Chapter 3

Differential equations

3.1 Problems DE-1

3.1.1 Topics of this homework:
Complex numbers and functions (ordering and algebra), complex power series, fundamental theorem of calculus (real and complex); Cauchy-Riemann conditions, multivalued functions (branch cuts and Riemann sheets)

3.1.2 Complex Power Series

Problem #1: In each case derive (e.g., using Taylor’s formula) the power series of \( w(s) \) about \( s = 0 \) and give the RoC of your series. If the power series doesn’t exist, state why! Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at \( s = 0 \).

- 1.1: \( 1/(1 - s) \)
  **Sol:** \( 1/(1 - s) = \sum_{n=0}^{\infty} s^n \), which converges for \( |s| < 1 \) (e.g., the RoC is \( |s| < 1 \)).

- 1.2: \( 1/(1 - s^2) \)
  **Sol:** \( 1/(1 - s^2) = \sum_{n=0}^{\infty} s^{2n} \), which converges for \( |s^2| < 1 \). (e.g., the RoC is \( |s| < 1 \)). One can also factor the polynomial, thus write it as: \( \frac{1}{(1-s)(1+s)} \). There are two poles, at \( s = \pm 1 \), and each has an RoC of 1.

- 1.3: \( 1/(1 + s^2) \)
  **Sol:** The resulting series is \( 1/(1 + s^2) = 0.5 \sum_{n=0}^{\infty} s^n ((-i)^n + (i)^n) \). The RoC is \( |s| < 1 \). We can see this by considering the poles of the function at \( s = \pm i \); both poles are 1 from \( s = 0 \), the point of expansion. An alternative is to write the function as \( 1/(1 - (is)^2) = \sum (is)^n \).

- 1.4: \( 1/s \)
  **Sol:** If you try to do a Taylor expansion at \( s = 0 \), the first term, \( w(0) \to \infty \). Thus, the Taylor series expansion in \( s \) does not exist.

- 1.5: \( 1/(1 - |s|^2) \)
  **Sol:** The imaginary part is zero. Thus the derivative of the imaginary part is zero. Thus the CR conditions cannot be obeyed.

**Problem #2:** Consider the function \( w(s) = 1/s \)

- 2.1: Expand this function as a power series about \( s = 1 \). Hint: Let \( 1/s = 1/(1 - 1 + s) = 1/(1 - (1 - s)) \).
  **Sol:** The power series is
  \[
  w(s) = \sum_{n=0}^{\infty} (-1)^n (s - 1)^n,
  \]
which converges for $|s - 1| < 1$.

To convince you this is correct, use the Matlab/Octave command `syms s; taylor(1/s, s, 'ExpansionPoint', 1)`, which is equivalent to the shorthand `syms s; taylor(1/s, s, 1)`. What is missing is the logic behind this expansion, given as follows: First move the pole to $z = -1$ via the Möbius "translation" $s = z + 1$, and expand using the Taylor series

$$\frac{1}{s} = \frac{1}{1 + z} = \sum_{n=0}^{\infty} (-z)^n.$$

Next back-substitute $z = s - 1$ giving

$$\frac{1}{s} = \sum_{n=0}^{\infty} (-1)^n(s - 1)^n.$$

It follows that the RoC is $|z| = |s - 1| < 1$, as provided by Matlab/Octave. ■

– 2.2: What is the RoC?

**Sol:** As stated in the solution of 2.1, $|s - 1| < 1$. ■

– 2.3: Expand $w(s) = 1/s$ as a power series in $s^{-1} = 1/s$ about $s^{-1} = 1$.

**Sol:** Let $z = s^{-1}$ and expand about 1: The solution is $w(z) = z$, which has a zero at 0 thus a pole at $\infty$. ■

– 2.4: What is the RoC?

**Sol:** $|s| > 0$ or $|z| < \infty$. ■

– 2.5: What is the residue of the pole?

**Sol:** The pole is at $\infty$. Since $w(s) = 1/s$ and applying the definition for the residue $c_{-1} = \lim_{s \to \infty} s(1/s) = 1$. Thus residue is 1. Note that it is the amplitude of the pole, which is 1. ■

**Problem #3: Consider the function $w(s) = 1/(2 - s)$**

– 3.1: Expand $w(s)$ as a power series in $s^{-1} = 1/s$. State the RoC as a condition on $|s^{-1}|$. Hint: Multiply top and bottom by $s^{-1}$.

**Sol:** $1/(2 - s) = -s^{-1}/(1 - 2s^{-1}) = -s^{-1} \sum 2^n s^{-n}$. The RoC is $|2/s| < 1$, or $|s| > 2$. ■

– 3.2: Find the inverse function $s(w)$. Where are the poles and zeros of $s(w)$, and where is it analytic?

**Sol:** Solving for $s(w)$ we find $2 - s = 1/w$ and $s = 2 - 1/w = (2w - 1)/w$. This has a pole at 0 and a zero at $w = 1/2$. The RoC is therefore from the expansion point out to, but not including $w = 0$. ■

**Problem #4: Summing the series**

The Taylor series of functions have more than one region of convergence.

– 4.1: Given some function $f(x)$, if $a = 0.1$, what is the value of $\sum f(a) = 1 + a + a^2 + a^3 + \cdots$?

Show your work. **Sol:** To sum this series, we may use the fact that

$$f(a) - af(a) = (1 + a + a^2 + a^3 + \cdots) - a(1 + a + a^2) = 1 + a(1 - 1) + a^2(1 - 1) + \cdots$$

This gives $(1 - a)f(a) = 1$, or $f(a) = 1/(1 - a)$. Now since $a = .1$, the sum is $1/(1 - 0.1) = 1.11$. ■

– 4.2: Let $a = 10$. What is the value of $\sum f(a) = 1 + a + a^2 + a^3 + \cdots$?

**Sol:** In this case the series clearly does not converge. To make it converge we need to write a formula for $y = 1/x$ rather than for $x$.

$$f(1/y) - f(1/y)/a = (1 + 1/a + 1/a^2 + 1/a^3 + \cdots) - 1/a(1 + 1/a + 1/a^2) = 1 + (1 - 1)/a + (1 - 1)/a^2 + \cdots$$

This gives $f(1/a) = -a^{-1}/(1 - a^{-1})$. Now since $a = 10$, the series sums to $f(10) = -0.1/(1 - 0.1) = -1/9$. ■
3.1.3 Cauchy-Riemann Equations

**Problem # 5:** For this problem \( j = \sqrt{-1}, s = \sigma + \omega j, \) and \( F(s) = u(\sigma, \omega) + jv(\sigma, \omega). \) According to the fundamental theorem of complex calculus (FTCC), the integration of a complex analytic function is independent of the path. It follows that the derivative of \( F(s) \) is defined as

\[
\frac{dF}{ds} = \frac{d}{ds} [u(\sigma, \omega) + jv(\sigma, \omega)].
\]  

(DE-1.1)

If the integral is independent of the path, then the derivative must also be independent of the direction:

\[
\frac{dF}{ds} = \frac{\partial F}{\partial \sigma} = \frac{\partial F}{\partial \omega}.
\]  

(DE-1.2)

The Cauchy-Riemann (CR) conditions

\[
\frac{\partial u(\sigma, \omega)}{\partial \sigma} = \frac{\partial v(\sigma, \omega)}{\partial \omega}, \quad \text{and} \quad \frac{\partial u(\sigma, \omega)}{\partial \omega} = -\frac{\partial v(\sigma, \omega)}{\partial \sigma}
\]

may be used to show where Equation DE-1.2 holds.

– 5.1: Assuming Equation DE-1.2 is true, use it to derive the CR equations.

**Sol:** First form the partial derivatives as indicated and then set the real and imaginary parts equal. This results in the two CR equations. ■

– 5.2: Merge the CR equations to show that \( u \) and \( v \) obey Laplace’s equations

\[
\nabla^2 u(\sigma, \omega) = 0 \quad \text{and} \quad \nabla^2 v(\sigma, \omega) = 0.
\]

**Sol:** Take partial derivatives with respect to \( \sigma \) and \( \omega \) and solve for one equation in each of \( u \) and \( v. \) ■

What can you conclude?

**Sol:** We can conclude that the real and imaginary parts of complex analytic functions must obey these conditions. ■

**Problem # 6:** Apply the CR equations to the following functions. State for which values of \( s = \sigma + i\omega \) the CR conditions do or do not hold (e.g., where the function \( F(s) \) is or is not analytic). Hint: Review where CR-1 and CR-2 hold.

– 6.1: \( F(s) = e^s \)

**Sol:** CR conditions hold everywhere. ■

– 6.2: \( F(s) = 1/s \)

**Sol:** CR conditions are violated at \( s = 0. \) The function is analytic everywhere except \( s = 0. \) ■

3.1.4 Branch cuts and Riemann sheets

**Problem # 7:** Consider the function \( w^2(z) = z. \) This function can also be written as \( w_{\pm}(z) = \sqrt{z_{\pm}}. \) Assume \( z = r e^{j\phi} \) and \( w(z) = \rho e^{j\phi/2} = \sqrt{r} e^{j\phi/2}. \)

– 7.1: How many Riemann sheets do you need in the domain \( (z) \) and the range \( (w) \) to fully represent this function as single-valued?

**Sol:** There is one sheet for \( z \) and two sheet for \( w = \pm \sqrt{z}. \) When any point in the domain \( z \) (being mapped to \( w(z) \)) crosses the \( z \) branch cut, the codomain (range) \( w_{\pm}(z) \) switches from the \( w_+ \) sheet to the \( w_- \) sheet. \( w(z) \) remains analytic on the cut. Look at Fig. 4.4 in Chap. 4 (p. 148) to see how this works. ■

– 7.2: Indicate (e.g., using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.

**Sol:** Above we show the mapping for the square root function \( w(z) = \sqrt{z_{\pm}} = \sqrt{r} e^{j\phi/2}. \) ■
7.3: Use zviz.m to plot the positive and negative square roots $+\sqrt{z}$ and $-\sqrt{z}$. Describe what you see.

**Sol:** The sheet for the positive root is shown in Fig 3.2 (page 106 of the Oct 24 version of the class notes.) Two view the two sheets use Matlab command zviz sqrt(W) -sqrt(W).

7.4: Where does zviz.m place the branch cut for this function?

**Sol:** Typically the cut is placed along the negative real z axis $\phi = \pm \pi$. This is Matlab’s/Octave’s default location. In the figure above, it has been placed along the positive real axis, $\phi = 0 = 2\pi$.

7.5: Must the branch cut necessarily be in this location?

**Sol:** No, it can be moved, at will. It must start from $z = 0$ and end at $|z| \to \infty$. The cut may be move when using zviz.m by multiplying $z$ by $e^{j\theta}$. For example, zviz $W = \sqrt{(j+z)}$ rotates the cut by $\pi/2$. The colors of $w(z)$ (angle maps to color) always 'jump' at the branch cut, as you make the transition across the cut.

**Problem # 8:** Consider the function $w(z) = \log(z)$. As in Problem 7, let $z = re^{j\phi}$ and $w(z) = re^{j\theta}$.

8.1: Describe with a sketch and then discuss the branch cut for $f(z)$.

**Sol:** From the plot of zviz $w(z) = \log(z)$ of Lecture 18, we see a branch cut going from $z = 0$ to $w = -\infty$. If we express $z$ in polar coordinates ($z = re^{j\phi}$), then

$$w(z) = \log(r + j\phi) = u(x, y) + v(x, y)j,$$

where $r(x, y) = |z| = \sqrt{x^2 + y^2}$ and $\phi = \angle z = \phi(x, y)$. Thus a zero in $w(z)$ appears at $z = 1 + 0j$, and only appears on the principle sheet of $z$ (between $[-\pi < \angle z = \phi < \pi]$), because this is the only place where $\phi = 0$. As the angle $\phi$ increases, the imaginary part of $w = \angle z$, which increases without bound. Thus $w$ is like a spiral stair case, or cork-screw. If $\rho = 1$ and $\phi \neq 0$, $w(r) = \log(1 + \phi)$ is not zero, since the angle is not zero.

8.2: What is the inverse of the function $z(f)$? Does this function have a branch cut? If so, where is it?

**Sol:** $z(w) = e^w$ is a single valued function, so a branch cut is not appropriate. Only multi-valued functions require a branch cut.

8.3: Using zviz.m, show that

$$\tan^{-1}(z) = -\frac{j}{2} \log \frac{j-z}{j+z}.$$  

(De-1.3)

In Fig. 4.1 (p. 134) these two functions are shown to be identical.

**Sol:** Use the Matlab commands zviz atan(Z) and zviz -(j/2)*log((j+Z)./(j-Z)).

8.4: Algebraically justify Eq. DE-1.3. Hint: Let $w(z) = \tan^{-1}(z)$ and $z(w) = \tan w = \sin w / \cos w$; then solve for $e^{wj}$.

**Sol:** Following the hint gives

$$z(w) = -\frac{e^{wj} - e^{-wj}}{e^{wj} + e^{-wj}} = -\frac{e^{2wj} - 1}{e^{2wj} + 1}.$$ 

Solving this bilinear equation for $e^{2wj}$ gives

$$e^{2wj} = \frac{1 + wz}{1 - wz} = \frac{j - z}{j + z}.$$ 

Taking the log and using our definition of $w(z)$ we find

$$w(z) = \tan^{-1}(z) = -\frac{j}{2} \log \frac{j-z}{j+z}.$$ 

3.1.5 A Cauer synthesis of any Brune impedance

**Problem # 9:** One may synthesize a transmission line (ladder network) from a positive real impedance $Z(s)$ by using the continued fraction method. To obtain the series and shunt impedance values, we can use a residue expansion. Here we shall explore this method.
- **9.1**: Starting from the Brune impedance \( Z(s) = \frac{1}{s+1} \), find the impedance network as a ladder network.

**Sol:** Taking the reciprocal we find the sum of two shunt admittances, and capacitor and resistor

\[ Y(s) = s + 1. \]

The the impedance is \( Z(s) = \frac{1}{s + 1} \).

- **9.2**: Use a residue expansion in place of the CFA floor function (Sec. 2.4.4, p. 30) for polynomial expansions. Find the residue expansion of \( H(s) = \frac{s^2}{s+1} \) and express it as a ladder network.

**Sol:** Verify that

\[ Z(s) = \frac{s^2}{s+1} = s - 1 + \frac{1}{s+1}. \]  

(DE-1.4)

Thus the Cauer synthesis is a series combination \( s - 1 \) (an inductor \( L = 1 \) and a resistor \( R = -1 \) ohms) and a shunt \( 1 || s \) (i.e., \( Y(s) = 1 + s \), a resistor of \( R = 1 \) in parallel with a capacitor \( C = 1 \)). It would appear that \( Z(s) \) is not a positive real impedance.

- **9.3**: Discuss how the series impedance \( Z(s, x) \) and shunt admittance \( Y(s, x) \) determine the wave velocity \( \kappa(s, x) \) and the characteristic impedance \( z_0(s, x) \) when (1) \( Z(s) \) and \( Y(s) \) are both independent of \( x \); (2) \( Y(s) \) is independent of \( x \), \( Z(s, x) \) depends on \( x \); (3) \( Z(s) \) is independent of \( x \), \( Y(s, x) \) depends on \( x \); and (4) both \( Y(s, x), Z(s, x) \) depend on \( x \).

**Sol:** In the most general case

\[ z_0(s, x) = \frac{Z(s, x)}{Y(s, x)} \]

and

\[ \kappa(s, x) = \sqrt{Z(s, x)Y(s, x)}. \]

The general equations for \( z_0(s, s) \) and \( \kappa(s, x) \) are given in Mason (1927), and discussed in Appendix D (p. 239). When \( z_0 \) and \( \kappa \) depend on \( x \), the area function \( A(x) \) of the WHEN will depend on \( x \). Thus the eigenfunction will critically depend on the characteristic impedance and the propagation function.

For example, \( \kappa(s) \) can be independent of the area because it cancels out in the product. This is called the case of constant \( k \) because the speed of sound is independent of the area function. It follows that the area function only depends on \( z_0(s, x) \).

This shows that a Cauer synthesis may be implemented with the residue expansion replacing the floor function in the CFA. This seems to solve Brune’s network synthesis problem.