Chapter 2

Algebraic Equations

2.1 Problems AE-1

Topics of this homework: Fundamental theorem of algebra, polynomials, analytic functions and their inverse, convolution, Newton's root finding method, Riemann zeta function. Deliverables: Answers to problems

Note: The term analytic is used in two different ways. (1) An analytic function is a function that may be expressed as a locally convergent power series; (2) analytic geometry refers to geometry using a coordinate system.

Polynomials and the fundamental theorem of algebra (FTA)

Problem # *1: A polynomial of degree N is defined as*

$$P_N(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N.$$

– 1.1: How many coefficients a_n does a polynomial of degree N have?

Sol: $N + 1$ ■

– 1.2: How many roots does $P_N(x)$ have?

Sol: N ■

Problem # *2: The* fundamental theorem of algebra *(FTA)*

– 2.1: State and then explain the FTA.

Sol: The FTA says that every polynomial has at least one root $x = x_r$. ■

– 2.2: Using the FTA, prove *your answer to question 1.2.* Hint: Apply the FTA to *prove* how many roots a polynomial $P_N(x)$ of order N has.

Sol: When a root is determined, it may be factored out, leaving a new polynomial of degree one less than the first. Specifically,

$$P_{N-1}(x) = \frac{P_N(x)}{x - x_r}.$$

Thus it follows that by a recursive application of this theorem, a polynomial has a number of roots equal to its degree. All the roots must be counted, including repeated and complex roots and roots at ∞. ■

Problem # *3: Consider the polynomial function $P_2(x) = 1 + x^2$ of degree $N = 2$ and the related function $F(x) = 1/P_2(x)$. What are the roots (e.g., zeros) $x_\pm$ of $P_2(x)$?* Hint: Complete the square on the polynomial $P_2(x) = 1 + x^2$ of degree 2, and find the roots.

Sol: Solving for the roots by setting $P_2(x) = 0$ gives $x_\pm^2 = -1$, leading to $x_\pm = \pm 1j$. ■

Problem # *4: $F(x)$ may be expressed as $(A, B, x_\pm \in \mathbb{C})$*

$$F(x) = \frac{A}{x - x_+} + \frac{B}{x - x_-}, \tag{AE-1.1}$$

where $x_\pm$ are the roots (zeros) of $P_2(x)$, which become the *poles* of $F(x)$; A and B are the *residues*. The expression for $F(x)$ is sometimes called a *partial fraction expansion* or *residue expansion*, and it appears in many engineering applications.

– 4.1:Find $A, B \in \mathbb{C}$ in terms of the roots $x_\pm$ of $P_2(x)$.
Sol: The fastest (i.e., easiest) way to find the constants A, B is to cross-multiply

$$\frac{1}{1+x^2} = \frac{A(x-x_-)+B(x-x_+)}{(x-x_+)(x-x_-)} = \frac{(A+B)x-(Ax_-+Bx_+)}{(x-x_+)(x-x_-)}$$

Since the numerator must equal 1, $B = -A$ and $A = 1/(x_+ - x_1)$.
In summary , in terms of the roots of Eq. AE-1.1

$$A = -B = \frac{1}{(x_+ - x_-)}, \quad \text{thus} \quad F(x) = \frac{1}{1+x^2} = \frac{1}{2\jmath}\left(\frac{1}{x-1\jmath} - \frac{1}{x+1\jmath}\right).$$

■

– 4.2: Verify your answers for A and B by showing that this expression for $F(x)$ is indeed equal to $1/P_2(x)$.
Sol: This is easily verified by cross-multiplying and simplifying. In the numerator the x terms cancel and Eq. AE-1.1 is recovered. ■

– 4.3: Give the values of the poles and zeros of $P_2(x)$.
Sol: The zeros are at $x_z = \pm\jmath$, and the poles are at $x_p = \pm\infty$ ■

– 4.4: Give the values of the poles and zeros of $F(x) = 1/P_2(x)$.
Sol: The poles are at $x_p = \pm\jmath$, and the zeros are at $x_z = \pm\infty$ ■

2.1.1 Analytic functions

Overview: Analytic functions are defined by infinite (power) series. The function $f(x)$ is said to be *analytic* at any value of constant $x = x_o$, where there exists a convergent power series

$$P(x) = \sum_{n=0}^{\infty} a_n (x-x_o)^n$$

such that $P(x_o) = f(x_o)$. The point $x = x_o$ is called the *expansion point*. The region around x_o such that $|x - x_o| < 1$ is called the *radius of convergence*, or region of convergence (RoC). The local power series for $f(x)$ about $x = x_o$ is defined by the Taylor series:

$$\begin{aligned} f(x) &\approx f(x_o) + \left.\frac{df}{dx}\right|_{x=x_o}(x-x_o) + \frac{1}{2!}\left.\frac{d^2f}{dx^2}\right|_{x=x_o}(x-x_o)^2 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n}{dx^n} f(x)\right|_{x=x_o} (x-x_o)^n. \end{aligned}$$

Two classic examples are the geometric series[1] where $a_n = 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \tag{AE-1.2}$$

and the exponential function where $a_n = 1/n!$, Eq. 3.2.11 (p. 70). The coefficients for both series may be derived from the Taylor formula.

Problem # *5: The geometric series*

– 5.1: What is the region of convergence *(RoC) for the power series Eq. AE-1.2 of $1/(1-x)$ given above—for example, where does the power series $P(x)$ converge to the function value $f(x)$? State your answer as a condition on x.* Hint: What happens to the power series when $x > 1$?
Sol: $|x| < 1$ because for $|x| \geq 1$, the power series diverges to infinity. ■

[1] The geometric series is *not* defined as the function $1/(1-x)$, it is defined as the series $1 + x + x^2 + x^3 + \cdots$, such that the ratio of consecutive terms is x.

– 5.2: In terms of the pole, what is the RoC for the geometric series in Eq. AE-1.2?
Sol: The nearest pole relative to the expansion point, at $x = 0$ is at at the nearest pole $x_p = 1$ to the expansion point at $x = 0$. Namely the RoC is 1 re 0. ■

– 5.3: How does the RoC relate to the location of the pole of $1/(1-x)$?
Sol: The pole is at $x = 1$, on the border of the RoC. The nearest pole relative to the expansion point, at $x = 0$ is at $x = 1$. Thus the RoC is 1. ■

– 5.4: Where are the zeros, if any, in Eq. AE-1.2?

Sol: There is a single zero at $x = \infty$. ■

– 5.5: Assuming x is in the RoC, prove that the geometric series correctly represents $1/(1-x)$ by multiplying both sides of Eq. AE-1.2 by $(1-x)$.
Sol:

$$\begin{aligned}
1 &= \frac{1-x}{1-x} && \text{for all } x \neq 1 \\
&= (1-x)(1+x+x^2+x^2\cdots), && |x| < 1 \\
&= (1+x+x^2+x^2\cdots) - x(1+x+x^2\cdots) \\
&= 1 + \cancel{(x+x^2+x^3\cdots)} - \cancel{(x+x^2+x^3\cdots)} \\
&= 1 && \text{for all } x.
\end{aligned}$$

The introduction of the pole introduces an added zero since $P_N(x)|_{x=1} = N$.

If one lets $z = 1/x$ the relation becomes

$$1 = \frac{1-z}{1-z},$$

which is valid for $z \neq 1$, which when expanded the RoC is $|z| < 1$, or $x > 1$. Once the removable pole and zero at $x = 1$ are cancelled, the solution is valid for all x. ■

Problem # *6: Use the geometric series to study the degree N polynomial. It is very important to note that all the coefficients c_n of this polynomial are 1.*

$$P_N(x) = 1 + x + x^2 + \cdots + x^N = \sum_{n=0}^{N} x^n. \tag{AE-1.3}$$

– 6.1: Prove that

$$P_N(x) = \frac{1-x^{N+1}}{1-x}. \tag{AE-1.4}$$

Sol:

$$\begin{aligned}
P_N(x) &= 1 + x + x^2 \cdots x^N \\
&= \sum_{n=0}^{\infty} x^n - \sum_{n=N+1}^{\infty} x^n \\
&= \sum_{n=0}^{\infty} x^n - x^{N+1} \sum_{n=0}^{\infty} x^n \\
&= (1-x^{N+1}) \sum_{n=0}^{\infty} x^n \\
&= \frac{1-x^{N+1}}{1-x}
\end{aligned}$$

■

– 6.2: What is the RoC for Eq. AE-1.3?
Sol: There is no pole; thus the RoC is ∞. This polynomial has N zeros. ■

– 6.3: What is the RoC for Eq. AE-1.4?
Sol: A polynomial has no RoC. ■

– 6.4: How many poles does $P_N(x)$ (Eq. AE-1.3) have? Where are they?
Sol: Since $P_N(x)$ is defined by Eq. AE-1.3, there is no poles at $x = 1$. However it still has a pole of order N at $x = \infty$. To show this, define $z = 1/x$ and study the zeros. ■

– 6.5: How many zeros does $P_N(x)$ (Eq. AE-1.4) have? State where are they in the complex plane.
Sol: $P_N(x)$ only has N zeros, at $s_z = \sqrt[N]{-1} = e^{\jmath 2\pi n/(N+1))}$ where $n = 1, 2, \ldots, N$. The zero at $s_z = 1$ ($n = 0$) of Eq. AE-1.4 exactly cancels with the pole at $s_p = 1$. This this zero-pole pair are referred to as a *removable singularity*. ■

– 6.6: Explain why Eqs. AE-1.3 and AE-1.4 have different numbers of poles and zeros.
Sol: The answer is very interesting. For Eq. AE-1.3, $P_N(s_r) = 0$ has N roots and we are not sure where they are. The numerator of Eq. AE-1.4 has $N + 1$ roots at $s_r = e^{\jmath 2\pi n/(N+1)}$ for $n = 0, 1, 2, \ldots N$. However for $n = 0$, $s_r = e^{\jmath 0/N} = 1$ is not a root, since $P_N(1) = N$. This root and the pole exactly cancel. All the roots $N + 1$ of Eq. AE-1.4 are known as *the roots of unity*, but the root at $n = 0$ is special because it cancels with the pole at $s = 1$. Given the roots of Eq. AE-1.4, we can see that the N roots of Eq. AE-1.3 are at $s_z = \sqrt[N]{-1} = e^{\jmath 2\pi n/(N+1))}$, with $n = 1, \ldots, N$ ($n \neq 0$). Perhaps even a bit clever. ■

– 6.7: Is the function $1/(1 - x)$ analytic outside of the RoC?
Sol: Yes, because it is analytice everwhere other than at the pole $x = 1$. ■

– 6.8: Extra credit. Evaluate $P_N(x)$ at $x = 0$ and $x = 0.9$ for the case of $N = 100$, and compare the result to that from Matlab.

```
%sum the geometric series and P_100(0.9)
clear all;close all;format long
N=100; x=0.9; S=0;
for n=0:N
S=S+x^n
end
P100=(1-x^(N+1))/(1-x);
disp(sprintf('S= %g, P100= %g, error= %g',S,P100, S-P100))
```

Sol: $P_N(0) = 1$ and $P_N(0.9) = \frac{1-.9^{N+1}}{1-0.9} = 9.999760947410010$. According to Matlab $P_{100}(0) = 1$ and $P_{100}(0.9) = 9.999760947410014$, with a difference of -3.55271×10^{-15} (i.e., -16×eps). ■

Problem # *7: The exponential series*

– 7.1: What is the RoC for the exponential series Eq. 3.2.11?
Sol: The exponential is convergent everywhere on the open real line. ■

– 7.2: Let $x = \jmath$ in Eq. 3.2.11, and write out the series expansion of e^x in terms of its real and imaginary parts.
Sol:

$$\begin{aligned}
e^{\jmath} &= \sum_{0}^{\infty} \frac{\jmath^n}{n!} \\
&= 1 + \jmath - \frac{1}{2!} - \jmath\frac{1}{3!} + \frac{1}{4!} + \jmath\frac{1}{5!} - \frac{1}{6!} + \cdots \\
&= \left(1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots\right) + \jmath\left(1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots\right) \\
&= \sum_{n=0,2,\ldots}^{\text{n even}} \frac{(-1)^n}{n!} + \jmath \sum_{n=1,3,\ldots}^{\text{n odd}} \frac{(-1)^n}{n!}.
\end{aligned}$$

■

– 7.3: Let $x = \jmath\theta$ in Eq. 3.2.11, and write out the series expansion of e^x in terms of its real and imaginary parts. How does your result relate to Euler's identity ($e^{\jmath\theta} = \cos(\theta) + \jmath\sin(\theta)$)?
Sol:

$$\begin{aligned} e^{\jmath\theta} &= \sum_0^\infty \frac{\jmath^n\theta^n}{n!} \\ &= 1 + \jmath\theta - \frac{\theta^2}{2!} - \jmath\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \jmath\frac{\theta^5}{5!} - \frac{\theta^6}{6!} + \cdots \\ &= \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)}_{\cos\theta} + \jmath \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)}_{\sin\theta} \\ &= \cos(\theta) + \jmath\sin(\theta). \end{aligned}$$

■

2.1.2 Inverse analytic functions and composition

Overview: It may be surprising, but every analytic function has an inverse function. Starting from the function ($x, y \in \mathbb{C}$)

$$y(x) = \frac{1}{1-x}$$

the inverse is

$$x = \frac{y-1}{y} = 1 - \frac{1}{y}.$$

Problem # *8: Consider the inverse function described above*

– 8.1: Where are the poles and zeros of $x(y)$?
Sol: The pole is at $y = 0$, and the zero is at $y = 1$. There are no poles or zeros at ∞ because $\lim_{y\to\pm\infty}(y-1)/y = 1$ ■

– 8.2: Where (for what condition on y) is $x(y)$ analytic?
Sol: It is analytic anywhere but the pole, at $y = 0$. ■

Problem # *9 Consider the exponential function $z(x) = e^x$ ($x, z \in \mathbb{C}$).*

– 9.1: Find the inverse $x(z)$.
Sol: Taking the natural log (ln) of both sides gives $x(z) = \ln(z)$. Thus the natural log is the inverse of the exponential. ■

– 9.2: Where are the poles and zeros of $x(z)$?
Sol: Their is a branch cut between $z = 0, -\infty$, and the zero is at $z = 1$. There are no poles. ■

Problem # *10: Composition.*

– 10.1: If $y(s) = 1/(1-s)$ and $z(s) = e^s$, compose these two functions to obtain $(y \circ z)(s)$. Give the expression for $(y \circ z)(s) = y(z(s))$. **Sol:**

$$(y \circ z)(s) = \frac{1}{1-e^s}$$

■

– 10.2: Where are the poles and zeros of $(y \circ z)(s)$?
Sol: Poles at $s = \jmath 2\pi n$, $n = \mathbb{Z}$. Zero at $\Re s = \sigma \to +\infty$. ■

– 10.3: Where (for what condition on x) is $(y \circ z)(x)$ analytic?
Sol: It is analytic everywhere except $x = 0$. ■

Eigen-analysis

Problem # *11: (4 pts) The vectorized eigen-equation for a matrix $\mathcal{A}$ is*

$$\boldsymbol{A}\boldsymbol{E} = \boldsymbol{E}\boldsymbol{\Lambda}. \tag{AE-1.5}$$

– 11.1: (4pt) Provide a formula for $\boldsymbol{A}^3$ in terms of the eigenvector $\boldsymbol{E}$ and eigenvalue $\boldsymbol{\Lambda}$ matricies.

Sol: To find powers of a matrix modify Eq. AE-1.5 by post multication by $\boldsymbol{E}$

$$\boldsymbol{A} = \boldsymbol{E}\boldsymbol{\Lambda}\boldsymbol{E}^{-1}.$$

Then

$$\boldsymbol{A}^3 = \boldsymbol{E}\boldsymbol{\Lambda}\boldsymbol{E}^{-1}\boldsymbol{E}\boldsymbol{\Lambda}\boldsymbol{E}^{-1}\boldsymbol{E}\boldsymbol{\Lambda}\boldsymbol{E}^{-1} = \boldsymbol{E}\boldsymbol{\Lambda}^3\boldsymbol{E}^{-1}.$$

■

– 11.2: (4 pts) Find the eigenvalues of the matrix, and find the roots, by completing the square, where $a, b, c, d \in \mathbb{C}$, and

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Sol: The definition of the eigenvalues is

$$\det \left| \boldsymbol{A} - \lambda \boldsymbol{I}_2 \right| = 0$$

which is

$$\det \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda - bc.$$

Completing the square

$$\left(\lambda - \frac{a+d}{2}\right)^2 - \left(\frac{a+2}{2}\right)^2 - bc = 0.$$

Thus

$$\lambda_\pm = \frac{a+d}{2} \pm \sqrt{bc + \left(\frac{a+d}{2}\right)^2}.$$

The eigenvalues are typically the damped resonant frequencies $\lambda_\pm = \sigma_o \pm \jmath\omega_o$ of a mechanical or electrical circuit. In these cases the radical is $\jmath\omega_0$ is the resonance radian frequency and $\jmath\omega_o \le 0$ is the resonant damping. This requires that the constants $\{a, b, c, d\} \ge 0$ and $\in \mathbb{R}$.

■

(4 pts) Convolution

Multiplying two short or simple polynomials is not demanding. However, if the polynomials have many terms, it can become tedious. For example, multiplying two 10th-degree polynomials is not something one would like to do every day.

An alternative is a method called *convolution*. The inverse of convolution is called *deconvolution*, which is equivalent to long-division of polynomials, also known as factoring polynomials (Sec. 3.4.1, p. 109-111). Newton's method is a reliable and accurately algorithm to extract roots from polynomials using term by term deconvolution. When the roots are well approximated by fractional numbers, the method is accurate to within computational accuracy. For example, if the root is $\pi \approx \hat{\pi}_{19} \equiv 817696623/260280919 \in \mathbb{F}$, as given by `rats(pi,19)`. $\hat{\pi}_{19}$ is the 64 bit machine's internal representation of π since $\pi - \hat{\pi}_{19} = 0$ (See text Fig. 2.6, p. 48).

Problem # *12: (4 pts) Convolution of sequences. Practice convolution (by hand!!) using a few simple examples. Manually evaluate the following convolutions. Show your work!*

– 12.1: (2 pts) Multiplying two polynomials is the same as convolving their coefficients.

Given

$$f(x) = x^3 + 3x^2 + 3x + 1 \leftrightarrow [1; 3, 3, 1]$$
$$g(x) = x^3 + 2x^2 + x + 2 \leftrightarrow [1; 2, 1, 2].$$

show that

$$f(x)g(x) = x^6 + 5x^5 + 10x^4 + 12x^3 + 11x^2 + 7x + 2 \leftrightarrow [1; 3, 3, 1] \star [1; 2, 1, 2].$$

Sol: Do the convolution $[1; 3, 3, 1] \star [1; 2, 1, 2]$. Reverse the first vector and run it across the second. This produces $[1, [3, 1] \cdot [1, 2], [1, 3, 3] \cdot [1, 2, 1] \cdots = [1; 5, 10, 12, 11, 7, 2]$. ■

– *12.2: (1 pts)* $[1; -1] \star [1; 2, 4, 7, 0]$

Sol: $[1; -1] \star [0; 1, 2, 4, 7, 0]$ = $[0; 1, 2, 4, 3, -7, 0, \ldots]$. = $[0, 1, 1, 2, 3, -7, 0, \ldots]$. ■

– *12.3: (1 pts)* $[1; 2, 1] \star [1; -1]$

Sol: $[1; 1, -1, -1]$ ■

Newton's root-finding method

Problem # *13: Use Newton's iteration to find the roots of the polynomial*

$$P_3(x) = 1 - x^3.$$

– *13.1: Draw a graph describing the first step of the iteration starting with* $x_0 = (1/2, 0)$.

Sol: Start with an (x, y) coordinate system and put points at and the vertex of $P_3(x)$. ■

– *13.2: Calculate* x_1 *and* x_2. *What number is the algorithm approaching?*

Sol: First we must find $P_3'(x) = -3x^2$. Thus the equation we must iterate is Eq. 3.1.14 (p. 56):

$$x_{n+1} = x_n + \frac{1 - x_n^3}{3x_n^2}.$$

Given a first guess for the root x_0, the next are $x_1 = x_0 + \frac{1-x_0^3}{3x_0^2}$ and $x_2 = x_1 + \frac{1-x_1^3}{3x_1^2}$. Note that if $x + 0$ is the root, then $x_1 = x_0$ and we are done. However, if $x_0 = 0$, then $x_1 = \infty$, since $x_0 = 0$ is a root of $P_3'(x)$. Thus we must not start at the roots of $P_n'(x_0) = 0$. ■

– *13.3: Does Newton's method work for* $P_2(x) = 1 + x^2$*? If so, why?* Hint: What are the roots in this case?

Sol: Here $P_2'(x) = +2x$; thus the iteration gives

$$x_{n+1} = x_n - \frac{1 + x_n^2}{2x_n}.$$

In this case the roots are $x_\pm = \pm 1\jmath$—namely, purely imaginary. The solution will converge for complex roots as long as the starting point is complex. If we start with a real number for x_0, and use real arithmetic, Newton's method fails because there is no way for the answer to become complex. Real in = Real out. ■

Problem # 14: *In this problem we consider the case of fractional roots, and take advantage of this fact during the itteration. Given that the roots are integers, composed of primes, we may uniquely identify the primes by factoring the numerator and denominator of the rational approximation of the root.*
The method is:

1. Start the Newton itteration
$$s_{n+1} = s_n - \frac{M(s_n)}{M'(s_n)}$$
2. Apply the CFA to the next output `rats`(s_{n+1})
3. Factor the Num and Dem of the CFA
4. Terminate when the factors converge

Using this method, show that we can find either the best possible fractional approximation to the roots (or even the exact roots, when the answer is within machine accuracy).

– *14.1: Find the roots of a Monic having coefficents $m_k \in \mathbb{F}$.*
Let
$$M_3(x) = (x - 254/17)(x - 2047/13)(x - 17/13)$$
In this case the root vector R becomes
$$R = [14.9412, 157.4615, 1.3077].$$
Verify that `rats(M)` returns the rational set of roots. **Sol:** In double precision this returns M_3. (Not sure what happens in single precision.) ■