1 Exercises DE-3

Topic of assignment:
Brune impedance, lattice transmission line analysis.

Brune Impedance

**Problem # 1(4 pts): Residue form**
A Brune impedance is defined as the ratio of the force $F(s)$ over the flow $V(s)$, and may be expressed in residue form as

$$Z(s) = c_0 + \sum_{k=1}^{K} \frac{c_k}{s - s_k} = \frac{N(s)}{D(s)}$$

(DE-1.1)

with

$$D(s) = \prod_{k=1}^{K} (s - s_k) \quad \text{and} \quad c_k = \lim_{s \to s_k} (s - s_k)D(s) = \prod_{n'=1}^{K-1} (s - s_{n'}).$$

The prime on index $n'$ means that $n = k$ is not included in the product.

– 1.1(1 pts): Take the Laplace transform ($\mathcal{LT}$) of Eq. DE-1.2 and find the total impedance $Z(s)$ of the mechanical circuit.

$$M \frac{d^2}{dt^2} x(t) + R \frac{d}{dt} x(t) + Kx(t) = f(t) \leftrightarrow (Ms^2 + Rs + K)X(s) = F(s).$$

(DE-1.2)

**Sol:** From the properties of the $\mathcal{LT}$ that $dx/dt \leftrightarrow sX(s)$, we find

$$f(t) \leftrightarrow F(s) = Ms^2X(s) + RsX(s) + KX(s).$$

In terms of velocity this is $(Ms + R + K/s)V(s) = F(s)$. Thus the circuit impedance is

$z(t) \leftrightarrow Z(s) = \frac{F}{V} = \frac{K + Rs + Ms^2}{s}.$

– 1.2(1 pt): What are $N(s)$ and $D(s)$ (e.g. Eq. DE-1.1)?

**Sol:** $D(s) = s$ and $N(s) = K + Rs + Ms^2$.

**Problem # 2:(14 pts) Train-mission-line** We wish to model the dynamics of a freight-train having $N$ such cars, and study the velocity transfer function under various load conditions.

As shown in Fig. 1, the train model consists of masses connected by springs.
Figure 1: Depiction of a train consisting of cars, treated as a mass $M$ and linkages, treated as springs of stiffness $K$ or compliance $C = 1/K$. Below it is the electrical equivalent circuit, for comparison. The mass is modeled as an inductor and the springs as capacitors to ground. The velocity is analogous to a current and the force $f_n(t)$ to the voltage $\phi_n(t)$. The length of each cell is $\Delta [m]$. The train may be accurately modeled as a transmission line (TL), since the equivalent electrical circuit is a lumped model of a TL.

Physical description:

Use the ABCD method to find the matrix representation of the system of Fig. 1. Define the force on the $n$th train car $f_n(t) \leftrightarrow F_n(\omega)$, and velocity $v_n(t) \leftrightarrow V_n(\omega)$.

Break the model into cells consisting of three elements: a series inductor representing half the mass ($M/2$), a shunt capacitor representing the spring ($C = 1/K$), and another series inductor representing half the mass ($L = M/2$), transforming the model into a cascade of symmetric ($A = D$) identical cell matrix $T(s)$.

- 2.1(4 pt): Find the elements of the ABCD matrix $T$ for the single cell that relate the input node 1 to output node 2

$$
\begin{bmatrix}
    F \\
    V
\end{bmatrix}_1 = T
\begin{bmatrix}
    F(\omega) \\
    -V(\omega)
\end{bmatrix}_2.
$$

Sol:

$$
T = \begin{bmatrix}
    1 & sM/2 \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    1 & 0 \\
    sC & 1
\end{bmatrix}
\begin{bmatrix}
    1 & sM/2 \\
    0 & 1
\end{bmatrix}

= \begin{bmatrix}
    1 + s^2MC/2 & (sM)(1 + s^2MC/4) \\
    sC & 1 + s^2MC/2
\end{bmatrix}
$$

- 2.2(6 pts): Express each element of $T(s)$ in terms of the complex Nyquist ratio $s/s_c < 1$ ($s = 2\pi j f$, $s_c = 2\pi j f_c$). The Nyquist sampling cutoff frequency $f_c$ is defined in terms the minimum number of cells (i.e., 2) of length $\Delta$ per wavelength:

The Nyquist sampling theorem says that there are at least two cars per wavelength (more than two time samples at the highest frequency). From the figure, the distance between cars $\Delta = c_o T_o [m]$ where

$$
c_o = \frac{1}{\sqrt{MC}}. \quad [m/s]
$$

The cutoff frequency obeys $f_c \lambda_c = c_o$ where the Nyquist wavelength is $\lambda_c = 2\Delta$. Therefore the Nyquist sampling condition is

$$
\omega < \omega_c = 2\pi f_c \leq \frac{2\pi c_o}{\lambda_c} = \frac{2\pi c_o}{2\Delta} = \frac{\pi}{\Delta\sqrt{MC}}. \quad [Hz]
$$
Reiterating what was said above: the system in Fig. 1 represents a transmission line having a wave speed of \( c_o = 1/\sqrt{MC} \) and characteristic impedance \( r_o = \sqrt{M/C} \). Each cell, composed of 2 masses \( M \) connected by one spring \( K \), has length \( \Delta \).

We wish to define the Nyquist frequency \( f_c \) such that the wavelength \( \lambda > 2\Delta \), where \( \Delta \) is the cell length. Using the formula for the wavelength in terms of the wave velocity and frequency we find

\[
\lambda = c_o / f_c = 2\Delta,
\]

thus we conclude that

\[
f < f_c = \frac{c_o}{2\Delta} = \frac{1}{2\Delta \sqrt{MC}}. \tag{DE-1.6}
\]

If we wish to have the system be accurate for a given frequency we may make the cell length \( \Delta \) smaller, while keeping the velocity constant (\( MC \) is held constant). Thus the characteristic resistance [ohms/unit length] \( r_o \) must change as \( f_c \to \infty \) and \( \Delta \to 0 \). We can either let \( M \to \infty \) and \( C \to 0 \) (their product remains constant), or the other way around. In one case \( r_o \to \infty \) and in the other case it goes to 0. ■

-2.3(4 pt): Use the property of the Nyquist sampling frequency \( f < f_c \) (Eq. DE-1.4) to remove higher order powers of frequency

\[
1 + \left( \frac{s}{s_c} \right)^2 \approx 1 \tag{DE-1.7}
\]

to determine a band-limited approximation of \( T(s) \).

\[
T = \begin{bmatrix}
1 + 2(s/s_c)^2 & sM(1 + (s/s_c)^2) \\
 sC & 1 + 2(s/s_c)^2
\end{bmatrix} 
\]

\[
\approx \begin{bmatrix}
1 & sM \\
 sC & 1
\end{bmatrix} 
\]

The approximation is highly accurate below the Nyquist cutoff frequency \( s < s_c \). Given any desired frequency \( f \), we can always make the cell size \( \Delta \) smaller by decreasing \( M \) and \( C \), while keeping \( f < f_c \) and the cell velocity constant (\( c_o = 1/\sqrt{MC} \)). Thus the Nyquist condition represents a computational bound, not a physical limitation. ■