal{B}\mathcal{C}$), since

$$(\mathcal{A} - \mathcal{D})^2 - (\mathcal{A} + \mathcal{D})^2 = -4\mathcal{A}\mathcal{D}.$$

The for the ABCD matrix the expression under the radical becomes

$$(\mathcal{A} - \mathcal{D})^2 + 4\mathcal{B}\mathcal{C} = \mathcal{A}^2 + \mathcal{D}^2 - 4\mathcal{A}\mathcal{D} + 4\mathcal{B}\mathcal{C}$$
$$= \mathcal{A}^2 + \mathcal{D}^2 - 4\Delta_T.$$

Rewriting the eigenvectors and eigenvalues in terms of Δ_T we find

$$e_\pm = \begin{pmatrix} \frac{1}{2\mathcal{C}} \left[(\mathcal{A} - \mathcal{D}) \mp \sqrt{(\mathcal{A} + \mathcal{D})^2 - 4\Delta_T} \right] \\ 1 \end{pmatrix} \tag{D.3}$$

and eigenvalues are

$$\lambda_\pm = \frac{1}{2} \begin{pmatrix} (\mathcal{A} + \mathcal{D}) - \sqrt{(\mathcal{A} + \mathcal{D})^2 - 4\Delta_T} \\ (\mathcal{A} + \mathcal{D}) + \sqrt{(\mathcal{A} + \mathcal{D})^2 - 4\Delta_T} \end{pmatrix}. \tag{D.4}$$

For the reversible transmission matrix:

When $\mathcal{A} = \mathcal{D}$ the transmission matrix is said to be reversible, thus

$$\lambda_{\pm} = \left(\begin{array}{c} \mathcal{A} - \sqrt{\mathcal{B}\mathcal{C}} \\ \mathcal{A} + \sqrt{\mathcal{B}\mathcal{C}} \end{array} \right). \tag{D.5}$$

Example: An application of the above results may be found in the Train problem (p. 183).

D.2   Special cases having symmetry

Each 2x2 matrix has four entries, each of which can be complex. This leads to 4x2=8 possible special symmetries (an eight-fold way), discussed next, in quasi-order of their importance. Each symmetry is related to properties of $\boldsymbol{T}$. For example if $\boldsymbol{T} = \boldsymbol{T}^{\dagger}$, the matrix is said to have Hermitian symmetry. When $\boldsymbol{T} = \boldsymbol{T}^{T}$ the matrix is symmetric, and when $\boldsymbol{T} = -\boldsymbol{T}^{T}$ is is said to be *skew-symmetric*. Each of these eight-fold symmetries corresponds to some sort of physical constraint, as discussed below.

D.2.1   Reversible systems

When the values of $\boldsymbol{T}$ on its diagonal are equal ($\mathcal{A} = \mathcal{D}$), the matrix is called *reversible*, and the eigenvectors and eigenvalues greatly simplify to

$$\boldsymbol{E} = \begin{bmatrix} -\sqrt{\frac{\mathcal{B}}{\mathcal{C}}} & +\sqrt{\frac{\mathcal{B}}{\mathcal{C}}} \\ 1 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \mathcal{A} - \sqrt{\mathcal{B}\mathcal{C}} & 0 \\ 0 & \boldsymbol{A} + \sqrt{\mathcal{B}\mathcal{C}} \end{bmatrix}. \tag{D.6}$$

Reversibility (P7) is a common symmetry. Its utility is in describing transmission lines as discussed in Sections 3.5 (p. 123), **??** and 4.5.4 (pp **??**-182).

D.2.2   Reciprocal systems

When the matrix is symmetric ($\mathcal{B} = \mathcal{C}$), the corresponding system is said to be *reciprocal*. Most physical systems are reciprocal. The determinant of the transmission matrix of a reciprocal network $\Delta_T = \mathcal{A}\mathcal{D} - \mathcal{B}\mathcal{C} = 1$. For example, electrical networks composed of inductors, capacitors and resistors are always reciprocal. It follows that the complex impedance matrix is symmetric (Van Valkenburg, 1964a).

Magnetic systems such as dynamic loudspeakers are anti-reciprocal, and correspondingly $\Delta_T = -1$. The impedance matrix of a loudspeaker is skew symmetric (Kim and Allen, 2013). All impedance matrices are either symmetric or anti-symmetric, depending on whether they are reciprocal (LRC networks) or anti-reciprocal (magnetic networks). These systems have complex eigenvalues with negative real parts, corresponding to lossy systems. In some sense, all of this follows from conservation of energy, but the precise general case is waiting for enlightenment. The impedance matrix is never Hermitian. It is easily proved that Hermitian matrices have real eigenvalues, which correspond to lossless networks. Any physical system of equations that has any type of loss cannot be Hermitian.

In summary, given a reciprocal system, the $\boldsymbol{T}$ matrix has $\Delta_T = 1$, and the corresponding impedance matrix is symmetric (*not* Hermitian).

D.3   Impedance matrix

As previously discussed in Section 3.5 (p. 123), the $\boldsymbol{T}$ matrix corresponding to an impedance matrix is

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \boldsymbol{Z}(s) \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{\mathcal{C}} \begin{bmatrix} \mathcal{A} & \Delta_T \\ 1 & \mathcal{D} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}.$$

Reciprocal systems have skew-symmetric impedance matrices, namely $z_{12} = z_{21}$. When the system is both reversible $\mathcal{A} = \mathcal{D}$ and reciprocal, the impedance matrix simplifies to

$$\boldsymbol{Z}(s) = \frac{1}{\mathcal{C}} \begin{bmatrix} \mathcal{A} & 1 \\ 1 & \mathcal{A} \end{bmatrix}.$$

For such systems there are only two degrees of freedom, $\mathcal{A}$ and $\mathcal{C}$. As discussed previously in Sect. 3.5 (p. 123), each of these has a physical meaning: $1/\mathcal{A}$ is the Thévenin source voltage given a voltage drive and $\mathcal{B}/\mathcal{C}$ is the Thévenin impedance.

D.3.1 Transmission matrices and symmetry

The transmission matrix fully characterizes a two-port network (Sect. 3.5, p. 123).   5

D.3.2 Hermitian symmetry

When a system is Hermitian its matrix is conjugate symmetric

$$Z(s) = Z^{\dagger}(s),$$

a stronger condition than reciprocal, but not the symmetric symmetry of a Brune impedance matrix. Impedance is not Hermitian unless the diagonal elements are real, but it does have *symmetric* symmetry.

In the case of a Hermitian matrix, the eigenvalues are always real. To show this start from the definition of an impedance eigen-equation ($\mathbf{V}$ is a vector of voltages, $\mathbf{I}$ a current vector and $\mathbf{Z}$ an impedance matrix)

$$\mathbf{V} = \mathbf{ZI} = \mathbf{I}\boldsymbol{\Lambda}; \qquad \mathcal{P} = \mathbf{V}'\mathbf{I} = \mathbf{I}'\boldsymbol{\Lambda}\mathbf{I},$$

where $\mathbf{Z}, \mathbf{I}, \mathbf{V}, \boldsymbol{\Lambda} \in \mathbb{C}$, $A = A^{\dagger}$ is a square conjugate-symmetric matrix, and $\mathbf{I}, \mathbf{V}$ are vectors of the size of $\mathbf{Z}$ and $\mathcal{P}$ is the power. Here $\mathbf{Z}^{\dagger}$ is the complex transpose (see Appendix A, p. 253).   Move this disc to HW

D.3.3 Double roots

For the 2x2 case of double roots the matrix has Jordan form

$$\boldsymbol{T} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

Then

$$\boldsymbol{T}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}.$$

This generalizes to nxn matrices having arbitrary combinations of degeneracies (multiple roots), as in   Needs work
symmetric (square) drums, for example ....

Octave vs. Matlab seeming conflict

Using Octave (version 4.2.2, June 21, 2018):

$$e_{1,1} = \frac{-b}{\frac{a}{2} - \frac{d}{2} + \frac{1}{2}\sqrt{a^2 - 2ad + 4bc + d^2}} = \frac{-2b}{(a-d) + \sqrt{(a-d)^2 + 4bc}}.$$

This looks very different from the Matlab output (above).   15
Here is what I get with the same command, from Matlab (June 21, 2018):

$$e_{1,1} = \frac{\frac{a}{2} + \frac{d}{2} - \frac{\sqrt{a^2 - 2\,a\,d + d^2 + 4\,b\,c}}{2}}{c} - \frac{d}{c} = \frac{1}{2c}\left[(a-d) - \sqrt{(a-d)^2 + 4\,b\,c}\right].$$

Here we derive the eigenmatrix $\boldsymbol{E}$, and eigenvalue matrix Λ given a 2x2 transmission matrix

$$\boldsymbol{T} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix},$$

such that $\boldsymbol{TE} = \boldsymbol{E}\Lambda$, using symbolic algebra methods, given by the Matlab/Octave's script

```
syms A B C D T E L %Use symbolic Matlab/Octave
   T=[A B;C D]      %Given matrix T
   [E,L]=eig(T)     %Find eigenvector matrix E and
5                    %eigenvalue matrix L
```

These results have been numerically verified to be the same using `CkEig.m`. Conclusions from the numerical experiment with ($A = [1, 2; 3, 5]$) are:

1. The Octave and Matlab symbolic formulas give the same numeric results for eigenmatrix and eigenvalues.

2. The symbolic and numeric results correctly diagonalize the matrix, that is $\Gamma = E^{-1}AE$.

3. The numeric equations are normalized to 1: $norm(e_{\pm}) = 1$.

4. The symbolic Eigenvectors are not normalized to 1, rather $E(2, :) = 1$.

5. When the numeric result is normalized by the lower element (making it 1) all results agree.

Appendix I

Tables of Laplace Transforms

Properties of the Laplace Transform

1. Time $t \in \mathbb{R}$ [s] and *Laplace frequency* [rad] are defined as $s = \sigma + \omega \jmath \in \mathbb{C}$.

2. Given a *Laplace transform* ($\mathcal{LT}$) pair $f(t) \leftrightarrow F(s)$, in the engineering literature, the *time domain* is always lower case [$f(t)$] and causal (i.e., $f(t < 0) = 0$) and the *frequency domain* is upper-case [e.g. $F(s)$]. Maxwell's venerable equations are the unfortunate exception to this otherwise universal rule.

3. The target time function $f(t < 0) = 0$ (i.e., it must be causal). The time limits are $0^- < t < \infty$. Thus the integral must start from slightly below $t = 0$ to integrate over a delta function at $t = 0$. For example if $f(t) = \delta(t)$, the integral must include both sides of the impulse. If you wish to include non-causal functions such as $\delta(t + 1)$, it is necessary to extend the lower time limit. In such cases simply set the lower limit of the integral to $-\infty$, and let the integrand ($f(t)$) determine the limits.

4. When taking the forward transform ($t \to s$), the sign of the exponential is negative. This is necessary to assure that the integral converges when the integrand $f(t) \to \infty$ as $t \to \infty$. For example, if $f(t) = e^t u(t)$ (i.e., without the negative σ exponent), the integral does not converge.

5. The limits on the integrals of the forward transform are $t : (0^-, \infty) \in \mathbb{R}$, and the reverse LTs are $[\sigma_o - \infty \jmath, \sigma_o + \infty \jmath] \in \mathbb{C}$. These limits will be further discussed in Section 4.5.2 (p. 179).

6. When taking the inverse Laplace transform, the normalization factor of $1/2\pi\jmath$ is required to cancel the $2\pi\jmath$ in the differential ds of the integral.

7. The frequency for the LT must be complex, and in general $F(s)$ is complex analytic for $\sigma > \sigma_o$. It follows that the real and imaginary parts of $F(s)$ are related. Given $\Re\{F(s)\}$ it is possible to find $\Im\{F(s)\}$ (Boas, 1987). More on this in Section 3.7 (p. 131).

8. To take the inverse Laplace transform, we must learn how to integrate in the complex s plane. This will be explained in Sections 4.4-4.5.2 (p. 165-179).

9. The Laplace step function is defined as

$$u(t) = \int_{-\infty}^{t} \delta(t)dt = \begin{cases} 1 & \text{if } t > 0 \\ \text{NaN} & t = 0 \\ 0 & \text{if } t < 0 \end{cases}.$$

Alternatively one could define $\delta(t) = du(t)/dt$.

10. It is easily shown that $u(t) \leftrightarrow 1/s$ by direct integration

$$F(s) = \int_0^\infty u(t)\, e^{-st} dt = -\left.\frac{e^{-st}}{s}\right|_o^\infty = \frac{1}{s}.$$

With the LT step $(u(t))$ there is no Gibbs ringing effect.

11. In many physical applications, the Laplace transform takes the form of a ratio of two polynomials. In such case the roots of the numerator polynomial are called the *zeros* while the roots of the denominator polynomial are called the *poles*. For example the LT of $u(t) \leftrightarrow 1/s$ has a pole at $s = 0$, which represents integration, since

$$u(t) \star f(t) = \int_{-\infty}^r f(\tau) d\tau \leftrightarrow \frac{F(s)}{s}.$$

12. The LT is quite different from the FT in terms of its analytic properties. For example, the step function $u(t) \leftrightarrow 1/s$ is complex analytic everywhere, except at $s = 0$. The FT of $1 \leftrightarrow 2\pi\widetilde{\delta}(\omega)$ is not analytic anywhere.

13. Dilated step function ($a \in \mathbb{R}$)

$$u(at) \leftrightarrow \int_{-\infty}^\infty u(at)e^{-st} dt = \frac{1}{a} \int_{-\infty}^\infty u(\tau)e^{-(s/a)\tau} d\tau = \frac{a}{|a|}\frac{1}{s} = \pm\frac{1}{s},$$

where we have made the change of variables $\tau = at$. The only effect that a has on $u(at)$ is the sign of t, since $u(t) = u(2t)$. However $u(-t) \neq u(t)$, since $u(t) \cdot u(-t) = 0$, and $u(t) + u(-t) = 1$, except at $t = 0$, where it is not defined.

Once complex integration in the complex plane has been defined (Section 4.2.2, p. 149), we can justify the definition of the inverse LT (Eq. 3.46).[1]

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[1] https://en.wikipedia.org/wiki/Laplace_transform#Table_of_selected_Laplace_transforms

Table I.1: *Functional relationships between Laplace transforms.*

$\mathcal{LT}$ functional properties	
$f(t) \star g(t) = \int_{t=0}^{t} f(t-\tau)g(\tau)d\tau \leftrightarrow F(s)G(s)$	convolution
$u(t) \star f(t) = \int_{0^{-}}^{t} f(t)dt \leftrightarrow \dfrac{F(s)}{s}$	convolution
$f(at)u(at) \leftrightarrow \dfrac{1}{a}F(\dfrac{s}{a}) \qquad a \in \mathbb{R} \neq 0$	scaling
$f(t)e^{-at}u(t) \leftrightarrow F(s+a)$	damped
$f(t-T)e^{-a(t-T)}u(t-T) \leftrightarrow e^{-sT}F(s+a)$	damped and delayed
$f(-t)u(-t) \leftrightarrow F(-s)$	reverse time
$f(-t)e^{-at}u(-t) \leftrightarrow F(a-s)$	time-reversed & damped
$\dfrac{d}{dt}f(t) = \delta'(t) \star f(t) \leftrightarrow sF(s)$	deriv

Additional transforms	
$\dfrac{\sin(t)u(t)}{t} \leftrightarrow \tan^{-1}(1/s)$	half-sync

Table I.2:   *Laplace transforms of $f(t), \delta(t),\ u(t),\ \mathrm{rect}(t), T_o, p, e, \in \mathbb{R}$ and $F(s), G(s), s, a \in \mathbb{C}$.   Given a Laplace transform $(\mathcal{LT})$ pair $f(t) \leftrightarrow F(s)$, the frequency domain will always be upper-case [e.g. $F(s)$] and the time domain lower case [$f(t)$] and causal (i.e., $f(t < 0) = 0$). An extended table of transforms is given in Table I.1 on page 281.*

$f(t) \leftrightarrow F(s)$	$t \in \mathbb{R};\ \ s, F(s) \in \mathbb{C}$	Name
$\delta(t) \leftrightarrow 1$		Dirac
$\delta(\lvert a \rvert t) \leftrightarrow \dfrac{1}{\lvert a \rvert}$	$a \neq 0$	time-scaled Dirac
$\delta(t - T_o) \leftrightarrow e^{-sT_o}$		delayed Dirac
$\delta(t - T_o) \star f(t) \leftrightarrow F(s)e^{-sT_o}$		–
$\displaystyle\sum_{n=0}^{\infty} \delta(t - nT_o) = \dfrac{1}{1 - \delta(t - T_o)} \leftrightarrow \dfrac{1}{1 - e^{-sT_o}} = \sum_{n=0}^{\infty} e^{-snT_o}$		one-sided impulse train
$u(t) \leftrightarrow \dfrac{1}{s}$		step
$u(-t) \leftrightarrow -\dfrac{1}{s}$		anti-causal step
$u(at) \leftrightarrow \dfrac{a}{s}$	$a \neq 0 \in \mathbb{R}$	dilated or reversed step
$e^{-at}u(t) \leftrightarrow \dfrac{1}{s + a}$	$a > 0 \in \mathbb{R}$	damped step
$\cos(at)u(t) \leftrightarrow \dfrac{1}{2}\left(\dfrac{1}{s - a} + \dfrac{1}{s + a}\right)$	$a \in \mathbb{R}$	cos
$\sin(at)u(t) \leftrightarrow \dfrac{1}{2\jmath}\left(\dfrac{1}{s - a} - \dfrac{1}{s + a}\right)$	$a \in \mathbb{C}$	"damped" sin
$u(t - T_o) \leftrightarrow \dfrac{1}{s}e^{-sT_o}$	$T_0 > 0 \in \mathbb{R}$	time delay
$\mathrm{rect}(t) = \dfrac{1}{T_o}\left[u(t) - u(t - T_o)\right] \leftrightarrow \dfrac{1}{T_o}\left(1 - e^{-sT_o}\right)$		rect-pulse
$u(t) \star u(t) = tu(t) \leftrightarrow 1/s^2$		ramp
$u(t) \star u(t) \star u(t) = \dfrac{1}{2}t^2 u(t) \leftrightarrow 1/s^3$		double ramp
$\dfrac{1}{\sqrt{t}}u(t) \leftrightarrow \sqrt{\dfrac{\pi}{s}}$		
$t^p u(t) \leftrightarrow \dfrac{\Gamma(p + 1)}{s^{p+1}}$	$\Re p > -1, q \in \mathbb{C}$	
$J_n(\omega_o t)\, u(t) \leftrightarrow \dfrac{\left(\sqrt{s^2 + \omega_o^2} - s\right)^n}{\omega_o^n \sqrt{s^2 + \omega_o^2}}$		

Table I.3: *The following table provides an extended table of Laplace Transforms. J_0, K_1 are Bessel functions of the first and second kind.*

$f(t) \leftrightarrow F(s)$		Name
$\delta(at) \leftrightarrow \dfrac{1}{a}$		$a \neq 0$; Time-scaled Dirac
$\delta(t + T_o) \leftrightarrow e^{sT_o}$		negative delay
$u(at) \leftrightarrow \dfrac{a}{s}$		$a \neq 0$; dilate
$u(-t) \leftrightarrow -\dfrac{1}{s}$		anti-causal step
$e^{at}u(-t) \leftrightarrow \dfrac{1}{-s + a}$		anticausal damped step
$\dfrac{d^{1/2}}{dt^{1/2}} f(t)u(t) \leftrightarrow \sqrt{s}F(s)$		"half" derivative
$\dfrac{d^{1/2}}{dt^{1/2}} u(t) \leftrightarrow \sqrt{s}$		"half" derivative
$\dfrac{1}{\sqrt{\pi t}} u(t) \leftrightarrow \dfrac{1}{\sqrt{s}}$		"half" integration
$\mathrm{erfc}(\alpha\sqrt{t}) \leftrightarrow \dfrac{1}{s}e^{-2\alpha\sqrt{s}}$	(Morse-Feshbach-II, p. 1582)	$\alpha > 0$; erfc
$J_o(at)u(t) \leftrightarrow \dfrac{1}{s^2 + a^2}$		Bessel
$J_1(t)u(t)/t \leftrightarrow \sqrt{s^2 + 1} - s$		
$J_1(t)u(t)/t + 2u(t) \leftrightarrow \sqrt{s^2 + 1} + s = e^{\sinh^{-1}(s)}$		
$\delta(t) + J_1(t)u(t)/t \leftrightarrow \sqrt{s^2 + 1}$		
$I_o(t)u(t) \leftrightarrow 1/\sqrt{s^2 - 1}$		
$u(t)/\sqrt{t+1} \leftrightarrow e^s\sqrt{\dfrac{\pi}{s}}\mathrm{erfc}(\sqrt{s})$		
$\sqrt{t}u(t) * \sqrt{1 + t}u(t) \leftrightarrow e^{s/2}K_1(s/2)/2s$		

Appendix L

Figure L.1: Page 21

| 1500BCE | | |0CE | |500 | |1000 | |1400|1650 |

Chinese
Babylonia
Pythagoreans
Euclid
Archimedes
Christ
Brahmagupta
Diophantus
Leonardo
Bhaskara Bombelli
al-Khawarizmi Copernicus
|830
Marco Polo
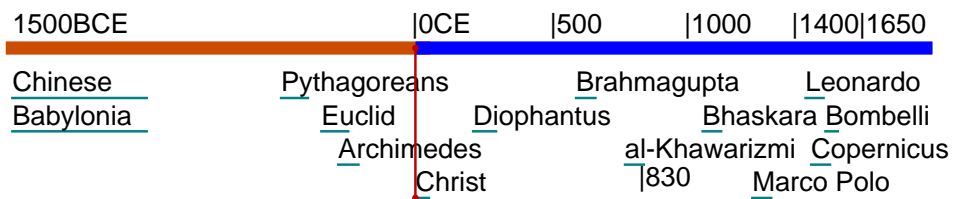
Figure L.2: Page 24

Chronological history: 16th to 19th centuries

| |1525 | |1600 | |1643 | |1700 | |1750 | |1800 | |1855 |

Bombelli
|1526
Galileo
|1564
Fermat
|1601
Descartes
|1596
Newton
Jacob Bernoulli
|1655
Johann Bernoulli
|1667
Euler
Daniel Bernoulli
d'Alembert
Gauss
Mozart
|1756
Beethoven

Figure L.3: Page 73

| |1640 | |1700 | |1750 | |1800 | |1850 | |1900 | |1950 |

Newton
Johann Bernoulli
Daniel Bernoulli
Euler
dAlembert
Mozart
Beethoven
Cauchy
Gauss
Stokes
Helmholtz
Kirchhoff
Kelvin
Riemann
Maxwell
Rayleigh
Heaviside
Poincare
Sommerfeld
Einstein
Brillouin

295

Figure L.4: Page 107

```
|1596      |1650     |1700     |1750     |1800

Descartes   Johann Bernoulli    Gauss
      Fermat              Euler         Cauchy
                       Daniel Bernoulli
                          dAlembert
         Newton             Lagrange
```

Figure L.5: Page 146

```
|1525        |1600          |1700          |1800        |1875   |1925

Bombelli            Newton                     Maxwell
       Descartes  Johann Bernoulli  Gauss         Hilbert
          Mersenne       Euler               Riemann
             Huygens   Daniel Bernoulli         Rayleigh
       Galileo      Jacob Bernoulli      Cauchy      Einstein
            Fermat            d'Alembert      Helmholtz
                               Mozart          US Civil War
```

Appendix M

Figure M.1: Page 10
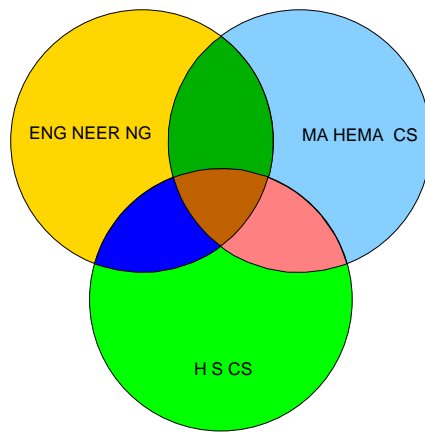
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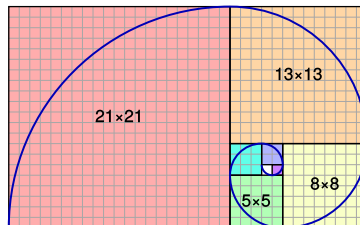
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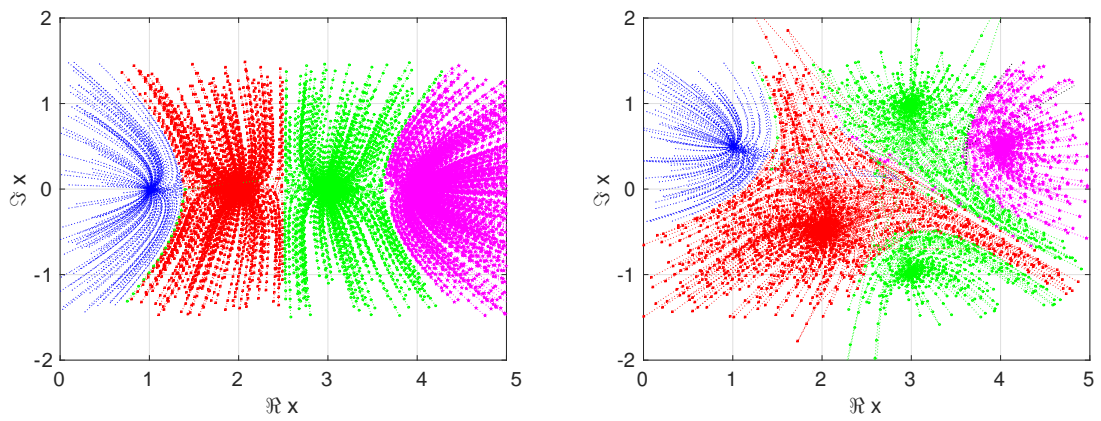
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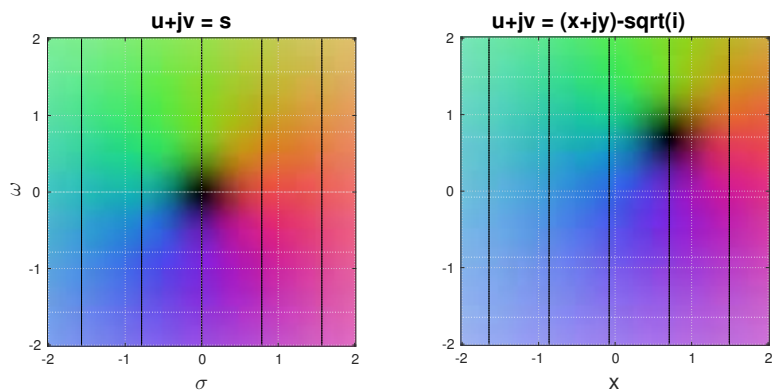
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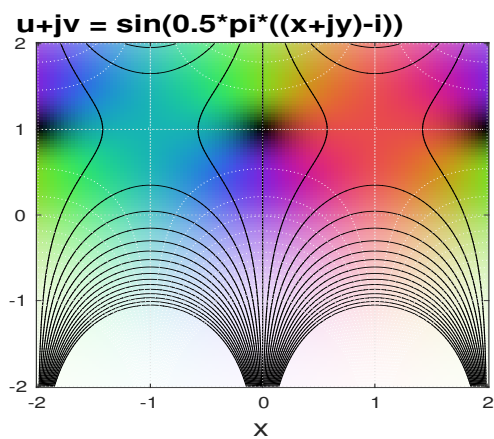
Figure M.6: Page 137

Figure M.7: Page 147

u+jv = atan((x+jy)) **u+jv = (i/2)*log((1-i*(x+jy))./(1+i*(x+jy)))**

Figure M.8: Page 161

u+jv = s.^2 **u+jv = -sqrt(-s)**
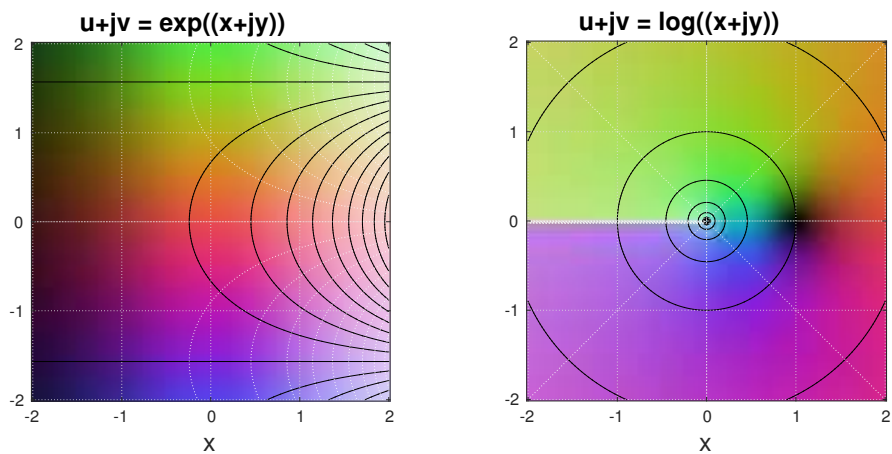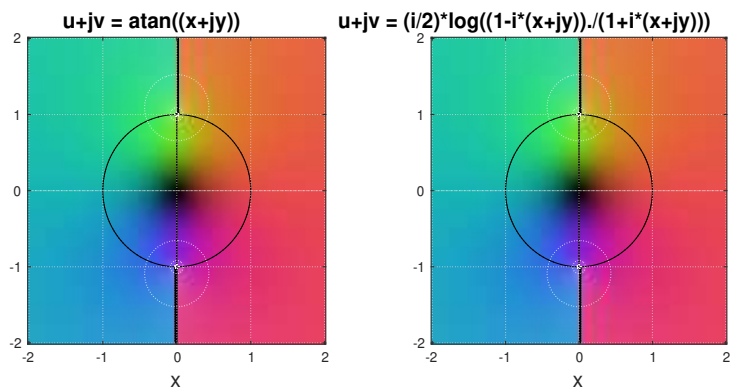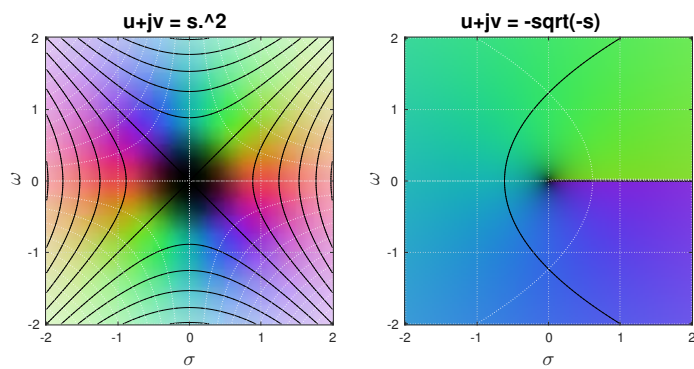
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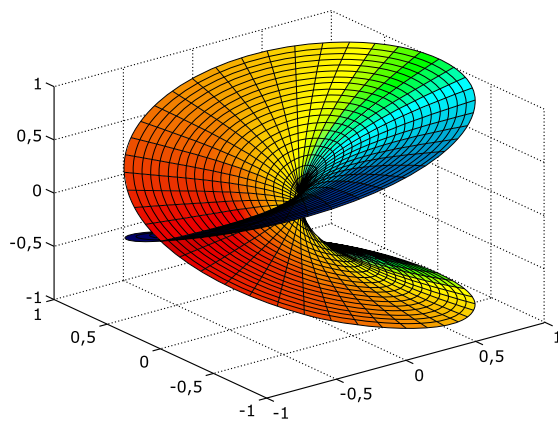
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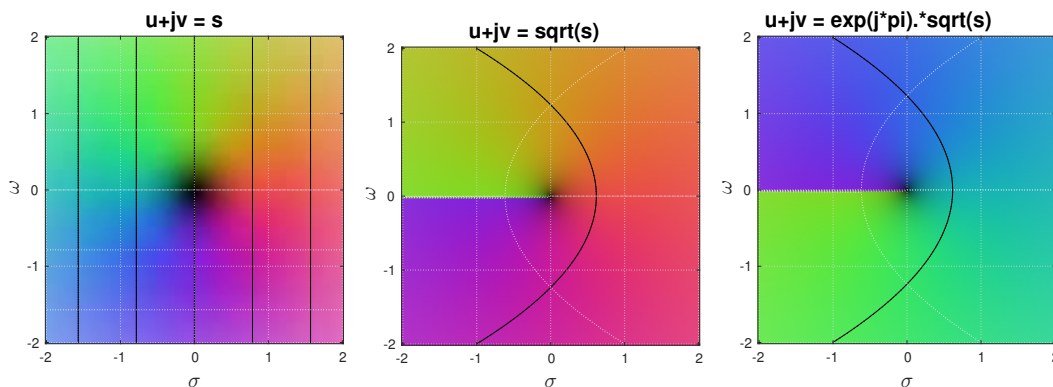
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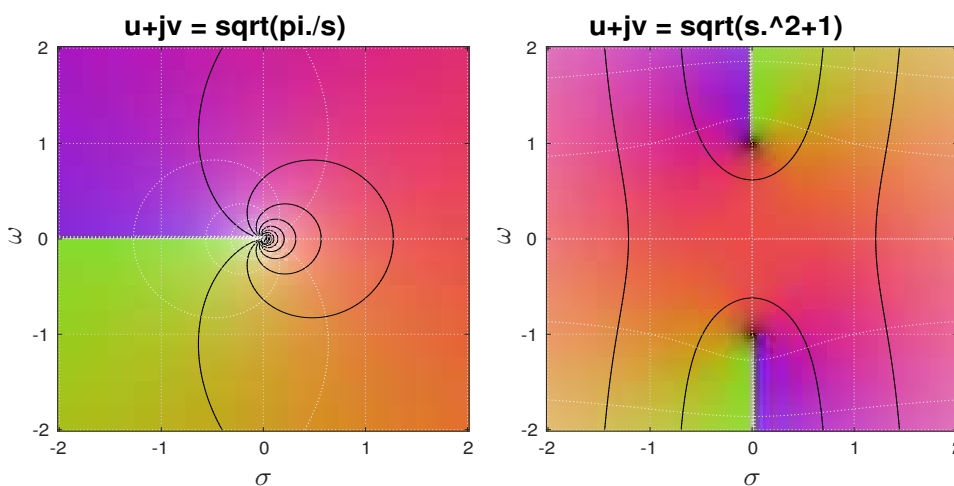
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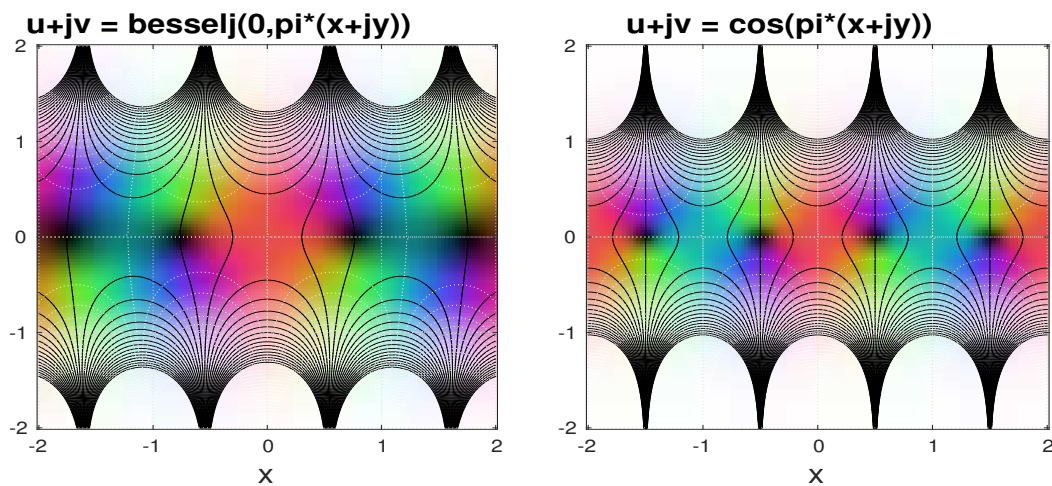
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Campbell22a (footnote 2): In that discussion, ... it is tacitly assumed that the line is either an actual
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