2.3 Problems AE-3

Topics of this homework:
Visualizing complex functions, bilinear/Möbius transformation, Riemann sphere.
Deliverables: Answers to problems

Two-port network analysis

Problem # 1: Perform an analysis of electrical two-port networks, shown in Fig. 3.6 (page 144). This can be a mechanical system if the capacitors are taken to be springs and inductors taken as mass, as in the suspension of the wheels of a car. In an acoustical circuit, the low-pass filter could be a car muffler. While the physical representations will be different, the equations and the analysis are exactly the same.

The definition of the ABCD transmission matrix ($T$) is
\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix}.
\]

The impedance matrix, where the determinant $\Delta_T = AD - BC$, is given by
\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} = \frac{1}{C} \begin{bmatrix}
A & \Delta_T \\
1 & D
\end{bmatrix} \begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}.
\]

- 1.1: Derive the formula for the impedance matrix (Eq. AE-3.2) given the transmission matrix definition (Eq. AE-3.1). Show your work.

Ans:

Problem # 2: Consider a single circuit element with impedance $Z(s)$.

- 2.1: What is the ABCD matrix for this element if it is in series?

Ans:

- 2.2: What is the ABCD matrix for this element if it is in shunt?

Ans:
Problem # 3: Find the ABCD matrix for each of the circuits of Fig. 3.6.

For each circuit, (i) show the cascade of transmission matrices in terms of the complex frequency \( s \in \mathbb{C} \), then (ii) substitute \( s = 1j \) and calculate the total transmission matrix at this single frequency.

- 3.1: Left circuit (let \( R_1 = R_2 = 10 \) kilo-ohms and \( C = 10 \) nano-farads)
  \textbf{Ans:} \\

- 3.2: Right circuit (use \( L \) and \( C \) values given in the figure), where the pressure \( P \) is analogous to the voltage \( V \), and the velocity \( U \) is analogous to the current \( I \).
  \textbf{Ans:} \\

- 3.3: Convert both transmission (ABCD) matrices to impedance matrices using Eq. AE-3.2. Do this for the specific frequency \( s = 1j \) as in the previous part (feel free to use Matlab/Octave for your computation).
  \textbf{Ans:} \\

- 3.4: Right circuit: Repeat the analysis as in question 3.3.
  \textbf{Ans:} \\

Algebra

Problem # 4: Fundamental theorem of algebra (FTA).

- 4.1: State the fundamental theorem of algebra (FTA).
  \textbf{Ans:}
(13 pts) Algebra with complex variables

Problem # 5: (7 pts) Order and complex numbers:
One can always say that $3 < 4$—namely, that real numbers have order. One way to view this is to take the difference and compare it to zero, as in $4 - 3 > 0$. Here we will explore how complex variables may be ordered. In the following define \( \{x, y\} \in \mathbb{R} \) and complex variable \( z = x + yj \in \mathbb{C} \).

- 5.1: Explain the meaning of \(|z_1| > |z_2|\).
  \textbf{Ans:} \\

- 5.2: If \( x_1, x_2 \in \mathbb{R} \) (are real numbers), define the meaning of \( x_1 > x_2 \).
  \textbf{Ans:} \\

- 5.3: Explain the meaning of \( z_1 > z_2 \).
  \textbf{Ans:} \\

- 5.4: (2 pts) What is the meaning of \(|z_1 + z_2| > 3|\)?
  \textbf{Ans:} \\

- 5.5: (2 pts) If time were complex, how might the world be different?
  \textbf{Ans:}
Problem # 6: (1 pt) It is frequently necessary to consider a function \( w(z) = u + v \) in terms of the real functions \( u(x, y) \) and \( v(x, y) \) (e.g. separate the real and imaginary parts). Similarly, we can consider the inverse \( z(w) = x + yj \), where \( x(u, v) \) and \( y(u, v) \) are real functions.

- 6.1: (1 pts) Find \( u(x, y) \) and \( v(x, y) \) for \( w(z) = 1/z \).

\[ \text{Ans:} \]

Problem # 7: (5 pts) Find \( u(x, y) \) and \( v(x, y) \) for \( w(z) = e^z \) with complex constant \( c \in \mathbb{C} \) for questions 7.1, 7.2, and 7.3:

- 7.1: \( c = e \)

\[ \text{Ans:} \]

- 7.2: \( c = 1 \) (recall that \( 1 = e^{±2\pi k} \) for \( k \in \mathbb{Z} \))

\[ \text{Ans:} \]

- 7.3: \( c = j \). Hint: \( j = e^{\pi/2 + j2\pi k}, \quad k \in \mathbb{Z} \).

\[ \text{Ans:} \]
Figure 2.2: This figure shows how to derive the Schwarz inequality, by finding the value of $\alpha = \alpha^*$ corresponding to $\min\limits_\alpha |E(\alpha)|$. It is identical to Fig. 3.5 on page 91.

Problem #8: The above figure shows three vectors for an arbitrary value of $\alpha \in \mathbb{R}$ and a specific value of $\alpha = \alpha^*$.

- 8.1: Find the value of $\alpha \in \mathbb{R}$ such that the length (norm) of $\vec{E}$ (i.e., $|\vec{E}| \geq 0$) is minimum. Show your derivation, not the answer ($\alpha = \alpha^*$).

Ans:

- 8.2: Find the formula for $|E(\alpha^*)|^2 \geq 0$. Hint: Substitute $\alpha^*$ into Eq. 3.5.9 (p. 92) and show that this results in the Schwarz inequality

$$|\vec{U} \cdot \vec{V}| \leq ||\vec{U}||||\vec{V}|.$$
Problem # 9: Geometry and scalar products

– 9.1: What is the geometrical meaning of the dot product of two vectors?

Ans:

– 9.2: Give the formula for the dot product of two vectors. Explain the meaning based on Fig. 3.4 (page 87).

Ans:

– 9.3: Write the formula for the dot product of two vectors \( \vec{U} \cdot \vec{V} \) in \( \mathbb{R}^n \) in polar form (e.g., assume the angle between the vectors is \( \theta \)).

Ans:

– 9.4: How is the Schwarz inequality related to the Pythagorean theorem?

Ans:
- 9.5: Starting from $||\vec{U} + \vec{V}||$, derive the triangle inequality

$$||\vec{U} + \vec{V}|| \leq ||\vec{U}|| + ||\vec{V}||.$$ 

**Ans:**

- 9.6: The triangle inequality $||\vec{U} + \vec{V}|| \leq ||\vec{U}|| + ||\vec{V}||$ is true for two and three dimensions: Does it hold for five-dimensional vectors?

**Ans:**

- 9.7: Show that the wedge product $\vec{U} \wedge \vec{V} \perp \vec{U} \cdot \vec{V}$.

**Ans:**