

Chapter 2

Algebraic Equations

2.1 Problems AE-1

Topics of this homework: Fundamental theorem of algebra, polynomials, analytic functions and their inverse, convolution, Newton's root finding method, Riemann zeta function. Deliverables: Answers to problems

Note: The term analytic is used in two different ways. (1) An analytic function is a function that may be expressed as a locally convergent power series; (2) analytic geometry refers to geometry using a coordinate system.

Polynomials and the fundamental theorem of algebra (FTA)

Problem # 1: A polynomial of degree N is defined as

$$P_N(x) = a_0 + a_1x + a_2x^2 + \cdots + a_Nx^N.$$

– 1.1: How many coefficients a_n does a polynomial of degree N have?

Ans:

– 1.2: How many roots does $P_N(x)$ have?

Ans:

Problem # 2: The fundamental theorem of algebra (FTA)

– 2.1: State and then explain the FTA.

Ans:

– 2.2: Using the FTA, prove your answer to question 1.2. Hint: Apply the FTA to prove how many roots a polynomial $P_N(x)$ of order N has.

Ans:

Problem # 3: Consider the polynomial function $P_2(x) = 1 + x^2$ of degree $N = 2$ and the related function $F(x) = 1/P_2(x)$. What are the roots (e.g., zeros) x_{\pm} of $P_2(x)$? Hint: Complete the square on the polynomial $P_2(x) = 1 + x^2$ of degree 2, and find the roots.

Ans:

Problem # 4: $F(x)$ may be expressed as $(A, B, x_{\pm} \in \mathbb{C})$

$$F(x) = \frac{A}{x - x_+} + \frac{B}{x - x_-}, \quad (\text{AE-1.1})$$

where x_{\pm} are the roots (zeros) of $P_2(x)$, which become the poles of $F(x)$; A and B are the residues. The expression for $F(x)$ is sometimes called a *partial fraction expansion* or *residue expansion*, and it appears in many engineering applications.

– 4.1: Find $A, B \in \mathbb{C}$ in terms of the roots x_{\pm} of $P_2(x)$.

Ans:

– 4.2: Verify your answers for A and B by showing that this expression for $F(x)$ is indeed equal to $1/P_2(x)$.

Ans:

– 4.3: Give the values of the poles and zeros of $P_2(x)$.

Ans:

– 4.4: Give the values of the poles and zeros of $F(x) = 1/P_2(x)$.

Ans:

2.1.1 Analytic functions

Overview: Analytic functions are defined by infinite (power) series. The function $f(x)$ is said to be *analytic* at any value of constant $x = x_o$, where there exists a convergent power series

$$P(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n$$

such that $P(x_o) = f(x_o)$. The point $x = x_o$ is called the *expansion point*. The region around x_o such that $|x - x_o| < 1$ is called the *radius of convergence*, or region of convergence (RoC). The local power series for $f(x)$ about $x = x_o$ is defined by the Taylor series:

$$\begin{aligned} f(x) &\approx f(x_o) + \left. \frac{df}{dx} \right|_{x=x_o} (x - x_o) + \frac{1}{2!} \left. \frac{d^2f}{dx^2} \right|_{x=x_o} (x - x_o)^2 + \dots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n}{dx^n} f(x) \right|_{x=x_o} (x - x_o)^n. \end{aligned}$$

Two classic examples are the geometric series¹ where $a_n = 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n, \quad (\text{AE-1.2})$$

and the exponential function where $a_n = 1/n!$, Eq. 3.2.11 (p. 68). The coefficients for both series may be derived from the Taylor formula.

Problem # 5: The geometric series

– 5.1: What is the region of convergence (RoC) for the power series Eq. AE-1.2 of $1/(1-x)$ given above—for example, where does the power series $P(x)$ converge to the function value $f(x)$? State your answer as a condition on x . Hint: What happens to the power series when $x > 1$?

Ans:

¹The geometric series is *not* defined as the function $1/(1-x)$, it is defined as the series $1 + x + x^2 + x^3 + \dots$, such that the ratio of consecutive terms is x .

– 5.2: In terms of the pole, what is the RoC for the geometric series in Eq. AE-1.2?

Ans:

– 5.3: How does the RoC relate to the location of the pole of $1/(1-x)$?

Ans:

– 5.4: Where are the zeros, if any, in Eq. AE-1.2?

Ans:

– 5.5: Assuming x is in the RoC, prove that the geometric series correctly represents $1/(1-x)$ by multiplying both sides of Eq. AE-1.2 by $(1-x)$.

Ans:

Problem # 6: Use the geometric series to study the degree N polynomial. It is very important to note that all the coefficients c_n of this polynomial are 1.

$$P_N(x) = 1 + x + x^2 + \cdots + x^N = \sum_{n=0}^N x^n. \quad (\text{AE-1.3})$$

– 6.1: Prove that

$$P_N(x) = \frac{1 - x^{N+1}}{1 - x}. \quad (\text{AE-1.4})$$

Ans:

– 6.2: What is the RoC for Eq. AE-1.3?

Ans:

– 6.3: What is the RoC for Eq. AE-1.4?

Ans:

– 6.4: How many poles does $P_N(x)$ (Eq. AE-1.3) have? Where are they?

Ans:

– 6.5: How many zeros does $P_N(x)$ (Eq. AE-1.4) have? State where are they in the complex plane.

Ans:

– 6.6: Explain why Eqs. AE-1.3 and AE-1.4 have different numbers of poles and zeros.

Ans:

– 6.7: Is the function $1/(1 - x)$ analytic outside of the RoC?

Ans:

– 6.8: *Extra credit. Evaluate $P_N(x)$ at $x = 0$ and $x = 0.9$ for the case of $N = 100$, and compare the result to that from Matlab.*

```
%sum the geometric series and P_100(0.9)
clear all;close all;format long
N=100; x=0.9; S=0;
for n=0:N
S=S+x^n
end
P100=(1-x^(N+1))/(1-x);
disp(sprintf('S= %g, P100= %g, error= %g',S,P100, S-P100))
```

Ans:

Problem # 7: The exponential series

– 7.1: *What is the RoC for the exponential series Eq. 3.2.11?*

Ans:

– 7.2: *Let $x = j$ in Eq. 3.2.11, and write out the series expansion of e^x in terms of its real and imaginary parts.*

Ans:

– 7.3: *Let $x = j\theta$ in Eq. 3.2.11, and write out the series expansion of e^x in terms of its real and imaginary parts. How does your result relate to Euler's identity ($e^{j\theta} = \cos(\theta) + j \sin(\theta)$)?*

Ans:

2.1.2 Inverse analytic functions and composition

Overview: It may be surprising, but every analytic function has an inverse function. Starting from the function $(x, y \in \mathbb{C})$

$$y(x) = \frac{1}{1-x}$$

the inverse is

$$x = \frac{y-1}{y} = 1 - \frac{1}{y}.$$

Problem # 8: Consider the inverse function described above

– 8.1: Where are the poles and zeros of $x(y)$?

Ans:

– 8.2: Where (for what condition on y) is $x(y)$ analytic?

Ans:

Problem # 9 Consider the exponential function $z(x) = e^x$ ($x, z \in \mathbb{C}$).

– 9.1: Find the inverse $x(z)$.

Ans:

– 9.2: Where are the poles and zeros of $x(z)$?

Ans:

Problem # 10: Composition.

– 10.1: If $y(s) = 1/(1-s)$ and $z(s) = e^s$, compose these two functions to obtain $(y \circ z)(s)$. Give the expression for $(y \circ z)(s) = y(z(s))$. **Ans:**

– 10.2: Where are the poles and zeros of $(y \circ z)(s)$?

Ans:

– 10.3: Where (for what condition on x) is $(y \circ z)(x)$ analytic?

Ans:

Convolution

Multiplying two short or simple polynomials is not demanding. However, if the polynomials have many terms, it can become tedious. For example, multiplying two 10th-degree polynomials is not something one would want to do every day.

An alternative is a method called *convolution*, as described in Sec. 3.4 (p. 78).

Problem # 11: Convolution of sequences. Practice convolution (by hand!!) using a few simple examples. Show your work!!! Check your solution using Matlab.

– 11.1: Convolve the sequence $\{0 \ 1 \ 1 \ 1 \ 1\}$ with itself.

Ans:

– 11.2: Calculate $\{1, 1\} \star \{1, 1\} \star \{1, 1\}$.

Ans:

Problem # 12: *Multiplying two polynomials is the same as convolving their coefficients.*

$$\begin{aligned}f(x) &= x^3 + 3x^2 + 3x + 1 \\g(x) &= x^3 + 2x^2 + x + 2.\end{aligned}$$

– 12.1: *In Octave/Matlab, compute $h(x) = f(x) \cdot g(x)$ in two ways: (1) use the commands `roots` and `poly`, and (2) use the convolution command `conv`. Confirm that both methods give the same result.*

Ans:

– 12.2: *What is $h(x)$?*

Ans:

Newton's root-finding method

Problem # 13: *Use Newton's iteration to find the roots of the polynomial*

$$P_3(x) = 1 - x^3.$$

– 13.1: *Draw a graph describing the first step of the iteration starting with $x_0 = (1/2, 0)$.*

Ans:

– 13.2: Calculate x_1 and x_2 . What number is the algorithm approaching?

Ans:

– 13.3: Does Newton's method work for $P_2(x) = 1 + x^2$? If so, why? Hint: What are the roots in this case?

Ans:

Riemann zeta function $\zeta(s)$

Definitions and preliminary analysis: The zeta function $\zeta(s)$ is defined by the complex analytic power series

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots .$$

This series converges, and thus is valid, only in the RoC given by $\Re s = \sigma > 1$, since there $|n^{-\sigma}| \leq 1$. To determine its formula in other regions of the s plane, one must extend the series via analytic continuation (see p. 67).

Euler product formula: As Euler first published in 1737, one may recursively factor out the leading prime term, which results in Euler's product formula.² Multiplying $\zeta(s)$ by the factor $1/2^s$ and subtracting from $\zeta(s)$ remove all the terms $1/(2n)^s$ (e.g., $1/2^s + 1/4^s + 1/6^s + 1/8^s + \cdots$)

$$\left(1 - \frac{1}{2^s}\right) \zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \cdots\right), \quad (\text{AE-1.5})$$

which results in

$$\left(1 - \frac{1}{2^s}\right) \zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \cdots . \quad (\text{AE-1.6})$$

Problem # 14: Questions about the Riemann zeta function.

– 14.1: What is the RoC for Eq. AE-1.6?

Ans:

²This is known as *Euler's sieve*, as distinguished from the Eratosthenes sieve.

– 14.2: Repeat the algebra of Eq. AE-1.5 using the lead factor of $1/3^s$.

Ans:

– 14.3: What is the RoC for Eq. AE-1.6?

Ans:

– 14.4: Repeat the algebra of Eq. AE-1.5 for all prime scale factors (i.e., $1/5^s$, $1/7^s$, ..., $1/\pi_k^s$, ...) to show that

$$\zeta(s) = \prod_{\pi_k \in \mathbb{P}} \frac{1}{1 - \pi_k^{-s}} = \prod_{\pi_k \in \mathbb{P}} \zeta_k(s), \quad (\text{AE-1.7})$$

where π_p represents the p th prime.

Ans:

– 14.5: Given the product formula, identify the poles of $\zeta_p(s)$ ($p \in \mathbb{Z}$), which is important for defining the RoC of each factor. For example, the p th factor of Eq. AE-1.7, expressed as an exponential, is

$$\zeta_p(s) \equiv \frac{1}{1 - \pi_p^{-s}} = \frac{1}{1 - e^{-sT_p}}, \quad (\text{AE-1.8})$$

where $T_p \equiv \ln \pi_p$.

Ans:

– 14.6: Plot Eq. AE-1.8 using z vs $i z$ for $p = 1$. Describe what you see.

Ans: