Problems AE-1

Polynomials and the fundamental theorem of algebra (FTA)

Problem #1: A polynomial of degree \( N \) is defined as

\[ P_N(x) = a_0 + a_1x + a_2x^2 + \cdots + a_Nx^N. \]

1.1: How many coefficients \( a_n \) does a polynomial of degree \( N \) have?
Sol: \( N + 1 \)

1.2: How many roots does \( P_N(x) \) have?
Sol: \( N \)

Problem #2: The fundamental theorem of algebra (FTA)

2.1: State and then explain the FTA.
Sol: The FTA says that every polynomial has at least one root \( x = x_r \).

2.2: Using the FTA, prove your answer to question 1.2. Hint: Apply the FTA to prove how many roots a polynomial \( P_N(x) \) of order \( N \) has.
Sol: When a root is determined, it may be factored out, leaving a new polynomial of degree one less than the first. Specifically,

\[ P_{N-1}(x) = \frac{P_N(x)}{x - x_r}. \]

Thus it follows that by a recursive application of this theorem, a polynomial has a number of roots equal to its degree. All the roots must be counted, including repeated and complex roots and roots at \( \infty \).

Problem #3: Consider the polynomial function \( P_2(x) = 1 + x^2 \) of degree \( N = 2 \) and the related function \( F(x) = 1/P_2(x) \). What are the roots (e.g., zeros) \( x_{\pm} \) of \( P_2(x) \)? Hint: Complete the square on the polynomial \( P_2(x) = 1 + x^2 \) of degree 2, and find the roots.
Sol: Solving for the roots by setting \( P_2(x) = 0 \) gives \( x_\pm = -1 \), leading to \( x_\pm = \pm 1j \).

Problem #4: \( F(x) \) may be expressed as \( (A, B, x_\pm \in \mathbb{C}) \)

\[ F(x) = \frac{A}{x - x_+} + \frac{B}{x - x_-}, \quad (1) \]

where \( x_\pm \) are the roots (zeros) of \( P_2(x) \), which become the poles of \( F(x) \); \( A \) and \( B \) are the residues. The expression for \( F(x) \) is sometimes called a partial fraction expansion or residue expansion, and it appears in many engineering applications.

4.1: Find \( A, B \in \mathbb{C} \) in terms of the roots \( x_\pm \) of \( P_2(x) \).
Sol: The fastest (i.e., easiest) way to find the constants \( A, B \) is to cross-multiply

\[ \frac{1}{1 + x^2} = \frac{A(x - x_-) + B(x - x_+)}{(x - x_+)(x - x_-)} = \frac{(A + B)x - (Ax_- + Bx_+)}{(x - x_+)(x - x_-)} \]

Since the numerator must equal 1, \( B = -A \) and \( A = 1/(x_+ - x_1) \).

In summary, in terms of the roots of Eq. 1

\[ A = -B = \frac{1}{(x_+ - x_-)}, \quad \text{thus} \quad F(x) = \frac{1}{1 + x^2} = \frac{1}{2j} \left( \frac{1}{x - 1j} - \frac{1}{x + 1j} \right). \]
– 4.2: Verify your answers for A and B by showing that this expression for \( F(x) \) is indeed equal to \( 1/P_2(x) \).

\textbf{Sol:} This is easily verified by cross-multiplying and simplifying. In the numerator the \( x \) terms cancel and Eq. 1 is recovered. ■

– 4.3: Give the values of the poles and zeros of \( P_2(x) \).

\textbf{Sol:} The zeros are at \( x_z = \pm j \), and the poles are at \( x_p = \pm \infty \) ■

– 4.4: Give the values of the poles and zeros of \( F(x) = 1/P_2(x) \).

\textbf{Sol:} The poles are at \( x_p = \pm j \), and the zeros are at \( x_z = \pm \infty \) ■

0.1 Analytic functions

\textbf{Overview:} Analytic functions are defined by infinite (power) series. The function \( f(x) \) is said to be \textit{analytic} at any value of constant \( x = x_o \), where there exists a convergent power series \( P(x) = \sum_{n=0}^{\infty} a_n(x-x_o)^n \)

such that \( P(x_o) = f(x_o) \). The point \( x = x_o \) is called the \textit{expansion point}. The region around \( x_o \) such that \( |x-x_o| < 1 \) is called the \textit{radius of convergence}, or region of convergence (RoC). The local power series for \( f(x) \) about \( x = x_o \) is defined by the Taylor series:

\[ f(x) \approx f(x_o) + \frac{df}{dx}\bigg|_{x=x_o} (x-x_o) + \frac{1}{2!} \frac{d^2f}{dx^2}\bigg|_{x=x_o} (x-x_o)^2 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n f(x)}{dx^n}\bigg|_{x=x_o} (x-x_o)^n. \]

Two classic examples are the geometric series\(^1\) where \( a_n = 1 \),

\[ \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \] (2)

and the exponential function where \( a_n = 1/n! \), Eq. ?? (p. ??). The coefficients for both series may be derived from the Taylor formula.

\textbf{Problem # 5: The geometric series}

– 5.1: What is the region of convergence (RoC) for the power series Eq. 2 of \( 1/(1-x) \) given above—for example, where does the power series \( P(x) \) converge to the function value \( f(x) \)? State your answer as a condition on \( x \). Hint: What happens to the power series when \( x > 1 \)?

\textbf{Sol:} \( |x| < 1 \) because for \( |x| \geq 1 \), the power series diverges to infinity. ■

– 5.2: In terms of the pole, what is the RoC for the geometric series in Eq. 2?

\textbf{Sol:} The nearest pole relative to the expansion point, at \( x = 0 \) is at the nearest pole \( x_p = 1 \) to the expansion point at \( x = 0 \). Namely the RoC is \( 1 \) re \( 0 \). ■

– 5.3: How does the RoC relate to the location of the pole of \( 1/(1-x) \)?

\textbf{Sol:} The pole is at \( x = 1 \), on the border of the RoC. The nearest pole relative to the expansion point, at \( x = 0 \) is at \( x = 1 \). Thus the RoC is \( 1 \). ■

– 5.4: Where are the zeros, if any, in Eq. 2?

\textbf{Sol:} There is a single zero at \( x = \infty \). ■

\(^1\)The geometric series is not defined as the function \( 1/(1-x) \), it is defined as the series \( 1 + x + x^2 + x^3 + \cdots \), such that the ratio of consecutive terms is \( x \).
5.5: Assuming \( x \) is in the RoC, prove that the geometric series correctly represents \( \frac{1}{1 - x} \) by multiplying both sides of Eq. 2 by \((1 - x)\).

**Sol:**

\[
1 = \frac{1 - x}{1 - x} \quad \text{for all } x \neq 1
\]

\[
= (1 - x)(1 + x + x^2 + x^3 + \cdots), \quad |x| < 1
\]

\[
= (1 + x + x^2 + x^3 + \cdots - x(1 + x + x^2 + \cdots))
\]

\[
= 1 + (x + x^2 + x^3 + \cdots) - (x + x^2 + x^3 + \cdots)
\]

\[
= 1 \quad \text{for all } x.
\]

The introduction of the pole introduces an added zero since \( P_N(x)|_{x=1} = N \).

If one lets \( z = 1/x \) the relation becomes

\[
1 = \frac{1 - z}{1 - z},
\]

which is valid for \( z \neq 1 \), which when expanded the RoC is \(|z| < 1 \), or \( x > 1 \). Once the removable pole and zero at \( x = 1 \) are cancelled, the solution is valid for all \( x \).

**Problem #6: Use the geometric series to study the degree \( N \) polynomial. It is very important to note that all the coefficients \( c_n \) of this polynomial are 1.**

\[
P_N(x) = 1 + x + x^2 + \cdots + x^N = \sum_{n=0}^{N} x^n. \quad (3)
\]

**6.1: Prove that**

\[
P_N(x) = \frac{1 - x^{N+1}}{1 - x}. \quad (4)
\]

**Sol:**

\[
P_N(x) = 1 + x + x^2 + \cdots x^N
\]

\[
= \sum_{n=0}^{\infty} x^n - \sum_{n=N+1}^{\infty} x^n
\]

\[
= \sum_{n=0}^{\infty} x^n - x^{N+1} \sum_{n=0}^{\infty} x^n
\]

\[
= (1 - x^{N+1}) \sum_{n=0}^{\infty} x^n
\]

\[
= \frac{1 - x^{N+1}}{1 - x}
\]

**6.2: What is the RoC for Eq. 3?**

**Sol:** There is no pole; thus the RoC is \( \infty \). This polynomial has \( N \) zeros.

**6.3: What is the RoC for Eq. 4?**

**Sol:** A polynomial has no RoC.

**6.4: How many poles does \( P_N(x) \) (Eq. 3) have? Where are they?**

**Sol:** Since \( P_N(x) \) is defined by Eq. 3, there is no poles at \( x = 1 \). However it still has a pole of order \( N \) at \( x = \infty \). To show this, define \( z = 1/x \) and study the zeros.

**6.5: How many zeros does \( P_N(x) \) (Eq. 4) have? State where are they in the complex plane.**

**Sol:** \( P_N(x) \) only has \( N \) zeros, at \( s_n = \sqrt[N+1]{1} = e^{2\pi n/(N+1)} \) where \( n = 1, 2, \ldots, N \). The zero at \( s_n = 1 \) (\( n = 0 \)) of Eq. 4 exactly cancels with the pole at \( s_p = 1 \). This this zero-pole pair are referred to as a removable singularity.
– 6.6: Explain why Eqs. 3 and 4 have different numbers of poles and zeros.

**Sol:** The answer is very interesting. For Eq. 3, \( P_N(s_r) = 0 \) has \( N \) roots and we are not sure where they are. The numerator of Eq. 4 has \( N + 1 \) roots at \( s_r = e^{j\pi n/(N+1)} \) for \( n = 0, 1, 2, \ldots N \). However for \( n = 0 \), \( s_r = e^{j0/N} = 1 \) is not a root, since \( P_N(1) = N \). This root and the pole exactly cancel. All the roots \( N + 1 \) of Eq. 4 are known as the roots of unity, but the root at \( n = 0 \) is special because it cancels with the pole at \( s = 1 \). Given the roots of Eq. 4, we can see that the \( N \) roots of Eq. 3 are at \( s_z = \sqrt{-1} = e^{j2\pi n/(N+1)} \), with \( n = 1, \ldots, N \) (\( n \neq 0 \)). Perhaps even a bit clever. ■

– 6.7: Is the function \( 1/(1 - x) \) analytic outside of the RoC?

**Sol:** Yes, because it is analytically everywhere other than at the pole \( x = 1 \). ■

– 6.8: Extra credit. Evaluate \( P_N(x) \) at \( x = 0 \) and \( x = 0.9 \) for the case of \( N = 100 \), and compare the result to that from Matlab.

```matlab
% sum the geometric series and P_100(0.9)
clear all; close all; format long
N=100; x=0.9; S=0;
for n=0:N
    S=S+x^n
end
P100=(1-x^(N+1))/(1-x);
disp(sprintf('S= %g, P100= %g, error= %g',S,P100, S-P100))
```

**Sol:** \( P_N(0) = 1 \) and \( P_N(0.9) = \frac{1-0.9^{N+1}}{1-0.9} = 9.999760947410010 \). According to Matlab \( P_{100}(0) = 1 \) and \( P_{100}(0.9) = 9.999760947410014 \), with a difference of \(-3.55271 \times 10^{-15} \) (i.e., -16\times eps). ■

#### Problem # 7: The exponential series

– 7.1: What is the RoC for the exponential series Eq. ??

**Sol:** The exponential is convergent everywhere on the open real line. ■

– 7.2: Let \( x = j \) in Eq. ??, and write out the series expansion of \( e^x \) in terms of its real and imaginary parts.

**Sol:**

\[
e^j = \sum_{n=0}^{\infty} \frac{j^n}{n!} = 1 + j - \frac{1}{2!} + j\frac{1}{3!} + \frac{1}{4!} - j\frac{1}{6} + \cdots
\]

\[
= \left( 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots \right) + j \left( \frac{1}{3!} - \frac{1}{5!} + \frac{1}{7!} - \cdots \right)
\]

\[
= \sum_{n=0,2,4,6,\ldots} \frac{(-1)^n}{n!} + j \sum_{n=1,3,5,\ldots} \frac{(-1)^n}{n!}
\]

■

– 7.3: Let \( x = j\theta \) in Eq. ??, and write out the series expansion of \( e^x \) in terms of its real and imaginary parts. How does your result relate to Euler’s identity \((e^j\theta = \cos(\theta) + j\sin(\theta))\)?

**Sol:**

\[
e^{j\theta} = \sum_{n=0}^{\infty} \frac{j^n\theta^n}{n!}
\]

\[
= 1 + j\theta - \frac{\theta^2}{2!} + j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} - j\frac{\theta^5}{5!} + \frac{\theta^6}{6!} + \cdots
\]

\[
= \left( 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \right) + j \left( \frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \frac{\theta^7}{7!} - \cdots \right)
\]

\[
= \cos(\theta) + j\sin(\theta).
\]

■
0.2 Inverse analytic functions and composition

Overview: It may be surprising, but every analytic function has an inverse function. Starting from the function 
\( y(x, y \in \mathbb{C}) = \frac{1}{1 - x} \)
the inverse is 
\( x = \frac{y - 1}{y} = 1 - \frac{1}{y} \).

Problem #8: Consider the inverse function described above

- 8.1: Where are the poles and zeros of \( x(y) \)?
  **Sol:** The pole is at \( y = 0 \), and the zero is at \( y = 1 \). There are no poles or zeros at \( \infty \) because \( \lim_{y \to \pm \infty} (y - 1)/y = 1 \).

- 8.2: Where (for what condition on \( y \)) is \( x(y) \) analytic?
  **Sol:** It is analytic anywhere but the pole, at \( y = 0 \).

Problem #9 Consider the exponential function \( z(x) = e^x \) \( (x, z \in \mathbb{C}) \).

- 9.1: Find the inverse \( x(z) \).
  **Sol:** Taking the natural log \((\ln)\) of both sides gives \( x(z) = \ln(z) \). Thus the natural log is the inverse of the exponential.

- 9.2: Where are the poles and zeros of \( x(z) \)?
  **Sol:** There is a branch cut between \( z = 0, -\infty \), and the zero is at \( z = 1 \). There are no poles.

Problem #10: Composition.

- 10.1: If \( y(s) = 1/(1 - s) \) and \( z(s) = e^s \), compose these two functions to obtain \((y \circ z)(s)\). Give the expression for \((y \circ z)(s) = y(z(s))\). **Sol:**
  \[ (y \circ z)(s) = \frac{1}{1 - e^s} \]

- 10.2: Where are the poles and zeros of \((y \circ z)(s)\)?
  **Sol:** Poles at \( s = j2\pi n, n = \mathbb{Z} \). Zero at \( \Re s = \sigma \to +\infty \).

- 10.3: Where (for what condition on \( x \)) is \((y \circ z)(x)\) analytic?
  **Sol:** It is analytic everywhere except \( x = 0 \).

Convolution

Multiplying two short or simple polynomials is not demanding. However, if the polynomials have many terms, it can become tedious. For example, multiplying two 10th-degree polynomials is not something one would want to do every day.

An alternative is a method called convolution, as described in Sec. ?? (p. ??).

Problem #11: Convolution of sequences. Practice convolution (by hand!!) using a few simple examples. Show your work!!! Check your solution using Matlab.

- 11.1: Convolve the sequence \{0 1 1 1\} with itself.
  **Sol:** \{0 0 1 2 3 4 3 2 1\}
- 11.2: Calculate \{1, 1\} \times \{1, 1\} \times \{1, 1\}.

\textbf{Sol:}

\{1, 1\} \times \{1, 2, 1\} = \{1, 3, 3, 1\}

- \textbf{Problem #12: Multiplying two polynomials is the same as convolving their coefficients.}

\[ f(x) = x^3 + 3x^2 + 3x + 1 \]
\[ g(x) = x^3 + 2x^2 + x + 2. \]

- 12.1: In Octave/Matlab, compute \( h(x) = f(x) \cdot g(x) \) in two ways: (1) use the commands \texttt{roots} and \texttt{poly}, and (2) use the convolution command \texttt{conv}. Confirm that both methods give the same result.

\textbf{Sol:} \( h(x) = [1, 3, 3, 1] \ast [1, 2, 1, 2]. \)

- 12.2: What is \( h(x) \)?

\textbf{Sol:} \( h(x) = x^6 + 5x^5 + 10x^4 + 12x^3 + 11x^2 + 7x + 2. \)

- \textbf{Newton’s root-finding method}

- \textbf{Problem #13: Use Newton’s iteration to find the roots of the polynomial}

\[ P_3(x) = 1 - x^3. \]

- 13.1: Draw a graph describing the first step of the iteration starting with \( x_0 = (1/2, 0) \).

\textbf{Sol:} Start with an \((x, y)\) coordinate system and put points at and the vertex of \( P_3(x) \).

- 13.2: Calculate \( x_1 \) and \( x_2 \). What number is the algorithm approaching?

\textbf{Sol:} First we must find \( P'_3(x) = -3x^2 \). Thus the equation we must iterate is Eq. ?? (p. ??):

\[ x_{n+1} = x_n + \frac{1 - x_n^3}{3x_n^2}. \]

Given a first guess for the root \( x_0 \), the next are \( x_1 = x_0 + \frac{1 - x_0^3}{3x_0^2} \) and \( x_2 = x_1 + \frac{1 - x_1^3}{3x_1^2} \). Note that if \( x + 0 \) is the root, then \( x_1 = x_0 \) and we are done. However, if \( x_0 = 0 \), then \( x_1 = \infty \), since \( x_0 = 0 \) is a root of \( P'_3(x) \). Thus we must not start at the roots of \( P'_n(x_0) = 0 \).

- 13.3: Does Newton’s method work for \( P_2(x) = 1 + x^2 \)? If so, why? Hint: What are the roots in this case?

\textbf{Sol:} Here \( P'_2(x) = +2x \); thus the iteration gives

\[ x_{n+1} = x_n - \frac{1 + x_n^2}{2x_n}. \]

In this case the roots are \( x_\pm = \pm 1 \gamma \)—namely, purely imaginary. The solution will converge for complex roots as long as the starting point is complex. If we start with a real number for \( x_0 \), and use real arithmetic, Newton’s method fails because there is no way for the answer to become complex. Real in = Real out.