2.3 Problems AE-3

Topics of this homework:

Visualizing complex functions, bilinear/Möbius transformation, Riemann sphere.
Deliverables: Answers to problems

Two-port network analysis

Problem # *1: Perform an analysis of electrical two-port networks, shown in Fig. 3.6 (page 144). This can be a mechanical system if the capacitors are taken to be springs and inductors taken as mass, as in the suspension of the wheels of a car. In an acoustical circuit, the low-pass filter could be a car muffler. While the physical representations will be different, the equations and the analysis are exactly the same.*

The definition of the ABCD *transmission matrix* ($\mathcal{T}$) is

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \tag{AE-3.1}$$

The *impedance matrix*, where the determinant $\Delta_{\mathcal{T}} = AD - BC$, is given by

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{\mathcal{C}} \begin{bmatrix} \mathcal{A} & \Delta_{\mathcal{T}} \\ 1 & \mathcal{D} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{AE-3.2}$$

– 1.1: Derive the formula for the impedance matrix (Eq. AE-3.2) given the transmission matrix definition (Eq. AE-3.1). Show your work.

Sol: The formula may be easily derived by re-arranging the equations from the matrix (Eq. AE-3.2). Begin with

$$V_1 = \mathcal{A}V_2 - \mathcal{B}I_2$$
$$I_1 = \mathcal{C}V_2 - \mathcal{D}I_2$$

From the second equation, we get

$$V_2 = \frac{1}{\mathcal{C}}I_1 + \frac{\mathcal{D}}{\mathcal{C}}I_2$$

which gives (upon substitution)

$$V_1 = \frac{\mathcal{A}}{\mathcal{C}}I_1 + \frac{\mathcal{A}\mathcal{D}}{\mathcal{C}}I_2 - \mathcal{B}I_2 = \frac{\mathcal{A}}{\mathcal{C}}I_1 + \left(\frac{\mathcal{A}\mathcal{D}}{\mathcal{C}} - \mathcal{B}\right)I_2$$

which yields the matrix equation

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \mathcal{A}/\mathcal{C} & (\mathcal{A}\mathcal{D}/\mathcal{C} - \mathcal{B}) \\ 1/\mathcal{C} & \mathcal{D}/\mathcal{C} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{\mathcal{C}} \begin{bmatrix} \mathcal{A} & \Delta_{\mathcal{T}} \\ 1 & \mathcal{D} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{AE-3.3}$$

■

Problem # *2: Consider a single circuit element with impedance $Z(s)$.*

– 2.1: What is the ABCD matrix for this element if it is in series?

Sol:

$$\begin{bmatrix} 1 & Z(s) \\ 0 & 1 \end{bmatrix}$$

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– 2.2: What is the ABCD matrix for this element if it is in shunt?

Sol:

$$\begin{bmatrix} 1 & 0 \\ 1/Z(s) & 1 \end{bmatrix}$$

■

Problem # *3: Find the ABCD matrix for each of the circuits of Fig. 3.6.*

For each circuit, (i) show the cascade of transmission matrices in terms of the complex frequency $s \in \mathbb{C}$, then (ii) substitute $s = 1\jmath$ and calculate the total transmission matrix at this single frequency.

– 3.1: Left circuit (let $R_1 = R_2 = 10$ kilo-ohms and $C = 10$ nano-farads)

Sol: Write the system in chain matrix form:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & Z_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y_C & 1 \end{bmatrix} \begin{bmatrix} 1 & Z_3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} 1 & Z_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC & 1 \end{bmatrix} \begin{bmatrix} 1 & Z_3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

Now we substitute the given values:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & 10^4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \jmath 10^{-8} & 1 \end{bmatrix} \begin{bmatrix} 1 & 10^4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} 1+\jmath 10^{-4} & 2\times 10^4 + \jmath \\ \jmath 10^{-8} & 1+\jmath 10^{-4} \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

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– 3.2: Right circuit (use L and C values given in the figure), where the pressure P is analogous to the voltage V, and the velocity U is analogous to the current I.

Sol: Write the system in chain matrix form:

$$\begin{bmatrix} P_1 \\ U_1 \end{bmatrix} = \begin{bmatrix} 1 & sL_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{sC_3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{sL_4} & 1 \end{bmatrix} \begin{bmatrix} P_2 \\ -U_2 \end{bmatrix}$$

Now we substitute the given values:

$$\begin{bmatrix} P_1 \\ U_1 \end{bmatrix} = \begin{bmatrix} 1 & \jmath \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2\jmath & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{3\jmath} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{4\jmath} & 1 \end{bmatrix} \begin{bmatrix} P_2 \\ -U_2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{4}{3}\jmath \\ \frac{19}{12}\jmath & \frac{5}{3} \end{bmatrix} \begin{bmatrix} P_2 \\ -U_2 \end{bmatrix}$$

I used Matlab/Octave to evaluate this `T` matrix:

```
a=[1 j;0 1];b=[1 0;2j 1];c=[1 1/3j; 0 1];d=[1 0;1/4j 1]; T=
a*b*c*d.
```

Finally I found $T(2,1)$ to be $19/12$ using the Matlab/Octave command: `rats(1.5833,6)`

■

– 3.3: Convert both transmission (ABCD) matrices to impedance matrices using Eq. AE-3.2. Do this for the specific frequency $s = 1\jmath$ as in the previous part (feel free to use Matlab/Octave for your computation).

Sol: Left circuit: Using the previous solution, and Matlab:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{\jmath 10^{-8}} \begin{bmatrix} 1+\jmath 10^{-4} & 1 \\ 1 & 1+\jmath 10^{-4} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

■

– 3.4: Right circuit: Repeat the analysis as in question 3.3.

Sol:

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{1}{1.5833\jmath} \begin{bmatrix} -\frac{2}{3} & 1 \\ 1 & \frac{5}{3} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

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Algebra

Problem # *4: Fundamental theorem of algebra (FTA).*

– 4.1: State the fundamental theorem of algebra (FTA).

Sol: There are multiple definitions of the FTA, which of course must be equivalent.
Here are three (equivalent) answers from Wikipedia

1. The fundamental theorem of algebra states that every non-constant single-variable polynomial with complex coefficients has at least one complex root. This may then be applied recursively till the degree is zero.

2. Every degree n polynomial with complex coefficients has, counted with multiplicity, exactly n roots. The equivalence of the two statements can be proven through the use of successive polynomial division.

3. The field of complex numbers is algebraically closed. Note: this one requires an understanding of the term *algebraically closed*.

Wikipedia warns:

> In spite of its name, there is no purely algebraic proof of the theorem, since any proof must use the completeness of the reals (or some other equivalent formulation of completeness), which is not an algebraic concept.

■

(13 pts) Algebra with complex variables

Problem # *5: (7 pts) Order and complex numbers:*
One can always say that $3 < 4$—namely, that real numbers have order. One way to view this is to take the difference and compare it to zero, as in $4 - 3 > 0$. Here we will explore how complex variables may be ordered. In the following define $\{x, y\} \in \mathbb{R}$ and complex variable $z = x + y\jmath \in \mathbb{C}$.

– *5.1: Explain the meaning of* $|z_1| > |z_2|$.
Sol: $|z| = \sqrt{x^2 + y^2}$ is the length of z, so the above expression says that a disk of radius $|z_1|$ contains a second disk of radius $|z_2|$. ■

– *5.2: If* $x_1, x_2 \in \mathbb{R}$ *(are* real *numbers), define the meaning of* $x_1 > x_2$.
Sol: This conditions is the same as $x_1 - x_2 > 0$. Order is meaningful on the real line, as a length. ■

– *5.3: Explain the meaning of* $z_1 > z_2$.
Sol: It makes no sense to order complex numbers. A complex number has both a length and an angle (it is the same as a vector). The concept of an angle extends the sign of a real number, making order impossible. To show this, place two points on a plane, and ask which is larger than the other. The order of the x and y components, each have order. Thus order cannot be defined. ■

– *5.4: (2 pts) What is the meaning of* $|z_1 + z_2| > 3$*?*
Sol: Define $z_3 = z_1 + z_2$. Then the problem becomes $|z_3| > 3$, which is a disk of radius $3 > 0$. Thus the solution is all values of $z_1 + z_2$ outside, but not including, a circle of radius 3. ■

– *5.5: (2 pts) If time were complex, how might the world be different?*
Sol: As best we know, time is real, thus it has the *order* property: the is a past, present and future. If time were complex this would not be the case. Thus if time were complex, the past could be the future. ■

Problem # *6: (1 pt) It is frequently necessary to consider a function* $w(z) = u + v\jmath$ *in terms of the real functions* $u(x, y)$ *and* $v(x, y)$ *(e.g. separate the real and imaginary parts). Similarly, we can consider the inverse* $z(w) = x + y\jmath$*, where* $x(u, v)$ *and* $y(u, v)$ *are real functions.*

– *6.1: (1 pts) Find* $u(x, y)$ *and* $v(x, y)$ *for* $w(z) = 1/z$.

Sol: Multiply by the complex conjugate $x - y\jmath$

$$w = \frac{1}{x + y\jmath} = \frac{x - y\jmath}{x^2 + y^2}$$

Therefore $u(x, y) = \frac{x}{x^2+y^2}$ and $v(x, y) = \frac{-y}{x^2+y^2}$. ■

Problem # *7: (5 pts) Find* $u(x, y)$ *and* $v(x, y)$ *for* $w(z) = c^z$ *with complex constant* $c \in \mathbb{C}$ *for questions 7.1, 7.2, and 7.3:*

– *7.1:* $c = e$
Sol: Since $u + iv = e^z = e^{x+y\jmath} = e^x(\cos y + \jmath \sin y)$,

$$u = e^x \cos y$$

and

$$v = e^x \sin y.$$

■

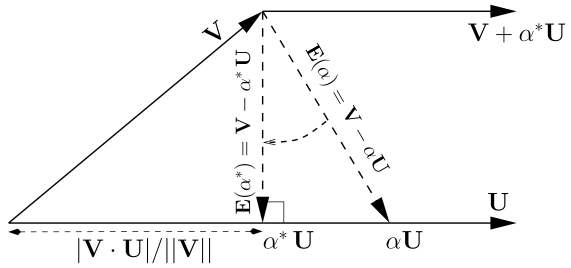

Figure 2.2: *This figure shows how to derive the Schwarz inequality, by finding the value of $\alpha = \alpha^*$ corresponding to $\min_\alpha[\boldsymbol{E}(\alpha)]$. It is identical to Fig. 3.5 on page 91.*

– *7.2: $c = 1$ (recall that $1 = e^{\pm \jmath 2\pi k}$ for $k \in \mathbb{Z}$*

Sol: From the general formula with $c = 1$

$$1^z = e^{z \log 1} e^{\jmath k 2\pi z} = e^0 e^{\jmath k 2\pi z} = e^{-yk2\pi} e^{-\jmath x k 2\pi}$$

where $k \in \mathbb{Z}$ is a signed counting integer. Thus $u = e^{-k2\pi y} \cos k2\pi x$ and $v = e^{-k2\pi y} \sin k2\pi x$. ■

– *7.3: $c = \jmath$.* Hint: $\jmath = e^{\jmath\pi/2 + \jmath 2\pi k}, \quad k \in \mathbb{Z}$.

Sol: $\jmath^\jmath = \left(e^{\jmath\pi/2+\jmath 2\pi m}\right)^\jmath = e^{\jmath\jmath\pi/2+\jmath\jmath 2\pi m} = e^{-\pi/2}\, e^{-2\pi m} = 0.2079\, e^{-2\pi m}$.

Thus for $m = 0$, $\jmath^z = \left(e^{\jmath\pi/2}\right)^z = e^{\jmath z\pi/2} = e^{\jmath(x+\jmath y)\pi/2} = e^{(\jmath x - y)\pi/2} = e^{-\pi y/2}(\cos(x\pi/2) + \jmath \sin(x\pi/2))$. ■

– *7.4: (2 pts) What is $\jmath^\jmath$?*

Sol: Since $\jmath = e^{\frac{\pi}{2}\jmath}$, then $\jmath^\jmath = e^{\frac{\pi}{2}\jmath\jmath} = e^{-\pi/2} \approx 0.20788$.

Expanding this in a continued-fraction expansion using Matlab's `rat(exp(-pi/2))` function gives $[0; 5, -5, -4, 3, -3, 3, \cdots]$. ■

Schwarz inequality

Problem # *8: The above figure shows three vectors for an arbitrary value of $\alpha \in \mathbb{R}$ and a specific value of $\alpha = \alpha^*$.*

– *8.1: Find the value of $\alpha \in \mathbb{R}$ such that the length (norm) of $\vec{E}$ (i.e., $||\vec{E}|| \geq 0$) is minimum. Show your derivation, not the answer ($\alpha = \alpha^*$).*

Sol: In Fig. **??** we see vectors V, U, and for reference, $V + \alpha^* U$. Also shown are scaled values of U, αU and $\alpha^* U$. The setup for the derivation is

$$||\boldsymbol{E}(a)||^2 = \boldsymbol{E} \cdot \boldsymbol{E} = (\vec{V} + \alpha\vec{U}) \cdot (\vec{V} + \alpha\vec{U}) \geq 0. \qquad \text{(AE-3.4)}$$

Minimize with respect to α.

When U is scaled by α^*, length $||E(\alpha^*)||$ is minimum, and $(V - \alpha^* U) \perp U$, namely vector $E(\alpha^*)$ is $\perp$ to vector U. This follows from $\frac{\partial}{\partial\alpha}||\vec{E}||^2 = \frac{\partial}{\partial\alpha}((\vec{V} + \alpha\vec{U}) \cdot (\vec{V} + \alpha\vec{U})) = 2(\vec{V} + \alpha\vec{U}) \cdot \vec{U} = 0$. Thus

$$\alpha^* = -\frac{\vec{V} \cdot \vec{U}}{||\vec{U}||^2}$$

■

– *8.2: Find the formula for $||\boldsymbol{E}(\alpha^*)||^2 \geq 0$.* Hint: Substitute α^* into Eq. 3.5.9 (p. 92) and show that this results in the Schwarz inequality

$$|\vec{U} \cdot \vec{V}| \leq ||\vec{U}||||\vec{V}||.$$

Sol: From Eq. 3.5.9

$$||\boldsymbol{V}||^2 + 2\alpha^* \boldsymbol{V} \cdot \boldsymbol{U} + (\alpha^*)^2\ ||\boldsymbol{U}||^2 \geq 0$$

Substituting α^* gives

$$||\boldsymbol{V}||^2||\boldsymbol{U}||^2 - 2(\boldsymbol{V} \cdot \boldsymbol{U})^2 + |\boldsymbol{U} \cdot \boldsymbol{V}|^2 \geq 0.$$

Simplifying

$$||\boldsymbol{V}||^2||\boldsymbol{U}||^2 \geq |\boldsymbol{U}\cdot\boldsymbol{V}|^2$$

and taking the square root (and swap order), gives the *Schwarz inequality*

$$|\vec{U}\cdot\vec{V}| \leq ||\vec{U}||||\vec{V}||.$$

■

Problem # *9: Geometry and scaler products*

– 9.1: What is the geometrical meaning of the dot product of two vectors?

Sol: The dot product of two vectors is the length of the $\perp$ projection of one vector on the other. According to the Schwarz inequality, this project length must be less than the product of the lengths of the two vectors. ■

– 9.2: Give the formula for the dot product of two vectors. Explain the meaning based on Fig. 3.4 (page 87).

Sol: $\vec{V}\cdot\vec{U} = ||\vec{V}||||\vec{U}||\cos\theta_{\vec{V},\vec{U}}$. $\vec{V}\cdot\vec{U} = ||\vec{V}||||\vec{U}||\cos\theta_{\vec{V},\vec{U}}$. It represents the amount of one vector going in the direction of the other. In a drawing, it is a projection of the one on the other, found by dropping the $\perp$ from the tip of one, on the other. ■

– 9.3: Write the formula for the dot product of two vectors $\vec{U}\cdot\vec{V}$ in $\mathbb{R}^n$ in polar form (e.g., assume the angle between the vectors is θ).

Sol: $\vec{U}\cdot\vec{V} = \sum_{i=1}^{n} a_i b_i (= ||\vec{U}||\;||\vec{V}||\cos(\theta))$. This last relationship defines the angle between two vectors. ■

– 9.4: How is the Schwarz inequality related to the Pythagorean theorem?

Sol: It says that for a right triangle, the case when $a = a^*$, the lengths of the two vectors must be greater than the projection of one on the other, unless they are co-linear (i.e., the angle between them is zero). ■

– 9.5: Starting from $||\boldsymbol{U}+\boldsymbol{V}||$, derive the triangle inequality

$$||\vec{U}+\vec{V}|| \leq ||\vec{U}|| + ||\vec{V}||.$$

Sol: $||\vec{U}+\vec{V}||^2 = (\vec{U}+\vec{V})\cdot(\vec{U}+\vec{V}) = ||\boldsymbol{U}||^2 + ||\boldsymbol{V}||^2 + 2\boldsymbol{U}\cdot\boldsymbol{V} \leq ||\boldsymbol{U}||^2 + ||\boldsymbol{V}||^2 + 2|\boldsymbol{U}\cdot\boldsymbol{V}|$ Using the Schwarz inequality we find $||\vec{U}+\vec{V}||^2 \leq ||\boldsymbol{U}||^2 + ||\boldsymbol{V}||^2 + 2||\boldsymbol{U}||\;||\boldsymbol{V}||$. Completing the square on the right gives $||\vec{U}+\vec{V}||^2 \leq (||\boldsymbol{U}|| + ||\boldsymbol{V}||)^2$. Final taking the square root gives the *triangle inequality*. ■

– 9.6: The triangle inequality $||\vec{U}+\vec{V}|| \leq ||\vec{U}|| + ||\vec{V}||$ *is true for two and three dimensions: Does it hold for five-dimensional vectors?*

Sol: It is true in any number of dimensions. ■

– 9.7: Show that the wedge product $\vec{U}\wedge\vec{V} \perp \vec{U}\cdot\vec{V}$.

Sol: $\vec{V}\wedge\vec{U} = ||\vec{V}||||\vec{U}||\sin\theta_{\vec{V},\vec{U}}$ while $\vec{V}\cdot\vec{U} = ||\vec{V}||||\vec{U}||\cos\theta_{\vec{V},\vec{U}}$. Thus they are perpendicular. This is true in any number of dimensions. See the discussion in the text on the wedge product. ■