# Chapter 3

# Differential equations

## 3.1 Problems DE-1

### 3.1.1 Topics of this homework:

Complex numbers and functions (ordering and algebra), complex power series, fundamental theorem of calculus (real and complex); Cauchy-Riemann conditions, multivalued functions (branch cuts and Riemann sheets)

### 3.1.2 Complex Power Series

**Problem #** *1: In each case derive (e.g., using Taylor's formula) the power series of $w(s)$ about $s = 0$ and give the RoC of your series. If the power series doesn't exist, state why!* Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at $s = 0$.

– *1.1:* $1/(1-s)$
**Sol:** $1/(1-s) = \sum_{n=0}^{\infty} s^n$, which converges for $|s| < 1$ (e.g., the RoC is $|s| < 1$) ■

– *1.2:* $1/(1-s^2)$
**Sol:** $1/(1-s^2) = \sum_{n=0}^{\infty} s^{2n}$, which converges for $|s^2| < 1$. (e.g., the RoC is $|s| < 1$). One can also factor the polynomial, thus write it as: $\frac{1}{(1-s)(1+s)}$. There are two poles, at $s = \pm 1$, and each has an RoC of 1. ■

– *1.3:* $1/(1+s^2)$.
**Sol:** The resulting series is $1/(1+s^2) = 0.5 \sum_{n=0}^{\infty} s^n((-i)^n + (i)^n)$. The RoC is $|s| < 1$. We can see this by considering the poles of the function at $s = \pm i$; both poles are 1 from $s = 0$, the point of expansion. An alternative is to write the function as $1/(1-(is)^2) = \sum(is)^n$. ■

– *1.4:* $1/s$
**Sol:** If you try to do a Taylor expansion at $s = 0$, the first term, $w(0) \to \infty$. Thus, the Taylor series expansion in $s$ does not exist. ■

– *1.5:* $1/(1-|s|^2)$
**Sol:** The imaginary part is zero. Thus the derivative of the imaginary part is zero. Thus the CR conditions cannot be obeyed. ■

**Problem #** *2: Consider the function $w(s) = 1/s$*

– *2.1: Expand this function as a power series about $s = 1$.* Hint: Let $1/s = 1/(1-1+s) = 1/(1-(1-s))$.
**Sol:** The power series is

$$w(s) = \sum_{n=0}^{\infty} (-1)^n (s-1)^n,$$

which converges for $|s-1| < 1$.

To convince you this is correct, use the Matlab/Octave command `syms s; taylor(1/s,s,'ExpansionPoint',1)`, which is equivalent to the shorthand `syms s; taylor(1/s,s,1)`. What is missing is the logic behind this expansion, given as follows: First move the pole to $z = -1$ via the Möbius "translation" $s = z + 1$, and expand using the Taylor series

$$\frac{1}{s} = \frac{1}{1+z} = \sum_{n=0}^{\infty}(-z)^n.$$

Next back-substitute $z = s - 1$ giving

$$\frac{1}{s} = \sum(-1)^n(s-1)^n.$$

It follows that the RoC is $|z| = |s-1| < 1$, as provided by Matlab/Octave. ■

– *2.2: What is the RoC?*
**Sol:** As stated in the solution of 2.1, $|s-1| < 1$. ■

– *2.3: Expand $w(s) = 1/s$ as a power series in $s^{-1} = 1/s$ about $s^{-1} = 1$.*
**Sol:** Let $z = s^{-1}$ and expand about 1: The solution is $w(z) = z$, which has a zero at 0 thus a pole at $\infty$. ■

– *2.4: What is the RoC?*
**Sol:** $|s| > 0$ or $|z| < \infty$. ■

– *2.5: What is the residue of the pole?*
**Sol:** The pole is at $\infty$. Since $w(s) = 1/s$ and applying the definition for the residue $c_{-1} = \lim_{s\to\infty} s(1/s) = 1$. Thus residue is 1. Note that it is the amplitude of the pole, which is 1. ■

### Problem # *3: Consider the function $w(s) = 1/(2-s)$*

– *3.1: Expand $w(s)$ as a power series in $s^{-1} = 1/s$. State the RoC as a condition on $|s^{-1}|$.* Hint: Multiply top and bottom by $s^{-1}$.
**Sol:** $1/(2-s) = -s^{-1}/(1-2s^{-1}) = -s^{-1}\sum 2^n s^{-n}$. The RoC is $|2/s| < 1$, or $|s| > 2$. ■

– *3.2: Find the inverse function $s(w)$. Where are the poles and zeros of $s(w)$, and where is it analytic?*
**Sol:** Solving for $s(w)$ we find $2 - s = 1/w$ and $s = 2 - 1/w = (2w-1)/w$. This has a pole at 0 and a zero at $w = 1/2$. The RoC is therefore from the expansion point out to, but not including $w = 0$. ■

### Problem # *4:Summing the series*

The Taylor series of functions have more than one region of convergence.

– *4.1: Given some function $f(x)$, if $a = 0.1$, what is the value of*

$$f(a) = 1 + a + a^2 + a^3 + \cdots ?$$

Show your work. **Sol:** To sum this series, we may use the fact that

$$f(a) - af(a) = (1 + a + a^2 + a^3 + \cdots) - a(1 + a + a^2) = 1 + a(1-1) + a^2(1-1) + \cdots$$

This gives $(1-a)f(a) = 1$, or $f(a) = 1/(1-a)$. Now since $a = .1$, the sum is $1/(1-0.1) = 1.11$. ■

– *4.2: Let $a = 10$. What is the value of*

$$f(a) = 1 + a + a^2 + a^3 + \cdots ?$$

**Sol:** In this case the series clearly does not converge. To make it converge we need to write a formula for $y = 1/x$ rather than for $x$.

$$f(1/y) - f(1/y)/a = (1 + 1/a + 1/a^2 + 1/a^3 + \cdots) - 1/a(1 + 1/a + a1/^2) = 1 + (1-1)/a + (1-1)/a^2 + \cdots$$

This gives $f(1/a) = -a^{-1}/(1-a^{-1})$. Now since $a = 10$, the series sums to $f(10) = -0.1/(1-0.1) = -1/9$. ■

### 3.1.3 Cauchy-Riemann Equations

**Problem # 5:** *For this problem $\jmath = \sqrt{-1}$, $s = \sigma + \omega\jmath$, and $F(s) = u(\sigma,\omega) + \jmath v(\sigma,\omega)$. According to the fundamental theorem of complex calculus (FTCC), the integration of a complex analytic function is independent of the path. It follows that the derivative of $F(s)$ is defined as*

$$\frac{dF}{ds} = \frac{d}{ds}\left[u(\sigma,\omega) + \jmath v(\sigma,\omega)\right]. \tag{DE-1.1}$$

If the integral is independent of the path, then the derivative must also be independent of the direction:

$$\frac{dF}{ds} = \frac{\partial F}{\partial \sigma} = \frac{\partial F}{\partial \jmath\omega}. \tag{DE-1.2}$$

The Cauchy-Riemann (CR) conditions

$$\frac{\partial u(\sigma,\omega)}{\partial \sigma} = \frac{\partial v(\sigma,\omega)}{\partial \omega} \quad \text{and} \quad \frac{\partial u(\sigma,\omega)}{\partial \omega} = -\frac{\partial v(\sigma,\omega)}{\partial \sigma}$$

may be used to show where Equation DE-1.2 holds.

*– 5.1: Assuming Equation DE-1.2 is true, use it to derive the CR equations.*

**Sol:** First form the partial derivatives as indicated and then set the real and imaginary parts equal. This results in the two CR equations. ■

*– 5.2: Merge the CR equations to show that $u$ and $v$ obey Laplace's equations*

$$\nabla^2 u(\sigma,\omega) = 0 \quad \text{and} \quad \nabla^2 v(\sigma,\omega) = 0.$$

**Sol:** Take partial derivatives with respect to $\sigma$ and $\omega$ and solve for one equation in each of $u$ and $v$. ■
What can you conclude?
**Sol:** We can conclude that the real and imaginary parts of complex analytic functions must obey these conditions. ■

**Problem # 6:** *Apply the CR equations to the following functions. State for which values of $s = \sigma + i\omega$ the CR conditions do or do not hold (e.g., where the function $F(s)$ is or is not analytic).* Hint: Review where CR-1 and CR-2 hold.

*– 6.1: $F(s) = e^s$*

**Sol:** CR conditions hold everywhere. ■

*– 6.2: $F(s) = 1/s$*

**Sol:** CR conditions are violated at $s = 0$. The function is analytic everywhere except $s = 0$. ■

### 3.1.4 Branch cuts and Riemann sheets

**Problem # 7:** *Consider the function $w^2(z) = z$. This function can also be written as $w_\pm(z) = \sqrt{z_\pm}$. Assume $z = re^{\phi\jmath}$ and $w(z) = \rho e^{\theta\jmath} = \sqrt{r}e^{\phi\jmath/2}$.*

*– 7.1: How many Riemann sheets do you need in the* domain *$(z)$ and the* range *$(w)$ to fully represent this function as single-valued?*
**Sol:** There is one sheet for $z$ and two sheet for $w = \pm\sqrt{z}$. When any point in the domain $z$ (being mapped to $w(z)$) crosses the $z$ branch cut, the codomain (range) $w_\pm(z)$ switches from the $w_+$ sheet to the $w_-$ sheet. $w(z)$ remains analytic on the cut. Look at Fig. 4.4 in Chap. 4 (p. 148) to see how this works. ■

*– 7.2: Indicate (e.g., using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.*
**Sol:** Above we show the mapping for the square root function $w(z) = \sqrt{z_\pm} = \sqrt{r}e^{\jmath\phi/2}$. ■

– *7.3: Use* `zviz.m` *to plot the positive and negative square roots* $+\sqrt{z}$ *and* $-\sqrt{z}$*. Describe what you see.*
**Sol:** The sheet for the positive root is shown in Fig 3.2 (page 106 of the Oct 24 version of the class notes.) Two view the two sheets use Matlab command `zviz sqrt(W) -sqrt(W)`. ■

– *7.4: Where does* `zviz.m` *place the branch cut for this function?*
**Sol:** Typically the cut is placed along the negative real $z$ axis $\phi = \pm\pi$. This is Matlab's/Octave's default location. In the figure above, it has been placed along the positive real axis, $\phi = 0 = 2\pi$. ■

– *7.5: Must the branch cut necessarily be in this location?*
**Sol:** No, it can be moved, at will. It must start from $z = 0$ and end at $|z| \to \infty$. The cut may be move when using `zviz.m` by multiplying $z$ by $e^{\jmath\phi_o}$. For example, `zviz W = sqrt(j*Z)` rotates the cut by $\pi/2$. The colors of $w(z)$ (angle maps to color) always 'jump' at the branch cut, as you make the transition across the cut. ■

**Problem #** *8: Consider the function* $w(z) = \log(z)$*. As in Problem 7, let* $z = re^{\phi\jmath}$ *and* $w(z) = \rho e^{\theta\jmath}$*.*

– *8.1: Describe with a sketch and then discuss the* branch cut *for* $f(z)$*.*
**Sol:** From the plot of `zviz w(z) = log(z)` of Lecture 18, we see a branch cut going from $w = 0$ to $w = -\infty$. If we express $z$ in polar coordinates ($z = re^{\jmath\phi}$), then

$$w(z) = \log(r) + \phi\jmath = u(x, y) + v(x, y)\jmath,$$

where $r(x, y) = |z| = \sqrt{x^2 + y^2}$ and $\phi = \angle z = \phi(x, y)$. Thus a zero in $w(z)$ appears at $z = 1 + 0\jmath$, and only appears on the principle sheet of $z$ (between $[-\pi < \angle z = \phi < \pi]$), because this is the only place where $\phi = 0$. As the angle $\phi$ increases, the imaginary part of $w = \angle z$, which increases without bound. Thus $w$ is like a spiral stair case, or cork-screw. If $\rho = 1$ and $\phi \neq 0$, $w(r) = log(1) + \phi\jmath$ is not zero, since the angle is not zero. ■

– *8.2: What is the inverse of the function* $z(f)$*? Does this function have a branch cut? If so, where is it?*
**Sol:** $z(w) = e^w$ is a single valued function, so a branch cut is not appropriate. Only multi-valued functions require a branch cut. ■

– *8.3: Using* `zviz.m`*, show that*

$$\tan^{-1}(z) = -\frac{\jmath}{2}\log\frac{\jmath - z}{\jmath + z}. \qquad \text{(DE-1.3)}$$

In Fig. 4.1 (p. 134) these two functions are shown to be identical.
**Sol:** Use the Matlab commands `zviz atan(Z)` and `zviz -(j/2)*log((j+Z)./(j-Z))`. ■

– *8.4: Algebraically justify Eq. DE-1.3.* Hint: Let $w(z) = \tan^{-1}(z)$ and $z(w) = \tan w = \sin w / \cos w$; then solve for $e^{w\jmath}$.
**Sol:** Following the hint gives

$$z(w) = -\jmath\frac{e^{w\jmath} - e^{-w\jmath}}{e^{w\jmath} + e^{-w\jmath}} = -\jmath\frac{e^{2w\jmath} - 1}{e^{2w\jmath} + 1}.$$

Solving this bilinear equation for $e^{2w\jmath}$ gives

$$e^{2w\jmath} = \frac{1 + z\jmath}{1 - z\jmath} = \frac{\jmath - z}{\jmath + z}$$

Taking the log and using our definition of $w(z)$ we find

$$w(z) = \tan^{-1}(z) = -\frac{\jmath}{2}\log\frac{\jmath - z}{\jmath + z}.$$

■

### 3.1.5 A Cauer synthesis of any Brune impedance

**Problem #** *9: One may synthesize a transmission line (ladder network) from a positive real impedance* $Z(s)$ *by using the continued fraction method. To obtain the series and shunt impedance values, we can use a residue expansion. Here we shall explore this method.*

*– 9.1: Starting from the Brune impedance $Z(s) = \frac{1}{s+1}$, find the impedance network as a ladder network.*

**Sol:** Taking the reciprocal we find the sum of two shunt admittances, and capacitor and resistor

$$Y(s) = s + 1.$$

The the impedance is $Z(s) = 1/(s+1)$. ■

*– 9.2: Use a residue expansion in place of the CFA floor function (Sec. 2.4.4, p. 30) for polynomial expansions. Find the residue expansion of $H(s) = s^2/(s+1)$ and express it as a ladder network.*

**Sol:** Verify that

$$Z(s) = s^2/(s+1) = s - 1 + 1/(s+1). \tag{DE-1.4}$$

Thus the Cauer synthesis is a series combination $s-1$ (an inductor $L=1$ and a resistor $R=-1$ ohms) and a shunt $1||s$ (i.e., $Y(s) = 1+s$, a resistor of $R=1$ in parallel with a capacitor $C=1$.) It would appear that $Z(s)$ is not a positive real impedance. ■

*– 9.3: Discuss how the series impedance $Z(s,x)$ and shunt admittance $Y(s,x)$ determine the wave velocity $\kappa(s,x)$ and the characteristic impedance $z_o(s,x)$ when (1) $Z(s)$ and $Y(s)$ are both independent of $x$; (2) $Y(s)$ is independent of $x$, $Z(s,x)$ depends on $x$; (3) $Z(s)$ is independent of $x$, $Y(s,x)$ depends on $x$; and (4) both $Y(s,x), Z(s,x)$ depend on $x$.*

**Sol:** In the most general case

$$z_o(s,x) = \sqrt{Z(s,x)/Y(s,x)}$$

and

$$\kappa(s,x) = \sqrt{Z(s,x)Y(s,x)}.$$

The general equations for $z_o(s,s)$ and $\kappa(s,x)$ are given in Mason (1927), and discussed in Appendix D (p. 239). When $z_o$ and $\kappa$ depend on $x$, the area function $A(x)$ of the WHEN will depend on $x$. Thus the eigenfunction will critically depend on the characteristic impedance and the propagation function.

For example, $\kappa(s)$ can be independent of the area because it cancels out in the product. This is called the case of *constant* $k$ because the speed of sound is independent of the area function. It follows that the area function only depends on $z_o(s,x)$.

This shows that a Cauer synthesis may be implemented with the residue expansion replacing the floor function in the CFA. This seems to solve Brune's network synthesis problem. ■