2.2 Problems AE-2

Topics of this homework:

Linear vs nonlinear systems of equations, Euclid's formula, Gaussian elimination, matrix permutations, Ohm's law, two-port networks,

Deliverables: Answers to problems

Gaussian elimination

Problem # *1: Gaussian elimination*

– *1.1: Find the inverse of*

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Sol:

$$A^{-1} = \frac{1}{3-8}\begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}.$$

■

– *1.2: Verify that* $A^{-1}A = AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Sol: Multiply them to show this. ■

Problem # 2: *Find the solution to the following 3×3 matrix equation $Ax = b$ by GE. Show your intermediate steps. You can check your work at each step using Octave/Matlab.*

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 8 \end{bmatrix}.$$

– 2.1 Show (i.e., verify) that the first GE matrix G_1, which zeros out all entries in the first column is given by

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Identify the elementary row operations that this matrix performs. **Sol:** Operate with GE matrix on A

$$\begin{aligned} G_1[A|b] &= \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 3 & 1 & 1 & 9 \\ 1 & -1 & 4 & 8 \end{array}\right] \\ &= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 4 & 6 \\ 0 & -2 & 5 & 7 \end{array}\right] \end{aligned}$$

It scales the first row by -3 and adds it to the second row, and scales the first row by -1 and adds it to the third row.
■

– 2.2 Find a second GE matrix, G_2, to put G_1A in upper triangular form. Identify the elementary row operations that this matrix performs.
Sol:

$$G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

which scales the second row by -1 and adds it to the third row. Thus we have

$$\begin{aligned} G_2G_1[A|b] &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 3 & 1 & 1 & 9 \\ 1 & -1 & 4 & 8 \end{array}\right] \\ &= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 4 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{aligned}$$

■

– 2.3 Find a third GE matrix G_3 that scales each row so that its leading term is 1. Identify the elementary row operations that this matrix performs.
Sol:

$$G_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which scales the second row by -1/2. Thus we have

$$\begin{aligned} G_3G_2G_1[A|b] &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 3 & 1 & 1 & 9 \\ 1 & -1 & 4 & 8 \end{array}\right] \\ &= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{aligned}$$

■

– 2.4: Find the last GE matrix, G_4, which subtracts a scaled version of row 3 from row 2, and scaled versions of rows 2 and 3 from row 1, such that you are left with the identity matrix ($G_4G_3G_2G_1A = I$).
Sol:

$$G_4 = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Two linear equations

Problem # *3 In this problem we transition from a general pair of equations*

$$f(x,y)=0$$
$$g(x,y)=0$$

to the important case of two linear equations

$$y=ax+b$$
$$y=\alpha x+\beta.$$

Note that to help keep track of the variables, roman coefficients (a,b) are used for the first equation and Greek (α,β) for the second.

– 3.1: What does it mean, graphically, if these two linear equations have (1) a unique solution, (2) a nonunique solution, or (3) no solution?
Sol: There are three possibilities:

1. When they have different slopes, they meet at one (x,y) point, which is the solution.
2. If the two lines are identical, any point on the line is a solution.
3. If they have the same slope but different intercepts (are parallel to each other) there is no solution.

■

– 3.2: Assuming the two equations have a unique solution, find the solution for x and y.
Sol: Since there must be one point where the two are equal, we may solve for that by setting the y values equal to each other:

$$ax+b=\alpha x+\beta$$

Thus

$$x=\frac{\beta-b}{a-\alpha}$$
$$y=a\frac{\beta-b}{a-\alpha}+b$$

■

– 3.3: When will this solution fail to exist (for what conditions on a, b, α, and β)?
Sol: As stated above, if they have the same slope $\alpha=a$ but different intercepts $\beta\neq b$, there is no solution. When $\beta=b$ and $\alpha=a$ every point on the line is a solution. ■

– 3.4: Write the equations as a 2×2 matrix equation of the form $A\vec{x}=\vec{b}$, where $\vec{x}=\{x,y\}^T$.
Sol:

$$\begin{bmatrix}1 & -a\\ 1 & -\alpha\end{bmatrix}\begin{bmatrix}y\\ x\end{bmatrix}=\begin{bmatrix}b\\ \beta\end{bmatrix}$$

■

– 3.5: Find the inverse of the 2×2 matrix, and solve the matrix equation for x and y.
Sol:

$$\begin{bmatrix}y\\ x\end{bmatrix}=\frac{1}{\Delta}\begin{bmatrix}-\alpha & a\\ -1 & 1\end{bmatrix}\begin{bmatrix}b\\ \beta\end{bmatrix}=\frac{1}{\Delta}\begin{bmatrix}-\alpha b+a\beta\\ -b+\beta\end{bmatrix}$$

where the determinant is $\Delta\equiv a-\alpha$. ■

– 3.6: Discuss the properties of the determinant of the matrix (Δ) in terms of the slopes of the two equations (a and α).
Sol: When the slopes are the same there is no solution and $\Delta=0$. Thus the matrix solution is consistent with the geometry. This is our first result in analytic geometry. ■

Problem # *4: The application of* linear functional relationships *between* two variables

We use 2×2 matrices to describe two-port networks, as discussed in Sec. 3.8 (p. 108). Transmission lines are a great example: Both voltage and current must be tracked as they travel along the line. Figure 3.8 (p. 112) shows an example segment of a transmission line.

Suppose you are given the following pair of linear relationships between the input (source) variables V_1 and I_1 and the output (load) variables V_2 and I_2 of the transmission line:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \jmath & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}.$$

– 4.1: Let the output (the load) be $V_2 = 1$ and $I_2 = 2$ (i.e., V_2/I_2 =1/2 $\{\Omega\}$). Find the input voltage and current, V_1 and I_1.

Sol: This case corresponds to

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \jmath & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1\jmath + 2 \\ 1 - 2 \end{bmatrix}$$

Thus $V_1 = 2 + 1\jmath$ and $I_1 = -1$. ■

– 4.2: Let the input (source) be $V_1 = 1$ and $I_1 = 2$. Find the output voltage and current, V_2 and I_2.

Sol: With the input specified the two equations are

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \jmath & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}.$$

To find the input we must invert the matrix ($\Delta = -j - 1$)

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \frac{1}{1+j} \begin{bmatrix} 1 & 1 \\ 1 & -\jmath \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

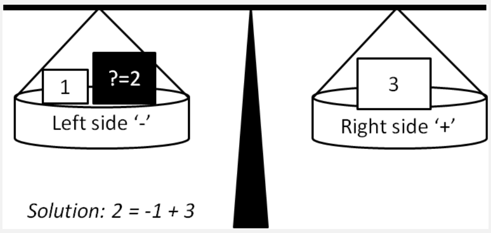
Thus $V_2 = 3/(1+\jmath) = 3(1-\jmath)/2$, $I_2 = (1-2\jmath)/(1+1\jmath) = -(1+3\jmath)/2$. The point of this exercise is that the two lines have a complex intersection point, not easily visualized. ■

Integer equations: applications and solutions

Any equation for which we seek only integer solutions is called a *Diophantine* equation.

Problem # *5: A practical example of using a Diophantine equation:*

"A merchant had a 40-pound weight that broke into 4 pieces. When the pieces were weighed, it was found that each piece was a whole number of pounds and that the four pieces could be used to weigh every integral weight between 1 and 40 pounds. What were the weights of the pieces?" - *Bachet de Bèziriac (1623)*[a].
Here, weighing is performed using a balance scale that has two pans, with weights on either pan. Thus, given weights of 1 and 3 pounds, one can weigh a 2-pound weight by putting the 1-pound weight in the same pan with the 2-pound weight, and the 3-pound weight in the other pan. Then the scale will be balanced. A solution to the four weights for Bachet's problem is $1 + 3 + 9 + 27 = 40$ pounds.



– *5.1: Show how the combination of 1-, 3-, 9-, and 27-pound weights can be used to weigh* $1, 2, 3, \ldots, 8, 28,$ *and* 40 *pounds of milk (or something else, such as flour). Assuming that the milk is in the* left *pan, provide the position of the weights using a negative sign* $-$ *to indicate the left pan and a positive sign* $+$ *to indicate the right pan. For example, if the left pan has 1 pound of milk, then 1 pound of milk in the right pan,* $+1$*, will balance the scales.*

Hint: It is helpful to write the answer in matrix form. Set the vector of values to be weighed equal to a matrix indicating the pan assignments, multiplied by a vector of the weights $[1, 3, 9, 27]^T$. The pan assignments matrix should contain only the values -1 (left pan), $+1$ (right pan), and 0 (leave out). You can indicate these using $-$, $+$, and blanks.
Sol: Any integer between 1 and 40 may be expanded using the weights 1, 3, 9, 27. Here is the problem stated in matrix form:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \ldots \\ 28 \\ \ldots \\ 40 \end{bmatrix} = \begin{bmatrix} + & & & \\ - & + & & \\ & + & & \\ + & + & & \\ - & - & + & \\ & - & + & \\ + & - & + & \\ - & & + & \\ \ldots & & & \\ + & & & + \\ \ldots & & & \\ + & + & + & + \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 9 \\ 27 \end{bmatrix}$$

The left column is the weight of the milk. The right-most column are the four weights. It should be clear that these four weights span the integers from 1-40 with binary weights. Each weight may be computed recursively from twice the sum of the previous weights +1, that is

$$W_{n+1} = 2W_n + 1 = 2^{n+1} \quad \text{since} \quad W_n = 2^n.$$

For example to get 26 we place weights 9+3+1 in the pan with 26, and get 27-1. For example 27 = 2*(9+3+1)+1 is the next weight. Recursively, the weights are 3=2*1+1, 9=2*(3+1)+1, 27=2*(9+3+1)+1. The next weight (not shown) would be: 81=2*(27+9+3+1)+1 = 2*40+1. ■

[a] Taken from Rotman (1996, p. 50)

Vector algebra in $\mathbb{R}^3$

Definitions of the scalar (also called a dot product) $\boldsymbol{A}\cdot\boldsymbol{B}$, cross $\boldsymbol{A}\times\boldsymbol{B}$ and triple product $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})$, may be found in Appendix A (p. 215), where $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$ in $\mathbb{R}^3\subset\mathbb{C}^3$), as shown in Fig. 3.4, p. 87. A fourth "double-cross" (☠) vector product is:[2]

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \alpha_o\boldsymbol{B} - \beta_o\boldsymbol{C}.$$

where $\alpha_o = \boldsymbol{A}\cdot\boldsymbol{C}$ and $\beta_o = \boldsymbol{A}\cdot\boldsymbol{B}$ (Note: $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) \neq (\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C}$).

Problem # *6:* Scalar product $\mathbf{A}\cdot\mathbf{B}$

– 6.1: If $\boldsymbol{A} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}$ and $\boldsymbol{B} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}} + b_z\hat{\mathbf{z}}$, write out the definition of $\mathbf{A}\cdot\mathbf{B}$.

Sol: See the definition in the above figure. $\boldsymbol{A}\cdot\boldsymbol{B} = a_xb_x + a_yb_y + a_zb_z$. In general: $\boldsymbol{A}\cdot\boldsymbol{B} = \sum_k A_kB_k$. ■

– 6.2: The dot product is often defined as $||\boldsymbol{A}||\,||\boldsymbol{B}||\cos(\theta)$, where $||\boldsymbol{A}|| = \sqrt{\boldsymbol{A}\cdot\boldsymbol{A}}$ and θ is the angle between $\boldsymbol{A},\boldsymbol{B}$. If $||\boldsymbol{A}|| = 1$, describe how the dot product relates to the vector $\mathbf{B}$.

Sol: See the definition in the above figure. The vector product is the portion of $\mathbf{B}$ in the direction of $\mathbf{A}$. ■

Problem # *7:* Vector (cross) product $\mathbf{A}\times\mathbf{B}$

– 7.1: If $\boldsymbol{A} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}$ and $\boldsymbol{B} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}} + b_z\hat{\mathbf{z}}$, write out the definition of $\mathbf{A}\times\mathbf{B}$.

Sol:

$$\boldsymbol{A}\times\boldsymbol{B} \equiv \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix} = \hat{\mathbf{x}}\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix} - \hat{\mathbf{y}}\begin{vmatrix}a_x & a_z\\ b_x & b_z\end{vmatrix} + \hat{\mathbf{z}}\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}.$$

■

– 7.2: Show that the cross product is equal to the area of the parallelogram formed by $\boldsymbol{A},\boldsymbol{B}$, namely $||\boldsymbol{A}||\,||\boldsymbol{B}||\sin(\theta)$, where $||\boldsymbol{A}|| = \sqrt{\boldsymbol{A}\cdot\boldsymbol{A}}$ and θ is the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$.

Sol: A parallelogram's area is equal to its base times its height. Therefore, let's say the base is length $||A||$, and the height $||B||\sin(\theta)$, which is the portion of $\mathbf{B}$ that is perpendicular to $\mathbf{A}$. ■

Problem # *8:* *Triple product $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})$*

Let $\boldsymbol{A} = [a_1, a_2, a_3]^T$, $\boldsymbol{B} = [b_1, b_2, b_3]^T$, $\boldsymbol{C} = [c_1, c_2, c_3]^T$ be three vectors in $\mathbb{R}^3$.

– 8.1: Describe why $|\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})|$ is the volume of parallelepiped generated by $\boldsymbol{A}$,$\boldsymbol{B}$, and $\boldsymbol{C}$.

Sol: Note that the norm of $\boldsymbol{B}\times\boldsymbol{C}$ is the area of the parallelogram generated by $\boldsymbol{C}$ and $\boldsymbol{B}$. Taking the dot product with $\mathbf{A}$ results in the volume of the corresponding parallelepiped (prism). So the absolute value of triple product is volume of parallelepiped. ■

– 8.2: Explain why three vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are in one plane if and only if the triple product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{0}$.

Sol: (triple product is zero) if and only if: (volume is zero), if and only if: (they are in the same plane) ■

Problem # *9: Given two vectors $\boldsymbol{A},\boldsymbol{B}$ in the $\hat{\mathbf{x}},\hat{\mathbf{y}}$ plane shown in Fig.3.4 with $\boldsymbol{B} = \hat{\mathbf{y}}$ (i.e., $||\boldsymbol{B}|| = 1$).*

– 9.1: Show that $\boldsymbol{A}$ may be split into two orthogonal parts, one in the direction of $\boldsymbol{B}$ and the other perpendicular ($\perp$) to $\boldsymbol{B}$. Hint: Express the vector products of $\boldsymbol{A}$ and $\boldsymbol{B}$ (dot and cross) in polar coordinates (Greenberg, 1988).

$$\begin{aligned}\boldsymbol{A} &= (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{B} + \boldsymbol{B}\times(\boldsymbol{A}\times\boldsymbol{B})\\ &= \boldsymbol{A}_{\|} + \boldsymbol{A}_{\perp}.\end{aligned}$$

Sol: Expressing the vector products in polar form makes this result transparent:

$$\boldsymbol{A}\cdot\boldsymbol{B} = ||\boldsymbol{A}||\cos(\theta) \quad \text{and} \quad \boldsymbol{A}\times\boldsymbol{B} = ||\boldsymbol{A}||\sin(\theta)$$

[2] Greenberg p. 694, Eq. 8.

The first quantity is in the direction of $\boldsymbol{B}$, while the second is in the direction $\boldsymbol{A}\times\boldsymbol{B}$, which is $\perp$ to $\boldsymbol{B}$. Thus

$$\begin{aligned}\boldsymbol{A} &= ||\boldsymbol{A}||\,(\boldsymbol{B}\cos(\theta) + \boldsymbol{A}\times\boldsymbol{B}\sin(\theta))\\ &= \boldsymbol{A}_{\|} \qquad\qquad\quad + \boldsymbol{A}_{\perp}.\end{aligned}$$

■

Ohm's Law

In general, impedance is defined as the ratio of a force to a flow. For electrical circuits, the voltage is the force and the current is the flow. Ohm's law states that the voltage across and the current through a circuit element are related by the impedance of that element (which may be a function of frequency). For resistors, the voltage over the current is called the *resistance* and is a constant (e.g., the simplest case is $V/I = R$). For inductors and capacitors, the voltage over the current is a frequency-dependent impedance (e.g., $V/I = Z(s)$, where s is the complex frequency $s \in \mathbb{C}$).

As shown in Table 3.2 (p. 111), the impedance concept also holds in mechanics and acoustics. In mechanics, the force is equal to the mechanical force on an element (e.g., a mass, dashpot, or spring) and the flow is the velocity. In acoustics, the force is pressure and the flow is the volume velocity or particle velocity of air molecules.

Case	Force	Flow	Impedance	units
Electrical	voltage (V)	current (I)	Z	Ohms [Ω]
Mechanics	force (F)	velocity (V)	Z	Mechanical Ohms [Ω]
Acoustics	pressure (P)	particle velocity (U)	Z	Acoustic Ohms [Ω]
Thermal	temperature (T)	heat-flux (J)	Z	Thermal Ohms [Ω]

Problem # *10: The resistance of an incandescent (filament) lightbulb, measured cold, is about 100 ohms. As the bulb lights up, the resistance of the metal filament increases.*
Ohm's law says that the current

$$\frac{V}{I} = R(T),$$

where T is the temperature. In the United States, the voltage is 120 volts (RMS) at 60 [Hz]. Find the current when the light is first switched on. **Sol:** Thus the current is

$$I = 120/R = 120/100 = 1.2. \qquad \text{[Amps]}$$

As the bulb heats up, the current rapidly drops, and the resistance increases. This typically takes less than a milliseconds [ms], which depends on the wattage of the light bulb. Such lightbulbs are *nonlinear.* These rules don't apply to LED bulbs. ■

Problem # *11: The power in watts is the product of the force and the flow. What is the power of the lightbulb of Problem 10?*
Sol: $P = V \cdot I = 120 \times 1.2 = 120 + 24 = 144$ [W]. ■

Problem # *12: State the impedance $Z(s)$ of each of the following circuit elements: (1) a resistor with resistance R, (2) an inductor with inductance L, and (3) a capacitor with capacitance C.*

Sol: (1) For the resistor, $Z = R$.

(2) For the inductor, $Z = sL$ with $s = \sigma + \omega\jmath$. Note the flux $\psi(t) = Li(t)$. The voltage $v(t)$ is the time derivative of the flux

$$v(t) = \frac{d\psi(t)}{dt} = L\frac{di(t)}{dt}.$$

(3) For the capacitor, $Z = 1/sC$. Note the charge $q(t) = Cv(t)$, thus the current $i(t)$ is the time derivative of the charge

$$i(t) = \frac{d}{dt}q(t) = C\frac{dv(t)}{dt}.$$

■

Problem # *13: Describe the temperature as a function of time when 1 kg of water, 1 kg of ice and 1 kgm of steam are place in an insulated chamber.*

Sol: In electrical terms this is the same problem of connecting two capacitors together. To solve that problem you need to know the size of the capacitors and the amount each is charge on each capacitor. Thus we know that $Q_1 = C_1V_2$ and $Q_1 = C_2V_2$.

To find the voltage when connecting two capacitors together we need to know the charge on each capacitor and its capacitance. We also need to know the resistance of the wire that connects the two capacitors. Once the switch is closed, the charge will flow between them,, until the voltages are equal. The problem is greatly confounded by the heat created as the current flows between the two capacitors. It is reasonable to ignore this heat loss and simply split the charge between the two capacitors by assuming the final voltage is the same, namely assume that once the system is in equilibrium, $Q_1C_1 = Q_2C2$, or $Q_1/Q_2 = C_2/C_1$.

For the case of water, the masses of water are equal, but the heat capacity of water and ice are different. To solve this problem we need to know the heat capacity of water in its various phases $C_{pl} = 4.218$ [kJ/kg C] for liquid water , $C_{pi} = 2.05$ [kJ/kg C] for ice, and $C_{pv} = 1.859$ [kJ/kg C] for vapor (steam) (Ambaum, 2010, p. xi). The triple point for water is where all three phases exist at the same time, which requires the application of pressure to the water of $P_{tp} = 0.6117$ [kPa] (6.03659×10^{-3} [atm]) at $T_{tp} = 0.01$°C.

This problem is not that different from conecting two car batteries together, which is a very good way to start a fire, due to the extremely low resistance of a car battery. ■

Problem # *14: Consider what happens at the triple point of water. As water freezes or thaws, the temperature remains constant at 0 (C°). Once all the water is frozen and more heat is removed, the temperature drops below 0°. As heat is added, water thaws but the temperature remains at 0°. Once all the ice has melted, what is the temperature as more heat is added?*

Model the triple point using a Zener diode, a resistor, and a capacitor. A Zener diode holds the voltage constant independent of current. For the case of water's triple point, the voltage represents the temperature of water at the triple point, clamped at 0 [C°]. The current represents the heat flux. The latent heat of water at the triple point is 32 Cal/gm. Thus as the temperature rises from below freezing, the water is clamped at 0° once the triple point is reached. At that point, adding more heat flux has no effect on the temperature until all the ice melts. Once the ice has melted, the temperature again begins to rise until it hits the boiling point, where it again stays at 100° until all the water has evaporated. **Sol:** Need a figure here showing how to model the triple point of water. The Heat capacity may be modeled by a capacitor, which is fixed at 0°as the capacitor discharges. Once it is empty, the temperature again begins to rise as the heat Q from the sun is added

$$T^\circ = mcQ.$$

Thus the required circuit needs to emulate this temperature behavior due to the latent heat of melting ice and boiling water into steam. ■

Nonlinear (quadratic) to linear equations

In the following problems we deal with algebraic equations in more than one variable that are not linear equations. For example, the circle $x^2 + y^2 = 1$ may be solved for $y(x) = \pm\sqrt{1 - x^2}$. If we let $z_+ = x + y\jmath = x + \jmath\sqrt{1 - x^2} = e^{\theta\jmath}$, we obtain the equation for half a circle ($y > 0$). The entire circle is described by the magnitude of z as $|z|^2 = (x+y\jmath)(x-y\jmath) = 1$.

Problem # *15: Give the curve defined by the equation:*

$$x^2 + xy + y^2 = 1$$

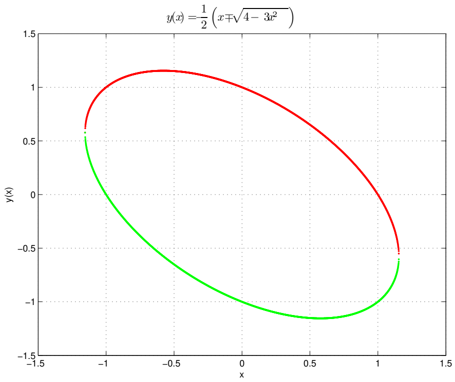
– *15.1: Find the function $y(x)$.*

Sol: Completing the square in y and solve for $y(x)$:

$$\begin{aligned}(y + x/2)^2 - x^2/4 + x^2 &= 1\\ (y + x/2)^2 &= 1 - \frac{3}{4}x^2\\ y + x/2 &= \pm\sqrt{\frac{4 - 3x^2}{4}}\\ y &= \frac{1}{2}\left(\pm\sqrt{4 - 3x^2} - x\right)\end{aligned}$$

■

– *15.2: Using Matlab/Octave, plot $y(x)$ and describe the graph.*
Sol:



Thus we find the equation is a rotated ellipse. ■

– *15.3: What is the name of this curve?*
Sol: It is an ellipse, rotated by 45 degrees. ■

– *15.4: Find the solution (in x, p, and q) to these equations:*

$$\begin{aligned} x + y &= p \\ xy &= q. \end{aligned}$$

Sol: Solve the first equation for y as $y = p - x$, and then substitute it into the second equation

$$x(p - x) = -x^2 + px = q.$$

Thus we find the quadratic

$$x^2 - px + q = 0$$

having roots given by completing the square

$$(x - p/2)^2 = (p/2)^2 - q.$$

resulting in $x = p/2 \pm \sqrt{(p/2)^2 - q}$, $y = p - x$.

Summary: Here we started with one linear and one quadratic (hyperbola). By the use of composition we found the roots. ■

– *15.5: Find an equation that is linear in y starting from equations that are quadratic (second-degree) in the two unknowns x and y:*

$$x^2 + xy + y^2 = 1 \qquad \text{(AE-2.1)}$$
$$4x^2 + 3xy + 2y^2 = 3. \qquad \text{(AE-2.2)}$$

Sol: The goal is to obtain a linear equation in y.
Method 1: remove xy term: Scale the upper equation by 3 and subtract from the lower:

$$\begin{aligned} 4x^2 + 3xy + 2y^2 &= 3 \\ 3x^2 + 3xy + 3y^2 &= 3 \end{aligned}$$

giving $x^2 - y^2 = 0$, or $x = \pm y$.
This results in the two equations

$$\begin{aligned} x^2 \qquad -y^2 &= 0 \\ x^2 + xy + y^2 &= 1 \end{aligned}$$

Adding these gives $2x^2 \pm x^2 = 1$, which is $3x^2 = 1$ and $x^2 = 1$. Thus the final solutions are $x = \pm y = \pm 1/\sqrt{3}$ and $x = \pm y = \pm 1$. ■

– 15.6: Compose the following two quadratic equations and describe the results.

$$\begin{aligned} x^2 + xy + y^2 &= 1 \\ 2x^2 + xy \quad &= 1 \end{aligned}$$

Sol: By isolating y from one of the two equations, we may remove it from the other equation, giving us a single 4^{th} degree equation in x:

$$x^2 + (1 - 2x^2) + (1 - 2x^2)^2/x^2 = 1$$

or

$$x^4 + x^2 - 2x^4 + 1 - 4x^2 + 4x^4 - x^2 = 0$$

Collecting terms

$$3x^4 - 4x^2 + 1 = 0$$

This is a quartic, but quadratic in x^2. Thus it may be solved for x^2 by the completion of squares

$$\begin{aligned} x^4 - \frac{4}{3}x^2 &= -\frac{1}{3} \\ \left(x^2 - \frac{2}{3}\right)^2 &= \frac{1}{3}\left(\frac{4}{3} - 1\right) \\ x^2 &= \frac{2}{3} \pm \frac{1}{3} \\ x &= \pm\frac{\sqrt{2 \pm 1}}{\sqrt{3}} = \pm\frac{1}{\sqrt{3}} \text{ and } \pm\frac{2}{\sqrt{3}} \end{aligned}$$

resulting in four roots. ■

Nonlinear intersection in analytic geometry

Euclid's formula for Pythagorean triplets (Eq. 2.5.6, p. 40) can be derived by intersecting a circle and a secant line. Consider the nonlinear equation of a unit circle having radius 1, centered at $(x, y) = (0, 0)$,

$$x^2 + y^2 = 1,$$

and the secant line through $(-1, 0)$,

$$y = t(x + 1),$$

a linear equation having slope t and intercept $x = -1$. If the slope $0 < t < 1$, the line intersects the circle at a second point (a, b) in the positive x, y quadrant. The goal is to find $a, b \in \mathbb{N}$ and then show that $c^2 = a^2 + b^2$. Since the construction gives a right triangle with short sides $a, b \in \mathbb{N}$, then it follows that $c \in \mathbb{N}$.

Problem # *16: Derive Euclid's formula*

– 16.1: Draw the circle and the line, given a positive slope $0 < t < 1$.
Sol: Sol in given in Fig. 3.7 ■

Problem # *17: Substitute $y = t(x + 1)$ (the line equation) into the equation for the circle, and solve for $x(t)$.*
Hint: Because the line intersects the circle at two points, you will get two solutions for x. One of these solutions is the trivial solution $x = -1$. **Sol:** $x(t) = (1 - t^2)/(1 + t^2)$ ■

– 17.1: Substitute the $x(t)$ you found back into the line equation, and solve for $y(t)$.
Sol: $y(t) = 2t/(1 + t^2)$ ■

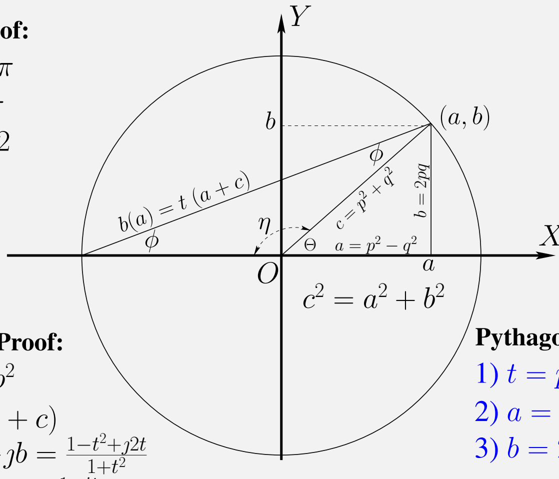
– 17.2: Let $t = q/p$ be a rational number, where p and q are integers. Find $x(p, q)$ and $y(p, q)$.
Sol: $x(p, q) = 2pq/(p^2 + q^2)$ and $y(p, q) = (p^2 - q^2)/(p^2 + q^2)$ ■

– 17.3: Substitute $x(p, q)$ and $y(p, q)$ into the equation for the circle, and show how Euclid's formula for the Pythagorean triples is generated.
Sol: Multiplying out gives $(p^2 + q^2) = (p^2 - q^2) + 2pq$ ■

For full points you must show that you understand the argument. Explain the meaning of the comment "magic happens" when t^4 cancels.

Euclidean Proof:

1) $2\phi + \eta = \pi$
2) $\eta + \Theta = \pi$
3) $\therefore \phi = \Theta/2$



Diophantus's Proof:

1) $c^2 = a^2 + b^2$
2) $b(a) = t(a + c)$
3) $\zeta(t) \equiv a + \jmath b = \frac{1-t^2+\jmath 2t}{1+t^2}$
4) $\zeta = |c|e^{i\theta} = |c|\frac{1+it}{1-it} = |c|(\cos(\theta) + i\sin(\theta))$

Pythagorean triplets:

1) $t = p/q \in \mathbb{Q}$
2) $a = p^2 - q^2$
3) $b = 2pq$
4) $c = p^2 + q^2$

Figure 2.1: *Derivation of Euclid's formula for the Pythagorean triplets (PT)* $[a, b, c]$*, based on a composition of a line, having a rational slope* $t = p/q \in \mathbb{F}$*, and a circle* $c^2 = a^2 + b^2$*,* $[a, b, c] \in \mathbb{N}$*. This analysis is attributed to Diophantus (Di·o·phan′·tus) (250 CE), and today such equations are called Diophantine (Di·o·phan′·tine)* equations. *PTs have applications in architecture and scheduling, and many other practical problems. Most interesting is their relation to Rydberg's formula for the eigenstates of the hydrogen atom (Appendix H).*