

Chapter 3

Differential equations

3.1 Problems DE-1

3.1.1 Topics of this homework:

Complex numbers and functions (ordering and algebra), complex power series, fundamental theorem of calculus (real and complex); Cauchy-Riemann conditions, multivalued functions (branch cuts and Riemann sheets)

3.1.2 Complex Power Series

Problem # 1: *In each case derive (e.g., using Taylor's formula) the power series of $w(s)$ about $s = 0$ and give the RoC of your series. If the power series doesn't exist, state why! Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at $s = 0$.*

– 1.1: $1/(1 - s)$

Ans:

– 1.2: $1/(1 - s^2)$

Ans:

– 1.3: $1/(1 + s^2)$.

Ans:

– 1.4: $1/s$

Ans:

– 1.5: $1/(1 - |s|^2)$

Ans:

Problem # 2: Consider the function $w(s) = 1/s$

– 2.1: Expand this function as a power series about $s = 1$. Hint: Let $1/s = 1/(1 - (1 - s)) = 1/(1 - (1 - s))$.

Ans:

– 2.2: What is the RoC?

Ans:

– 2.3: Expand $w(s) = 1/s$ as a power series in $s^{-1} = 1/s$ about $s^{-1} = 1$.

Ans:

– 2.4: What is the RoC?

Ans:

– 2.5: What is the residue of the pole?

Ans:

Problem # 3: Consider the function $w(s) = 1/(2 - s)$

– 3.1: Expand $w(s)$ as a power series in $s^{-1} = 1/s$. State the RoC as a condition on $|s^{-1}|$. Hint: Multiply top and bottom by s^{-1} .

Ans:

– 3.2: Find the inverse function $s(w)$. Where are the poles and zeros of $s(w)$, and where is it analytic?

Ans:

Problem # 4: Summing the series

The Taylor series of functions have more than one region of convergence.

– 4.1: Given some function $f(x)$, if $a = 0.1$, what is the value of

$$f(a) = 1 + a + a^2 + a^3 + \dots?$$

Show your work. Ans:

– 4.2: Let $a = 10$. What is the value of

$$f(a) = 1 + a + a^2 + a^3 + \dots?$$

Ans:

3.1.3 Cauchy-Riemann Equations

Problem # 5: For this problem $j = \sqrt{-1}$, $s = \sigma + \omega j$, and $F(s) = u(\sigma, \omega) + jv(\sigma, \omega)$. According to the fundamental theorem of complex calculus (FTCC), the integration of a complex analytic function is independent of the path. It follows that the derivative of $F(s)$ is defined as

$$\frac{dF}{ds} = \frac{d}{ds} [u(\sigma, \omega) + jv(\sigma, \omega)]. \quad (\text{DE-1.1})$$

If the integral is independent of the path, then the derivative must also be independent of the direction:

$$\frac{dF}{ds} = \frac{\partial F}{\partial \sigma} = \frac{\partial F}{\partial j\omega}. \quad (\text{DE-1.2})$$

The Cauchy-Riemann (CR) conditions

$$\frac{\partial u(\sigma, \omega)}{\partial \sigma} = \frac{\partial v(\sigma, \omega)}{\partial \omega} \quad \text{and} \quad \frac{\partial u(\sigma, \omega)}{\partial \omega} = -\frac{\partial v(\sigma, \omega)}{\partial \sigma}$$

may be used to show where Equation DE-1.2 holds.

– 5.1: Assuming Equation DE-1.2 is true, use it to derive the CR equations.

Ans:

– 5.2: Merge the CR equations to show that u and v obey Laplace's equations

$$\nabla^2 u(\sigma, \omega) = 0 \quad \text{and} \quad \nabla^2 v(\sigma, \omega) = 0.$$

Ans:

What can you conclude?

Ans:

Problem # 6: Apply the CR equations to the following functions. State for which values of $s = \sigma + i\omega$ the CR conditions do or do not hold (e.g., where the function $F(s)$ is or is not analytic). Hint: Review where CR-1 and CR-2 hold.

– 6.1: $F(s) = e^s$

Ans:

$$- 6.2: F(s) = 1/s$$

Ans:

3.1.4 Branch cuts and Riemann sheets

Problem # 7: Consider the function $w^2(z) = z$. This function can also be written as $w_{\pm}(z) = \sqrt{z_{\pm}}$. Assume $z = re^{i\phi}$ and $w(z) = \rho e^{i\theta} = \sqrt{r}e^{i\phi/2}$.

– 7.1: How many Riemann sheets do you need in the domain (z) and the range (w) to fully represent this function as single-valued?

Ans:

– 7.2: Indicate (e.g., using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.

Ans:

– 7.3: Use `zviz.m` to plot the positive and negative square roots $+\sqrt{z}$ and $-\sqrt{z}$. Describe what you see.

Ans:

– 7.4: Where does `zviz.m` place the branch cut for this function?

Ans:

– 7.5: Must the branch cut necessarily be in this location?

Ans:

Problem # 8: Consider the function $w(z) = \log(z)$. As in Problem 7, let $z = re^{i\phi}$ and $w(z) = \rho e^{i\theta}$.

– 8.1: Describe with a sketch and then discuss the branch cut for $f(z)$.

Ans:

– 8.2: What is the inverse of the function $z(f)$? Does this function have a branch cut? If so, where is it?

Ans:

– 8.3: Using *zviz.m*, show that

$$\tan^{-1}(z) = -\frac{j}{2} \log \frac{j-z}{j+z}. \quad (\text{DE-1.3})$$

In Fig. 4.1 (p. 134) these two functions are shown to be identical.

Ans:

– 8.4: Algebraically justify Eq. DE-1.3. Hint: Let $w(z) = \tan^{-1}(z)$ and $z(w) = \tan w = \sin w / \cos w$; then solve for e^{wj} .

Ans:

3.1.5 A Cauer synthesis of any Brune impedance

Problem # 9: One may synthesize a transmission line (ladder network) from a positive real impedance $Z(s)$ by using the continued fraction method. To obtain the series and shunt impedance values, we can use a residue expansion. Here we shall explore this method.

– 9.1: Starting from the Brune impedance $Z(s) = \frac{1}{s+1}$, find the impedance network as a ladder network.

Ans:

– 9.2: Use a residue expansion in place of the CFA floor function (Sec. 2.4.4, p. 30) for polynomial expansions. Find the residue expansion of $H(s) = s^2/(s+1)$ and express it as a ladder network.

Ans:

– 9.3: Discuss how the series impedance $Z(s, x)$ and shunt admittance $Y(s, x)$ determine the wave velocity $\kappa(s, x)$ and the characteristic impedance $z_o(s, x)$ when (1) $Z(s)$ and $Y(s)$ are both independent of x ; (2) $Y(s)$ is independent of x , $Z(s, x)$ depends on x ; (3) $Z(s)$ is independent of x , $Y(s, x)$ depends on x ; and (4) both $Y(s, x)$, $Z(s, x)$ depend on x .

Ans: