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## 2.3 Problems AE-3

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### Topics of this homework:

Visualizing complex functions, bilinear/Möbius transformation, Riemann sphere.

Deliverables: Answers to problems

### Two-port network analysis

**Problem # 1:** *Perform an analysis of electrical two-port networks, shown in Fig. 3.9 (page 104). This can be a mechanical system if the capacitors are taken to be springs and inductors taken as mass, as in the suspension of the wheels of a car. In an acoustical circuit, the low-pass filter could be a car muffler. While the physical representations will be different, the equations and the analysis are exactly the same.*

The definition of the ABCD transmission matrix ( $\mathcal{T}$ ) is

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \quad (\text{AE-3.1})$$

The impedance matrix, where the determinant  $\Delta_{\mathcal{T}} = AD - BC$ , is given by

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{\mathcal{C}} \begin{bmatrix} \mathcal{A} & \Delta_{\mathcal{T}} \\ 1 & \mathcal{D} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \quad (\text{AE-3.2})$$

– 1.1: *Derive the formula for the impedance matrix (Eq. AE-3.2) given the transmission matrix definition (Eq. AE-3.1). Show your work.*

**Ans:**

**Problem # 2:** *Consider a single circuit element with impedance  $Z(s)$ .*

– 2.1: *What is the ABCD matrix for this element if it is in series?*

**Ans:**

– 2.2: *What is the ABCD matrix for this element if it is in shunt?*

**Ans:**

**Problem # 3:** Find the ABCD matrix for each of the circuits of Fig. 3.9.

For each circuit, (i) show the cascade of transmission matrices in terms of the complex frequency  $s \in \mathbb{C}$ , then (ii) substitute  $s = 1j$  and calculate the total transmission matrix at this single frequency.

– 3.1: Left circuit (let  $R_1 = R_2 = 10$  kilo-ohms and  $C = 10$  nano-farads)

**Ans:**

– 3.2: Right circuit (use  $L$  and  $C$  values given in the figure), where the pressure  $P$  is analogous to the voltage  $V$ , and the velocity  $U$  is analogous to the current  $I$ .

**Ans:**

– 3.3: Convert both transmission (ABCD) matrices to impedance matrices using Eq. AE-3.2. Do this for the specific frequency  $s = 1j$  as in the previous part (feel free to use Matlab/Octave for your computation).

**Ans:**

– 3.4: Right circuit: Repeat the analysis as in question 3.3.

**Ans:**

## Algebra

**Problem # 4:** Fundamental theorem of algebra (FTA).

– 4.1: State the fundamental theorem of algebra (FTA).

**Ans:**

**Algebra with complex variables****Problem # 5: Order and complex numbers:**

One can always say that  $3 < 4$ —namely, that real numbers have order. One way to view this is to take the difference and compare it to zero, as in  $4 - 3 > 0$ . Here we will explore how complex variables may be ordered. Define the complex variable  $z = x + yj \in \mathbb{C}$ .

– 5.1: Explain the meaning of  $|z_1| > |z_2|$ .

**Ans:**

– 5.2: If  $x_1, x_2 \in \mathbb{R}$  (are real numbers), define the meaning of  $x_1 > x_2$ . Hint: Take the difference.

**Ans:**

– 5.3: Explain the meaning of  $z_1 > z_2$ .

**Ans:**

– 5.4: If time were complex, how might the world be different?

**Ans:**

**Problem # 6: It is sometimes necessary to consider a function  $w(z) = u + vj$  in terms of the real functions  $u(x, y)$  and  $v(x, y)$  (e.g. separate the real and imaginary parts). Similarly, we can consider the inverse  $z(w) = x + yj$ , where  $x(u, v)$  and  $y(u, v)$  are real functions.**

– 6.1: Find  $u(x, y)$  and  $v(x, y)$  for  $w(z) = 1/z$ .

**Ans:**

**Problem # 7:** Find  $u(x, y)$  and  $v(x, y)$  for  $w(z) = c^z$  with complex constant  $c \in \mathbb{C}$  for questions 7.1, 7.2, and 7.3:

– 7.1:  $c = e$

**Ans:**

– 7.2:  $c = 1$  (recall that  $1 = e^{jk2\pi k}$  for  $k = 0, 1, 2, \dots$ )

**Ans:**

– 7.3:  $c = j$ . Hint:  $j = e^{j\pi/2 + j2\pi m}$ ,  $m \in \mathbb{Z}$ .

**Ans:**

– 7.4: Find  $u(x, y)$  for  $w(z) = \sqrt{z}$ .

**Ans:**

**Problem # 8:** Convolution of an impedance  $z(t)$  and its inverse  $y(t)$ :

In the frequency domain a Brune impedance is defined as the ratio of a numerator polynomial  $N(s)$  to a denominator polynomial  $D(s)$ .

– 8.1: Consider a Brune impedance defined by the ratio of numerator and denominator polynomials,  $Z(s) = N(s)/D(s)$ . Since the admittance  $Y(s)$  is defined as the reciprocal of the impedance, the product must be 1. If  $z(t) \leftrightarrow Z(s)$  and  $y(t) \leftrightarrow Y(s)$ , it follows that  $z(t) \star y(t) = \delta(t)$ . What property must  $n(t) \leftrightarrow N(s)$  and  $d(t) \leftrightarrow D(s)$  obey for this to be true?

**Ans:**

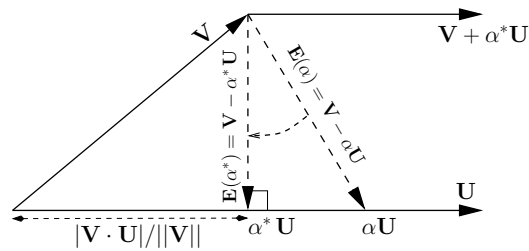


Figure 2.4: This figure shows how to derive the Schwarz inequality, by finding the value of  $\alpha = \alpha^*$  corresponding to  $\min_{\alpha} [E(\alpha)]$ . It is identical to Fig. 3.5 on page 88.

– 8.2: The definition of a minimum phase function is that it must have a causal inverse. Show that every impedance is minimum phase.

**Ans:**

## Schwarz inequality

**Problem # 9:** The above figure shows three vectors for an arbitrary value of  $\alpha \in \mathbb{R}$  and a specific value of  $\alpha = \alpha^*$ .

– 9.1: Find the value of  $\alpha \in \mathbb{R}$  such that the length (norm) of  $\vec{E}$  (i.e.,  $\|\vec{E}\| \geq 0$ ) is minimum. Show your derivation, not the answer ( $\alpha = \alpha^*$ ).

**Ans:**

– 9.2: Find the formula for  $\|\vec{E}(\alpha^*)\|^2 \geq 0$ . Hint: Substitute  $\alpha^*$  into Eq. 3.5.9 (p. 89) and show that this results in the Schwarz inequality

$$|\vec{U} \cdot \vec{V}| \leq \|\vec{U}\| \|\vec{V}\|.$$

**Ans:**

**Problem # 10: Geometry and scalar products**

– 10.1: What is the geometrical meaning of the dot product of two vectors?

**Ans:**

– 10.2: Give the formula for the dot product of two vectors. Explain the meaning based on Fig. 3.4 (page 84).

**Ans:**

– 10.3: Write the formula for the dot product of two vectors  $\vec{U} \cdot \vec{V}$  in  $\mathbb{R}^n$  in polar form (e.g., assume the angle between the vectors is  $\theta$ ).

**Ans:**

– 10.4: How is the Schwarz inequality related to the Pythagorean theorem?

**Ans:**

– 10.5: Starting from  $\|\mathbf{U} + \mathbf{V}\|$ , derive the triangle inequality

$$\|\vec{U} + \vec{V}\| \leq \|\vec{U}\| + \|\vec{V}\|.$$

**Ans:**

– 10.6: The triangle inequality  $\|\vec{U} + \vec{V}\| \leq \|\vec{U}\| + \|\vec{V}\|$  is true for two and three dimensions: Does it hold for five-dimensional vectors?

**Ans:**

– 10.7: Show that the wedge product  $\vec{U} \wedge \vec{V} \perp \vec{U} \cdot \vec{V}$ .

**Ans:**

## Probability

### Problem # 11: Basic terminology of experiments

– 11.1: What is the mean of a trial, and what is the average over all trials?

**Ans:**

– 11.2: What is the expected value of a random variable  $X$ ?

**Ans:**

– 11.3: What is the standard deviation about the mean?

**Ans:**

– 11.4: What is the definition of information of a random variable?

Ans:

– 11.5: How do you combine events? Hint: If the event is the flip of a biased coin, the events are  $H = p$ ,  $T = 1 - p$ , so the event is  $\{p, 1 - p\}$ . To solve the problem, you must find the probabilities of two independent events.

Ans:

– 11.6: What does the term independent mean in the context of question 11.5? Give an example.

Ans:

– 11.7: Define odds.

Ans: