Chapter 4

Vector differential equations

4.1 Problems VC-1

4.1.1 Topics of this homework:
Vector algebra and fields in $\mathbb{R}^3$, gradient and scalar Laplacian operators, definitions of divergence and curl, Gauss’s (divergence) and Stokes’s (curl) laws, system classification (postulates).

4.1.2 Scalar fields and the $\nabla$ operator

Problem # 1: Let $T(x, y) = x^2 + y$ be an analytic scalar temperature field in two dimensions (single-valued $\in \mathbb{R}^2$).

- 1.1: Find the gradient of $T(x)$ and make a sketch of $T$ and the gradient.

Ans:

- 1.2: Compute $\nabla^2 T(x)$ to determine whether $T(x)$ satisfies Laplace’s equation.

Ans:

- 1.3: Sketch the iso-temperature contours at $T = -10, 0, 10$ degrees.
Ans:

– 1.4: The heat flux$^1$ is defined as $\mathbf{J}(x, y) = -\kappa(x, y) \nabla T$, where $\kappa(x, y)$ is a constant that denotes thermal conductivity at the point $(x, y)$. Given that $\kappa = 1$ everywhere (the medium is homogeneous), plot the vector $\mathbf{J}(x, y) = -\nabla T$ at $x = 2$, $y = 1$. Be clear about the origin, direction, and length of your result.

Ans:

– 1.5: Find the vector $\perp$ to $\nabla T(x, y)$—that is, tangent to the iso-temperature contours. Hint: Sketch it for one $(x, y)$ point (e.g., 2, 1) and then generalize.

Ans:

– 1.6: The thermal resistance $R_T$ is defined as the potential drop $\Delta T$ over the magnitude of the heat flux $|\mathbf{J}|$. At a single point the thermal resistance is

$$R_T(x, y) = -\nabla T/|\mathbf{J}|.$$  

How is $R_T(x, y)$ related to the thermal conductivity $\kappa(x, y)$?

Ans:

Problem # 2: Acoustic wave equation

Note: In this problem, we will work in the frequency domain.

$^1$The heat flux is proportional to the change in temperature times the thermal conductivity $\kappa$ of the medium.
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- 2.1: The basic equations of acoustics in one dimension are

\[- \frac{\partial}{\partial x} P = \rho_0 s \mathbf{V} \quad \text{and} \quad - \frac{\partial}{\partial x} \mathbf{V} = \frac{s}{\eta_0 P_0} P.\]

Here \( P(x, \omega) \) is the pressure (in the frequency domain), \( \mathbf{V}(x, \omega) \) is the volume velocity (the integral of the velocity over the wavefront with area \( A \)), \( s = \sigma + \omega \mathbf{j} \), \( \rho_0 = 1.2 \) is the specific density of air, \( \eta_0 = 1.4 \), and \( P_0 \) is the atmospheric pressure (i.e., \( 10^5 \) Pa). Note that the pressure field \( P \) is a scalar (pressure does not have direction), while the volume velocity field \( \mathbf{V} \) is a vector (velocity has direction).

We can generalize these equations to three dimensions using the \( \nabla \) operator

\[- \nabla P = \rho_0 s \mathbf{V} \quad \text{and} \quad - \nabla \cdot \mathbf{V} = \frac{s}{\eta_0 P_0} P.\]

- 2.2: Starting from these two basic equations, derive the scalar wave equation in terms of the pressure \( P \),

\[\nabla^2 P = \frac{s^2}{c_0^2} P,\]

where \( c_0 \) is a constant representing the speed of sound.

**Ans:**

- 2.3: What is \( c_0 \) in terms of \( \eta_0 \), \( \rho_0 \), and \( P_0 \)?

**Ans:**

- 2.4: Rewrite the pressure wave equation in the time domain using the time derivative property of the Laplace transform [e.g., \( \frac{dx}{dt} \leftrightarrow s X(s) \)]. For your notation, define the time–domain signal using a lowercase letter, \( p(x, y, z, t) \leftrightarrow P \).

**Ans:**
4.1.3 Vector fields and the $\nabla$ operator

4.1.4 Vector algebra

Problem # 3: Let $\mathbf{R}(x, y, z) \equiv x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$.

- 3.1: If $a$, $b$, and $c$ are constants, what is $\mathbf{R}(x, y, z) \cdot \mathbf{R}(a, b, c)$? 
  Ans:

- 3.2: If $a$, $b$, and $c$ are constants, what is $\frac{d}{dt} (\mathbf{R}(x, y, z) \cdot \mathbf{R}(a, b, c))$? 
  Ans:

Problem # 4: Find the divergence and curl of the following vector fields:

- 4.1: $\mathbf{v} = \hat{x} + \hat{y} + 2\hat{z}$ 
  Ans:

- 4.2: $\mathbf{v}(x, y, z) = x\hat{x} + xy\hat{y} + z^2\hat{z}$ 
  Ans:

- 4.3: $\mathbf{v}(x, y, z) = x\hat{x} + xy\hat{y} + \log(z)\hat{z}$ 
  Ans:
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\[ 4.4: \mathbf{v}(x, y, z) = \nabla \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \]

Ans:

4.1.5 Vector and scalar field identities

**Problem # 5:** Find the divergence and curl of the following vector fields:

\[- 5.1: \mathbf{v} = \nabla \phi, \text{ where } \phi(x, y) = xe^y \]

Ans:

\[- 5.2: \mathbf{v} = \nabla \times \mathbf{A}, \text{ where } \mathbf{A} = x\mathbf{\hat{x}} + y\mathbf{\hat{y}} + z\mathbf{\hat{z}} \]

Ans:

\[- 5.3: \mathbf{v} = \nabla \times \mathbf{A}, \text{ where } \mathbf{A} = y\mathbf{\hat{x}} + x^2\mathbf{\hat{y}} + z\mathbf{\hat{z}} \]

Ans:

\[- 5.4: \text{For any differentiable vector field } \mathbf{V}, \text{ write two vector calculus identities that are equal to zero.} \]

Ans:
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- 5.5: What is the most general form a vector field may be expressed in, in terms of scalar \( \Phi \) and vector \( \mathbf{A} \) potentials?

\textbf{Ans:}

\textbf{Problem # 6: Perform the following calculations. If you can state the answer without doing the calculation, explain why.}

- 6.1: Let \( \mathbf{v} = \sin(x)\hat{x} + y\hat{y} + z\hat{z} \). Find \( \nabla \cdot (\nabla \times \mathbf{v}) \).

\textbf{Ans:}

- 6.2: Let \( \mathbf{v} = \sin(x)\hat{x} + y\hat{y} + z\hat{z} \). Find \( \nabla \times (\nabla \sqrt{\mathbf{v} \cdot \mathbf{v}}) \)

\textbf{Ans:}

- 6.3: Let \( \mathbf{v}(x, y, z) = \nabla (x + y^2 + \sin(\log(z))) \). Find \( \nabla \times \mathbf{v}(x, y, z) \).

\textbf{Ans:}

4.1.6 Integral theorems

\textbf{Problem # 7: For each of the following problems, in a few words, identify either Gauss’s or Stokes’s law, define what it means, and explain the formula that follows the question.}

- 7.1: What is the name of this formula?

\[ \int_{\mathcal{S}} \hat{n} \cdot \mathbf{v} \, dA = \int_{\mathcal{V}} \nabla \cdot \mathbf{v} \, dV. \]
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Ans:

- 7.2: What is the name of this formula?

\[ \int_S (\nabla \times \mathbf{V}) \cdot d\mathbf{S} = \int_C \mathbf{V} \cdot d\mathbf{R} \]

Give one important application. **Ans:**

- 7.3: Describe a key application of the vector identity

\[ \nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}. \]

**Ans:**